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Area-minimizing integral currents with movable boundary parts of prescribed mass

by

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ABSTRACT. — We generalize the thread problem for minimal surfaces to higher dimensions using the framework of integral currents.

Key words : Integral currents, minimizing area, minimal surface, free boundary, mass.

RÉSUMÉ. — On généralise le « problème fil » pour surfaces minimales aux dimensions plus hautes en utilisant le cadre de courants intégrals.

0. INTRODUCTION

The classical *thread problem* for minimal surfaces in $\mathbb{R}^3$ can be formulated as follows: For a given rectifiable Jordan arc Γ and a movable arc Σ of fixed length attached to the endpoints of Γ one wants to find a surface $\mathcal{M}$ of least area among all surfaces spanning this configuration.

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For a detailed description of the problem and a list of relevant literature on related soap-film experiments we refer the reader to the recent paper by Dierkes, Hildebrandt and Lewy [DHL].

One can easily construct examples where the *thread* Σ "crosses" the *wire* Γ (for planar "S"-shaped Γ) or "sticks" to it in a subarc of positive length (if for instance Γ has the shape of a long "U"). In other words, the solution surface $\mathcal{M}$ may consist of several disconnected components and there may be parts of Σ and Γ which do not belong to $\partial\mathcal{M}$. In fact this represents the main difficulty for the existence proof, at least in the parametric approach of [AHW], [N1]-[N3] and [DHL].

Nitsche ([N1]-[N3]) proved that the nonselfintersecting components of $\Sigma \sim \Gamma$ are actually smooth arcs of constant curvature. Dierkes, Hildebrandt and Lewy [DHL] established the real analyticity of these arcs.

Alt [AHW] was able to prove that the parts of Σ which attach to regular parts of Γ in subarcs of positive length have to do this tangentially. Moreover he could show, if a solution surface consists of several disconnected components, all regular parts of $\Sigma \sim \Gamma$ necessarily have the *same* curvature.

The present work is concerned with a more general approach to the *thread problem* which, due to its generality in handling the existence problem, does not enable one to determine *a priori* the topological type of the solution surfaces as was done by Alt [AHW] in his existence proof.

For a start we would like to allow Γ to be disconnected. Γ may for instance consist of several oriented arcs or even closed curves. A suitable generalization of the classical problem would then be to seek a surface $\mathcal{M}$ of minimal area among all oriented surfaces $\mathcal{S}$ such that $\partial\mathcal{S}-\Gamma$ is prescribed, where in subtracting Γ form $\partial\mathcal{S}$ we take orientations into account. If Γ consists of several *wire* arcs we do not prescribe the way in which our *threads* have to be connected to the endpoints of Γ. Also, rather than prescribing the length of each single piece of *thread*, we only keep the total length of $\Sigma=\partial\mathcal{M}-\Gamma$ fixed. As there is no obvious way of excluding the possibility of Σ having higher multiplicity we may as well allow Γ to have arbitrary integer multiplicity.

In section 1 we give a precise formulation of the problem for arbitrary dimension and codimension using the framework of integral currents. We then solve the existence problem (Theorem 1.4).

Section 2 is concerned with properties of the *thread* related to the above mentioned results ([AHW], [DHL], [N1]-[N3]). We generalize the Lagrange multiplier techniques used in [DHL] to obtain control of the first variation of Σ (Theorem 2.3 and Corollary 2.5). In fact we show that Σ has bounded generalized mean curvature away from its boundary $\partial\Sigma$. This implies in particular that Σ only coincides with parts of Γ which have bounded generalized mean curvature. Moreover this establishes a weak tangential property of Σ at points on Γ.

Proposition 2.7 states that all free regular parts of Σ are of class C^∞ and have the same constant mean curvature and that, in constrast to the higher multiplicity Plateau problem (*cf.* [WB]), a *thread* with higher integer multiplicity cannot locally bound several distinct sheets of minimal surfaces unless the *thread* itself has zero mean curvature. By "free parts" of Σ we not only mean $\Sigma \sim \Gamma$ but also those sections of Σ supported in Γ where the multiplicity of $\partial\mathcal{M}$ is not smaller than the multiplicity of Γ. A simple example where a "free" Σ is supported in Γ is obtained by letting $\mathcal{M}$ be an oriented annulus with multiplicity two, and Σ be the inner circle counted with multiplicity one.

If however locally near a point of Σ

$$\partial\mathcal{M} = c\,\Gamma$$

for some $c \in [0, 1)$, the mean curvature of Σ need no longer be constant. Nevertheless it cannot exceed the mean curvature of the free parts of Σ.

As Theorem 2.3 holds without any major conditions imposed on Γ one can show that also the decomposable components of any local decomposition of Σ have bounded generalized mean curvature. This leads to some partial regularity results for the two dimensional *thread problem:* Theorem 3.1 states that one dimensional stationary *threads* consist of straightline segments which do not intersect, thus suggesting a natural condition for the existence of a Lagrange multiplier as in Theorem 2.3.

In Theorem 3.3 we show that the *thread* Σ consists of $C^{1,1}$-arcs which do not cross each other. If several pieces of *thread* have a point in common they must have the same tangent at this point. It is tempting to conjecture that one dimensional *threads* are completely regular.

Finally we derive a monotonicity formula for the two dimensional problem, from which the existence of area-minimizing tangent cones immediately follows.

We would like to thank Prof. S. Hildebrandt for directing our attention to this problem.

1. THE VARIATIONAL PROBLEM

For detailed information on geometric measure theory the reader is referred to [FH] and [SL]. We shall follow the notation used in [SL].

Let U be an open subset of $\mathbb{R}^{n+k}$. We denote the class of n-dimensional integral currents in U by

$$I_{n,\,\mathrm{loc}}(U) = \{S \in \mathcal{D}_n(U)/S,\ \partial S \text{ integer multiplicity}\}$$

and

$$I_n(U) = \{S \in I_{n,\,\mathrm{loc}}(U)/\mathbf{M}(S) + \mathbf{M}(\partial S) < \infty\}.$$

1.1. Definition

$T \in I_{n,\,\mathrm{loc}}(U)$ *is called a minimizer of the thread problem with respect to* $\Gamma \in I_{n-1,\,\mathrm{loc}}(U)$ *if*

$$\mathbf{M}_W(T) \leqq \mathbf{M}_W(S)$$

whenever $W \subset\subset U$ *is open and* $S \in I_{n,\,\mathrm{loc}}(U)$ *satisfies*

$$\operatorname{spt}(S-T) \subset W$$

as well as

$$\mathbf{M}_W(\partial S - \Gamma) = \mathbf{M}_W(\partial T - \Gamma).$$

1.2. Remark

(1) We shall sometimes refer to $\Sigma = \partial T - \Gamma$ as the *free* or *thread-boundary part* and to Γ as the *fixed* or *wire-boundary part* of T although neither $\operatorname{spt}\Sigma$ nor $\operatorname{spt}\Gamma$ has to be totally contained in $\operatorname{spt}\partial T$; in fact we may have

$$\mu_\Sigma(\operatorname{spt}\Gamma \sim \operatorname{spt}\partial T) > 0.$$

(2) A minimizer T of the thread problem obviously minimizes mass also in the usual sense, that is among all comparison surfaces which agree with T along its boundary ∂T.

1.3. Proposition

A minimizer in the sense of 1.1 *still satisfies*

$$\mathbf{M}_W(T) \leqq \mathbf{M}_W(S)$$

even if we only assume that the inequality

$$\mathbf{M}_W(\partial S - \Gamma) \leqq \mathbf{M}_W(\partial T - \Gamma)$$

holds for surfaces $S \in I_{n,\,\mathrm{loc}}(U)$ *satisfying* $\operatorname{spt}(S-T) \subset W$.

Proof. — Suppose there exists an $R \in I_{n,\,\mathrm{loc}}(U)$ which satisfies $\operatorname{spt}(R-T) \subset W$,

$$\mathbf{M}_W(\partial R - \Gamma) < \mathbf{M}_W(\partial T - \Gamma)$$

and

$$\mathbf{M}_W(R) < \mathbf{M}_W(T).$$

Obviously we can always find an integral current $Q \in I_n(W)$ such that $\operatorname{spt}Q \cap (\operatorname{spt}R \cup \operatorname{spt}\Gamma) = \varnothing$, $\operatorname{spt}Q \subset\subset W$,

$$\mathbf{M}_W(Q) < \mathbf{M}_W(T) - \mathbf{M}_W(R)$$

and

$$\mathbf{M_W}(\partial Q) = \mathbf{M_W}(\partial T - \Gamma) - \mathbf{M_W}(\partial R - \Gamma).$$

R + Q then furnishes an admissible comparison surface in the sense of 1.1 with the property

$$\mathbf{M_W}(R + Q) < \mathbf{M_W}(T)$$

thus contradicting the minimality of T. ■

We are now going to establish the existence of a nontrivial minimizer.

Let $\Gamma \in I_{n-1}(\mathbb{R}^{n+k})$ have compact support. Define

$$d_\Gamma = \inf\{\mathbf{M}(Q)/Q \in I_{n-1}(\mathbb{R}^{n+k}) \text{ s. t. } \partial Q = \partial\Gamma\}$$

and suppose $\mathbf{M}(\Gamma) > d_\Gamma$.

1.4. Theorem

Let $d_\Gamma \leqq L < \mathbf{M}(\Gamma)$. Then there exists a nontrivial compactly supported surface $T \in I_n(\mathbb{R}^{n+k})$ which minimizes mass among all surfaces $S \in I_n(\mathbb{R}^{n+k})$ with the property $\mathbf{M}(\partial S - \Gamma) = L$.

1.5. Remark

Every minimizer of 1.4 also minimizes mass in the sense of Definition 1.1.

Proof of 1.4. We set

$$A(\Gamma, L) = \{S \in I_n(\mathbb{R}^{n+k})/\mathbf{M}(\partial S - \Gamma) \leqq L\}.$$

Obviously $L < \mathbf{M}(\Gamma)$ implies $0 \notin A(\Gamma, L)$. Since $\mathbf{M}(\Gamma) > d_\Gamma$ there exists a compactly supported $Q \in I_{n-1}(\mathbb{R}^{n+k})$ which is different from Γ and satisfies $\partial Q = \partial\Gamma$ as well as $\mathbf{M}(Q) = d_\Gamma$. (Use [SL], 34.1 for instance.) The integral cone $R = 0 \mathbin{⋕} (\Gamma - Q)$ then satisfies $\mathbf{M}(\partial R - \Gamma) = \mathbf{M}(Q) = d_\Gamma$. From $d_\Gamma \leqq L$ we conclude that $A(\Gamma, L)$ is nonempty.

We now proceed in a similar way as in [SL, 34.1]. Let $(T_j) \subset A(\Gamma, L)$, $j \geqq 1$, be a minimizing sequence, that is

$$\lim_{j \to \infty} \mathbf{M}(T_j) = \inf\{\mathbf{M}(S)/S \in A(\Gamma, L)\}.$$

Since Γ has compact support we may assume that $\operatorname{spt}\Gamma \subset \mathbf{B_R}(0)$ for some $R > 0$, where $\mathbf{B_R}(0)$ denotes an open ball in $\mathbb{R}^{n+k}$. Let $f: \mathbb{R}^{n+k} \to \overline{\mathbf{B_R}(0)}$ be the nearest point retraction form $\mathbb{R}^{n+k}$ onto $\overline{\mathbf{B_R}(0)}$. It follows from the fact that $\operatorname{Lip} f = 1$ and $f = \mathrm{id}$ in $\overline{\mathbf{B_R}(0)}$ that

$$\mathbf{M}(f_\# T_j) \leqq \mathbf{M}(T_j)$$
$$\mathbf{M}(\partial f_\# T_j - \Gamma) = \mathbf{M}(f_\#(\partial T_j - \Gamma)) \leqq \mathbf{M}(\partial T_j - \Gamma) \leqq L$$

and

$$\operatorname{spt} f_{\#} T_j \subset \overline{B_R(0)}.$$

Hence we may assume without loss of generality that

$$\operatorname{spt} T_j \subset \overline{B_R(0)}, \qquad j \geqq 1.$$

The assumption $\mathbf{M}(\Gamma) < \infty$ combined with $\mathbf{M}(\partial T_j - \Gamma) \leqq L$ $(j \geqq 1)$ yields

$$\sup_{j \geqq 1} (\mathbf{M}(T_j) + \mathbf{M}(\partial T_j)) < \infty.$$

By the compactness theorem for integral currents ([SL, 27.3]) we can select a subsequence [again denoted by (T_j)] which converges in $\mathscr{D}_n(\mathbb{R}^{n+k})$ to an integral current $T \in I_n(\mathbb{R}^{n+k})$ which satisfies

$$\operatorname{spt} T \subset \overline{B_R(0)}.$$

The lower-semicontinuity of the mass implies

$$\mathbf{M}(T) \leqq \varliminf_{j \to \infty} \mathbf{M}(T_j)$$

and

$$\mathbf{M}(\partial T - \Gamma) \leqq \varliminf_{j \to \infty} \mathbf{M}(\partial T_j - \Gamma) \leqq L$$

so that in fact

$$\mathbf{M}(T) = \inf \{\mathbf{M}(S)/S \in A(\Gamma, L)\}.$$

It remains to show that $\mathbf{M}(\partial T - \Gamma) = L$. In order to establish this (*cf.* [AHW; 3.4]) we first recall that for every $x_0 \in \operatorname{spt} T \sim \operatorname{spt} \partial T$ we have

$$\mathbf{M}(T \llcorner B_\rho(x_0)) \leqq c\,\rho^n, \qquad \forall\, \rho < \operatorname{dist}(x_0, \operatorname{spt} \partial T)$$

where the constant depends on $\mathbf{M}(T)$ and x_0. (This is an immediate consequence of the interior monotonicity formula for mass-minimizing currents.) We can therefore conclude that for every $\varepsilon > 0$ there exists a number $\tau > 0$ such that

$$\mathbf{M}(\partial(T \llcorner B_\tau(x_0))) \leqq \varepsilon.$$

[The slice $\partial(T \llcorner B_\tau(x_0))$ is well-defined for $\mathscr{L}^1$-a. e. $\tau > 0$.] Indeed if this was false the coarea-formula would immediately yield that for some $\varepsilon > 0$

$$\varepsilon\rho < \int_0^\rho \mathbf{M}(\partial(T \llcorner B_\tau(x_0))\, d\tau \leqq \mathbf{M}(T \llcorner B_\rho(x_0)) \leqq c\,\rho^n$$

holds for every $\rho < \operatorname{dist}(x_0, \operatorname{spt} \partial T)$.

Suppose now that $\mathbf{M}(\partial T - \Gamma) < L$. As above we can find a ball $B_\tau(x_0)$ about some $x_0 \in \operatorname{spt} T \sim \operatorname{spt} \partial T$ such that

$$\mathbf{M}(\partial(T \llcorner B_\tau(x_0))) \leqq L - \mathbf{M}(\partial T - \Gamma).$$

The surface $T' = T - (T \llcorner B_\tau(x_0))$ then satisfies

$$M(T') < M(T)$$

and

$$M(\partial T' - \Gamma) \leqq L$$

thus contradicting the minimality of T in $A(\Gamma, L)$. ■

1.6. Proposition

Let $T \in I_n(\mathbb{R}^{n+k})$ *be minimizing with respect to* $\Gamma \in I_{n-1}(\mathbb{R}^{n+k})$ *in the sense of* Theorem 1.4. *Then*

$$\operatorname{spt} T \subset \operatorname{conv}(\operatorname{spt} \Gamma).$$

Proof. — We modify a well-known argument used in the case of the ordinary problem of mass-minimizing.

Since the convex hull of $\operatorname{spt}\Gamma$ is the intersection of all balls in $\mathbb{R}^{n+k}$ which contain $\operatorname{spt}\Gamma$ it suffices to show that $\operatorname{spt}\Gamma \subset \overline{B_R(x_0)}$ implies $\operatorname{spt} T \subset \overline{B_R(x_0)}$. By translating and scaling we may assume without loss of generality that $x_0 = 0$ and $R = 1$. Let $f: \mathbb{R}^{n+k} \to \overline{B_1(0)}$ be defined by $f(x) = x$ for $|x| < 1$, $f(x) = |x|^{-1} x$ for $|x| \geqq 1$. Since $\operatorname{Lip} f \leqq 1$ and $f_\# \Gamma = \Gamma$ we infer as in the proof of Theorem 1.4

$$M(f_\# T) \leqq M(T)$$
$$M(\partial f_\# T - \Gamma) \leqq M(\partial T - \Gamma)$$

which in view of the minimality of T implies

$$M(T) = M(f_\# T).$$

Using this, the fact that $f_\# T \llcorner B_1(0) = T \llcorner B_1(0)$ and the area-formula

$$M(f_\# T) = M(f_\# T \llcorner B_1(0)) + \int_{\mathbb{R}^{n+k} \sim B_1(0)} \left| \vec{T}(x) \wedge \frac{x}{|x|} \right| |x|^{-n} \, d\mu_T(x)$$

we obtain

$$\int_{\mathbb{R}^{n+k} \sim B_1(0)} \left(\left| \vec{T}(x) \wedge \frac{x}{|x|} \right| |x|^{-n} - 1 \right) d\mu_T(x) = 0.$$

Since $|\vec{T}(x)| = 1$ for μ_T-a. e. $x \in \mathbb{R}^{n+k}$ we conclude

$$\mu_T(\mathbb{R}^{n+k} \sim \overline{B_1(0)}) = 0. \quad ■$$

The following decomposition property of T and restriction property of Σ is going to play a central role in section 2.

1.7. Proposition

Let $T \in I_n(U)$ *be a minimizer of the thread problem with respect to* $\Gamma \in I_{n-1}(U)$.

(1) *Suppose the free boundary part* $\Sigma = \partial T - \Gamma$ *is decomposed inside* $W_0 \subset\subset U$ *in the following way:*

$$\Sigma = \Sigma' + \Sigma''$$
$$\mathbf{M}_{W_0}(\Sigma) = \mathbf{M}_{W_0}(\Sigma') + \mathbf{M}_{W_0}(\Sigma'').$$

Then

$$\mathbf{M}_{W_0}(T) \leqq \mathbf{M}_{W_0}(S)$$

for every $S \in I_{n, \text{loc}}(U)$ *satisfying* $\operatorname{spt}(S - T) \subset W_0$ *and*

$$\mathbf{M}_{W_0}(\partial S - \Gamma') = \mathbf{M}_{W_0}(\Sigma')$$

where $\Gamma' = \partial T - \Sigma'$ *is the new fixed boundary part.*

(2) *Suppose* T *can be decomposed inside* $W_0 \subset\subset U$ *in the following way:*

$$\begin{aligned} T &= T' + T'', \qquad & \mathbf{M}_{W_0}(T) &= \mathbf{M}_{W_0}(T') + \mathbf{M}_{W_0}(T'') \\ \Gamma &= \Gamma' + \Gamma'', & \Sigma' &= \partial T' - \Gamma',\ \Sigma'' = \partial T'' - \Gamma'' \\ \Sigma &= \Sigma' + \Sigma'', & \mathbf{M}_{W_0}(\Sigma) &= \mathbf{M}_{W_0}(\Sigma') + \mathbf{M}_{W_0}(\Sigma''). \end{aligned}$$

Then T' *and* T'' *are minimizers of the thread problem in* W_0 *with respect to* Γ' *and* Γ'' *respectively.*

Proof.

(1) We have

$$\begin{aligned} \mathbf{M}_{W_0}(\partial S - \Gamma) &\leqq \mathbf{M}_{W_0}(\partial S - \Gamma') + \mathbf{M}_{W_0}(\Sigma'') \\ &= \mathbf{M}_{W_0}(\Sigma') + \mathbf{M}_{W_0}(\Sigma'') \\ &= \mathbf{M}_{W_0}(\Sigma) = \mathbf{M}_{W_0}(\partial T - \Gamma). \end{aligned}$$

From Prop. 1.3 we obtain

$$\mathbf{M}_{W_0}(T) \leqq \mathbf{M}_{W_0}(S).$$

(2) Let $S \in I_{n, \text{loc}}(U)$ satisfy $\operatorname{spt}(S - T') \subset W_0$ and

$$\mathbf{M}_{W_0}(\partial S - \Gamma') = \mathbf{M}_{W_0}(\partial T' - \Gamma') = \mathbf{M}_{W_0}(\Sigma').$$

Then we check as in the proof of part (1) that $S'' = S + T''$ is an admissible comparison surface for T. This implies

$$\mathbf{M}_{W_0}(T) \leqq \mathbf{M}_{W_0}(S'') \leqq \mathbf{M}_{W_0}(S) + \mathbf{M}_{W_0}(T'').$$

From the mass-additivity of T′ and T″ in W_0 we conclude

$$\mathbf{M}_{W_0}(T') \leqq \mathbf{M}_{W_0}(S). \quad \blacksquare$$

2. THE FIRST VARIATION OF THE THREAD

The *first variation of the mass* of $S \in I_{n,\,\mathrm{loc}}(U)$ is given by (*cf.* [AW], [SL])

$$\delta S(X) = \int \mathrm{div}_S X \, d\mu_S$$

where $X \in C_c^1(U; \mathbb{R}^{n+k})$.

We define the *support of* δS in U by

$$\mathrm{spt}\,\delta S = \{x \in U / \forall\, \rho > 0,\ \exists\, X_\rho \in C_c^1(B_\rho(x); \mathbb{R}^{n+k}) \text{ s. t. } \delta S(X_\rho) \neq 0\}.$$

In order to obtain some control on the first variation of the *thread-boundary* Σ introduced in section 1 we shall have to make use of the following crucial lemma.

2.1. Lemma

Let $T \in I_{n,\,\mathrm{loc}}(U)$ *be a minimizer of the thread problem with respect to* $\Gamma \in I_{n-1,\,\mathrm{loc}}(U)$.

Then the inequality

$$(21)\quad |\delta T(X)\,\delta\Sigma(Y) - \delta T(Y)\,\delta\Sigma(X)| \leqq |\delta\Sigma(Y)| \int |X \wedge \vec{\Gamma}|\, d\mu_\Gamma + |\delta\Sigma(X)| \int |Y \wedge \vec{\Gamma}|\, d\mu_\Gamma$$

holds for every $X \in C_c^1(V; \mathbb{R}^{n+k})$ *and* $Y \in C_c^1(W; \mathbb{R}^{n+k})$ *whenever* $V, W \subset\subset U \sim \mathrm{spt}\,\partial\Gamma$ *are disjoint open sets.*

The proof of Lemma 2.1 is based on Lagrange multiplier techniques used in [HW] and [DHL]. We give a slight generalization of Lemma 2 of [DHL] for the case where some nondifferentiable functions are involved.

2.2. Lemma

Let $f(s, t)$, $g(s, t)$ *be real-valued functions of* $(s, t) \in [-s_0, s_0] \times [-t_0, t_0]$, $s_0 > 0$, $t_0 > 0$ *which split in the form*

$$f(s, t) = f_0 + f_1(s) + \bar{f}_1(s) + f_2(t) + \bar{f}_2(t)$$
$$g(s, t) = g_0 + g_1(s) + g_2(t)$$

where f_0, g_0 *are constants and*

$$f_1(0)=\bar{f}_1(0)=f_2(0)=\bar{f}_2(0)=g_1(0)=g_2(0)=0.$$

Suppose g_2 *is continuous in* $[-t_0, t_0]$, *the derivatives* $f_1'(0)$, $f_2'(0)$, $g_1'(0)$, $g_2'(0)$ *exist and* $g_2'(0)=1$.

Suppose furthermore that

$$f_0=f(0, 0) \leqq f(s, t)$$

for every $(s, t) \in [-s_0, s_0] \times [-t_0, t_0]$ *such that* $g(s, t)=g_0$.

Then

$$(2.2) \qquad \left| f_1'(0)-f_2'(0)g_1'(0) \right| \leqq \overline{\lim_{s \to 0}} \left| \frac{\bar{f}_1(s)}{s} \right| + \overline{\lim_{t \to 0}} \left| \frac{\bar{f}_2(t)}{t} \right| \left| g_1'(0) \right|.$$

Proof. — We refer the reader to Lemma 2 of [DHL]. The auxiliary function $\tau(s)$ defined there depends only on g_1 and g_2. One then immediately verifies that the difference quotient expressions corresponding to the left hand side of (2.2) can be estimated by difference quotient terms involving $\bar{f}_1$ and $\bar{f}_2$. ■

Proof of Lemma 2.1. — Let (φ_s), $s \in [-s_0, s_0]$ be a one-parameter family of diffeomorphisms of U which leave the boundary of Γ fixed, that is $\varphi_0 = \mathrm{id}$ and $\mathrm{spt}(\varphi_s - \mathrm{id}) \subset\subset \mathrm{V} \subset\subset \mathrm{U} \sim \mathrm{spt}\, \partial\Gamma$ for $s \in [-s_0, s_0]$. Suppose furthermore that φ_s satisfies

$$(2.3) \qquad \mathbf{M}_{\mathrm{V}}(\varphi_{s\#}\Sigma)=\mathbf{M}_{\mathrm{V}}(\Sigma).$$

Then

$$\mathrm{T}_s = \varphi_{s\#}\mathrm{T} - \varphi_{\#}([\![(0, s)]\!] \times \Gamma)$$

is an admissible comparison surface for T in V. Indeed we have $\mathrm{spt}(\mathrm{T}-\mathrm{T}_s) \subset\subset \mathrm{V}$ and

$$(2.4) \qquad \begin{aligned} \partial \mathrm{T}_s - \Gamma &= \partial(\varphi_{s\#}\mathrm{T} - \varphi_{\#}([\![(0, s)]\!] \times \Gamma) - \Gamma) \\ &= \varphi_{s\#}\Sigma + \varphi_{s\#}\Gamma - \partial\varphi_{\#}([\![(0, s)]\!] \times \Gamma) - \Gamma \\ &= \varphi_{s\#}\Sigma + \varphi_{s\#}\Gamma - \varphi_{\#}\Gamma + \Gamma - \Gamma \\ &= \varphi_{s\#}\Sigma. \end{aligned}$$

Here we used the homotopy formula for currents taking $\mathrm{spt}(\varphi_s - \mathrm{id}) \cap \mathrm{spt}\, \partial\Gamma = \emptyset$ into account.

In particular, (2.4) yields $\mathbf{M}(\partial \mathrm{T}_s - \Gamma) = \mathbf{M}(\partial \mathrm{T} - \Gamma)$ which by the minimality of T implies

$$(2.5) \qquad \begin{aligned} \mathbf{M}_{\mathrm{V}}(\mathrm{T}) &\leqq \mathbf{M}_{\mathrm{V}}(\mathrm{T}_s) \\ &\leqq \mathbf{M}_{\mathrm{V}}(\varphi_{s\#}\mathrm{T}) + \mathbf{M}_{\mathrm{V}}(\varphi_{\#}([\![(0, s)]\!] \times \Gamma)). \end{aligned}$$

Suppose $\varphi_s(x)=x+s\mathrm{X}$ where $\mathrm{X}\in \mathrm{C}_c^1(\mathrm{V};\ \mathbb{R}^{n+k})$. Then we compute as in ([BJ], Lemma 3.1)

$$\begin{aligned}\mathbf{M}(\varphi_\#([\![(0,s)]\!]\times\Gamma)) &= \int_0^s\int |\dot{\varphi}_\tau(x)\wedge(d_x\varphi_\tau)_\#(\vec{\Gamma}(x))|\,d\mu_\Gamma(x)\,d\tau\\ &= \int_0^s\int |\mathrm{X}\wedge\vec{\Gamma}(x)+\mathrm{X}\wedge\tau^{n-1}(\mathrm{DX}(x))_\#(\vec{\Gamma}(x))|\,d\mu_\Gamma(x)\,d\tau\end{aligned}$$

which implies

$$\overline{\lim_{s\to 0}}\left|\frac{\mathbf{M}(\varphi_\#([\![(0,s)]\!]\times\Gamma))}{s}\right|=\int|\mathrm{X}\wedge\vec{\Gamma}|\,d\mu_\Gamma. \tag{2.6}$$

Let now V, W be two disjoint open sets which are compactly contained in $\mathrm{U}\sim\operatorname{spt}\partial\Gamma$ and choose variation vectorfields $\mathrm{X}\in\mathrm{C}_c^1(\mathrm{V};\ \mathbb{R}^{n+k})$ and $\mathrm{Y}\in\mathrm{C}_c^1(\mathrm{W};\ \mathbb{R}^{n+k})$. Let $\Omega \subset\subset \mathrm{U}$ be an open set such that $\mathrm{V}\cup\mathrm{W} \subset\subset \Omega$. For one-parameter deformations

$$\varphi_s(x)=x+s\mathrm{X}(x),\qquad \psi_t(x)=x+t\mathrm{Y}(x),$$

$(s,t)\in[-s_0,s_0]\times[-t_0,t_0]$, we define

$$\begin{gathered}f_0=\mathbf{M}_\Omega(\mathrm{T}),\qquad g_0=\mathbf{M}_\Omega(\Sigma)\\ f_1(s)=\mathbf{M}_\mathrm{V}(\varphi_{s\#}\mathrm{T})-\mathbf{M}_\mathrm{V}(\mathrm{T})\\ \bar{f}_1(s)=\mathbf{M}_\mathrm{V}(\varphi_\#([\![(0,s)]\!]\times\Gamma))\\ f_2(t)=\mathbf{M}_\mathrm{W}(\psi_{t\#}\mathrm{T})-\mathbf{M}_\mathrm{W}(\mathrm{T})\\ \bar{f}_2(t)=\mathbf{M}_\mathrm{W}(\psi_\#([\![(0,t)]\!]\times\Gamma))\\ g_1(s)=\mathbf{M}_\mathrm{V}(\varphi_{s\#}\Sigma)-\mathbf{M}_\mathrm{V}(\Sigma)\\ g_2(t)=\mathbf{M}_\mathrm{W}(\psi_{t\#}\Sigma)-\mathbf{M}_\mathrm{W}(\Sigma)\end{gathered}$$

and $f(s,t)$, $g(s,t)$ as in Lemma 2.2. Let

$$\mathrm{T}_{s,t}=\varphi_{s\#}\mathrm{T}-\varphi_\#([\![(0,s)]\!]\times\Gamma)+\psi_{t\#}\mathrm{T}-\psi_\#([\![(0,t)]\!]\times\Gamma).$$

From the definition of φ_s and ψ_t we infer

$$\operatorname{spt}(\mathrm{T}_{s,t}-\mathrm{T}) \subset\subset \Omega.$$

Furthermore we derive from (2.4)

$$\mathbf{M}_\Omega(\partial\mathrm{T}_{s,t}-\Gamma)=\mathbf{M}_\mathrm{V}(\varphi_{s\#}\Sigma)+\mathbf{M}_\mathrm{W}(\psi_{t\#}\Sigma)+\mathbf{M}_{\Omega\sim(\mathrm{V}\cup\mathrm{W})}(\Sigma).$$

For those $(s,t)\in[-s_0,s_0]\times[-t_0,t_0]$ which satisfy $g(s,t)=g_0$ we have

$$\mathbf{M}_\mathrm{V}(\varphi_{s\#}\Sigma)+\mathbf{M}_\mathrm{W}(\psi_{t\#}\Sigma)=\mathbf{M}_\mathrm{V}(\Sigma)+\mathbf{M}_\mathrm{W}(\Sigma).$$

This implies [for such (s,t)]

$$\mathbf{M}_\Omega(\partial\mathrm{T}_{s,t}-\Gamma)=\mathbf{M}_\Omega(\partial\mathrm{T}-\Gamma)$$

which establishes $T_{s,t}$ as an admissible comparison surface. As in (2.5) we conclude

$$\begin{aligned}\mathbf{M}_\Omega(T) \leqq \mathbf{M}_\Omega(T_{s,t}) \\ \leqq \mathbf{M}_V(\varphi_{s\#}T)+\mathbf{M}_W(\psi_{t\#}T)+\mathbf{M}_V(\varphi_\#([\![(0,s)]\!]\times\Gamma)) \\ +\mathbf{M}_W(\psi_\#([\![(0,t)]\!]\times\Gamma))+\mathbf{M}_{\Omega\sim(V\cup W)}(T).\end{aligned}$$

In view of the definition of f_1, $\bar{f}_1$, f_2 and $\bar{f}_2$ this implies for (s, t) satisfying $g(s, t)=g_0$

$$0 \leqq f_1(s)+\bar{f}_1(s)+f_2(t)+\bar{f}_2(t)$$

which is equivalent to

$$f(0, 0) \leqq f(s, t)$$

for every (s, t) s.t. $g(s, t)=g_0$. Moreover

$$f_1(0)=\bar{f}_1(0)=f_2(0)=\bar{f}_2(0)=g_1(0)=g_2(0)=0$$

and all the differentiability and continuity requirements of Lemma 2.2 are satisfied.

In case $\delta\Sigma(X)=0$ for all $X\in C_c^1(U\sim \operatorname{spt}\partial\Gamma; \mathbb{R}^{n+k})$ the statement of Lemma 2.1 holds trivially. Hence we may assume $Y\in C_c^1(W; \mathbb{R}^{n+k})$ satisfies $\delta\Sigma(Y)\neq 0$ and set $Y'=\delta\Sigma(Y)^{-1}Y$. This gives $\delta\Sigma(Y')=1$ which by the definition of g_2 represents the condition $g_2'(0)=1$.

We can now apply Lemma 2.2, the definition of first variation to f_1, f_2, g_1, g_2 and (2.6) to $\bar{f}_1$ and $\bar{f}_2$ to arrive at

$$|\delta T(X)-\delta T(Y')\,\delta\Sigma(X)| \leqq \int |X\wedge\vec{\Gamma}|\,d\mu_\Gamma+|\delta\Sigma(X)|\int|Y'\wedge\vec{\Gamma}|\,d\mu_\Gamma$$

for $X\in C_c^1(V; \mathbb{R}^{n+k})$ and $Y'=\delta\Sigma(Y)^{-1}Y\in C_c^1(W; \mathbb{R}^{n+k})$ which completes the proof of (2.1). ■

We now turn to establishing the main result of this paper.

2.3. Theorem

Let $T\in I_{n,\mathrm{loc}}(U)$ *be a minimizer of the thread problem with respect to* $\Gamma\in I_{n-1,\mathrm{loc}}(U)$.

Suppose

(A1) $$\operatorname{spt}\delta\Sigma \sim \operatorname{spt}\partial\Gamma \neq \varnothing$$

(A2) *There exists a point* $x_0\in\operatorname{spt}\Sigma\sim\operatorname{spt}\partial\Gamma$, *a radius* $\rho<\operatorname{dist}(x_0, \operatorname{spt}\partial\Gamma)$ *and a local decomposition*

$$T \llcorner B_\rho(x_0)=T_0 \llcorner B_\rho(x_0)+(T-T_0)\llcorner B_\rho(x_0)$$

satisfying $T_0 \in I_{n,\text{loc}}(U)$,

$$(1)\quad \begin{cases} \mathbf{M}(T \llcorner B_\rho(x_0)) = \mathbf{M}(T_0 \llcorner B_\rho(x_0)) + \mathbf{M}((T - T_0) \llcorner B_\rho(x_0)) \\ \mathbf{M}(\Sigma \llcorner B_\rho(x_0)) = \mathbf{M}(\Sigma_0 \llcorner B_\rho(x_0)) + \mathbf{M}((\Sigma - \Sigma_0) \llcorner B_\rho(x_0)) \end{cases}$$

for $\Sigma_0 = \partial T_0$ *and*

$$(2)\qquad x_0 \in \operatorname{spt} \delta T_0.$$

Then we can find a number $\lambda_\Sigma \in (0, \infty)$ *such that*

$$(2.7)\qquad |\delta T(X) + \lambda_\Sigma \delta\Sigma(X)| \leqq \int |X \wedge \vec{\Gamma}| \, d\mu_\Gamma$$

holds for every $X \in C^1_c(U \sim \operatorname{spt} \partial\Gamma; \mathbb{R}^{n+k})$, *where* λ_Σ *is given by*

$$(2.8)\qquad \delta T_0(X) + \lambda_\Sigma \delta\Sigma_0(X) = 0$$

for every $X \in C^1_c(B_\rho(x_0); \mathbb{R}^{n+k})$.

Moreover (2.8), *at any point of* $\operatorname{spt}\Sigma \sim \operatorname{spt}\partial\Gamma$ *satisfying* (A2) *and for any possible decomposition at such a point, is valid with the same* $\lambda_\Sigma > 0$.

2.4. Remark

(1) If (A1) is not satisfied Σ is a stationary *thread* away from $\partial\Sigma = -\partial\Gamma$. For the structure of such boundaries we refer to Corollary 2.10 and Theorem 3.1.

(2) Although in the codimension one case, *i.e.* $U \subset \mathbb{R}^{n+1}$ condition (A2) can be verified under reasonably weak hypotheses it nevertheless appears to be a rather artificial assumption which one would hope, could be removed altogether.

In fact if $U \subset \mathbb{R}^{n+1}$ it suffices to assume the existence of at least one regular point of $\operatorname{spt}\Sigma \sim \operatorname{spt}\partial\Gamma$ in the sense of Proposition 2.7 (1).

Proof of Theorem 2.3. — We first prove (2.7) assuming

$$(B2)\qquad \operatorname{spt}\delta T \sim \operatorname{spt}\Gamma \neq \varnothing.$$

From Remark 1.2 (2) and ([BJ], Lemma 3.1) we infer

$$(2.9)\qquad |\delta T(X)| \leqq \int |X \wedge \overrightarrow{\partial T}| \, d\mu_{\partial T}$$

for every $X \in C^1_c(U; \mathbb{R}^{n+k})$. In particular, the representation formula for δT (*cf.* [SL], Chapt. 8])

$$(2.10)\qquad \delta T(X) = \int \nu_{\partial T} \cdot X \, d\mu_{\partial T}$$

holds for $X \in C_c^1(U; \mathbb{R}^{n+k})$, where $\nu_{\partial T}$ is a $\mu_{\partial T}$-measurable vectorfield in U satisfying $|\nu_{\partial T}| \leqq 1$ $\mu_{\partial T}$-a. e. Assumption (B2) implies that

$$\mu_{\partial T}(\{x \in \operatorname{spt} \Sigma \sim \operatorname{spt} \Gamma / \nu_{\partial T}(x) \neq 0\}) > 0. \tag{2.11}$$

Hence we may select three points $x_1, x_2, x_3 \in \operatorname{spt} \delta T \sim \operatorname{spt} \Gamma$, radii $\rho_i < \operatorname{dist}(x_i, \operatorname{spt} \Gamma)$ s. t. $B_{\rho_i}(x_i) \cap B_{\rho_j}(x_j) = \varnothing$ for $i \neq j$ $(i, j = 1, 2, 3)$ and variation vectorfields $X_i \in C_c^1(B_{\rho_i}(x_i); \mathbb{R}^{n+k})$ which satisfy

$$\delta T(X_i) \neq 0, \qquad i = 1, 2, 3. \tag{2.12}$$

From (A1) we obtain the existence of a point $x_0 \in \operatorname{spt} \delta\Sigma \sim \operatorname{spt} \partial\Gamma$, a radius $\rho_0 < \operatorname{dist}(y_0, \operatorname{spt} \partial\Gamma)$ and a vectorfield $Y_0 \in C_c^1(B_{\rho_0}(y_0); \mathbb{R}^{n+k})$ such that

$$\delta\Sigma(Y_0) \neq 0. \tag{2.13}$$

We may assume $B_{\rho_0}(y_0) \cap B_{\rho_i}(x_i) = \varnothing$ for $i = 1, 2, 3$. Otherwise, by virtue of (2.11), we can choose different $x_i \in \operatorname{spt} \delta T \sim \operatorname{spt} \Gamma$ and $\rho_i > 0$.

Applying now (2.1) to the pairs X_i, Y_0 for $i = 1, 2, 3$ we obtain

$$|\delta T(X_i)\,\delta\Sigma(Y_0) - \delta T(Y_0)\,\delta\Sigma(X_i)| \leqq |\delta\Sigma(X_i)| \int |Y_0 \wedge \vec{\Gamma}|\, d\mu_\Gamma.$$

Hence from (2.12) and (2.13) we deduce

$$\delta\Sigma(X_i) \neq 0, \qquad i = 1, 2, 3. \tag{2.14}$$

If we apply (2.1) to the pairs X_i, X_3 for $i = 1, 2$ and take (2.14) into account we derive

$$\delta T(X_3) - \frac{\delta T(X_1)}{\delta\Sigma(X_1)}\,\delta\Sigma(X_3) = \delta T(X_3) - \frac{\delta T(X_2)}{\delta\Sigma(X_2)}\,\delta\Sigma(X_3)$$

which implies, in view of (2.14) again,

$$\frac{\delta T(X_1)}{\delta\Sigma(X_1)} = \frac{\delta T(X_2)}{\delta\Sigma(X_2)}.$$

At this stage we define

$$\lambda_\Sigma = -\frac{\delta T(X_1)}{\delta\Sigma(X_1)} \neq 0. \tag{2.15}$$

An arbitrary vectorfield $X \in C_c^1(U \sim \operatorname{spt} \partial\Gamma; \mathbb{R}^{n+k})$ we decompose as follows: $X = X^{(1)} + X^{(2)}$, where $X^{(i)} = X\,\eta^{(i)}$ $(i = 1, 2)$ and $\eta^{(i)} \in C^\infty(U)$ satisfies $\operatorname{spt} \eta^{(i)} \cap B_{\rho_i}(x_i) = \varnothing$, $0 \leqq \eta^{(i)} \leqq 1$ and $\eta^{(1)} + \eta^{(2)} = 1$.

Using (2.1) again, this time with X_i, $X^{(i)}$ $(i = 1, 2)$, we obtain

$$|\delta T(X^{(i)}) + \lambda_\Sigma\,\delta\Sigma(X^{(i)})| \leqq \int |X^{(i)} \wedge \vec{\Gamma}|\, d\mu_\Gamma$$

for $i=1, 2$ which in turn establishes (2.7). Note that

$$\delta T(X)+\lambda_\Sigma \delta\Sigma(X)=0 \tag{2.16}$$

holds for all $X \in C_c^1(U \sim \operatorname{spt}\Gamma; \mathbb{R}^{n+k})$.

Before we prove the result under the general assumption we want to show that (2.16) implies $\lambda_\Sigma > 0$.

We already know $\lambda_\Sigma \neq 0$ [see (2.15)]. Suppose $\lambda_\Sigma < 0$. Select a variation $Y \in C_c^1(U \sim \operatorname{spt}\Gamma; \mathbb{R}^{n+k})$ satisfying $\delta\Sigma(Y) < 0$. (2.16) then yields $\delta T(Y) < 0$. If we let (ψ_t) denote the one-parameter family of deformations generated by Y this implies that for some small $t > 0$ we have

$$\mathbf{M}_{\operatorname{spt} Y}(\psi_{t\#} T) < \mathbf{M}_{\operatorname{spt} Y}(T)$$

and

$$\mathbf{M}_{\operatorname{spt} Y}(\psi_{t\#} \Sigma) < \mathbf{M}_{\operatorname{spt} Y}(\Sigma)$$

which in view of Proposition 1.3 contradicts the minimality of T.

Suppose now that condition (A2) holds instead of (B2).

By virtue of Proposition (1.7) (2) and (A2) (1) T_0 minimizes the *thread problem* in $B_\rho(x_0)$ with respect to $\Gamma=0$. Hence in view of (A2) (2) [which for T_0 reduces to condition (B2)] and (2.11) we may select two points $x_1, x_2 \in \operatorname{spt}\delta T_0 \cap \operatorname{spt}\Sigma_0$ and radii ρ_1, ρ_2 such that $B_{\rho_1}(x_1) \cap B_{\rho_2}(x_2)=\varnothing$ and $B_{\rho_1}(x_1) \cup B_{\rho_2}(x_2) \subset\subset B_\rho(x_0)$.

For $i=1, 2$ we define

$$\begin{gathered} T_i = T-(T-T_0) \llcorner B_{\rho_i}(x_i) \\ \Gamma_i=\Gamma-\Gamma \llcorner B_{\rho_i}(x_i) \\ \Sigma_i=\partial T_i-\Gamma_i \\ U_i=(U \sim \overline{B_{\rho_i}(x_i)}) \cup B_{\rho_i/2}(x_i) \end{gathered} \tag{2.17}$$

such that

$$\begin{gathered} T_i=T_0 \quad \text{in } B_{\rho_i}(x_i), \\ T_i=T \quad \text{in } U \sim \overline{B_{\rho_i}(x_i)} \\ \Sigma_i=\Sigma_0 \quad \text{in } B_{\rho_i}(x_i), \\ \Sigma_i=\Sigma \quad \text{in } U \sim \overline{B_{\rho_i}(x_i)}. \end{gathered} \tag{2.18}$$

We infer from (A2) (1) that for $i=1, 2$ the pair T_i, $T-T_i$ (replacing T', T'') satisfies the conditions of Proposition 1.7 (2) for every open $W \subset\subset U_i$. Hence T_i is a minimizer of the *thread problem* in U_i with respect to Γ_i. Due to the choice of x_1 and x_2 we have for $i=1, 2$ in U_i

$$\operatorname{spt}\delta T_i \sim \operatorname{spt}\Gamma_i \neq \varnothing. \tag{2.19}$$

Moreover, in view of (A1) and (2.11) applied to T_0 we may assume x_i and ρ_i to be chosen such that

$$\text{spt}\,\delta\Sigma_i \sim \text{spt}\,\partial\Gamma_i \neq \varnothing \tag{2.20}$$

for $i=1, 2$.

Therefore T_i satisfies the conditions (A1) and (B2). From (2.7), (2.16) and (2.18) we derive

$$|\delta T_i(X)+\lambda_\Sigma^i\,\delta\Sigma_i(X)| \leqq \int |X \wedge \vec{\Gamma}_i|\,d\mu_{\Gamma_i} \tag{2.21}$$

for every $X \in C_c^1(U_i \sim \text{spt}\,\partial\Gamma_i; \mathbb{R}^{n+k})$ where $\lambda_\Sigma^i > 0$ is defined by

$$\delta T_0(X)+\lambda_\Sigma^i\,\delta\Sigma_0(X)=0 \tag{2.22}$$

for every $X \in C_c^1(B_{\rho_i/2}(x_i); \mathbb{R}^{n+k})$ $(i=1, 2)$.

The identity (2.22) and $x_i \in \text{spt}\,\delta T_0 \cap \text{spt}\,\Sigma_0$ for $i=1, 2$ imply that $x_i \in \text{spt}\,\delta\Sigma_0$. Therefore T_0, which minimizes the *thread problem* in $B_\rho(x_0)$ with respect to $\Gamma=0$, also satisfies (A1) and (B2) there, such that (2.7) is applicable to T_0. This establishes (2.8) for every $X \in C_c^1(B_\rho(x_0); \mathbb{R}^{n+k})$. Hence $\lambda_\Sigma^1=\lambda_\Sigma^2$.

From (2.21) we now obtain in particular

$$|\delta T(X)+\lambda_\Sigma\,\delta\Sigma(X)| \leqq \int |X \wedge \vec{\Gamma}|\,d\mu_\Gamma \tag{2.23}$$

for every $X \in C_c^1(U \sim \text{spt}\,\partial\Gamma; \mathbb{R}^{n+k})$ satisfying $\text{spt}\,X \cap B_{\rho_i}(x_i)=\varnothing$, where $i=1, 2$.

If $X \in C_c^1(U \sim \text{spt}\,\partial\Gamma; \mathbb{R}^{n+k})$ is arbitrary, we decompose it as in the first part of the proof and apply (2.23) to arrive at inequality (2.7).

It remains to show that λ_Σ is independent of x_0 and T_0.

Suppose that we have two decompositions at x_0, that is (A2) holds for T_0 replaced by T_0^1 and T_0^2 respectively. From (2.8) we obtain

$$\delta T_0^i(X)+\lambda_\Sigma^i\,\delta\Sigma_0^i(X)=0 \tag{2.24}$$

for some $\lambda_\Sigma^i>0$ $(i=1, 2)$ and for every $X \in C_c^1(B_\rho(x_0); \mathbb{R}^{n+k})$. Pick $y_i \in \text{spt}\,\delta T_0^i$ and radii σ_i $(i=1, 2)$ such that $B_{\sigma_1}(y_1) \cap B_{\sigma_2}(y_2)=\varnothing$ and $B_{\sigma_1}(y_1) \cup B_{\sigma_2}(y_2) \subset\subset B_\rho(x_0)$. Then (2.24) implies $y_i \in \text{spt}\,\delta\Sigma_0^i$.

Define

$$T_{1,2}=T_0^1 \llcorner B_{\sigma_1}(y_1)+T_0^2 \llcorner B_{\sigma_2}(y_2).$$

In view of (A2) (1), for T_0^1 and T_0^2 respectively, $T_{1,2}$ and $T-T_{1,2}$ satisfy the conditions of Proposition 1.7 (2) in $U_{1,2}=B_{\sigma_1/2}(y_1) \cup B_{\sigma_2/2}(y_2)$. Thus $T_{1,2}$ is a minimizer of the *thread problem* in $U_{1,2}$ with respect to $\Gamma=0$.

Moreover since $y_i \in \operatorname{spt} \delta T_{1,2} \cap \operatorname{spt} \delta \Sigma_{1,2}$ $(i=1, 2)$, where $\Sigma_{1,2} = \partial T_{1,2}$, (A1) and (A2) are satisfied, which enables us to apply (2.7). Thus

$$\delta T_{1,2}(X) + \lambda_\Sigma^{1,2}\, \delta \Sigma_{1,2}(X) = 0 \tag{2.25}$$

for every $X \in C_c^1(U_{1,2}; \mathbb{R}^{n+k})$ where $\lambda_\Sigma^{1,2} > 0$.

By the definition of $T_{1,2}$ this reduces to

$$\delta T_0^i(X) + \lambda_\Sigma^{1,2}\, \delta \Sigma_0^i(X) = 0 \tag{2.26}$$

for $X \in C_c^1(B_{\sigma_i/2}(y_i); \mathbb{R}^{n+k})$, $i=1, 2$.

The fact that $y_i \in \operatorname{spt} \delta \Sigma_0^i$ implies the existence of vectorfields $Y_i \in C_c^1(B_{\sigma_i/2}(y_i); \mathbb{R}^{n+k})$ which satisfy $\delta \Sigma_0^i(Y_i) \neq 0$. Applying now (2.24) and (2.26) to Y_i $(i=1, 2)$ yields $\lambda_\Sigma^{1,2} = \lambda_\Sigma^1 = \lambda_\Sigma^2$.

For decomposition components at distinct points of $\operatorname{spt} \Sigma \sim \operatorname{spt} \partial \Gamma$ the same argument obviously works.

This completes the proof of the theorem. ■

2.5. Corollary

Let $T \in I_{n, \mathrm{loc}}(U)$ *satisfy the assumptions of* Theorem 2.3. *Suppose that* Γ *additionally satisfies*

(A3) (1) *For every* $x_0 \in \operatorname{spt} \Gamma \sim \operatorname{spt} \partial \Gamma$ *there exists a radius* $\rho(x_0) < \operatorname{dist}(x_0, \operatorname{spt} \partial \Gamma)$ *and a constant* $c(x_0)$ *such that for every* $x \in B_{\rho(x_0)}(x_0)$ *and* $\rho < \rho(x_0) - |x - x_0|$

$$\mu_\Gamma(B_\rho(x_0)) \leqq c(x_0)\, \rho^{n-2+\beta}$$

for some $\beta > 0$.

(2) *For every* $W \subset\subset U \sim \operatorname{spt} \partial \Gamma$ *there is a constant* $c(W)$ *such that*

$$|\theta_\Gamma \llcorner W| \leqq c(W), \qquad \mu_\Gamma\text{-a. e.} \quad \textit{in } W$$

where θ_Γ *is the multiplicity function of* Γ.

Then Σ *has bounded generalized mean curvature* H_Σ, *in fact*

$$\int \operatorname{div}_\Sigma X\, d\mu_\Sigma = -\int H_\Sigma \cdot X\, d\mu_\Sigma \tag{2.27}$$

for every $X \in C_c^1(U \sim \operatorname{spt}\partial\Gamma;\ \mathbb{R}^{n+k})$, *where* H_Σ *satisfies*

$$(2.28)\qquad |H_\Sigma \llcorner W| \leqq \frac{c(W)}{\lambda_\Sigma}, \qquad \mu_\Sigma\text{-a. e.}\quad \text{in } W$$

for every $W \subset\subset U \sim \operatorname{spt}\partial\Gamma$, *where* $c(W)$ *depends on* W *only.*

Proof. – We combine (2.7) and (2.9) to obtain

$$|\delta\Sigma(X)| \leqq \frac{1}{\lambda_\Sigma}\left(\int |X \wedge \vec{\Gamma}|\, d\mu_\Gamma + \int |X \wedge \vec{\partial T}|\, d\mu_{\partial T}\right)$$

for $X \in C_c^1(U \sim \operatorname{spt}\partial\Gamma;\ \mathbb{R}^{n+k})$, which in view of the fact that $\mu_{\partial T} \leqq \mu_\Gamma + \mu_\Sigma$ yields

$$|\delta\Sigma(X)| \leqq \frac{1}{\lambda_\Sigma}\int |X|\, d\mu_\Sigma + \frac{2}{\lambda_\Sigma}\int |X|\, d\mu_\Gamma$$

for every $X \in C_c^1(U \sim \operatorname{spt}\partial\Gamma;\ \mathbb{R}^{n+k})$.

We now proceed as in ([SL] 17.6) to obtain for every $x \in B_{\rho(x_0)}(x_0)$ and $\mathscr{L}^1$-a. e. $\rho \leqq \rho(x_0) - |x - x_0|$

$$\frac{d}{d\rho}(\rho^{1-n}\mu_\Sigma(B_\rho(x_0))) \geqq -\frac{1}{\lambda_\Sigma}\rho^{1-n}\mu_\Sigma(B_\rho(x_0)) - \frac{2}{\lambda_\Sigma}\rho^{1-n}\mu_\Gamma(B_\rho(x_0))$$

which by (A3) (1) implies

$$\frac{d}{d\rho}(e^{\lambda_\Sigma^{-1}\rho}\rho^{1-n}\mu_\Sigma(B_\rho(x_0))) \geqq -\frac{2}{\lambda_\Sigma}c(x_0)\, e^{\lambda_\Sigma^{-1}\rho}\rho^{\beta-1}.$$

Integrating we arrive at

$$e^{\lambda_\Sigma^{-1}\sigma}\sigma^{1-n}\mu_\Sigma(B_\sigma(x_0)) \leqq e^{\lambda_\Sigma^{-1}\rho}\rho^{1-n}\mu_\Sigma(B_\rho(x_0)) + \frac{1}{\lambda_\Sigma}c(x_0, \beta)(\rho^\beta - \sigma^\beta)$$

for $0 < \sigma < \rho \leqq \rho(x_0) - |x - x_0|$.

Hence, we can check as in ([SL], Cor. 17.8) that $\theta^{n-1}(\mu_\Sigma, .)$ is uppersemicontinuous and we can apply ([SL], 17.9 (i)) to conclude $\theta_\Sigma(x) \geqq 1$ for every $x \in \operatorname{spt}\Sigma \sim \operatorname{spt}\partial\Gamma$. (Recall that $\theta_\Sigma \geqq 1$ μ_Σ-a. e. since Σ is an integer multiplicity current.) Using this in combination with (A3) (2) we infer from the definition of μ_Σ and μ_Γ that

$$\mu_\Gamma(\operatorname{spt}\Sigma \cap W) \leqq c(W)\,\mu_\Sigma(W)$$

for any $W \subset\subset U \sim \operatorname{spt}\partial\Gamma$.

Thus we can differentiate μ_Γ with respect to μ_Σ to obtain

$$|\delta\Sigma(X)| \leqq \frac{3}{\lambda_\Sigma} c(W) \int |X|\, d\mu_\Sigma$$

for any $X \in C_c^1(W; \mathbb{R}^{n+k})$, which in turn implies the result. ■

2.6. Remark

(1) Since $\Sigma = \partial T$ in $U \sim \operatorname{spt}\Gamma$ and $\Sigma = -\Gamma$ in $U \sim \operatorname{spt}\partial T$ we have $|H_\Sigma(x)| \leqq 1/\lambda_\Sigma$ for μ_Σ-a. e. $x \in U \sim (\operatorname{spt}\Gamma \cap \operatorname{spt}\partial T)$.

(2) One easily checks that (A3) holds (with $\beta = 1$) in case Γ locally corresponds to an oriented embedded $C^{0,1}$-submanifold of $\mathbb{R}^{n+k}$ with multiplicity m_Γ.

2.7. Proposition

Let $T \in I_{n,\mathrm{loc}}(U)$ *be a minimizer of the thread problem with respect to* Γ *satisfying* (A1) *and assume now that* $U \subset \mathbb{R}^{n+1}$.

Suppose x_0 *is a regular point of* $\operatorname{spt}\Sigma \sim \operatorname{spt}\partial\Gamma$ *and* $\rho < \operatorname{dist}(x_0, \operatorname{spt}\partial\Gamma)$ *such that*

$$\Gamma \llcorner B_\rho(x_0) = m_\Gamma [\![M_\Sigma \cap B_\rho(x_0)]\!],\ m_\Gamma \in \mathbb{Z}^+ \cup \{0\}$$
$$\partial T \llcorner B_\rho(x_0) = m_{\partial T} [\![M_\Sigma \cap B_\rho(x_0)]\!],\ m_{\partial T} \in \mathbb{Z} \sim \{m_\Gamma\}$$

where M_Σ *is an* $(n-1)$*-dimensional embedded, oriented* C^1*-submanifold of* $\mathbb{R}^{n+1}$.

(1) *If* $m_{\partial T} \notin [0, m_\Gamma]$ M_Σ *is actually of class* C^∞ *and (for some smaller* $\rho > 0$)

(2.29) $$T \llcorner B_\rho(x_0) = m_{\partial T} [\![M_T \cap B_\rho(x_0)]\!] + m_0 [\![M_0 \cap B_\rho(x_0)]\!]$$

where M_T *is an oriented embedded minimal hypersurface of* $\mathbb{R}^{n+1}$ *with boundary* M_Σ, m_0 *is a nonnegative integer and* M_0 *is an oriented, embedded real-analytic minimal hypersurface without boundary which contains* M_T.

Moreover, the mean curvature vector H_Σ *of* M *satisfies* $|H_\Sigma| = 1/\lambda_\Sigma$ (λ_Σ *is the Lagrange multiplier of* Theorem 2.3). *In fact we have*

(2.30) $$\int_{M_\Sigma} \operatorname{div}_{M_\Sigma} X\, d\mathcal{H}^{n-1} = -\frac{1}{\lambda_\Sigma} \int_{M_\Sigma} \nu_{\partial T} \cdot X\, d\mathcal{H}^{n-1}$$

for all $X \in C_c^1(B_\rho(x_0); \mathbb{R}^{n+1})$, *where* $\nu_{\partial T}$ *is the outer unit normal vector of* M_Σ *with respect to* M_T.

Note in particular that all regular parts of Σ *have the* same *constant mean curvature.*

(2) *If* $0 \leqq m_{\partial T} < m_\Gamma$ *and condition* (A2) *of* Theorem (2.3) *holds in* $U \sim \overline{B_\rho(x_0)}$, M_Σ *is of class* $C^{1,\alpha}$ *for any* $\alpha < 1$ *and the generalized mean curvature vector* H_Σ *of* M_Σ *satisfies* $|H_\Sigma| \leqq \dfrac{1}{\lambda_\Sigma}$.

(3) *If* M_Σ *is stationary, i. e. when* (A1) *is not satisfied T may be supported by several distinct sheets of smooth surfaces with boundary* M_Σ.

Proof. — Suppose first of all that $x_0 \in \operatorname{spt}\Sigma \sim \operatorname{spt}\Gamma$. In this case we may assume $m_{\partial T} = m_\Sigma > 0$ and

$$\Sigma \llcorner B_\rho(x_0) = m_\Sigma [\![M_\Sigma \cap B_\rho(x_0)]\!].$$

From the local decomposition theorem in [WB] we infer

$$(2.31)\qquad \begin{aligned} T \llcorner B_\rho(x_0) &= \sum_{i=1}^{m_\Sigma} T_i \llcorner B_\rho(x_0) \\ \mathbf{M}(T \llcorner B_\rho(x_0)) &= \sum_{i=1}^{m_\Sigma} \mathbf{M}(T_i \llcorner B_\rho(x_0)) \end{aligned}$$

where each T_i satisfies $\partial T_i = \dfrac{1}{m_\Sigma}\Sigma$.

We want to show that $x_0 \in \operatorname{spt} \delta T_i$ for every $1 \leqq i \leqq m_\Sigma$. Since $\partial T_i = \dfrac{1}{m_\Sigma}\Sigma$ and (2.31) holds we can obviously apply Proposition 1.7 (2) again to derive that each $T_i \llcorner B_\rho(x_0)$ is a minimizer of the *thread problem* (in $B_{\rho/2}(x_0)$ say) with respect to $\Gamma = 0$.

If $x_0 \notin \operatorname{spt} \delta T_i$ we can find a radius $\sigma > 0$ such that $T_i \llcorner B_\sigma(x_0)$ is stationary. Hence the usual monotonicity formula holds for T_i at x_0 (*cf.* [SL], Chapt. 4). This and the fact that ∂T is regular in a neighbourhood of x_0 yields for small enough $\sigma > 0$

$$\frac{\mathbf{M}(T_i \llcorner B_\sigma(x_0))}{\sigma^n} + \frac{\mathbf{M}(\partial T_i \llcorner B_\sigma(x_0))}{\sigma^{n-1}} \leqq c$$

where c is independent of σ.

The fact that T_i locally minimizes mass in the ordinary sense with respect to ∂T_i and the compactness theorem for mass-minimizing currents ([SL], Chapt. 7), then imply the existence of a mass-minimizing tangent cone C_i at x_0. Obviously $\partial C_i = [\![T_{x_0} M_\Sigma]\!]$, where $T_{x_0} M_\Sigma$ denotes the oriented tangent space of M_Σ at x_0. By ([HS], Chapt. 11) C_i has to be the sum of an oriented n-dimensional halfplane of multiplicity one and possibly a hyperplane of arbitrary multiplicity containing this halfplane. Hence $\delta C_i \neq 0$.

On the other hand the lower-semicontinuity of the first variation with respect to varifold-convergence and the fact that T_i was assumed to be stationary in $B_\sigma(x_0)$ implies the stationarity of C_i and thus leads to a contradiction. Hence we conclude $x_0 \in \operatorname{spt} \delta T_i$.

Because each T_i satisfies (A2) and since (A1) holds T we may now apply Theorem 2.3, in particular (2.8) with T_0 replaced by T_i, to deduce

$$\delta T_i(X) + \frac{\lambda_\Sigma}{m_\Sigma} \delta\Sigma(X) = 0, \qquad 1 \leqq i \leqq m_\Sigma \tag{2.32}$$

for every $X \in C_c^1(B_\rho(x_0); \mathbb{R}^{n+1})$ (ρ slightly smaller than above).

Combining (2.10) and (2.32) we obtain

$$\delta\Sigma(X) = -\frac{1}{\lambda_\Sigma} \int \nu_{\partial T_i} \cdot X \, d\mu_\Sigma, \qquad 1 \leqq i \leqq m_\Sigma \tag{2.33}$$

for all $X \in C_c^1(B_\rho(x_0); \mathbb{R}^{n+1})$, where the $\nu_{\partial T_i}$ are $\mathscr{H}^{n-1}$-measurable and satisfy $|\nu_{\partial T_i}| \leqq 1$ $\mathscr{H}^{n-1}$-a. e. Standard regularity theory for C^1-solutions of the prescribed mean curvature system implies that $M_\Sigma \cap B_\rho(x_0)$ is of class $C^{1,\alpha}$ for any $\alpha < 1$ (and smaller radius $\rho > 0$). The boundary regularity theory for mass-minimizing currents (*cf.* [HS]) then yields (again for some smaller $\rho > 0$) that either

$$T \llcorner B_\rho(x_0) = m_\Sigma [\![M_T \cap B_\rho(x_0)]\!] + m_0 [\![M_0 \cap B_\rho(x_0)]\!]$$

where M_0 is an oriented, embedded real analytic minimal hypersurface without boundary which contains M_T and m_0 is a nonnegative integer, (M_T like the M_{T_i} below) or

$$T_i \llcorner B_\rho(x_0) = [\![M_{T_i} \cap B_\rho(x_0)]\!], \qquad 1 \leqq i \leqq m_\Sigma$$

where each M_{T_i} is an oriented, embedded minimal $C^{1,\alpha}$-hypersurface with boundary M_Σ.

In both cases the representation vector $\nu_{\partial T_i}$ for $\delta\Sigma$ in (2.33) is given by the exterior normal of M_Σ with respect to M_T and M_{T_i}, and is of class $C^{0,\alpha}$. We furthermore deduce from (2.33) that $\nu_{\partial T_i} = \nu_{\partial T_j}$ for $i \neq j$ which by virtue of the Hopf-boundary point lemma for minimal surfaces implies $M_{T_i} = M_{T_j}$ for $i \neq j$.

Moreover standard regularity theory implies $M_\Sigma \cap B_\rho(x_0) \in C^{2,\alpha}$. A standard "boot-strapping" argument then leads to the C^∞-regularity of M_Σ.

Since the above line of argument is applicable at every point in $M_\Sigma \cap B_\rho(x_0)$ (for the original radius $\rho > 0$) our conclusion also holds for the original ball $B_\rho(x_0)$.

Let us now assume $x_0 \in \operatorname{spt}\Gamma$ and $m_\Gamma \geqq 1$. Suppose $m_{\partial T} \notin [0, m_\Gamma)$. (If $m_{\partial T} = m_\Gamma$, $\Sigma \llcorner B_\rho(x_0) = 0$.) We again decompose

$$T \llcorner B_\rho(x_0) = \sum_{i=1}^{|m_{\partial T}|} T_i \llcorner B_\rho(x_0)$$

where the $T_i \llcorner B_\rho(x_0)$ are additive in mass and satisfy

$$\partial T_i \llcorner B_\rho(x_0) = \frac{m_{\partial T}}{|m_{\partial T}|} [\![M_\Sigma \cap B_\rho(x_0)]\!], \qquad 1 \leqq i \leqq m_\Sigma.$$

One easily checks that for $1 \leqq i \leqq |m_{\partial T}|$ and $\Sigma_i = \partial T_i$

$$\mathbf{M}(\Sigma \llcorner B_\rho(x_0)) = \mathbf{M}(\Sigma_i \llcorner B_\rho(x_0)) + \mathbf{M}((\Sigma - \Sigma_i) \llcorner B_\rho(x_0)).$$

Thus, as above, each $T_i \llcorner B_\rho(x_0)$ is [in view of Prop. 1.7 (2)] a minimizer of the thread problem in $B_\rho(x_0)$ with respect to $\Gamma = 0$. [In case $m_{\partial T} < 0$ even T minimizes the *thread problem* in $B_\rho(x_0)$ with respect to $\Gamma = 0$ since then $\mathbf{M}(\Sigma \llcorner B_\rho(x_0)) = \mathbf{M}(\partial T \llcorner B_\rho(x_0)) + \mathbf{M}(\Gamma \llcorner B_\rho(x_0))$.] As before we show $x_0 \in \operatorname{spt} \delta T_i$, $1 \leqq i \leqq |m_{\partial T}|$ which again enables us to apply (2.8) in order to deduce

$$\delta T_i(X) \pm \lambda_\Sigma \delta [\![M_\Sigma]\!](X) = 0, \qquad 1 \leqq i \leqq |m_{\partial T}|$$

depending on whether $m_{\partial T}$ is positive or negative. As this identity corresponds to (2.32) the same argument as before can be applied.

It remains to discuss the case where $0 \leqq m_{\partial T} < m_\Gamma$. Define

$$T' = T - T \llcorner B_\sigma(x_0)$$
$$\Gamma' = \Gamma - \Gamma \llcorner B_\sigma(x_0)$$
$$U' = (U \sim \overline{B_\sigma(x_0)}) \cup B_{\sigma/2}(x_0)$$

where $\sigma \leqq \rho$ is chosen such that the assumptions (A1) and (A2) still hold in U' [(A2) was assumed to be valid in $U \sim \overline{B_\rho(x_0)}$]. Since $\partial T' = 0$ in $B_{\sigma/2}(x_0)$ the conditions of Proposition 1.7 (2) are trivially satisfied for T' and $\Sigma' = \partial T' - \Gamma'$. Hence T' minimizes the *thread problem* in U' with respect to Γ'. Applying (2.7) we conclude

$$|\delta T'(X) + \lambda_\Sigma \delta \Sigma'(X)| \leqq \int |X \wedge \vec{\Gamma}'| \, d\mu_{\Gamma'}$$

for every $X \in C_c^1(U' \sim \operatorname{spt} \partial\Gamma'; \mathbb{R}^{n+1})$ where $\lambda_\Sigma > 0$ is determined by

$$T' \llcorner (U' \sim \overline{B_\rho(x_0)}) = T \llcorner (U \sim \overline{B_\rho(x_0)}).$$

Since $\Sigma' \llcorner B_\sigma(x_0) = -\Gamma' \llcorner B_\sigma(x_0)$ and $T' \llcorner B_\sigma(x_0) = 0$ we obtain

$$\left| \int \operatorname{div}_{M_\Sigma} X \, d\mathscr{H}^{n-1} \right| \leqq \frac{1}{\lambda_\Sigma} \int |X| \, d\mathscr{H}^{n-1}$$

for all $X \in C_c^1(B_\sigma(x_0); \mathbb{R}^{n+1})$.

The above argument works for every point in $M_\Sigma \cap B_\rho(x_0)$ with λ_Σ being determined by $T \llcorner (U \sim \overline{B_\rho(x_0)})$. This completes the proof. ■

In view of Proposition 2.7 (2) we define the set along which the *thread* Σ "sticks" to the *wire* Γ by

2.8. Definition

$$S_\Gamma = \{x \in \operatorname{spt} \Sigma \sim \operatorname{spt} \partial\Gamma / \exists \rho \in (0, \operatorname{dist}(x, \operatorname{spt} \partial\Gamma))$$

and

$$c \in [0, 1) \text{ s. t. } \partial T \llcorner B_\rho(x_0) = c(\Gamma \llcorner B_\rho(x_0))\}.$$

We are going to show that unless Σ is stationary away from its boundary the first variation of Σ does not vanish at all, except possibly along S_Γ.

2.9. Corollary

Let $T \in I_{n,\text{loc}}(U)$ *be a minimizer of the thread problem with respect to* $\Gamma \in I_{n-1,\text{loc}}(U)$, *where* $U \subset \mathbb{R}^{n+1}$.

Suppose $\operatorname{reg} \Gamma$ *is dense in* $\operatorname{spt} \Gamma$.

(1) *If* (A1) *of* Theorem 2.3 *is satisfied we have*

$$\operatorname{spt} \Sigma \sim (S_\Gamma \cup \operatorname{spt} \partial\Gamma) \subset \operatorname{spt} \delta\Sigma \tag{2.34}$$

(2) *If additionally* (A2) *and* (A3) *hold we have*

$$\operatorname{spt} \Sigma \sim (S_\Gamma \cup \operatorname{spt} \partial\Gamma) \subset \operatorname{spt} \delta T. \tag{2.35}$$

Proof. — (1) Let $x_0 \in \operatorname{spt} \Sigma \sim (S_\Gamma \cup \operatorname{spt} \partial\Gamma)$ and suppose there exists a $\rho < \operatorname{dist}(x_0, \operatorname{spt} \partial\Gamma)$ such that

$$\delta\Sigma(X) = 0, \qquad \forall X \in C_c^1(B_\rho(x_0); \mathbb{R}^{n+1})$$

where we may assume that $\rho < \operatorname{dist}(x_0, S_\Gamma)$. From Allard's regularity theorem ([AW], [SL], Chapt. 5) we see that inside $B_\rho(x_0)$ the set $\operatorname{reg} \Sigma$ is dense in $\operatorname{spt} \Sigma$. Using this and the assumption on $\operatorname{reg} \Gamma$ we may assume without loss of generality that

$$\partial T \llcorner B_\rho(x_0) = m_{\partial T} [\![M_\Sigma \cap B_\rho(x_0)]\!]$$
$$\Gamma \llcorner B_\rho(x_0) = m_\Gamma [\![M_\Sigma \cap B_\rho(x_0)]\!], \qquad m_\Gamma \in \mathbb{Z}^+ \cup \{0\}$$

where $m_{\partial T} \notin [0, m_\Gamma)$ since $x_0 \notin S_\Gamma$. M_Σ is a real-analytic $(n-1)$-dimensional oriented embedded minimal submanifold of $\mathbb{R}^{n+1}$.

On the other hand we obtain, using (A1) and Proposition 2.7 (1), that M_Σ has nonzero constant mean curvature, which is a contradiction.

(2) Suppose $x_0 \in \operatorname{spt}\Sigma \sim (S_\Gamma \cup \operatorname{spt}\partial\Gamma)$ and there exists a $\rho < \operatorname{dist}(x_0, \operatorname{spt}\partial\Gamma \cup S_\Gamma)$ such that

$$\delta T(X) = 0, \qquad \forall X \in C_c^1(B_\rho(x_0); \mathbb{R}^{n+1}). \tag{2.34}$$

Since (A1), (A2) and (A3) hold, we can apply Corollary 2.5 to deduce that the generalized mean curvature of Σ is bounded in every open set $W \subset\subset U \sim \operatorname{spt}\partial\Gamma$. Using again Allard's theorem we obtain that inside $B_\rho(x_0)$ the set $\operatorname{reg}\Sigma$ must be dense in $\operatorname{spt}\Sigma$. In view of the additional assumption $\operatorname{reg}\Gamma = \operatorname{spt}\Gamma$ we may proceed as in part (1) of the proof. Proposition 2.7 (1) [in particular (2.29)] and the divergence theorem for regular minimal submanifolds with boundary then imply $\delta T \llcorner B_\rho(x_0) \neq 0$ thus contradicting (2.34).

2.10. Corollary

Let $T \in I_{n,\mathrm{loc}}(U)$ *be a minimizer of the thread problem with respect to* $\Gamma \in I_{n-1,\mathrm{loc}}(U)$, *where* $U \subset \mathbb{R}^{n+1}$.

Suppose condition (A1) *is* not *satisfied, that is we have*

$$\delta\Sigma(X) = 0, \qquad \forall X \in C_c^1(U \sim \operatorname{spt}\partial\Gamma; \mathbb{R}^{n+1}). \tag{2.35}$$

In case $\operatorname{spt}\Sigma \subset \operatorname{spt}\Gamma$ *we furthermore assume that* $(\operatorname{reg}\Gamma \cap \operatorname{spt}\Sigma) \sim S_\Gamma \neq \emptyset$.

Suppose we have the following local decomposition of Σ*: Let* $x_0 \in \operatorname{spt}\Sigma \sim \operatorname{spt}\partial\Gamma$, $\rho < \operatorname{dist}(x_0, \operatorname{spt}\partial\Gamma)$ *and* $\Sigma_0 \in I_{n-1,\mathrm{loc}}(U)$ *satisfy*

$$\begin{aligned} &\Sigma \llcorner B_\rho(x_0) = \Sigma_0 \llcorner B_\rho(x_0) + (\Sigma - \Sigma_0) \llcorner B_\rho(x_0). \\ &\mathbf{M}(\Sigma \llcorner B_\rho(x_0)) = \mathbf{M}(\Sigma_0 \llcorner B_\rho(x_0)) + \mathbf{M}((\Sigma - \Sigma_0) \llcorner B_\rho(x_0)) \\ &\partial\Sigma_0 \llcorner B_\rho(x_0) = 0 \end{aligned} \tag{2.36}$$

Then

$$\delta\Sigma_0(X) = 0, \qquad \forall X \in C_c^1(B_\rho(x_0); \mathbb{R}^{n+1}). \tag{2.37}$$

Proof. — Let us suppose $x_0 \in \operatorname{spt}\delta\Sigma_0$.

If $\operatorname{spt}\Sigma \sim \operatorname{spt}\Gamma \neq \emptyset$ we can choose (by Allard's theorem) a point $x_1 \in \operatorname{reg}\Sigma \sim \operatorname{spt}\Gamma$ and $\sigma < \operatorname{dist}(x_1, \operatorname{spt}\Gamma)$ such that

$$\Sigma \llcorner B_\sigma(x_1) = m_\Sigma [\![M_\Sigma \cap B_\sigma(x_1)]\!] \tag{2.38}$$

where M_Σ is an $(n-1)$-dimensional oriented, embedded real analytic minimal submanifold of $\mathbb{R}^{n+1}$.

If $\operatorname{spt} \Sigma \subset \operatorname{spt}\Gamma$ we select $x_1 \in (\operatorname{reg}\Gamma \cap \operatorname{spt}\Sigma) \sim S_\Gamma$ and $\sigma < \operatorname{dist}(x_1, \operatorname{spt}\partial\Gamma \cup S_\Gamma)$. Again by Allard's theorem we may assume $x_1 \in \operatorname{reg}\Sigma$ such that

$$\begin{aligned} \partial T \llcorner B_\sigma(x_1) &= m_{\partial T} [\![M_\Sigma \cap B_\sigma(x_1)]\!] \\ \Gamma \llcorner B_\sigma(x_1) &= m_\Gamma [\![M_\Sigma \cap B_\sigma(x_1)]\!], \ m_\Gamma \in \mathbb{Z}^+ \cup \{0\} \end{aligned} \tag{2.39}$$

where $m_{\partial T} \notin [0, m_\Gamma)$ and M_Σ is as in (2.38). [(2.38) is a special case of (2.39).] We may also assume $x_1 \neq x_0$ and choose σ, ρ s.t. $B_\rho(x_0) \cap B_\sigma(x_1) = \emptyset$. (Note that $x_1 \in \operatorname{spt}\delta\Sigma_0$ would imply $x_1 \notin \operatorname{reg}\Sigma$.)

Define

$$\begin{aligned} \Gamma' &= \Gamma + (\Sigma - \Sigma_0) \llcorner B_\rho(x_0) \\ \Sigma' &= \partial T - \Gamma'. \end{aligned}$$

We then have

$$\begin{aligned} \Sigma' \llcorner B_\rho(x_0) &= \Sigma_0 \llcorner B_\rho(x_0) \\ \Sigma' \llcorner B_\sigma(x_1) &= \Sigma \llcorner B_\sigma(x_1) \\ \Gamma' \llcorner B_\sigma(x_1) &= \Gamma \llcorner B_\sigma(x_1). \end{aligned} \tag{2.40}$$

Using (2.36) and Proposition 1.7 (1) we conclude that T is a minimizer of the *thread problem* in $B_\rho(x_0) \cup B_\sigma(x_1)$ with respect to Γ' as new fixed boundary part. Furthermore (2.40) and the choice of x_0 imply $\operatorname{spt}\delta\Sigma' \sim \operatorname{spt}\partial\Gamma' \neq \emptyset$. Applying Proposition 2.7 (1) to T in $B_\sigma(x_1)$ we derive that M_Σ has nonzero constant mean curvature which gives a contradiction to (2.39).

2.11. Remark

Corollary 2.10 holds in arbitrary codimension if additionally require $\operatorname{spt}\delta T \sim \operatorname{spt}\Gamma \neq \emptyset$. Indeed, by virtue of (2.11) we can always find a point $x_1 \in \operatorname{spt}\delta T \sim \operatorname{spt}\Gamma$ different from x_0. Let $B_\sigma(x_1)$ and $B_\rho(x_0) \cup \operatorname{spt}\Gamma$ be disjoint. As in the proof of Corollary 2.10 T minimizes the *thread problem* in $B_\rho(x_0) \cup B_\sigma(x_1)$ with respect to Γ', where now $\Gamma' \llcorner B_\sigma(x_1) = 0$. Let $X_0 \in C_c^1(B_\rho(x_0); \mathbb{R}^{n+k})$ satisfy $\delta\Sigma_0(X_0) \neq 0$. From (2.1) applied to T and

Σ' in $B_\rho(x_0) \cup B_\sigma(x_1)$ we then infer [in view of (2.40) and $\Gamma' \llcorner B_\sigma(x_1)=0$]

$$|\delta T(X)\,\delta\Sigma_0(X_0) - \delta T(X_0)\,\delta\Sigma(X)| \leqq |\delta\Sigma(X)| \int |X_0 \wedge \vec{\Gamma}|\, d\mu_\Gamma$$

for every $X \in C_c^1(B_\sigma(x_1); \mathbb{R}^{n+k})$. The stationarity of Σ in $B_\sigma(x_1)$ and the fact that $\delta\Sigma_0(X_0) \neq 0$ contradict the choice of $x_1 \in \operatorname{spt} \delta T$.

The next Corollary of Theorem 2.3 is valid for arbitrary codimension.

2.12. Corollary

Let $T \in I_{n, \mathrm{loc}}(U)$ *satisfy the assumptions of* Theorem 2.3. *Suppose* $\Sigma \llcorner B_\rho(x_0)$ *decomposes as in* (2.36) *with* Σ_0 *satisfying* $\delta\Sigma_0 \llcorner B_\rho(x_0) \neq 0$.

Then for $\Gamma_0 = \Gamma + \Sigma - \Sigma_0$ *the inequality*

$$|\delta T(X) + \lambda_\Sigma \delta\Sigma_0(X)| \leqq \int |X \wedge \vec{\Gamma}_0|\, d\mu_{\Gamma_0} \tag{2.41}$$

holds for every $X \in C_c^1(B_\rho(x_0); \mathbb{R}^{n+k})$ *where* λ_Σ *is the Lagrange multiplier of* Theorem 2.3.

If we additionally assume (A3) (2.41) *implies that the generalized mean curvature vector* H_{Σ_0} *of* Σ_0 *satisfies*

$$|H_{\Sigma_0}| \leqq \frac{1}{\lambda_\Sigma} c(x_0, \rho, \Gamma), \qquad \mu_\Sigma\text{-a. e.} \quad \text{in } B_\rho(x_0) \tag{2.42}$$

where $c(x_0, \rho, \Gamma)$ *depends on* x_0, ρ *and the constant* $c(B_\rho(x_0))$ *of condition* (A3) (2) (*see* Cor. 2.5).

2.13. Remark

If $U \subset \mathbb{R}^{n+1}$ we can employ Proposition 2.7 to show that $|H_{\Sigma_0} \llcorner \operatorname{reg} \Sigma_0| \leqq \frac{1}{\lambda_\Sigma}$. Here "regular" refers to the parts of Σ_0 where ∂T is also regular (as in Prop. 2.7).

Proof of Corollary 2.12. — Taking (2.11) into account we can find a point x_1 different from x_0 such that (A2) holds at x_1. We assumed that

$$\delta\Sigma_0 \llcorner B_\rho(x_0) \neq 0. \tag{2.43}$$

We now choose $\sigma \in (0, \operatorname{dist}(x_1, \operatorname{spt} \partial\Gamma))$ such that $B_\sigma(x_1) \cap B_\rho(x_0) = \emptyset$. Let Γ' and Σ' be defined as in the proof of Corollary 2.10. T then minimizes the *thread problem* in $B_\sigma(x_1) \cup B_\rho(x_0)$ with respect to Γ' and $\Sigma' = \partial T - \Gamma'$.

Furthermore (A1) and (A2) hold in $B_\sigma(x_1) \cup B_\rho(x_0)$ [due to assumption (2.43), the choice of x_1 and the definition of Σ']. Theorem 2.3 then yields

$$|\delta T(X) + \lambda_\Sigma \delta\Sigma'(X)| \leqq \int |X \wedge \vec{\Gamma}'| \, d\mu_{\Gamma'}$$

for every $X \in C_c^1(B_\rho(x_0) \cup B_\sigma(x_1); \mathbb{R}^{n+k})$ which reduces to

$$|\delta T(X) + \lambda_\Sigma \delta\Sigma_0(X)| \leqq \int |X \wedge \vec{\Gamma}_0| \, d\mu_{\Gamma_0}$$

for every $X \in C_c^1(B_\rho(x_0); \mathbb{R}^{n+k})$.

Let us now assume that Γ satisfies assumption (A3). From Corollary 2.5 we infer

$$|H_\Sigma \llcorner B_\rho(x)| \leqq c(x_0, \rho, \Gamma), \qquad \mu_\Sigma\text{-a. e.} \quad \text{in } B_\rho(x_0).$$

[We denote all constants depending on x_0, ρ, Γ by $c(x_0, \rho, \Gamma)$.] Hence we can use the monotonicity formula [for $\Sigma \llcorner B_\rho(x_0)$] and ([SL], 17.9) to verify that Σ satisfies (A3) (with $\beta = 1$) in $B_\rho(x_0)$. Applying the same argument as in the proof of Corollary 2.5 we derive

$$\mu_\Sigma(B_\rho(x_0) \cap \operatorname{spt}\Sigma_0 \cap W) \leqq c(x_0, \rho, \Gamma)\, \mu_{\Sigma_0}(B_\rho(x_0) \cap W), \qquad \forall\, W \subset\subset B_\rho(x_0)$$

(using the definition of μ_Σ, μ_{Σ_0} and the fact that the monotonicity formula for Σ yields $\theta_\Sigma \leqq c(x_0, \rho, \Gamma)$ $\mathcal{H}^{n-1}$-a. e. in $B_\rho(x_0)$). Similarly we obtain in view of $\mu_{\Gamma_0} \leqq \mu_\Gamma + \mu_\Sigma + \mu_{\Sigma_0}$

$$\mu_{\Gamma_0}(B_\rho(x_0) \cap \operatorname{spt}\Sigma_0 \cap W) \leqq c(x_0, \rho, \Gamma)\, \mu_\Sigma(B_\rho(x_0) \cap \operatorname{spt}\Sigma_0 \cap W) + \mu_{\Sigma_0}(B_\rho(x_0) \cap W)$$

for every $W \subset\subset B_\rho(x_0)$.

Altogether we conclude

$$\mu_{\Gamma_0}(B_\rho(x_0) \cap \operatorname{spt}\Sigma_0 \cap W) \leqq c(x_0, \rho, \Gamma)\, \mu_{\Sigma_0}(B_\rho(x_0) \cap W), \qquad \forall\, W \subset\subset B_\rho(x_0)$$

which enables us to derive (2.42) from (2.41) as in the proof of Corollary 2.5 by differentiating μ_{Γ_0} with respect to μ_{Σ_0}. ■

3. PARTIAL REGULARITY FOR THE TWO DIMENSIONAL THREAD PROBLEM

3.1. Theorem

Let $T \in I_{2,\,\mathrm{loc}}(U)$ *be a minimizer of the thread problem with respect to* $\Gamma \in I_{1,\,\mathrm{loc}}(U)$, *where* $U \subset \mathbb{R}^3$.

Suppose

$$\delta\Sigma(X) = 0$$

for every $X \in C_c^1(U \sim \operatorname{spt} \partial\Gamma;\ \mathbb{R}^3)$.

In case $\operatorname{spt} \Sigma \subset \operatorname{spt} \Gamma$ *we furthermore assume*

$$(\operatorname{reg} \Gamma \cap \operatorname{spt} \Sigma) \sim S_\Gamma \neq \varnothing.$$

Then

$$\operatorname{sing} \Sigma \sim \operatorname{spt} \partial\Gamma = \varnothing. \tag{3.1}$$

3.2. Remark

Theorem 3.1. suggests sufficient conditions for assumption (A1) to hold.

In the simplest case (*see* also [DHL]), for instance if $\Gamma = m_\Gamma [\![\gamma]\!]$ where γ is a rectifiable Jordan arc in $\mathbb{R}^3$ with endpoints P_1 and P_2 then (A1) is satisfied if we assume

$$\mathbf{M}(\Sigma) > m_\Gamma \operatorname{dist}(P_1, P_2). \tag{3.2}$$

Proof of Theorem 3.1. — By exploiting the special structure of one dimensional stationary varifolds ([AA], Chapt. 3) we obtain that for every $x_0 \in \operatorname{spt} \Sigma \sim \operatorname{spt} \partial\Gamma$ there exists a $\rho < \operatorname{dist}(x_0, \operatorname{spt} \partial\Gamma)$ and a positive integer $N(x_0)$ such that

$$\Sigma \llcorner B_\rho(x_0) = \sum_{i=1}^{N(x_0)} m_i [\![l_i \cap B_\rho(x_0)]\!]$$

where $m_i \in \mathbb{Z}^+$ and the l_i denote piecewise linear curves through x_0 (singular only at x_0) without endpoints in $B_\rho(x_0)$. By virtue of Corollary 2.10, any local decomposition of Σ which does not introduce boundary points consists of stationary components only. Obviously this implies

$$\Sigma \llcorner B_\rho(x_0) = m [\![l \cap B_\rho(x_0)]\!]$$

where $m \in \mathbb{Z}^+$ and l is a line through x_0.

Thus every connected component of $\operatorname{spt} \Sigma$ has to be a line segment.

3.3. Remark

The Theorem holds for arbitrary codimension if we additionally require $\operatorname{spt}\delta T \sim \operatorname{spt}\Gamma \neq \varnothing$ (as in Remark 2.11).

3.4. Theorem

Let $T \in I_{2,\mathrm{loc}}(U)$ *satisfy the assumptions of* Corollary 2.5.

Then for every point $x_0 \in \operatorname{spt}\Sigma \sim \operatorname{spt}\partial\Gamma$ *there exists a radius* $\rho < \operatorname{dist}(x_0, \operatorname{spt}\partial\Gamma)$ *and a positive integer* $N(x_0)$ *such that*

$$\Sigma \llcorner B_\rho(x_0) = \sum_{i=1}^{N(x_0)} m_i [\![\sigma_i \cap B_\rho(x_0)]\!] \tag{3.3}$$

where $m_i \in \mathbb{Z}^+$ *and each* σ_i *is an embedded oriented* $C^{1,1}$*-curve through* x_0 *without endpoints in* $B_\rho(x_0)$. *Moreover all* σ_i *have the* same tangent *at* x_0.

Proof. – Let $x_0 \in \operatorname{spt}\Sigma \sim \operatorname{spt}\partial\Gamma$, $\rho \in (0, \operatorname{dist}(x_0, \operatorname{spt}\partial\Gamma))$. The decomposition theorem of ([FH], 4.2.25) implies

$$\begin{aligned} \Sigma \llcorner B_\rho(x_0) &= \sum_{i=1}^{\infty} [\![\sigma_i \cap B_\rho(x_0)]\!] \\ \mathbf{M}(\Sigma \llcorner B_\rho(x_0)) &= \sum_{i=1}^{\infty} L(\sigma_i \cap B_\rho(x_0)) \end{aligned} \tag{3.4}$$

where each σ_i is an embedded Lipschitz curve parametrized by arc length and L denotes the length of a curve.

Corollary 2.12 (in particular 2.42)

$$|H(\sigma_i) \llcorner B_{\rho_0}(x_0)| \leqq c(x_0, \rho_0, \Gamma), \qquad \mu_\Sigma\text{-a. e.}$$

where $\rho_0 < \operatorname{dist}(x_0, \operatorname{spt}\partial\Gamma)$ is fixed. $H(\sigma_i)$ denotes the generalized curvature of $[\![\sigma_i]\!]$. Using ([SL], Lemma 19.1) we may choose some $\rho \leqq \rho_0$ small enough depending on $c(x_0, \rho_0, \Gamma)$ such that $\overline{B_\rho(x_0)}$ does not contain any closed σ_i.

Moreover each σ_i has to be of class $C^{1,1}$. Indeed, since the σ_i are parametrized by arc length, the first variation formula for $[\![\sigma_i]\!]$ reduces to

$$\int \sigma_i' \eta' \, dt = \int H(\sigma_i) \eta \, dt$$

for all $\eta \in C_c^{0,1}(0, L(\sigma_i \cap B_\rho(x_0)))$.

Since $x_0 \in \operatorname{spt}\Sigma$ we can find for every $\rho_j \leqq \rho$ ($j \geqq 1$) a curve σ_j intersecting $B_{\rho_j}(x_0)$. Because there are no closed σ_j inside $\overline{B_\rho(x_0)}$, each σ_j has to intersect $\partial B_\rho(x_0)$ at least twice, which implies (by the continuity of the σ_j)

$$L(\sigma_j \cap B_\rho(x_0)) \geqq \rho$$

for large enough j. Hence (3.4) and the fact that $\mathbf{M}(\Sigma \llcorner \mathrm{B}_\rho(x_0)) < \infty$ imply that there are only finitely many σ_j contained in $\mathrm{B}_\rho(x_0)$. If we choose ρ small enough we can even ensure that there exists an $\mathrm{N}(x_0) \in \mathbb{Z}^+$ such that

$$\Sigma \llcorner \mathrm{B}_\rho(x_0) = \sum_{i=1}^{\mathrm{N}(x_0)} m_i [\![\sigma_i \cap \mathrm{B}_\rho(x_0)]\!],$$

where each σ_i contains x_0 and coinciding curves are counted with multiplicities.

We can the employ the decomposition argument of Corollary 2.13 to conclude that the tangents of all σ_i at x_0 have to agree. Otherwise we could find a decomposition of Σ consisting of components which are not even differentiable at x_0.

We are now able to prove a monotonicity formula for T at points of $\operatorname{spt} \Sigma \sim \operatorname{spt} \partial \Gamma$.

3.5. Proposition

Let T *satisfy the assumptions of* Theorem 3.4. *Let* Γ *be supported in an oriented embedded Jordan arc of class* $\mathrm{C}^{1,\alpha}$.

Then for every $x_0 \in \operatorname{spt} \Sigma \sim \operatorname{spt} \partial \Gamma$ *we can find a radius* $\rho(x_0) < \operatorname{dist}(x_0, \operatorname{spt} \partial \Gamma)$ *such that for every* $0 < \sigma < \rho \leqq \rho(x_0)$

$$(3.5) \quad \rho^{-2} \mathbf{M}(\mathrm{T} \llcorner \mathrm{B}_\rho(x_0)) - \sigma^{-2} \mathbf{M}(\mathrm{T} \llcorner \mathrm{B}_\sigma(x_0)) \geqq \int_{\mathrm{B}_\rho(x_0) \sim \mathrm{B}_\sigma(x_0)} r^{-2} (1 - |\nabla^{\mathrm{T}} r|) \, d\mu_{\mathrm{T}} - \frac{c}{\alpha} (\rho^\alpha - \sigma^\alpha)$$

where c depends only on the $\mathrm{C}^{1,\alpha}$*-norm and the multiplicity of* Γ.

Note in particular that (3.5) *is independent of* Σ.

Proof. — Let $x_0 = 0$. If $\rho(0)$ is small enough we can, for $\mathscr{L}^1$-a.e. $\rho < \rho(0)$, i.e. for those ρ s.t. $\partial(\Gamma \llcorner \mathrm{B}_\rho(x_0))$ is well defined (note that the following argument holds for arbitrary dimension), find a bi-Lipschitz-homeomorphism g_ρ in $\mathrm{B}_\rho(0)$ satisfying $g_\rho|_{\partial \mathrm{B}_\rho(0)} = \mathrm{id}$ and

$$g_{\rho\#}(\Gamma \llcorner \mathrm{B}_\rho(0)) = 0 \mathbin{⨳} \partial(\Gamma \llcorner \mathrm{B}_\rho(0))$$

where $0 \mathbin{⨳} \partial(\Gamma \llcorner \mathrm{B}_\rho(0))$ denotes the cone over $\partial(\Gamma \llcorner \mathrm{B}_\rho(0))$. (We can, for instance, look at $\operatorname{spt}(\Gamma \llcorner \mathrm{B}_\rho(0))$ as a graph over $\operatorname{spt}(0 \mathbin{⨳} \partial(\Gamma \llcorner \mathrm{B}_\rho(0)))$.) For $t \in [0, 1]$ let $h_\rho(t, x) = t g_\rho(x) + (1-t) x$ and define

$$\mathrm{T}_\rho = -h_{\rho\#}([\![(0, 1)]\!] \times (\Gamma \llcorner \mathrm{B}_\rho(0))).$$

From ([SL], 26.23) we obtain

$$\mathbf{M}(T_\rho) \leqq (1+\sup_{B_\rho} |Dg_\rho|)\,\mathrm{dist}(\mathrm{spt}(\Gamma \llcorner B_\rho(0)), \mathrm{spt}(0 \times\!\!\!\!\times \partial(\Gamma \llcorner B_\rho(0)))) \cdot \mathbf{M}(\Gamma \llcorner B_\rho(0))$$

which, since $\mathrm{spt}\,\Gamma \in C^{1,\alpha}$, implies

$$\mathbf{M}(T_\rho) \leqq c\,\rho^{n+\alpha} \tag{3.6}$$

where c depends on the $C^{1,\alpha}$-norm and the multiplicity of Γ.

Suppose now that

$$\mu_T(\partial B_\rho(0)) = 0$$

and that the slices $\langle T, r, \rho \rangle$ and $\partial(\partial T \llcorner B_\rho(0))$ are defined. (This holds for $\mathscr{L}^1$-a. e. ρ.)

Define

$$S_\rho = 0 \times\!\!\!\!\times \langle T, r, \rho \rangle + T_\rho + T \llcorner (U \sim \overline{B_\rho(0)}).$$

We obviously have for every $\varepsilon > 0$

$$\mathrm{spt}(S_\rho - T) \subset\subset B_{\rho+\varepsilon}(0).$$

Furthermore

$$\partial(0 \times\!\!\!\!\times \langle T, r, \rho \rangle) = \langle T, r, \rho \rangle + 0 \times\!\!\!\!\times \partial(\Sigma \llcorner B_\rho(0)) + 0 \times\!\!\!\!\times \partial(\Gamma \llcorner B_\rho(0))$$

$$\partial(T \llcorner (U \sim \overline{B_\rho(0)})) = \partial T \llcorner (U \sim \overline{B_\rho(0)}) - \langle T, r, \rho \rangle$$

$$\partial T_\rho = \Gamma \llcorner B_\rho(0) - 0 \times\!\!\!\!\times \partial(\Gamma \llcorner B_\rho(0))$$

which gives

$$\partial S_\rho - \Gamma = 0 \times\!\!\!\!\times \partial(\Sigma \llcorner B_\rho(0)) + \Sigma \llcorner (U \sim \overline{B_\rho(0)}).$$

Hence for every $\varepsilon > 0$ we have (set $B_\rho = B_\rho(0)$)

$$\mathbf{M}_{B_{\rho+\varepsilon}}(\partial S_\rho - \Gamma) = \mathbf{M}_{B_\rho}(0 \times\!\!\!\!\times \partial(\Sigma \llcorner B_\rho)) + \mathbf{M}_{B_{\rho+\varepsilon} \sim B_\rho}(\Sigma \llcorner (U \sim \overline{B_\rho})).$$

Using the special local structure of one dimensional *threads* given in (3.3) of Theorem 3.4 which implies that for small enough ρ $0 \times\!\!\!\!\times \partial(\Sigma \llcorner B_\rho)$ is supported in a finite number of line segments we obtain

$$\mathbf{M}_{B_{\rho+\varepsilon}}(\partial S_\rho - \Gamma) \leqq \mathbf{M}_{B_{\rho+\varepsilon}}(\partial T - \Gamma).$$

Applying Proposition 1.3 we derive

$$\mathbf{M}_{B_{\rho+\varepsilon}(0)}(T) \leqq \mathbf{M}_{B_{\rho+\varepsilon}(0)}(S_\rho).$$

Since $\mu_T(B_\rho(0)) = 0$ we can let ε tend to 0 to conclude

$$\mathbf{M}(T \llcorner B_\rho(0)) \leqq \mathbf{M}(0 \times\!\!\!\!\times \langle T, r, \rho \rangle) + \mathbf{M}(T_\rho)$$

which by (3.6) and the definition of $0 \times\!\!\!\!\times \langle T, r, \rho \rangle$ implies

$$\mathbf{M}(T \llcorner B_\rho(0)) \leqq \frac{\rho}{2}\mathbf{M}(\langle T, r, \rho \rangle) + c\,\rho^{2+\alpha}.$$

The coarea-formula yields for $\mathscr{L}^1$-a. e. $\rho > 0$

$$\rho^{-2}\mathbf{M}(\langle T, r, \rho\rangle) = \rho^{-2}\frac{d}{d\rho}\mathbf{M}(T \llcorner B_\rho(0)) - \frac{d}{d\rho}\int_{B_\rho(0)} r^{-2}(1-|\nabla^T r|)\,d\mu.$$

Hence we obtain in the usual way

$$\frac{d}{d\rho}(\rho^{-2}\mathbf{M}(T \llcorner B_\rho(0))) \geqq \frac{d}{d\rho}\int_{B_\rho} r^{-2}(1-|\nabla^T r|)\,d\mu_T - 2c\rho^{\alpha-1}.$$

The result follows by integration. ■

3.6. Remark

The monotonicity formula remains valid if we assume that in a neighbourhood of each point $x_0 \in \operatorname{spt}\Gamma$ Γ is supported in a finite number of $C^{1,\alpha}$-arcs which intersect at x_0. We only have to check that an estimate like (3.6) still holds in this case for some current T_ρ connecting $\Gamma \llcorner B_\rho(x_0)$ to the cone over $\partial(\Gamma \llcorner B_\rho(x_0))$.

3.7. Corollary

Let T *and* Γ *satisfy the assumptions of* Theorem 3.4. *Then at each point* $x_0 \in \operatorname{spt}\Sigma \sim \operatorname{spt}\partial\Gamma$ *there exists a mass-minimizing tangent cone* C (*with* "*vertex*" 0) *such that*

$$\partial C = m_\Sigma(x_0)[\![l_\Sigma]\!] + m_\Gamma[\![l_\Gamma]\!]$$

where l_Σ, l_Γ *are the tangent directions of* Σ *and* Γ *at* x_0, m_Γ *is the multiplicity of* Γ *and* $m_\Sigma(x_0) = \sum_{i=1}^{N(x_0)} m_i$,

Proof. — As in ([SL], Chapt. 7).

3.8. Remark

$\partial(C \llcorner B_1(0))$ is given by a combination of great circles and great circle segments with multiplicities which has boundary

$$m_\Sigma(x_0)[\![l_\Sigma \cap \partial B_1(0)]\!] + m_\Gamma[\![l_\Gamma \cap \partial B_1(0)]\!].$$

Note that in view of the interior regularity of C the curves involved are disjoint except at the endpoints of $l_\Sigma \cap B_1(0)$ and $l_\Gamma \cap B_1(0)$.

If in particular $x_0 \in \operatorname{spt}\Sigma \sim \operatorname{spt}\Gamma$, the tangent cone C either will be supported in the union of halfplanes with boundary l_Σ or is a plane containing l_Σ with some multiplicity p on one side of l_Σ and $m_\Sigma(x_0)+p$ on the other side of l_Σ.

If $x_0 \in \operatorname{spt}\Gamma \sim \operatorname{spt}\partial\Gamma$ the cone C may have (possibly in addition to full planes and halfplanes bounded by l_Σ and/or l_Γ) decomposable components supported in the union of the two oriented regions into which the plane spanned by l_Σ and l_Γ is divided by the lines l_Σ and l_Γ.

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