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Univalent solutions of elliptic systems of Heinz-Lewy type

by

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ABSTRACT. — Under consideration are homeomorphisms $u=(u^1(x^1, x^2), u^2(x^1, x^2))$ with finite Dirichlet integral which solve binary, quasilinear elliptic systems (3) with quadratic growth in the gradient of the solution mapping. Regularity results are derived under minimal assumptions on the coefficients of the system. The non-vanishing of the Jacobian is shown for the Heinz-Lewy system (1) together with an *a priori* estimate from below under suitable normalizations. This involves proving an asymptotic expansion for real-valued functions $\varphi(x)$ satisfying the differential inequality (2).

Key words : Quasilinear elliptic systems, regularity, *a priori* estimates, local behavior, univalent mapings, Jacobian, integrals of Cauchy type, two-dimensional theory.

RÉSUMÉ. — On considère des homéomorphismes $u=(u^1(x^1, x^2), u^2(x^1, x^2))$ dont l'intégrale de Dirichlet est finie et qui résolvent certains systèmes elliptiques quasilinéaires. On démontre alors des résultats de régulants. Dans le cas particulier du système des Heinz-Lewy, on démontre la non-nullité du Jacobien ainsi qu'une estimation *a priori* par le dessous.

Classification A.M.S. : 35 J 50, 35 B 40, 35 B 45, 35 D 10, 30 C 99, 30 E 20.

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INTRODUCTION

In a series of classical works ([8]-[17]), E. Heinz has studied univalent solutions u, v of quasilinear elliptic systems in the x, y-plane,

$$\Delta u = h_1(u,v)\,|Du|^2 + h_2(u,v)\,Du\cdot Dv + h_3(u,v)\,|Dv|^2 + h_4(u,v)\,Du \wedge Dv$$
$$\Delta v = \tilde{h}_1(u,v)\,|Du|^2 + \tilde{h}_2(u,v)\,Du\cdot Dv + \tilde{h}_3(u,v)\,|Dv|^2 + \tilde{h}_4(u,v)\,Du \wedge Dv,$$

where $Du \wedge Dv$ is the Jacobian of u, v. The detailed analysis of this system was initiated by H. Lewy ([20], [21]), who discovered it in connection with Monge-Ampère equations. A notable case is the Darboux system [14], which arises when introducing conjugate isothermal parameters for a surface of positive Gauss curvature. The Euler-Lagrange equations for a harmonic mapping between two-dimensional Riemannian manifolds form another related system ([18], [19]).

In the present paper we shall study a slightly more general binary system in two independent variables. In view of the applications that we have in mind (*see* [22]), we shall consider homeomorphisms $u=(u^1, u^2)$ with finite Dirichlet integral, which solve quasilinear elliptic systems of the form

$$\text{(1)} \qquad L\,u^k = c^{\alpha\beta}\,h^k_{ij}(u)\,D_\alpha u^i\,D_\beta u^j \qquad (k=1,2)$$

in a domain Ω in the $x=(x^1, x^2)$-plane. Here

$$L = -\frac{1}{a(u)}\,D_\alpha(a(u)\,D_\alpha), \qquad \begin{bmatrix} c^{11}, & c^{12} \\ c^{21}, & c^{22} \end{bmatrix} = \begin{bmatrix} c & d \\ -d & c \end{bmatrix}$$

with real constants c, d.

In the first part we derive $C^{1,\mu}$-regularity and a priori estimates for solutions of an even more general system, if $a(u)$ is Hölder continuous with exponent μ, $0<\mu<1$ (Theorem 1). It seems as if this is essentially known, surely of course if $a(u)$ is differentiable. Indeed, Theorem 1 could basically be derived from Theorems 1.3 and 1.5 of Chapter VI of Giaquinta's monograph [5]. However, since there is probably no readable source for Theorem 1, we present a self-contained proof, incorporating some of Heinz's arguments in a nonlinear Campanato technique ([2], [3]) which was filtered out of various chapters of [5].

The second section is of a preparatory character. We study the local behavior of real-valued functions $\varphi(x)$ satisfying the differential inequality

$$\text{(2)} \qquad |D_\alpha(a^{\alpha\beta}(x)\,D_\beta\,\varphi)| \leq C(|\varphi| + |D\varphi|),$$

when the coefficients $a^{\alpha\beta}$ are Hölder continuous and are subject to the normalization $a^{11}(0)=a^{22}(0)$, $a^{12}(0)=a^{21}(0)=0$. The main result, Theorem 2, states that $\varphi(x)=o(|x|^n)$ for some $n\in\mathbb{N}$ implies the existence of

$$\lim_{|z|\to 0}\frac{\varphi_z}{z^n} \qquad (z=x^1+ix^2),$$

which in turn implies an asymptotic expansion for $\varphi(x)$. This generalizes results of Carleman [4] and Hartman-Wintner [7], in particular [7], Theorem 1*, which applies for differentiable coefficients. We rely on the classical ideas for the rather involved proof.

In the third section we show the non-vanishing of the Jacobian of homeomorphic solutions of the Heinz-Lewy system (1) and give an indirect *a priori* estimate from below by appropriately modifying the proofs of Heinz in ([11], [15]). This is accomplished if $a(u)$ is Hölder continuous and if the $h^k_{ij}(u)$'s are Lipschitz continuous. We only have to establish the connection between the system (1) and the differential inequality (2) of the second part, an argument which goes back to H. Lewy.

1. REGULARITY OF UNIVALENT SOLUTIONS

In this section we shall study quasilinear elliptic systems of diagonal form with quadratic growth in the gradient of the solution mapping $u(x)=(u^1(x^1, x^2), u^2(x^1, x^2))$,

$$-D_\alpha(a^{\alpha\beta}(x, u) D_\beta u^k)=f^k(x, u, Du) \qquad (k=1,2). \tag{3}$$

The following assumptions will be imposed:

Assumption (*A*2). — Suppose that u is a homeomorphism onto its image $u(\Omega)$ of class $H^{1,2}(\Omega, \mathbb{R}^2)$. Furthermore let M, N be real numbers such that

$$|u(x)| \leqq M \quad (x\in\Omega), \qquad \int_\Omega |Du|^2\, dx \leqq N.$$

Assumption (*A*2). — (i) The coefficients $a^{\alpha\beta}(x, u)$ of the leading part are Hölder continuous functions on $\Omega\times\mathbb{R}^2$ with exponent μ, $0<\mu<1$, and there are numbers λ, Λ, L, such that

$$\lambda|\xi|^2 \leqq a^{\alpha\beta}(x, u(x))\,\xi_\alpha\xi_\beta \leqq \Lambda|\xi|^2$$

for all $x\in\Omega$, $\xi\in\mathbb{R}^2$, and

$$\frac{|a^{\alpha\beta}(x', u')-a^{\alpha\beta}(x'', u'')|}{(|x'-x''|^2+|u'-u''|^2)^{\mu/2}} \leqq L$$

for $x', x''\in\Omega$, $x'\neq x''$, where $u'=u(x')$, $u''=u(x'')$.

(ii) The lower order term $f(x, u(x), Du(x))$ *is a measurable* $\mathbb{R}^2$- valued function on Ω satisfying, for some constants a, b,

$$|f(x, u(x), Du(x))| \leqq a|Du(x)|^2+b$$

for all $x\in\Omega$.

The principal result of this paragraph can then be stated as

THEOREM 1. — *The first derivatives of* $u(x)$ *are Hölder continuous in the interior of* Ω *with exponent* μ, $u\in C^{1,\mu}_{loc}(\Omega)$. *Moreover, the Hölder norm on any compact subset* Ω' *of* Ω *can be estimated in the form*

$$\|u\|_{C^{1,\mu}(\Omega')}\leqq C,$$

where C *depends only on the parameters* μ, λ, Λ, L, a, b, M, N *and* $\operatorname{dist}(\Omega', \partial\Omega)$.

The proof of this result will be accomplished in four steps. First we establish a modulus of continuity for $u(x)$ in the interior of Ω, essentially applying the Courant-Lebesque lemma [6], Lemma 3.1.

In the following, we shall work only in the interior of Ω, particularly in discs $D_R=D_R(x_0)$ of radius R centered at $x_0\in\Omega$. Radii are always assumed to be less than $\min\{1, \operatorname{dist}(x_0, \partial\Omega)\}$. Constants named C are $\geqq 1$. They may change from line to line and may depend on all available parameters.

LEMMA 1. — *The modulus of continuity of* $u(x)$ *can be estimated in the form*

$$\operatorname*{osc}_{D_R} u\leqq 2\sqrt{\frac{\pi N}{\log 1/R}}.$$

Proof. — By changing to polar coordinates about x_0,

$$\int_R^{\sqrt{R}}\int_0^{2\pi}|u_\theta|^2\,d\theta\frac{d\rho}{\rho}\leqq\int_{D_{\sqrt{R}}\setminus D_R}|Du|^2\,dx\leqq N.$$

Hence there is an R^*, $R\leqq R^*\leqq\sqrt{R}$, such that $u(R^*, .)$ is absolutely continuous on $[0,2\pi]$ and

$$\int_0^{2\pi}|u_\theta(R^*,\theta)|^2\,d\theta\leqq\frac{2N}{\log 1/R}.$$

We conclude that for all θ_1, θ_2 with $0\leqq\theta_1\leqq\theta_2\leqq 2\pi$,

$$|u(R^*,\theta_2)-u(R^*,\theta_1)|\leqq\int_{\theta_1}^{\theta_2}|u_\theta(R^*,\theta)|\,d\theta\leqq 2\sqrt{\frac{\pi N}{\log 1/R}}.$$

The statement of the lemma then follows from the univalency, resp. the homeomorphic character of the mapping u.

LEMMA 2. — *For all* σ, $0<\sigma<1$, *there is a radius* R_0, *which depends only on the available parameters, such that for all* R, $0<R\leqq R_0$,

$$\int_{D_R}|Du|^2\,dx\leqq CR^{2\sigma}.$$

The Dirichlet growth lemma therefore implies

COROLLARY. — $u(x)\in C^{\sigma}_{loc}(\Omega)$ for all σ, $0<\sigma<1$.

Proof of Lemma 2. — We freeze the coefficients $A^{\alpha\beta}(x)=a^{\alpha\beta}(x, u(x))$ by rewriting the system (3) in the form

$$-D_\alpha(A^{\alpha\beta}(x_0)D_\beta u^k)=f^k+D_\alpha g^{\alpha k},\tag{4}$$
$$g^{\alpha k}(x)=(A^{\alpha\beta}(x)-A^{\alpha\beta}(x_0))D_\beta u^k.$$

By testing (4) with $w=u-v$, where $v=(v^1(x), v^2(x))$ is the solution of the Dirichlet problem

$$-D_\alpha(A^{\alpha\beta}(x_0)D_\beta v)=0 \quad \text{in } D_R, \qquad v=u \quad \text{on } \partial D_R,$$

we estimate

$$\int_{D_\rho}|Dw|^2\,dx\leqq C\left\{\int_{D_R}|w|\,|f|\,dx+\int_{D_R}|g-g_R|^2\,dx\right\},\tag{5}$$

where g_R is the mean of g over D_R. Combined with the estimates

$$\int_{D_\rho}|Dv|^2\,dx\leqq C(\lambda,\Lambda)\left(\frac{\rho}{R}\right)^2\int_{D_R}|Dv|^2\,dx,\tag{6}$$

which hold for all ρ, R, $0<\rho\leqq R<\operatorname{dist}(x_0, \partial\Omega)$, (5) in turn yields

$$\int_{D_\rho}|Du|^2\,dx\leqq C\left\{\left(\frac{\rho}{R}\right)^2\int_{D_R}|Du|^2\,dx+\int_{D_R}|w|\,|f|\,dx+\int_{D_R}|g-g_R|^2\,dx\right\}.$$

By the maximum principle,

$$\sup_{D_R}|w|\leqq\operatorname*{osc}_{D_R}u.$$

On the other hand,

$$\operatorname*{osc}_{D_R}A^{\alpha\beta}(x)\leqq C((\operatorname*{osc}_{D_R}u)^\mu+R^\mu).$$

Accordingly

$$\int_{D_\rho}|Du|^2\,dx\leqq C\left\{\left(\left(\frac{\rho}{R}\right)^2+\operatorname*{osc}_{D_R}u+(\operatorname*{osc}_{D_R}u)^{2\mu}+R^{2\mu}\right)\int_{D_R}|Du|^2\,dx+R^2\right\}.$$

By Lemma 1, if R is smaller than some R_0, $R_0<\operatorname{dist}(x_0, \partial\Omega)$, which depends only on λ, Λ, μ, L, a, b, M, then the quantity

$$\operatorname*{osc}_{D_R}u+(\operatorname*{osc}_{D_R}u)^{2\mu}+R^{2\mu}$$

is so small that we can iterate the inequalities

$$\int_{D_\rho}|Du|^2\,dx\leqq C\left\{\left(\left(\frac{\rho}{R}\right)^2+\varepsilon\right)\int_{D_R}|Du|^2\,dx+R^{2\sigma}\right\}$$

for all σ, $0<\sigma<1$, to obtain, by the Calculus Lemma 2.1 of [5],

$$\int_{D_\rho} |Du|^2\,dx \leqq C\left\{\left(\frac{\rho}{R}\right)^{2\sigma}\int_{D_R}|Du|^2\,dx+\rho^{2\sigma}\right\}\leqq \frac{C}{R^{2\sigma}}\rho^{2\sigma}$$

for $0<\rho\leqq R\leqq R_0$ as required.

LEMMA 3. — *For all* σ, $0<\sigma<\min\{\mu, 1/2\}$, *there are radii* $R_0=R_0(\sigma)$, *such that for all* R, $0<R\leqq R_0$,

$$\int_{D_R} |Du-(Du)_R|^2\,dx \leqq CR^{2+2\sigma}.$$

COROLLARY. — $u(x)\in C^{1,\sigma}_{loc}(\Omega)$ *for all* σ, $0<\sigma<\min\{\mu, 1/2\}$.

This follows from Campanato's characterization of the Hölder classes. In particular $Du(x)$ is bounded.

Proof of Lemma 3. — As in the proof of Lemma 2, using the inequalities

$$\int_{D_\rho} |Dv-(Dv)_\rho|^2\,dx \leqq C\left(\frac{\rho}{R}\right)^4\int_{D_R}|Dv-(Dv)_R|^2\,dx$$

instead of (6), we have

$$\text{(7)}\quad \int_{D_\rho} |Du-(Du)_\rho|^2\,dx \leqq C\left\{\left(\frac{\rho}{R}\right)^4\int_{D_R}|Du-(Du)_R|^2\,dx + \int_{D_R}|w|\,|f|\,dx+\int_{D_R}|g-g_R|^2\,dx\right\}.$$

Incorporating Lemma 2 and its corollary, we estimate for all $\sigma<1$,

$$\int_{D_\rho} |Du-(Du)_\rho|^2\,dx \leqq C\left\{\left(\frac{\rho}{R}\right)^4\int_{D_R}|Du-(Du)_R|^2\,dx+R^{(2+2\mu')\sigma}\right\},$$

whenever $R\leqq R_0=R_0(\sigma)$, where $\mu'=\min\{\mu, 1/2\}$. The iteration argument then yields for all σ, $0<\sigma<\mu'$,

$$\int_{D_\rho} |Du-(Du)_\rho|^2\,dx \leqq \frac{C}{R^{2+2\sigma}}\rho^{2+2\sigma}$$

for all $0<\rho\leqq R\leqq R_0$, which proves the lemma.

LEMMA 4. — *There exists a radius* R_0 *such that for all* R, $0<R\leqq R_0$,

$$\int_{D_R} |Du-(Du)_R|^2\,dx \leqq CR^{2+2\mu}.$$

Proof. – Going back to (5), we use one of Poincaré's inequalities to estimate for $\varepsilon > 0$

$$\int_{D_R} |Dw|^2 dx \leqq C\left\{\int_{D_R} |w|\,|f|\,dx + \int_{D_R} |g-g_R|^2 dx\right\}$$
$$\leqq C\left\{\varepsilon\int_{D_R} |Dw|^2 dx + \frac{1}{\varepsilon} R^{2+2\mu}\right\}.$$

The statement is then an immediate consequence of (7).

Theorem 1 follows as a corollary to Lemma 4.

2. LOCAL BEHAVIOR OF SOLUTIONS OF DIFFERENTIAL INEQUALITIES

Let $\varphi(x) \in C^1(\Omega)$ be a real-valued function satisfying the differential inequality

$$|D_\alpha(a^{\alpha\beta}(x) D_\beta \varphi)| \leqq C(|\varphi| + |D\varphi|) \tag{8}$$

in a domain Ω in the $x = (x^1, x^2)$-plane which contains the origin, *i. e.*,

$$-D_\alpha(a^{\alpha\beta} D_\beta \varphi) = W, \qquad |W| \leqq C(|\varphi| + |D\varphi|),$$

in the weak sense. The coefficients $a^{\alpha\beta}$ depend only on x and are Hölder continuous satisfying Assumption (A2) part (i). Without loss of generality, we make the normalization $a^{11}(0) = a^{22}(0)$, $a^{12}(0) = a^{21}(0) = 0$. The crucial result of this section is the following

THEOREM 2. – *If $\varphi(x) = o(|x|^n)$ as $|x| \to 0$ for some $n \in \mathbb{N}$, then*

$$\lim_{\substack{|z| \to 0 \\ z \neq 0}} \frac{\varphi_z}{z^n} = \frac{1}{2} \lim_{\substack{|x| \to 0 \\ x \neq 0}} \frac{\varphi_{x^1} - i\varphi_{x^2}}{(x^1 + ix^2)^n}$$

exists.

In order to prove this result, we first note that only differential inequalities of the form

$$|D_\alpha(a D_\alpha \varphi)| \leqq C(|\varphi| + |D\varphi|), \qquad a \in C^\mu(\Omega), \tag{9}$$

have to be considered. We approximate a by a differentiable function $\tilde{a}$ for $x \neq 0$, and then we modify Hartman-Wintner's proof of the case $a(x) \equiv 1$ [7], Theorem 1.

LEMMA 5. – *Let $a \in C^\mu(\Omega)$ for some μ, $0 < \mu < 1$, which satisfies*

$$0 < \lambda \leqq a(x) \leqq \Lambda < +\infty \qquad (x \in \Omega),$$
$$\frac{|a(x') - a(x'')|}{|x' - x''|^\mu} \leqq L \qquad (x' \neq x'').$$

Then there exists a disc $D = D_{R_0}(0)$, $0 < R_0 = R_0(\mu, \lambda, \Lambda, L) < \operatorname{dist}(0, \partial\Omega)$, *such that for each* ζ, $0 < |\zeta| < R_0$, *there is a function* $\tilde{a} \in C^{\mu}(D) \cap C^1(D \setminus \{0\})$ *such that*

$$\tilde{a}(0) = a(0), \qquad \tilde{a}(\zeta) = a(\zeta),$$
$$\frac{\lambda}{2} \leqq \tilde{a}(x) \leqq 2\Lambda \qquad (x \in D),$$
$$\frac{|\tilde{a}(x') - \tilde{a}(x'')|}{|x' - x''|^{\mu}} \leqq C \qquad (x' \neq x''),$$
$$|D\tilde{a}(x)| \leqq C|x|^{\mu - 1} \qquad (x \neq 0),$$

where C *depends only on* μ, λ, Λ, L *and* $\operatorname{dist}(0, \partial\Omega)$.

Proof. — For $\varepsilon > 0$ set

$$\varphi_{\varepsilon}(x) = \int_{\Omega} \Phi_{\varepsilon}(y - x)\, a(y)\, dy.$$

Here

$$\Phi_{\varepsilon}(x) = \frac{1}{\varepsilon^2} \Phi\left(\frac{x}{\varepsilon}\right),$$

where Φ is a C^{∞}-function such that $\Phi(x) \geqq 0$, $\int \Phi(x)\, dx = 1$, $\Phi(x) \equiv 0$ for $|x| \geqq 1$. Then let

$$a_{\varepsilon}(x) = \varphi_{\varepsilon}(x) + \frac{(a(\zeta) - \varphi_{\varepsilon}(\zeta)) - (a(0) - \varphi_{\varepsilon}(0))}{|\zeta|^{\mu}} |x|^{\mu} + (a(0) - \varphi_{\varepsilon}(0)).$$

If R_0 and ε_0 are sufficiently small, depending only on μ, λ, Λ, L and dist $(0, \partial\Omega)$, then

$$\tilde{a}(x) = a_{\varepsilon_0}(x) \qquad (x \in D_{R_0})$$

satisfies all the stated properties.

Proof of Theorem 2. — By rewriting the coefficients $a^{\alpha\beta}(x)$ as

$$a^{\alpha\beta}(x) = a(x) A^{\alpha\beta}(x), \qquad a^2(x) = \det[a^{\alpha\beta}(x)],$$

we can solve the Cauchy-Riemann-Beltrami system

$$D_1 \xi^1 = A^{2k} D_k \xi^2$$
$$D_2 \xi^1 = -A^{1k} D_k \xi^2,$$

i. e., we can make a conformal change of variables

$$x = (x^1, x^2) \rightarrow \xi = (\xi^1, \xi^2)$$

in a neighborhood of the origin in order to write the differential inequality (8) in the form (9).

We change to complex notation as indicated in the statement and we assume that $\varphi \in C^2$ for the sake of exposition. Let $\tilde{a}$, R_0 and ζ be as in Lemma 5. Then

$$-D_\alpha(\tilde{a} D_\alpha \varphi) = D_\alpha((a-\tilde{a}) D_\alpha \varphi) + W.$$

Now

$$4(\tilde{a}\varphi_z)_{\bar{z}} = D_\alpha(\tilde{a} D_\alpha \varphi) + i(D_1 \varphi D_2 \tilde{a} - D_2 \varphi D_1 \tilde{a}),$$

and therefore

$$\begin{aligned} -4(\tilde{a}\varphi_z)_{\bar{z}} &= D_\alpha((a-\tilde{a}) D_\alpha \varphi) + i(D_1 \tilde{a} D_2 \varphi - D_2 \tilde{a} D_1 \varphi) + W \\ &= \tilde{W}. \end{aligned}$$

Multiplying by

$$h(z) = z^{-k}(z-\zeta)^{-1}, \qquad k \in \{1, \ldots, n\},$$

and integrating over $\Omega_\varepsilon = D_R \setminus (D_\varepsilon \cup D_\varepsilon(\zeta))$, it follows that

$$2i \int_{\partial\Omega_\varepsilon} \tilde{a}\varphi_z h\, dz = \int_{\Omega_\varepsilon} \tilde{W} h\, dx.$$

If

$$\varphi_z = o(|z|^{k-1}) \qquad (|z| \to 0), \tag{10}$$

then we can let $\varepsilon \to 0$ to obtain

$$\begin{aligned} 4\pi a(\zeta)\varphi_z(\zeta)\zeta^{-k} = &-2i \int_{\partial D_R} \tilde{a}\varphi_z h\, dz + \int_{D_R} (\tilde{a}-a) D_\alpha \varphi D_\alpha h\, dx \\ &+ \int_{\partial D_R} (a-\tilde{a}) D_\nu \varphi h\, ds + i \int_{D_R} (D_1 \tilde{a} D_2 \varphi - D_2 \tilde{a} D_1 \varphi) h\, dx + \int_{D_R} W h\, dx. \end{aligned} \tag{11}$$

For fixed $n \geqq 1$, it will be shown by induction over k, $1 \leqq k \leqq n$, that (10) holds. First, (10) is true for $k=1$. Suppose now that (10) holds for a k, $1 \leqq k < n$. By Lemma 5, all integrals are absolutely convergent. In particular, there is a constant C, which depends only on the data, such that

$$\begin{aligned} |D\varphi(\zeta)|\,|\zeta|^{-k} \leqq C \Big\{ &\int_{\partial D_R} |D\varphi|\,|z|^{-k}|z-\zeta|^{-1}\, ds \\ &+ \int_{D_R} |\varphi|\,|z|^{-k}|z-\zeta|^{-1}\, dx \\ &+ \int_{D_R} |D\varphi|(|z|^{-k-1+\mu}|z-\zeta|^{-1} + |z|^{-k}|z-\zeta|^{-2+\mu})\, dx \Big\}. \end{aligned} \tag{12}$$

This inequality is multiplied by $|\zeta - z_0|^{-2+\mu}$ and then integrated over D_R. By virtue of

$$(13)\qquad |(z-\zeta)(\zeta-z_0)|^{-\gamma} \leqq C(\gamma)\,|z-z_0|^{-\gamma}(|z-\zeta|^{-\gamma}+|\zeta-z_0|^{-\gamma})$$

for $\gamma>0$, we estimate after relabelling

$$(1-CR^{\mu})\int_{D_R}|D\varphi|\,|z|^{-k}|z-\zeta|^{-2+\mu}\,dx$$

$$(14)\qquad \leqq CR^{\mu}\left\{\int_{\partial D_R}|D\varphi|\,|z|^{-k}|z-\zeta|^{-1}\,ds + \int_{D_R}|\varphi|\,|z|^{-k}|z-\zeta|^{-1}\,dx + \int_{D_R}|D\varphi|\,|z|^{-k-1+\mu}|z-\zeta|^{-1}\,dx\right\}.$$

We now multiply (12) by $|\zeta|^{-1+\mu}|\zeta-z_0|^{-1}$ and integrate over D_R to obtain similarly, but using (13) twice,

$$(1-CR^{\mu})\int_{D_R}|D\varphi|\,|z|^{-k-1+\mu}|z-\zeta|^{-1}\,dx$$

$$(15)\qquad \leqq CR^{\mu}\left\{\int_{\partial D_R}|D\varphi|\,|z|^{-k}|z-\zeta|^{-1}\,ds + \int_{D_R}|\varphi|\,|z|^{-k}|z-\zeta|^{-1}\,dx\right\}.$$

If R is so small that $CR^{\mu}\leqq 1/2$, then we can combine (14) and (15) to arrive at

$$\int_{D_R}|D\varphi|(|z|^{-k-1+\mu}|z-\zeta|^{-1}+|z|^{-k}|z-\zeta|^{-2+\mu})\,dx$$

$$(16)\qquad \leqq CR^{\mu}\left\{\int_{\partial D_R}|D\varphi|\,|z|^{-k}|z-\zeta|^{-1}\,ds + \int_{D_R}|\varphi|\,|z|^{-k}|z-\zeta|^{-1}\,dx\right\}.$$

The R.H.S. is O(1) as $|\zeta|\to 0$, hence, by (12), $\varphi_z(\zeta)\zeta^{-k}=O(1)$. In fact

$$(17)\qquad \lim_{\substack{|\zeta|\to 0\\ \zeta\neq 0}} \varphi_z(\zeta)\zeta^{-k}$$

exists by virtue of (11) and (12). If $k<n$, this limit is zero which completes the induction. Finally, by arguing as above and making use of (12) for $k=n$, we infer that the limit (17) also exists for $k=n$ as required.

By putting $\zeta=0$ in (16), one can derive the well-known unique continuation principle, namely $\varphi(x)=o(|x|^n)$ $(|x|\to 0)$ for all $n\in\mathbb{N}$ implies $\varphi(x)\equiv 0$. A consequence is the following extension of Hartman-Wintner's

Theorem 2* of [7]:

COROLLARY 1. — *Let the assumptions of Theorem 2 be satisfied. Then either* $\varphi(x) \equiv 0$ *or there exists a non-negative integer m such that*

$$\lim_{\substack{|z| \to 0 \\ z \neq 0}} \frac{\varphi_z(z)}{z^m} \neq 0.$$

We shall need this theorem in the case $n \geqq 1$, where then obviously $m \geqq 1$. The following corollaries are essentially identical with [15], Hilfssatz 1 and [11], Hilfssatz 2:

COROLLARY 2. — *If* $n \geqq 1$ *and* $\varphi(x) \not\equiv 0$, *then* $\varphi(z)$ *has an asymptotic expansion of the form*

$$\varphi(z) = \operatorname{Re}(A z^{m+1}) + o(|z|^{m+1}) \qquad (|z| \to 0), \tag{18}$$

where $A \neq 0$ *and* $m \geqq 1$.

COROLLARY 3. — *Let* $\{\varphi^{(k)}(x)\}_{k=1}^{\infty}$ *be a family of* C^1*-solutions to the differential inequality* (8), *where* C *is independent of* k. *Assume that*

$$\varphi^{(k)}(x) \to \varphi(x), \qquad D\varphi^{(k)}(x) \to D\varphi(x)$$

uniformly in D_R $(k \to \infty)$. *Let* $\varphi(x) = o(|x|)$ *as* $|x| \to 0$ *and assume that* $D\varphi^{(k)}(x) \neq 0$ *in* D_R *for all* $k \in \mathbb{N}$. *Then* $\varphi(x) \equiv 0$.

Proof. — If $\varphi(x) \not\equiv 0$, then by Corollary 1,

$$\lim_{\substack{|z| \to 0 \\ z \neq 0}} \frac{\varphi_z(z)}{z^m} = A \neq 0.$$

Put $\psi(z) = A z^m$. Then there are k_0, R_0 such that

$$|\varphi_z^{(k_0)}(z) - \psi(z)| < |\psi(z)|$$

for $|z| = R_0$. Hence $\varphi_z^{(k_0)}$ must have a zero in D_{R_0} by the homotopy invariance of the winding number.

3. NON-VANISHING OF THE JACOBIAN

We consider solutions $u(x) = (u^1(x^1, x^2), u^2(x^1, x^2))$ of the system

$$L u^k = c^{\alpha\beta} h_{ij}^k(u) D_\alpha u^i D_\beta u^j \qquad (k = 1, 2), \tag{19}$$

where

$$L = -\frac{1}{a(u)} D_\alpha(a(u) D_\alpha), \begin{bmatrix} c^{11} & c^{12} \\ c^{21} & c^{22} \end{bmatrix} = \begin{bmatrix} c & d \\ -d & c \end{bmatrix}$$

with real constants c, d. With regards to the coefficients $a(u)$, $h_{ij}^k(u)$ $(i, j, k=1,2)$, we make the following

Assumption (A3). — (i) $a(u) \in C^\mu(\mathbb{R}^2)$ for some exponent μ, $0<\mu<1$, so that

$$0<\lambda \leqq a(u(x)) \leqq \Lambda < +\infty,$$
$$\frac{|a(u')-a(u'')|}{|u'-u''|^\mu} \leqq L$$

for $u'=u(x') \neq u''=u(x'')$, x, x', $x'' \in \Omega$.

(ii) The coefficients $h_{ij}^k(u)$ are Lipschitz continuous with respect to u, so that

$$|h_{ij}^k(u)| \leqq M_0 \qquad (u \in \mathbb{R}^2),$$
$$\frac{|h_{ij}^k(u')-h_{ij}^k(u'')|}{|u'-u''|} \leqq M_1 \qquad (u' \neq u'').$$

Note that one only needs to consider the matrix

$$[c^{\alpha\beta}]=[\varepsilon^{\alpha\beta}]=\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

The following result generalizes Hilfssatz 3 of [11] and goes back to H. Lewy.

PROPOSITION 1. — *There exists a* disc $D_\delta = D_\delta(0)$ *in the u-plane,* $\delta=\delta(M_0)$, *such that for each* $\xi=(\xi^1, \xi^2) \in \mathbb{R}^2$, $|\xi|=1$, *there is a function* $\Phi \in C^2(D_\delta)$ *with* $\Phi(0)=0$, $D\Phi(0)=\xi$, $\|\Phi\|_{C^2(D_\delta)} \leqq C$ *and* $\varphi(x)=\Phi(u(x))$ *satisfies a differential inequality of the form*

$$|L\varphi| \leqq C(|\varphi|+|D\varphi|)$$

in $|x|<R$, *for any* $u \in C^1(D_R)$ *solving* (19) *with* $u(0)=0$, $|u(x)|<\delta$, $|Du| \leqq K$.

Proof. — Assume that $u \in C^2$ for convenience only and let $g=g(u)$ be a real-valued function. Consider

$$\varphi(x)=\Phi(u(x))=u^1-g(u^2)$$

and calculate

$$L\varphi=\varepsilon^{\alpha\beta}(h_{ij}^1-g' h_{ij}^2) D_\alpha u^i D_\beta u^j + g'' |Du^2|^2$$
$$= b^\alpha D_\alpha \varphi + c |Du^2|^2.$$

Here we used the fact that $Du^1 = D\varphi + g' Du^2$, and we let $b^\alpha(x)$, $\alpha=1, 2$, denote certain bounded expressions in $\varepsilon^{\alpha\beta}$, h_{ij}^k, g', $D\varphi$, Du^2. The term c can easily be computed as

$$c(u^1, u^2, g', g'')=g''+\sum_{i,j=1}^{2}(h_{ij}^1(u)-g' h_{ij}^2(u))(g')^{p_{ij}},$$

where

$$p_{ij}=\begin{cases}2 & (i=j=1)\\ 1 & (i=1, j=2 \text{ or } i=2, j=1).\\ 0 & (i=j=2)\end{cases}$$

Now solve the differential equation

$$c(g, t, g', g'')=0$$

subject to the initial conditions $g(0)=g'(0)=0$ and use the above calculation to estimate

$$\begin{aligned}|\mathrm{L}\varphi| &\leqq \mathrm{C}\Big(|\mathrm{D}\varphi| + \Big|\sum_{i,j=1}^{2}((h_{ij}^1(u^1, u^2)-h_{ij}^1(g(u^2), u^2)\\ &\qquad -g'(h_{ij}^2(u^1, u^2)-h_{ij}^2(g(u^2), u^2)))(g')^{p_{ij}}\Big|\Big)\\ &\leqq \mathrm{C}(|\mathrm{D}\varphi| + |u^1-g(u^2)|)\\ &\leqq \mathrm{C}(|\varphi| + |\mathrm{D}\varphi|).\end{aligned}$$

The statement then follows from the observation that for any orthogonal matrix T, $\tilde{u}(x)=\mathrm{T}u(x)$ solves a system of the form (19), which satisfies Assumption (A3) with suitable constants $\tilde{\mathrm{M}}_0$, $\tilde{\mathrm{M}}_1$.

We have now provided all that's necessary to derive the following

MAIN THEOREM. — *Let u be a homeomorphism from $\bar{\mathrm{D}}$ onto $\bar{\mathrm{D}}$, $\mathrm{D}=\mathrm{D}_1(0)$, of class $\mathrm{H}^{1,2}$ with $u(0)=0$, which solves the Heinz-Lewy system* (19). *Suppose that the assumptions* (A1), (A3) *are satisfied.*

Then $u(x)\in \mathrm{C}_{\mathrm{loc}}^{1,\mu}(\Omega)$ and the Jacobian

$$J(u(x))=\mathrm{D}_1 u^1 \mathrm{D}_2 u^2-\mathrm{D}_1 u^2 \mathrm{D}_2 u^1$$

does not vanish in Ω. The following estimates hold in any compact subset Ω' of Ω:

$$\|u\|_{\mathrm{C}^{1,\mu}(\Omega')}\leqq \mathrm{C}, \qquad \mathrm{J}(u(x))\geqq c>0,$$

where the constants C, c *depend only on the data* μ, λ, Λ, L, M_0, M_1, M, N *and* $\operatorname{dist}(\Omega', \partial\Omega)$.

The non-vanishing of the Jacobian is shown by contradiction as in [15], pp. 88-89. Let us give an outline of the proof for the sake of completeness: Assume that $0\in\Omega$, $u(0)=0$ and $\mathrm{J}(0)=0$. Then there is a solution $\xi=(\xi^1, \xi^2)$, $|\xi|=1$, to

$$\mathrm{D}_1 u^1(0)\xi^1+\mathrm{D}_1 u^2(0)\xi^2=0$$
$$\mathrm{D}_2 u^1(0)\xi^1+\mathrm{D}_2 u^2(0)\xi^2=0.$$

By Proposition 1, there is a mapping $\Phi(u)$ with $\mathrm{D}\Phi(0)=\xi$, such that $\varphi(x)=\Phi(u(x))$ satisfies a differential inequality of the form (9), and hence, by Corollary 2 to Theorem 2, φ has an asymptotic expansion of the form

(18) with $A \neq 0$ and $m \geqq 1$. This is true because $\varphi(x) \not\equiv 0$ by the univalency of the mapping

$$\varphi(x) + i(\xi^1 u^2(x) - \xi^2 u^1(x))$$

in a neighborhood of 0. An argument of Berg's [1], p. 314, yields the asymptotic expansion (18) with $m = 0$, a contradiction.

The estimate for the Jacobian is derived indirectly by a compactness argument as in [11], pp. 142-143, incorporating pp. 138-139, and using Corollary 3 of Theorem 2 of the previous section.

Let us finally remark that the Lipschitz condition on the coefficients h_{ij}^k can be weakened to the effect that only certain combinations of the h_{ij}^k's need to be Lipschitz continuous.

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