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Periodic and heteroclinic orbits for a periodic hamiltonian system

by

Paul H. RABINOWITZ

Center for the Mathematical Sciences
University of Wisconsin,
Madison, Wisconsin 53706, U.S.A.

ABSTRACT. — Consider the Hamiltonian system:

(★) $$\ddot{q}+V'(q)=0$$

where $q=(q_1, \ldots, q_n)$ and V is periodic in q_i, $1 \leqq i \leqq n$. It is known that (★) then possesses at least $n+1$ equilibrium solutions. Here we (*a*) give criteria for V so that (★) has non-constant periodic solutions and (*b*) prove the existence of multiple heteroclinic orbits joining maxima of V.

Key words : Hamiltonian system, periodic solution, heteroclinic solutions.

RÉSUMÉ. — On considère le système hamiltonien

(★) $$\ddot{q}+V'(q)=0$$

où $q=(q_1, \ldots, q_n)$ et V est périodique en q. On sait qu'il existe n points d'équilibre au moins. Nous donnons ici des conditions sur V pour que (★) ait des solutions périodiques non constantes et des trajectoires hétéroclines joignant les maxima de V.

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1. INTRODUCTION

Several recent papers ([1]-[9]) have studied the existence of multiple periodic solutions of second order Hamiltonian systems which are both forced periodically in time and depend periodically on the dependent variables. In particular consider

$$\ddot{q}+V_q(t, q)=f(t) \tag{1.1}$$

where $q=(q_1, \ldots, q_n)$, $V \in C^1(\mathbf{R}\times\mathbf{R}^n, \mathbf{R})$, is τ periodic in t and is also T_i periodic in q_i, $1 \leqq i \leqq n$. The continuous function f is assumed to be τ periodic in t and

$$[f] \equiv \frac{1}{\tau}\int_0^\tau f(s)\,ds=0.$$

It was shown in [1], [2], [5], [9] that under these hypotheses, (1.1) possesses at least $n+1$ "distinct" solutions. Note that whenever $q(t)$ is a periodic solution of (1.1), so is $q(t)+(k_1 T_1, \ldots, k_n T_n)$ for any $k=(k_1, \ldots, k_n) \in Z^n$. This observation leads us to define Q and q to be equivalent solutions of (1.1) if $Q-q=(k_1 T_1, \ldots, k_n T_n)$ with $k \in Z^n$. Thus "distinct" as used above means there are at least $n+1$ distinct equivalence classes of periodic solutions of (1.1).

Suppose now that V is independent of t and $f \equiv 0$ so (1.1) becomes

$$\ddot{q}+V'(q)=0. \tag{HS}$$

Then the above result applies for any $\tau>0$ seemingly giving a large number of periodic solutions of (HS). However due to the periodicity of V in its arguments, V can be considered as a function on T^n. Since the Ljusternik-Schirelmann category of T^n in itself is $n+1$, a standard result gives at least $n+1$ critical points of V on T^n, each of which is an equilibrium solution of (HS). These solutions are τ periodic solutions of (HS). For example, for the simple pendulum $n=1$ and (HS) becomes

$$\ddot{q}+\sin q=0. \tag{1.2}$$

Studying (1.2) in the phase plane shows that if $\tau \leqq 2\pi$, the only periodic solutions are the equilibrium solutions $q \equiv 0$ and $q \equiv \pm\pi$ (modulo 2π). Moreover for $\tau > 2\pi$, there are k nonequilibrum solutions where k is the largest integer such that $\frac{\tau}{k} > 2\pi$. (There is exactly one solution having minimal period τ/j, $1 \leqq j \leqq k$.) The phase plane analysis also shows that (1.2) possesses a pair of heteroclinic orbits joining $-\pi$ and π.

Our goal in this note is twofold. First in section 2, criteria will be given on V so that (HS) possesses nontrivial τ periodic solutions, the results just mentioned for (1.2) appearing as special cases. Our main results are in section 3 where the existence of heteroclinic orbits of (HS) is established. The arguments used in section 2-3 are variational in nature. The multiplicity results of section 2 depend on a theorem of Clark [10] and those of section 3 involve a minimization argument.

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2. MULTIPLE SOLUTIONS OF (HS)

This section deals with the existence of multiple periodic solutions of (HS). Assume V satisfies

$$(V_1) \qquad V \in C^1(\mathbf{R}, \mathbf{R}^n)$$

and

$$(V_2) \qquad \text{V is periodic in } q_i \text{ with period } T_i,\ 1 \leqq i \leqq n.$$

As was noted in the Introduction, (V_1)-(V_2) imply that V has at least $n+1$ distinct critical points and these provide $n+1$ equilibrium solutions of (HS). By rescaling time, (HS) is replaced by

$$\ddot{q} + \lambda^2 V'(q) = 0 \tag{2.1}$$

and we study the number of 2π periodic solutions of (2.1) as a function of $\lambda = \tau/2\pi$.

Assume further that

$$(V_3) \qquad V(q) = V(-q) \quad \text{for} \quad q \in \mathbf{R}^n$$

as in the one dimensional example (1.2). Suppose (V_1)-(V_3) hold and q is a solution of (2.1) such that $q'(0) = 0$ and $q\left(\frac{\pi}{2}\right) = 0$. If q is extended beyond $\left[0, \frac{\pi}{2}\right]$ as an even function about 0 and an odd function about $\frac{\pi}{2}$, the resulting function is a 2π periodic solution of (2.1). Moreover the only constant function of this form is $q \equiv 0$. To exploit these observations to obtain 2π periodic solutions of (2.1), let E denote the set of functions on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ which are even about 0, vanish at $\pm\frac{\pi}{2}$, and possess square integrable first derivatives. As norm in E, we take

$$\|q\|^2 = \int_{-\pi/2}^{\pi/2} |\dot{q}(t)|^2 \, dt. \tag{2.2}$$

Set

$$(2.3)\qquad I(q)=\int_{-\pi/2}^{\pi/2}\left[\frac{1}{2}|\dot{q}(t)|^2-\lambda^2 V(q(t))\right]dt.$$

Since I is even, critical points of I occur in antipodal pairs $(-q, q)$. It is easily verified that (V_1)-(V_3) imply $I\in C^1(E, \mathbf{R})$ and critical points of I in E are classical solutions of (2.1) with $q'(0)=0$ and $q\left(\frac{\pi}{2}\right)=0$. See e. g. [10]. Hence by above remarks q extends to a 2π periodic solution of (2.1). Thus we are interested in the number of critical points of I in E.

Since (HS) or (2.1) only determine V up to an additive constant, by (V_1)-(V_2), it can be assumed that the minimum of V is 0 and occurs at 0. Therefore $V\geqq 0$, $I(0)=0$, and 0 is a critical value of I with 0 as a corresponding critical point. Thus lower bounds for the number of critical points of I having negative critical values (as a function of λ) provides estimates on the number of nontrivial periodic solutions of (HS). Suppose that

(V_4) V is twice continuously differentiable at 0 and $V''(0)$ is nonsingular.

Then $V''(0)$ is positive definite and Clark's Theorem [10] can be used to estimate the number of critical points of I.

To be more precise, let $a_1, \ldots, a_n$ be an orthogonal set of eigenvectors of $V''(0)$ with corresponding eigenvalues α_j, $1\leqq j\leqq n$. Note that the function $(\cos kt)\,a_j$, $k\in\mathbf{N}$ and odd, $1\leqq j\leqq n$ form an orthogonal basis for E. If $q\in E$,

$$q=\sum b_{kj}(\cos kt)\,a_j$$

and

$$(2.4)\qquad \int_{-\pi/2}^{\pi/2}\left[\frac{1}{2}|\dot{q}|^2-\frac{\lambda^2}{\alpha}V''(0)\,q.q\right]dt=\frac{\pi}{4}\sum(k^2-\lambda^2\alpha_j)\,|b_{kj}|^2\,|a_j|^2.$$

Let $\mu_{kj}(\lambda)=k^2-\lambda^2\alpha_j$. For λ sufficiently small, $\mu_{kj}(\lambda)>0$ for all k, j, but as λ increases, the number of negative μ_{kj} increases. For each λ, let $l(\lambda)$ denote the number of negative μ_{jk}.

THEOREM 2.5. — *Suppose* V *satisfies* (V_1)-(V_4). *Then* (2.1) *possess at least* $l(\lambda)$ *distinct pairs of nontrivial* 2π *periodic solutions.*

Proof: It was already observed above that $I\in C^1(E, \mathbf{R})$ and it is easy to see that I satisfies the Palais-Smale condition (PS) on E, (*see* e. g. [10]). Let E_l denote the span of the set of functions $(\cos kt)\,a_j$ such that $\mu_{kj}<0$.

Then E_l is l dimensional and for $q \in E_l$ with $\|q\| = \rho$, by (2.4) for small ρ:

$$(2.6) \qquad I(q) = \frac{\pi}{4} \sum (k^2 - \lambda^2 \alpha_j) |b_{kj} a_j|^2 + o(\rho^2) \\ \leqq -\delta_l \rho^2 + o(\rho^2)$$

where $\delta_l > 0$ (*see* e. g. [10] for a similar computation). Therefore for $\rho = \rho(\lambda)$ sufficiently small, $I(q) < 0$ for $q \in E_l$ and $\|q\| = \rho$. A result of Clark ([10], Theorem 9.1) states:

PROPOSITION 2.7. — Let E be a real Banach space and $I \in C^1(E, \mathbf{R})$ with $I(0) = 0$, I even, bounded from below, and satisfy (PS). If there is a set $K \subset E$ which is homeomorphic to S^{l-1} by an odd map and $\sup_K I < 0$, then I possesses at least l distinct pairs of critical points with corresponding negative critical values.

Since I is bounded from below via (V_2) and K can be taken to be a sphere of radius ρ in E_l, it is clear from earlier remarks that Proposition 2.7 is applicable here and Theorem 2.5 is proved.

3. HETEROCLINIC ORBITS

In this section, the existence of connecting orbits for (HS) will be studied. Assume again that (V_1)-(V_2) hold. They imply that V has a global maximum, $\bar{V}$, on $\mathbf{R}^n$. Let

$$\mathscr{M} = \{ \xi \in \mathbf{R}^n \mid V(\xi) = \bar{V} \}.$$

To begin further assume that

(V_5) $\qquad \mathscr{M}$ consists only of isolated points.

Hypothesis (V_5) implies that $\mathscr{M}$ contains only finitely many points in bounded subsets of $\mathbf{R}_n$. Note also that (V_5) holds if $V \in C^2(\mathbf{R}, \mathbf{R}^n)$ and $V''(\xi)$ is nonsingular whenever $\xi \in \mathscr{M}$. This is the case e. g. for (1.2) where $\mathscr{M} = \{ \pi + 2j\pi \mid j \in \mathbf{Z} \}$.

If $q \in C(\mathbf{R}, \mathbf{R}^n)$ and

$$\lim_{t \to \infty} q(t) \quad \text{exists,}$$

we denote this limit by $q(\infty)$. A similar meaning is attached to $q(-\infty)$. Our main goal in this section is to prove that (V_1), (V_2), (V_5) imply that for each $\beta \in \mathscr{M}$, there are at least 2 heteroclinic orbits of (HS) joining β to $\mathscr{M} \setminus \{ \beta \}$, at least one of which emanates from β and at least one of which terminates at β. We will also establish a stronger result for a generic setting.

The existence proof involves a series of steps. Consider the functional

$$(3.1)\qquad I(q)=\int_{-\infty}^{\infty}\left[\frac{1}{2}|\dot{q}(t)|^2-V(q(t))\right]dt.$$

Formally critical points of I are solutions of (HS). We will find critical points by minimizing I over an appropriate class of sets and showing that there are enough minimizing functions with the properties we seek. Hypotheses (V_1), (V_2), and (V_5) will always be assumed for the results below.

To begin, it can be assumed without loss of generality that $0\in\mathcal{M}$, $\beta=0$, and $V(0)=0$. Therefore $-V(x)\geqq 0$ for all $x\in\mathbf{R}^n$ and $-V(x)>0$ if $x\notin\mathcal{M}$. Set

$$E\equiv\left\{q\in W^{1,2}_{\text{loc}}(\mathbf{R},\mathbf{R}^n)\,\middle|\,\int_{-\infty}^{\infty}|\dot{q}(t)|^2\,dt<\infty\right\}.$$

Taking

$$(3.2)\qquad \|q\|^2\equiv\int_{-\infty}^{\infty}|\dot{q}(t)|^2\,dt+|q(0)|^2$$

as a norm in E makes E a Hilbert space. Note that $q\in E$ implies $q\in C(\mathbf{R},\mathbf{R}^n)$. For $\xi\in\mathcal{M}\setminus\{0\}$ and $\varepsilon>0$, define $\Gamma_\varepsilon(\xi)$ to be the set of $q\in E$ satisfying

$$(3.3)\qquad \begin{aligned}&\text{(i) } q(-\infty)=0\\ &\text{(ii) } q(\infty)=\xi\\ &\text{(iii) } q(t)\notin B_\varepsilon(\mathcal{M}\setminus\{0,\xi\})\quad\text{for all } t\in\mathbf{R}.\end{aligned}$$

Here for $A\subset\mathbf{R}^n$,

$$B_\varepsilon(A)=\{x\in\mathbf{R}^n\mid\ |x-A|<\varepsilon\},$$

i.e. $B_\varepsilon(A)$ is an open ε-neighborhood of A. We henceforth assume

$$(3.4)\qquad \varepsilon<\frac{1}{3}\min_{\xi\in\mathcal{M}\setminus\{0\}}|\xi|\equiv\gamma.$$

Then it is easy to see that $\Gamma_\varepsilon(\xi)$ is nonempty for all $\xi\,\varepsilon\,\mathcal{M}$. E. g. if $q(t)\equiv 0$, $t\leqq 0$, q is piecewise linear for $t\in[0,1]$, $q(t)\notin B_\varepsilon(\mathcal{M}\setminus\{0,\xi\})$, and $q(t)\equiv\xi$ for $t\geqq 1$, then $q(t)\in\Gamma_\varepsilon(\xi)$. Finally define

$$(3.5)\qquad c_\varepsilon(\xi)\equiv\inf_{q\in\Gamma_\varepsilon(\xi)} I(q).$$

It will be shown that for ε sufficiently small, there is some $\xi\in\mathcal{M}\setminus\{0\}$ such that $c_\varepsilon(\xi)$ is a critical value of I and the infimum is achieved for some $q\in\Gamma_\varepsilon(\xi)$ which is a desired heteroclinic orbit.

Let

$$\alpha_\varepsilon \equiv \min_{x \notin B_\varepsilon(\mathcal{M})} -V(x).$$

Then $\alpha_\varepsilon > 0$. The following lemma gives a useful estimate which will be applied repeatedly later.

LEMMA 3.6. — *Let $w \in E$. Then for any $r < s \in \mathbf{R}$ such that $w(t) \notin B_\varepsilon(\mathcal{M})$ for $t \in [r, s]$,*

$$\text{(3.7)} \qquad I(w) \geqq \sqrt{2\alpha_\varepsilon}\,|w(r) - w(s)|.$$

Proof. — Let $l \equiv |w(r) - w(s)|$ *and* $\tau \equiv |r - s|$. *then*

$$\text{(3.8)} \qquad l = \left| \int_r^s \dot{w}(t)\,dt \right| \leqq \int_r^s |\dot{w}(t)|\,dt \leqq \tau^{1/2} \left(\int_r^s |\dot{w}(t)|^2\,dt \right)^{1/2}.$$

Moreover since $V \leqq 0$ and $w(t) \notin B_\varepsilon(\mathcal{M})$ in $[r, s]$,

$$\text{(3.9)} \qquad I(w) \geqq \frac{l^2}{2\tau} - \int_r^s V(q(t)\,dt \geqq \frac{l^2}{2\tau} + \alpha_\varepsilon \tau \equiv \varphi(\tau).$$

The minimum of φ occurs for $\tau = \left(\dfrac{l^2}{2\alpha_\varepsilon} \right)^{1/2}$ so (3.9) yields (3.7).

Remark 3.10. — (i) (3.8) shows that l in (3.7) can be replaced by the length of the curve $w(t)$ in $[r, s]$. (ii) The above argument implies (3.7) holds with l replaced by a finite sum of lengths of intervals if $w(t) \notin B_\varepsilon(\mathcal{M})$ for t lying in these intervals. (iii) If $w \in E$ and $I(w) < \infty$, (ii) shows that $w \in L^\infty(\mathbf{R}, \mathbf{R}^n)$. In fact more is true as the next result shows:

PROPOSITION 3.11. — *If $w \in E$ and $I(w) < \infty$, there exist $\xi, \eta \in \mathcal{M}$ such that $\xi = w(-\infty)$ and $\eta = w(\infty)$.*

Proof. — Since $w \in L^\infty(\mathbf{R}, \mathbf{R}^n)$ by Remark 3.10 (iii), $A(w)$, the set of accumulation points of $w(t)$ as $t \to -\infty$, is nonempty. Suppose that there exists a $\delta > 0$ such that $w(t) \notin B_\delta(\mathcal{M})$ for all t near $-\infty$. Then

$$I(w) \geqq \int_{-\infty}^{\rho} -V(w(t))\,dt$$

for any $\rho \in \mathbf{R}$ shows $I(w) = \infty$ contrary to hypothesis. Hence $A(w)$ contain some $\xi \in \mathcal{M}$. We claim $\xi = w(-\infty)$. If not, there is a $\delta > 0$, a sequence $t_i \to -\infty$ as $i \to \infty$ with $w(t_i) \in B_{\delta/2}(\xi)$. Thus the curve $w(t)$ must intersect $\partial B_{\delta/2}(\xi)$ and $\partial B_\delta(\xi)$ infinitely often as $t \to -\infty$. Remark 3.10 (ii) then implies $I(w) \geqq \sqrt{2\alpha_{\delta/2}}\,\dfrac{\delta}{2} j$ for any $j \in \mathbf{N}$ contrary to $I(w) < \infty$.

The next step towards proving our existence result is the following:

PROPOSITION 3.12. – *For each* $\varepsilon \in (0, \gamma)$ *and* $\xi \in \mathcal{M} \setminus \{0\}$, *there exists* $q \equiv q_{\varepsilon, \xi} \in \Gamma_\varepsilon(\xi)$ *such that* $I(q_{\varepsilon, \xi}) = c_\varepsilon(\xi)$, i. e. $q_{\varepsilon, \xi}$ *minimizes* $I|_{\Gamma_\varepsilon(\xi)}$.

Proof. – Let (q_m) be a minimizing sequence for (3.5). By the form of I, the norm in E, and Remark 3.10 (iii), (q_m) is a bounded sequence in E. Therefore passing to a subsequence if necessary, there is a $q \in E$ such that q_m converges to q in E (weakly) and in L^∞_{loc}.

We claim

$$I(q) < \infty. \tag{3.13}$$

Indeed let $-\infty < \sigma < s < \infty$. For $w \in E$, set

$$\Phi(\sigma, s, w) = \int_\sigma^s \left[\frac{1}{2} |\dot{w}(t)|^2 - V(w(t)) \right] dt. \tag{3.14}$$

Then the first term on the right hand side of (3.14) is weakly lower semicontinuous on E and the second term is weakly continuous on E. Therefore $\Phi(\sigma, s, .)$ is weakly lower semicontinuous on E. Since (q_m) is a minimizing sequence for I, there is a $K > 0$ depending on ε and ξ but independent of t and s such that

$$K \geqq I(q_m) \geqq \Phi(\sigma, s, q_m). \tag{3.15}$$

Therefore

$$K \geqq \inf_{w \in \Gamma_\varepsilon(\xi)} I(w) = \varliminf_{m \to \infty} I(q_m) \geqq \varliminf_{m \to \infty} \Phi(\sigma, s, q_m) \geqq \Phi(\sigma, s, q). \tag{3.16}$$

Since $q \in E$ and σ, s are arbitrary, (3.16) implies $V(q) \in L^1(\mathbf{R}, \mathbf{R}^n)$, (3.13) holds, and

$$I(q) \leqq \inf_{w \in \Gamma_\varepsilon(\xi)} I(w).$$

Thus once we know $q \in \Gamma_\varepsilon(\xi)$, it follows that q minimizes $I|_{\Gamma_\varepsilon(\xi)}$.

Next we claim $q(-\infty) = 0$ and $q(\infty) = \xi$. Since $I(q) < \infty$, by Proposition 3.11, there are $\eta, \zeta \in \mathcal{M}$ such that $q(-\infty) = \eta$ and $q(\infty) = \zeta$. Since $q_m(t) \notin B_\varepsilon(\mathcal{M} \setminus \{0, \xi\})$ for all $t \in \mathbf{R}$ and $q_m \to q$ in L^∞_{loc}, $q(t) \notin B_\varepsilon(\mathcal{M} \setminus \{0, \xi\})$ for all $t \in \mathbf{R}$. Therefore $\eta, \zeta \in \{0, \xi\}$. For each $m \in \mathbf{N}$, since $q_m \in \Gamma_\varepsilon(\xi)$, there is a $t_m^- \in \mathbf{R}$ such that $q_m(t_m^-) \in \partial B_\varepsilon(0)$ and $q_m(t) \in B_\varepsilon(0)$ for $t < t_m^-$. Now if $w \in E$, so is $w_\theta(t) \equiv w(t - \theta)$ for each $\theta \in \mathbf{R}$ and $I(w_\theta) = I(w)$. Therefore it can be assumed that $t_m^- = 0$ for all $m \in \mathbf{N}$. Consequently $q_m(t) \in B_\varepsilon(0)$ for all $t < 0$. Therefore $q(t) \in \bar{B}_\varepsilon(0)$ for all $t < 0$ and $\eta \in \{0, \xi\} \cap \bar{B}_\varepsilon(0) = \{0\}$, *i. e.* $\eta = 0$.

Next to see that $q(\infty) = \xi$, note that $q(\infty) = 0$ or ξ. Suppose that $q(\infty) = 0$. We will show that this is impossible. Choose $\delta > 0$ so that

$4\delta < \varepsilon$ and

$$2\delta^2 + \max_{|x| \leqq 2\delta} (-V(x)) < \frac{\varepsilon}{4}\sqrt{2\alpha_{\varepsilon/2}}. \tag{3.17}$$

Since the left hand side of (3.17) goes to 0 as $\delta \to 0$, such a δ certainly exists. If $q(\infty) = 0$, there is a $t_\delta > 0$ such that $q(t) \in B_\delta(0)$ for all $t > t_\delta$. Since $q_m(t) \to q(t)$ uniformly for $t \in [0, t_\delta]$, for m sufficiently large, $q_m(t_\delta) \in B_{2\delta}(0)$. Recalling that $q_m(0) \in \partial B_\varepsilon(0)$, by Lemma 3.6,

$$I(q_m) \geqq \sqrt{2\alpha_{\varepsilon/2}}.\varepsilon/2 + \int_{t_\delta}^{\infty} \left[\frac{1}{2}|\dot{q}_m|^2 - V(q_m)\right] dt. \tag{3.17'}$$

Define

$$\begin{aligned} Q_m(t) &= 0, \quad t \leqq t_\delta - 1 \\ &= (t - (t_\delta - 1))\, q_m(t_\delta), \quad t \in [t_\delta - 1, t_\delta] \\ &= q_m(t), \quad t > t_\delta. \end{aligned}$$

Then $Q_m \in \Gamma_\varepsilon(\xi)$ and by (3.17')

$$\begin{aligned} I(Q_m) &= \int_{t_\delta - 1}^{t_\delta} \left[\frac{1}{2}|q_m(t_\delta)|^2 - V(Q_m(t))\right] dt \\ &\quad + \int_{t_\delta}^{\infty} \left[\frac{1}{2}|\dot{q}_m|^2 - V(q_m)\right] dt \\ &\leqq \frac{1}{2}(2\delta)^2 + \max_{|t| \leqq 2\delta} -V(t) + I(q_m) - \sqrt{2\alpha_{\varepsilon/2}}.\frac{\varepsilon}{2} \\ &< I(q_m) - \frac{\varepsilon}{4}\sqrt{2\alpha_{\varepsilon/2}}. \end{aligned}$$

But this implies

$$\begin{aligned} \inf_{w \in \Gamma_\varepsilon(\xi)} I(w) &\leqq \varliminf_{m \to \infty} I(Q_m) \leqq \varliminf_{m \to \infty} I(q_m) - \frac{\varepsilon}{4}\sqrt{\alpha_{\varepsilon/2}} \\ &= \inf_{w \in \Gamma_\varepsilon(\xi)} I(w) - \frac{\alpha}{4}\sqrt{2\alpha_{\varepsilon/2}}. \end{aligned}$$

which is impossible.

Let $\mathscr{T} = \mathscr{T}(\varepsilon, \xi) \equiv \{\sigma \in \mathbf{R} \mid q_{\varepsilon\xi}(\sigma) \in \partial B_\varepsilon(\mathscr{M} \setminus \{0, \xi\})\}$.

PROPOSITION 3.18. — *$q_{\varepsilon, \xi}$ is a classical solution of* (HS) *on* $\mathbf{R} \setminus \mathscr{T}$.

Proof. — Let $\sigma \in \mathbf{R} \setminus \mathscr{T}$. Then σ lies in a maximal open interval $\mathcal{O} \subset \mathbf{R} \setminus \mathscr{T}$. Let $\varphi \in C_0^\infty(\mathbf{R}, \mathbf{R}^n)$ such that the support of φ lies in $\mathcal{O}$. Then for δ sufficiently small, $q + \delta\varphi \in \Gamma_\varepsilon(\xi)$ (with $q \equiv q_{\varepsilon, \xi}$). Since q minimizes I on $\Gamma_\varepsilon(\xi)$, it readily follows that

$$I'(q)\varphi \equiv \int_{-\infty}^{\infty} [\dot{q}.\dot{\varphi} - V'(q).\varphi]\, dt = 0 \tag{3.19}$$

for all such φ. Fixing $r, s \in \mathcal{O}$ with $r<s$ and noting that (3.19) holds for all $\varphi \in W_0^{1,2}([r, s], \mathbf{R}^n)$, we see that q is a weak solution of the equation

$$(3.20) \qquad \begin{cases} \ddot{w}+V'(q)=0, & r<t<s \\ w(r)=q(r), & w(s)=q(s). \end{cases}$$

Consider the inhomogeneous linear system:

$$(3.21) \qquad \begin{cases} \ddot{u}+V'(q)=0, & r<t<s \\ u(r)=q(r), & u(s)=q(s). \end{cases}$$

This system possesses a unique C^2 solution which can be written down explicitly. Therefore from (3.21),

$$(3.22) \qquad \int_r^s [\dot{u}.\dot{\varphi}-V'(q).\varphi]\,dt=0$$

for all $\varphi \in W_0^{1,2}([r, s], \mathbf{R}^n)$. Comparing (3.19) and (3.22) yields

$$(3.23) \qquad \int_r^s (\dot{q}-\dot{u}).\dot{\varphi}\,dt=0$$

for all $\varphi \in W_0^{1,2}([r, s], \mathbf{R}^n)$ and since $q-u$ belongs to this space, it follows that $q \equiv u$ on $[r, s]$. In particular $q \in C^2([r, s], \mathbf{R}^n)$. Since r and s are arbitrary in $\mathcal{O}$, $q \in C^2(\mathbf{R} \setminus \mathcal{T}, \mathbf{R}^n)$ and satisfies (HS) there. Thus the Proposition is proved.

COROLLARY 3.24. — $\dot{q}_{\varepsilon,\xi}(t) \to 0$ as $t \to \pm\infty$.

Proof. — By Proposition 3.18, $q=q_{\varepsilon,\xi}$ is a solution of (HS) for $|t|$ large. Since (HS) is a Hamiltonian system

$$(3.25) \qquad H(t) \equiv \frac{1}{2}|\dot{q}(t)|^2+V(q(t)) \equiv \text{Const.}$$

for large t, e. g. $H(t) \equiv \rho$ for $t \geqq \bar{t}$. Now

$$(3.26) \qquad \begin{aligned} I(q) &\geqq \int_{\bar{t}}^{\infty} \left[\frac{1}{2}|\dot{q}(t)|^2-V(q(t))\right] dr \\ &= \int_{\bar{t}}^{\infty} [H(t)-2\,V(q(t))]\,dt \end{aligned}$$

and $V(q(.)) \in L^1$, so it follows that $\rho=0$. Since $q(t) \to \xi$ and $V(q(t)) \to 0$ as $t \to \infty$, (3.25) shows $\dot{q}(t) \to 0$ as $t \to \infty$ and similarly as $t \to -\infty$.

The above results show that functions $q_{\varepsilon,\xi}$ are candidates for heteroclinic orbits of (HS) emanating from 0. It remains to show that for appropriate choices of ε and ξ there actually are such orbits of (HS). That there is at least one follows next.

Let

$$c_\varepsilon \equiv \inf_{\xi \in \mathcal{M} \setminus \{0\}} c_\varepsilon(\xi).$$

By (3.7), only finitely many $c_\varepsilon(\xi)$ are candidates for the infimum and hence it is achieved by say $c_\varepsilon(\zeta) = \mathrm{I}(q_{\varepsilon,\,\zeta})$ where $\zeta = \zeta(\varepsilon)$. Choosing a sequence $\varepsilon_j \to 0$, by (3.7) again, it can be assumed that $\zeta(\varepsilon_j)$ is independent of j so $\zeta(\varepsilon_j) \equiv \zeta$.

PROPOSITION 3.27. — *For j sufficiently large, $q_{\varepsilon_j,\,\zeta}$ is a heteroclinic orbit of* (HS) *joining* 0 *and* ζ.

Proof. — Let $q_j \equiv q_{\varepsilon_j,\,\zeta}$. By the definition of $\Gamma_\varepsilon(\zeta)$, Proposition 3.18, and Corollary 3.24, it suffices to show that for large j, $q_j(t) \notin \partial \mathrm{B}_{\varepsilon_j}(\mathcal{M} \setminus \{0, \zeta\})$ *for all* $t \in \mathbf{R}$. If not, there is a sequence of $j's \to \infty$, $\eta_j \in \mathcal{M} \setminus \{0, \zeta\}$, and $t_j \in \mathbf{R}$ such that $q_j(t) \in \partial \mathrm{B}_{\varepsilon_j}(\eta_j)$ and $q_j(t) \notin \partial \mathrm{B}_{\varepsilon_j}(\eta_j)$ for $t < t_j$. By (3.7) again, the set of possible $n_j's$ is finite so passing to a subsequence if necessary, $\eta_j \equiv \eta$. Two possibilities now arise.

Case i. — There is a subsequence of $j's \to \infty$ such that $q_j(t) \notin \overline{\mathrm{B}_{\varepsilon_j}(\xi)}$ for $t < t_j$,

and

Case ii. — For every $j \in \mathbf{N}$, there is a $\tau_j < t_j$ such that $q_j(\tau_j) \in \partial \mathrm{B}_{\varepsilon_j}(\xi)$.

If Case i occurs, along the corresponding sequence of $j's$, define a family of new functions:

$$\begin{aligned} \mathrm{Q}_j(t) &= q_j(t), \qquad t \leqq t_j \\ &= (t - t_j)\,\eta + (1 - (t - t_j))\, q_{\varepsilon_j}(t_j), \qquad t \in [t_j,\, t_j + 1] \\ &= \eta, \qquad t > t_j + 1. \end{aligned}$$

Then $\mathrm{Q}_j \in \Gamma_{\varepsilon_j}(\eta)$ and

$$(3.28)\quad \mathrm{I}(q_j) - \mathrm{I}(\mathrm{Q}_j) = \int_{t_j}^{\infty} \left[\frac{1}{2} |\dot{q}_j(t)|^2 - \mathrm{V}(q_j(t)) \right] dt - \int_{t_j}^{t_j+1} \left[\frac{1}{2} |\dot{\mathrm{Q}}_j(t)|^2 - \mathrm{V}(\mathrm{Q}_j(t)) \right] dt.$$

Since the curves q_j intersect $\partial \mathrm{B}_{\varepsilon_1}(\eta)$ and $\partial \mathrm{B}_{\varepsilon_1}(\xi)$ in the interval $[t_j, \infty)$, by (3.7) and (3.28).

$$(3.29)\quad \mathrm{I}(q_j) - \mathrm{I}(\mathrm{Q}_j) \geqq \sqrt{2\,\alpha_{\varepsilon_1}}\,\gamma - \frac{1}{2} |\eta - q_j(t_j)|_+^2 \int_0^1 \mathrm{V}(\mathrm{Q}_j(t - t_j))\, dt.$$

As $j \to \infty$, the second and third terms on the right hand side of (3.29) $\to 0$. Hence for large j, $c_{\varepsilon_j} = \mathrm{I}(q_j) > \mathrm{I}(Q_j)$, a contradiction. Case ii can be eliminated by a similar but simpler argument.

Combining the above propositions, we have

THEOREM 3.30. — *If* V *satisfies* (V_1), (V_2), *and* (V_5), *for each* $\beta \in \mathcal{M}$, (HS) *has at least two heteroclinic orbits connecting* β *to* $\mathcal{M} \setminus \{\beta\}$, *one of which originates at* β *and one of which terminates at* β.

Proof. — We need only prove the last assertion. But it sollows immediately on observing that if $q(t)$ joins β to ξ, $q(-t)$ is a solution joining ξ to β. Alternatively, and this would be useful for time dependent versions of (HS) which are not time reversible, observe that the arguments given above work equally well for curves w in E for which $w(\infty)=0$ and $w(-\infty) \in \mathcal{M} \setminus \{0\}$.

Remark 3.31. — A. Weinstein has informed us of the following conjecture which has been attributed to Lyapunov [11] by Kozlov ([12]-[13]): consider a system of Lagrange's equations in the form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0 \tag{3.32}$$

where the Lagrangian has the form $K(q, \dot{q}) - V(q)$ with K positive definite quadratic in $\dot{q}$. Then any isolated equilibrium solution of (3.32) for which V does not have a local minimum is unstable. Some special cases are proved in [12]-[13] and the references cited there. Theorem 3.30 establishes the result for $K = \frac{1}{2}|\dot{q}|^2$ when the equilibrium is a strict local maximum for V, e.g. at $q=0$ since V can be redefined outside of a neighborhood of 0 so as to satisfy (V_1), (V_2) and (V_5). Thus Theorem 3.30 gives an orbit of (HS) emanating from 0 and which leaves a neighborhood of 0. The proof of Theorem 3.30 also is valid for a more general class of kinetic energy terms $K = K(q, \dot{q})$ satisfying $(V_1)-(V_2)$ and possessing appropriate definiteness properties. Thus the conjecture can also be obtained for a more general situation.

Next the multiplicity of heteroclinic orbits emanating from each $\beta \in \mathcal{M}$ will be studied in the simplest possible setting. Suppose V satisfies

(V_5') $\qquad \mathcal{M}/T^n$ is a singleton.

By (V_5'), we mean that $\mathcal{M}$ consists only of the translates as given by (V_2) of a single point which without loss of generality we can take to be 0.

Next let $\mathcal{B}$ denote the set of $\xi \in \mathcal{M} \setminus \{0\}$ such that for some $\varepsilon \in (0, \gamma)$, $c_\varepsilon(\xi)$ corresponds to a connecting orbit of (HS) joining 0 and ξ. $\mathcal{B}$ is nonempty by Theorem 3.30. Let Λ denote the set of finite linear combinations over $\mathbf{Z}$ of elements of $\mathcal{B}$. Then Λ is a lattice in $\mathbf{R}^n$.

PROPOSITION 3.33. — $\Lambda = \mathcal{M}$.

Proof. — If not, $\mathscr{S} \equiv \mathscr{M} \setminus \Lambda \neq \varnothing$. For each $\varepsilon \in (0, \gamma)$, choose $\xi_\varepsilon \in \mathscr{S}$ such that

$$c_\varepsilon(\xi_\varepsilon) = \inf_{\zeta \in \mathscr{S}} c_\varepsilon(\zeta). \tag{3.34}$$

By Proposition 3.12 and (3.6), this infinimum is achieved and there is such a ξ_ε and corresponding $q_\varepsilon \equiv q_{\varepsilon, \xi_\varepsilon} \in \Gamma_\varepsilon(\xi_\varepsilon)$ such that $I(q_\varepsilon) = c_\varepsilon(\xi_\varepsilon)$. We claim that as in Proposition 3.27, for ε sufficiently small,

$$q_\varepsilon(t) \notin \partial B_\varepsilon(\mathscr{M} \setminus \{0, \xi_\varepsilon\}) \quad \text{for all } t \in \mathbf{R} \tag{3.35}$$

and therefore by Proposition 3.18 and Corollary 3.24, q_ε is a connecting orbit of (HS) joining 0 and ξ. Hence $\xi_\varepsilon \in \mathscr{B}$ and *a fortiori* Λ, a contradiction. Thus $\Lambda = \mathscr{M}$.

To verify (3.35), suppose to the contrary that there exists $\eta_\varepsilon \in \mathscr{M} \setminus \{0, \xi_\varepsilon\}$ and $t_\varepsilon \in \mathbf{R}$ such that $q_\varepsilon(t_\varepsilon) \in \partial B_\varepsilon(\eta_\varepsilon)$. Either (*a*) $\eta_\varepsilon \in \mathscr{S}$ or (*b*) $\xi_\varepsilon - \eta_\varepsilon \in \mathscr{S}$ for if both belong to Λ, so does their sum, ξ_ε, contrary to the choice of ξ_ε. Within case (*a*), as in Proposition 3.27, two further possibilities arise:

(i) $$q_\varepsilon(t) \notin \overline{B_\varepsilon(\xi)} \text{ for } t < t_\varepsilon$$

or

(ii) there is $a\, \tau_\varepsilon < t_\varepsilon$ such that $q_\varepsilon(\tau_\varepsilon) \in \partial B_\varepsilon(\xi)$.

In case (*a*) (i) occurs, define

$$\begin{aligned} Q(t) &= q_\varepsilon(t), \qquad t \leqq t_\varepsilon \\ &= (t - t_\varepsilon)\eta_\varepsilon + (1 - (t - t_\varepsilon))\, q_\varepsilon(t_\varepsilon), \qquad t \in (t_\varepsilon, t_\varepsilon + 1) \\ &= \eta_\varepsilon, \qquad t \geqq t_\varepsilon + 1. \end{aligned}$$

Then $Q \in \Gamma_\varepsilon(\eta_\varepsilon)$ and

$$I(Q) - I(q_\varepsilon) = \int_{t_\varepsilon}^{t_\varepsilon + 1} \left[\frac{1}{2}|\eta_\varepsilon - q_\varepsilon(t_\varepsilon)|^2 - V(Q)\right] dt - \int_{t_\varepsilon}^{\infty} \left[\frac{1}{2}|\dot{q}_\varepsilon|^2 - V(q_\varepsilon)\right] dt. \tag{3.36}$$

The first term on the right hand side of (3.36) approaches 0 as $\varepsilon \to 0$ while, as in Proposition 3.27, the second exceeds a (fixed) multiple of γ in magnitude uniformly for small ε. Hence for ε small, $I(Q) < I(q_\varepsilon)$ and consequently $c_\varepsilon(\eta_\varepsilon) < c_\varepsilon(\xi_\varepsilon)$ contrary to the choice of ξ_ε. Thus (*a*) (i) is not possible. If case (*a*) (ii) occurs a simple comparison argument shows that for ε small, q_ε does not minimize I on $\Gamma_\varepsilon(\xi_\varepsilon)$, a contradiction.

Next suppose case (*b*) occurs. Two further possibilities must be considered here:

(iii) $q_\varepsilon(t) \notin \overline{B_\varepsilon(0)}$ for $t \geqq t_\varepsilon$

(iv) there is a $\sigma_\varepsilon > t_\varepsilon$ such that $q_\varepsilon(\sigma_\varepsilon) \in \partial B_\varepsilon(0)$. For case (*b*) (iii), define

$$\begin{aligned} Q(t) &= 0, \qquad t \leqq t_\varepsilon - 1 \\ &= (t - t_\varepsilon - 1)(q_\varepsilon(t_\varepsilon) - \eta_\varepsilon), \qquad t \in (t_\varepsilon - 1, t_\varepsilon) \\ &= q_\varepsilon(t) - \eta_\varepsilon, \qquad t \geqq t_\varepsilon. \end{aligned}$$

Then $Q \in \Gamma_\varepsilon(\xi_\varepsilon - \eta_\varepsilon)$ and

$$(3.37) \quad I(Q) - I(q_\varepsilon) = \int_{t_\varepsilon - 1}^{t_\varepsilon} \left[\frac{1}{2} | q_\varepsilon(t_\varepsilon) - \eta_\varepsilon |^2 - V(Q) \right] dt - \int_{-\infty}^{t_\varepsilon} \left[\frac{1}{2} | \dot{q}_\varepsilon |^2 - V(q_\varepsilon - \eta_\varepsilon) \right] dt$$

via (V_2). As in (3.36), for ε small, the right hand side of (3.37) is negative so $c_\varepsilon(\xi_\varepsilon - \eta_\varepsilon) < c_\varepsilon(\xi_\varepsilon)$, contrary to the choice of ξ_ε. Lastly a simple comparison argument shows that if (*b*) (iv) occured, q_ε would not minimize I on $\Gamma_\varepsilon(\xi_\varepsilon)$. The proof is complete.

Finally observing that if $\Lambda = \mathscr{M}$, there must be at least *n* distinct heteroclinic orbits of (HS) emanating from 0, we have

THEOREM 3.38. — *If* V *satisfies* (V_1), (V_2) *and* (V_5'), *for any* $\beta \in \mathscr{M}$, (HS) *has at* $4n$ *heteroclinic orbits joining* β *to* $\mathscr{M} \setminus \{\beta\}$, $2n$ *of which originate at* β *and* $2n$ *of which terminate at* β.

Proof. — Without loss of generality, we can take $\beta = 0$. Proposition 3.30 yields *n* heteroclinic orbits of (HS) corresponding to linearly independent members of Λ which join 0 to $\mathscr{M} \setminus \{0\}$. If $q(t)$ is one of these which joins 0 to ξ, then $q(-t) - \xi$ joins 0 to $-\xi$. The proof of Theorem 3.33 gives *n* additional orbits terminating at 0.

Remark 3.39. — If (V_5') is replaced by (V_5), Theorem 3.35 is probably no longer true although we suspect that some points in $\mathscr{M}$ are the origin of multiple heteroclinic orbits.

Remark 3.40. — A variant of Proposition 3.33 which is more iterative in nature can be given as follows: Let $\mathscr{B}_1$ denote the set of those $\xi \in \mathscr{M} \setminus \{0\}$ such that $c_\varepsilon(\xi) = c_\varepsilon$ for some $\varepsilon \in (0, \gamma)$. Let Λ_1 denote the span of $\mathscr{B}_1$ over Z. The arguments of Proposition 3.33 show either $\Lambda_1 = \mathscr{M}$ or for ε sufficiently small

$$\inf_{\xi \in \mathscr{M} \setminus \Lambda_1} c_\varepsilon(\xi)$$

corresponds to a heteroclinic orbit of (HS) with terminal point in $\mathscr{M} \setminus \Lambda_1$. Supplement $\mathscr{B}_1$ by these new orbits calling the result $\mathscr{B}_2$ and set Λ_2 equal to the span of $\mathscr{B}_2$ over Z. Continuing this process yields at least *n* heteroclinic orbits emanating from 0 in at most *n* steps.

Remark 3.41. — An interesting open question for (HS) when (V_1), (V_2), hold is whether there exist heteroclinic orbits joining non-maxima of V. Equation (1.2) shows there won't be any joining minima of V in general.

Remark 3.42. — An examination of the proof of Theorem 3.30 shows that hypothesis (V_2) plays no major role other than to ensure that $\mathcal{M}$ contains at least two points and there is no problem in dealing with $\mathcal{M}$ near infinity in $\mathbf{R}^n$. Thus the above arguments immediately yield:

THEOREM 3.43. — *If* V *satisfies* (V_1), (V_5),

(V_6) $\qquad$ *$\mathcal{M}$ contains at least two points,*

and

(V_7)
$$\overline{\lim_{|q|\to\infty}}\ V(q)<\bar{V},$$

then each $\beta\in\mathcal{M}$ *contains at least two heteroclinic orbits joining* β *to* $\mathcal{M}\setminus\{\beta\}$, *one originating at* β *and one terminating at* β.

Remark 3.44. — It is also possible to allow V to approach $\bar{V}$ as $|q|\to\infty$ but then some assumptions must be made about the rate of approach.

For our final result, (HS) is considered under a weaker version of (V_5). Certainly some form of (V_5) is needed. E. g. if $V'\equiv 0$, $q(t)\equiv\zeta$ is a solution of (HS) for all $\zeta\in\mathbf{R}^n$ and there exist no connecting orbits. Moreover if $\mathcal{M}$ possesses an accumulation point, ζ, which is the limit of isolated points in $\mathcal{M}$, the methods used above do not give a heteroclinic orbit emanating from ζ since $c_\varepsilon\to 0$ as $\varepsilon\to 0$. Of course there may still be connecting orbits that can be obtained by other means.

The earlier theory does carry over to the following setting:

Theorem 3.45. — *Suppose* V *satisfies* (V_1), (V_2), *and*

(V_8) $\qquad$ *β is an isolated point in $\mathcal{M}$ and $\mathcal{M}\setminus\{\beta\}\neq\varnothing$.*

Then there exists a solution w *of* (HS) *such that* $w(-\infty)=\beta$ *and* $w(t)\to\mathcal{M}\setminus\{\beta\}$ *as* $t\to\infty$.

Proof. — We will sketch the proof. Again without loss of generality $\beta=0$ and $V(0)=0$. Set

$$\Lambda=\{q\in E\,|\,q(-\infty)=0 \text{ and } q(t)\to\mathcal{M}\setminus\{0\} \text{ as } t\to\infty\}.$$

Define

$$c\equiv\inf_{q\in\Lambda} I(q). \tag{3.46}$$

We claim c is a critical value of I and any corresponding critical point, q, is a solution of (HS) of the desired type. The first step in the proof is to show that if $w\in E$ and $I(w)<\infty$, then $w(t)\to\mathcal{M}$ as $t\to\pm\infty$. This is done by the argument of Proposition 3.11. Next let (q_m) be a minimizing sequence for (3.46). It converges weakly in E to q. A slightly modified version of the argument of Proposition 3.12 shows $I(q) < \infty$, $q \in \Lambda$, and q minimizes I over Λ. Finally the arguments of Proposition 3.18 and Corollary 3.24 imply that q is a C^2 solution of (HS) emanating from β and approaching $\mathcal{M} \setminus \{\beta\}$ as $t \to \infty$.

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