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Variational problems on classes of rearrangements and multiple configurations for steady vortices

by

G. R. BURTON

School of Mathematical Sciences, University of Bath,
Claverton Down, Bath, BA2 7AY, U.K.

ABSTRACT. — A Mountain Pass Lemma is proved for a convex functional restricted to the class $\mathscr{F}$ of rearrangements of a fixed L^p function. Together with results on maximization and minimization relative to $\mathscr{F}$, this proves the existence of a least four solutions for a problem on the steady configurations of a vortex in an ideal fluid.

Key words : Rearrangements, variational problems, convex functionals, Mountain Pass Lemma, multiple solutions, vortices, ideal fluids.

RÉSUMÉ. — On prouve un Lemme du type « Mountain Pass » pour une fonctionnelle convexe restreinte à la classe $\mathscr{F}$ des réarrangements d'une fonction seule en L^p. On l'applique conjointement à des résultats sur la maximisation et minimisation relatifs à $\mathscr{F}$, à la démonstration de l'existence d'au moins quatre solutions pour un problème qui regarde les configurations stationnaires d'un tourbillon dans un fluide parfait.

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1. INTRODUCTION

We are concerned here with the existence of stationary points of a variational functional Ψ relative to the class $\mathscr{F} \subset L^p$ of all rearrangements of some fixed function f_0 in an L^p space. The existence of maximizers for convex Ψ was studied by the author [5]. A particular convex Ψ was studied by McLeod [10] using different methods; he obtained an existence theorem for maximizers, and both existence and non-existence results for minimizers, according to the choice of f_0.

The main result of the present paper is a version of the Mountain Pass Lemma for a convex Ψ relative to $\mathscr{F}$. In addition we prove a result on the existence of maximizers and minimizers without the assumption of convexity.

We apply our results to a problem on steady 2-dimensional ideal fluid flow confined by a solid boundary. The functions in $\mathscr{F}$ then represent possible configurations of a specified distribution of vorticity in the fluid, and the functional Ψ represents kinetic energy. The principle that stationary points of Ψ relative to $\mathscr{F}$ represent steady flows is a modern formulation, following Benjamin [3], of Kelvin's ideas [9]; Arnol'd [2] has given the general principle for unsteady flows. We consider flow in a dumb-bell-shaped region and prove, for a suitable f_0, the existence of at least four solutions; two of these are local maximizers for Ψ, one is a minimizer, and one is constructed by the Mountain Pass Lemma. The existence of multiple solutions in a region of this shape was envisaged in Kelvin's paper. Dumb-bell-shaped regions have also been shown by Schaeffer [15] to yield nonuniqueness for a somewhat different variational problem arising in plasma physics.

The plan of the paper is as follows. In Section 2, by way of preliminaries, we study properties of the set $\mathscr{F}$ and of its weak closure $\overline{\mathscr{F}}$, and study the maximization of linear functionals relative to $\mathscr{F}$; this foreshadows the prominent role played by convexity in the theory. The Mountain Pass Lemma and results on nonlinear maximization and minimization are stated and proved in Section 3 using the results of Section 2. The application to fluid dynamics is given in Section 4.

2. LINEAR MAXIMIZATION RELATIVE TO REARRANGEMENTS

The purpose of this Section is to collect some known results on rearrangements, and to prove some new ones. For the most part it addresses the prerequisites of Section 3, but some material is presented simply for completeness.

DEFINITIONS AND NOTATION. — If (Ω, μ) and (Ω', μ') are positive measure spaces and $\mu(\Omega')=\mu(\Omega)<\infty$, measurable functions $f:\Omega\to\mathbb{R}$ and $g:\Omega'\to\mathbb{R}$ are called *rearrangements* of one another if $\mu(f^{-1}[\beta,\infty))=\mu'(g^{-1}[\beta,\infty))$ for every real β. If (Ω, μ) is a finite positive measure space, $\omega=\mu(\Omega)$ and $f:\Omega\to\mathbb{R}$ is a measurable function, then there is a decreasing function $f^{\Delta}:[0,\omega]\to\bar{\mathbb{R}}$ that is a rearrangement of f when $[0,\omega]$ is endowed with Lebesgue measure, and f^{Δ} is unique except for its values at its discontinuities. The assumption that f is finite-valued ensures that f^{Δ} can take infinite values only at 0 and ω.

If (Ω, μ) and (Ω', μ) are measure spaces, a map $\rho:\Omega\to\Omega'$ will be called a *measure-preserving transformation* if for every μ'-measurable set $A\subset\Omega'$, the inverse image $\rho^{-1}(A)$ is μ-measurable and $\mu(\rho^{-1}(A))=\mu'(A)$. If additionally ρ has an inverse, and the inverse is a measure-preserving transformation, we call ρ a *measure-preserving bijection*.

A measure space (Ω, μ) will be called a *measure interval* if there is a measure-preserving bijection from Ω to the interval $[0, \mu(\Omega)]$. For example, any Lebesgue measurable set in $\mathbb{R}^N$, with any finite measure that is absolutely continuous with respect to N-dimensional Lebesgue measure is a measure interval; this can be deduced from Royden [11], p. 270, Theorem 9. Measure intervals are a subclass of the finite separable nonatomic measure spaces considered in [5]. All the results we state for measure intervals can be generalized to finite separable nonatomic measure spaces, but we have not considered it worthwhile introducing the machinery required in the more general context. It will be important to observe that any measurable subset of a measure interval is again a measure interval.

Two measures on the same set, and having the same measurable sets, are called *equivalent* if each is absolutely continuous with respect to the other.

If (Ω, μ) is a positive measure space, $1\leq p<\infty$, if q is the conjugate exponent of p, we identify the dual space of $L^p(\mu)$ with $L^q(\mu)$, and if

$f \in L^p(\mu)$ and $g \in L^q(\mu)$ we write

$$\langle f, g \rangle = \int_\Omega fg\, d\mu.$$

If (Ω, μ) is a positive measure space, $1 \leqq p < \infty$ and $F \subset L^p(\mu)$, we denote by $\bar{F}$ the closure of F in the weak topology.

When Ω is an open set in $\mathbb{R}^N$, we denote by $W^m(\Omega)$ the space of real locally (Lebesgue) integrable functions on Ω whose distributional partial derivatives of orders $1, \ldots, m$ are locally integrable functions.

The first lemma is common knowledge; we supply the proof for completeness.

LEMMA 2.1. – *Let (Ω, μ) and (Ω', μ') be positive measure spaces with $\mu(\Omega) = \mu'(\Omega') = \omega < \infty$, let $f: \Omega \to \mathbb{R}$ and $g: \Omega' \to \mathbb{R}'$ be measurable functions, and suppose f is a rearrangement of g. Then*

(i) *For every Borel set $A \subset \mathbb{R}$ we have $\mu(f^{-1}(A)) = \mu'(g^{-1}(A))$.*

(ii) *If $\varphi: \mathbb{R} \to \mathbb{R}$ is Borel measurable then $\varphi \circ f$ is a rearrangement of $\varphi \circ g$.*

(iii) *If $f \in L^1(\mu)$ then $g \in L^1(\mu')$ and*

$$\int_\Omega f\, d\mu = \int_{\Omega'} g\, d\mu'.$$

(iv) *If $1 \leqq p < \infty$ and $f \in L^p(\mu)$ then $g \in L^p(\mu')$ and $\|f\|_p = \|g\|_{p'}$*

Proof. – (i) The family $\mathcal{S}$ of Borel sets $A \subset \mathbb{R}$ that satisfy $\mu(f^{-1}(A)) = \mu'(g^{-1}(A))$ is closed under complementation and under countable disjoint unions, and $[\beta, \infty) \in \mathcal{S}$ for all $\beta \in \mathbb{R}$, hence $\mathcal{S}$ contains all Borel sets. (ii) If β is real then $(\varphi \circ f)^{-1}[\beta, \infty) = f^{-1}(\varphi^{-1}[\beta, \infty))$ and $\varphi^{-1}[\beta, \infty)$ is a Borel set, so (i) can be applied to give

$$\mu((\varphi \circ f)^{-1}[\beta, \infty)) = \mu'((\varphi \circ g)^{-1}[\beta, \infty)).$$

(iii) The positive and negative parts of f and g can be considered separately, so suppose f and g are non-negative and for $s \geqq 0$ define

$$F(s) = \{x \in \Omega \mid f(x) \geqq s\}$$
$$G(s) = \{x \in \Omega' \mid g(x) \geqq s\}.$$

Then by Fubini's Theorem

$$\int_\Omega f\, d\mu = \int_\Omega \int_0^{f(x)} ds\, d\mu(x) = \int_0^\infty \int_{F(s)} d\mu\, ds = \int_0^\infty \mu(F(s))\, ds$$
$$= \int_0^\infty \mu'(G(s))\, ds = \int_{\Omega'} g\, d\mu'.$$

(iv) follows from (ii) and (iii). □

LEMMA 2.2. — *Let* (Ω, μ) *be a measure interval,* $1 \leqq p < \infty$, *let* $f_0 \in L^p(\mu)$ *and let* $\mathscr{F}$ *be the set of rearrangements of* f_0 *on* Ω. *Then*

(i) $\bar{\mathscr{F}}$ *is convex, so* $\bar{\mathscr{F}}$ *equals the closed convex hull of* $\mathscr{F}$.

(ii) $\bar{\mathscr{F}}$ *is weakly sequentially compact.*

(iii) $\bar{\mathscr{F}}$ *with the weak topology is metrizable.*

Remarks. — The cases $1 < p < \infty$ and $p=1$ of (i) follow from work of Brown [4] and Ryff [14] respectively, on Markov operators. The author [5] rediscovered (i) and gave a direct proof. The case $1 < p < \infty$ of (ii) follows trivially from (i), and the case $p=1$ follows readily from the Dunford-Pettis criterion for weak compactness in L^1. A countable family of continuous linear functionals separates points of $\bar{\mathscr{F}}$, so (iii) follows from (ii).

LEMMA 2.3. — *Let* (Ω, μ) *be a measure interval, let* $\omega = \mu(\Omega)$, *let* $1 \leqq p < \infty$, *let* $f_0 \in L^p(\mu)$, *and let* $\mathscr{F}$ *be the set of all rearrangements of* f_0 *on* Ω. *Then* $\mathscr{F}$ *is the set of extreme points of* $\bar{\mathscr{F}}$, *and*

$$\bar{\mathscr{F}} = \left\{ v \in L^1(\mu) \,\middle|\, \int_0^s v^\Delta \leqq \int_0^s f_0^\Delta \text{ for } 0 < s < \omega \text{ and } \int_\Omega v\, d\mu = \int_\Omega f_0\, d\mu \right\}.$$

Remarks. — This result is due to Ryff ([12], [13]), who showed when $p=1$ that the set defined in the curly brackets is weakly compact and convex, and that its set of extreme points is $\mathscr{F}$, so the result follows from Lemma 2.2 (i) and the Krein-Mil'man Theorem. The case $1 < p < \infty$ is easily deduced from the case $p=1$.

LEMMA 2.4. — *Let* (Ω, μ) *be a finite positive measure space, let* $\omega = \mu(\Omega)$, *let* $1 \leqq p < \infty$, *let* $f_0 \in L^p(\mu)$, *let* $\mathscr{F}$ *be the set of rearrangements of* f_0 *on* Ω, *let* q *be the conjugate exponent of* p *and let* $g \in L^q(\mu)$. *Then*

(i) *For all* $f \in \mathscr{F}$ *we have* $\langle f, g \rangle \leqq \int_0^\omega f_0^\Delta g^\Delta$.

(ii) *Suppose there exists an increasing function* φ *such that* $f^* = \varphi \circ g \in \mathscr{F}$. *Then*

(*a*) $\langle f^*, g \rangle = \int_0^\omega f_0^\Delta g^\Delta$.

(*b*) If $\{f_n\}_{n=1}^\infty$ *is a maximizing sequence for* $\langle \cdot, g \rangle$ *relative to* $\mathscr{F}$ *then* $\| f_n - f^* \|_p \to 0$ *as* $n \to \infty$.

(*c*) f^* *is the unique maximizer of* $\langle \cdot, g \rangle$ *relative to* $\bar{\mathscr{F}}$.

(iii) *If* (Ω, μ) *is a measure interval, then there is a measure-preserving transformation* $\rho : \Omega \to [0, \omega]$ *such that* $g = g^\Delta \circ \rho$. *Then* $f^* = f_0^\Delta \circ \rho \in \mathscr{F}$ *and satisfies*

$$\langle f^*, g \rangle = \int_0^\omega f_0^\Delta g^\Delta.$$

(iv) *If* (Ω, μ) *is a measure interval and* $\langle\,.\,, g\rangle$ *has only one maximizer* f^* *relative to* $\mathscr{F}$, *then there is an increasing function* φ *such that* $f^* = \varphi \circ g$ *almost everywhere in* Ω.

Remarks. — (i) is well-known in many versions, and its origins can be traced to Hardy *et al.* [8]; one proof that is valid in a general measure space is given in [5]. The author [5] proved (ii) and (iv), and Ryff [12] proved (iii).

LEMMA 2.5. — *Let* (Ω, μ) *be a finite positive measure space,* $1 \leqq p < \infty$, *let* q *be the conjugate exponent of* p, *let* $f_0 \in L^p(\mu)$, *and let* $\mathscr{F}$ *be the set of rearrangements of* f_0 *on* Ω. *For all* $g \in L^q(\Omega)$ *define* $\sigma(g)$ *to be the supremum of* $\langle\,.\,, g\rangle$ *relative to* $\mathscr{F}$. *Let* G *be the set of* $g \in L^q(\mu)$ *such that* $\varphi \circ g \in \mathscr{F}$ *for some increasing function* φ, *and let* $k(g) = \varphi \circ g$ *for this* φ. *Then*

(i) $|\sigma(g) - \sigma(h)| \leqq \|f_0\|_p \|g-h\|_q$ *for all* g *and* $h \in L^q(\mu)$.

(ii) $k : G \to \mathscr{F}$ *is strongly continuous.*

Proof. — (i) Let g, $h \in L^q(\mu)$, let $\varepsilon > 0$ and choose $f \in \mathscr{F}$ satisfying

$$\langle f, g\rangle \geqq \sigma(g) - \varepsilon.$$

Then

$$\sigma(h) \geqq \langle f, h\rangle \geqq \langle f, g\rangle - \|f\|_p \|g-h\|_q \geqq \sigma(g) - \varepsilon - \|f\|_p \|g-h\|_q.$$

Since $\|f\|_p = \|f_0\|_p$, letting ε tend to zero we obtain

$$\sigma(h) \geqq \sigma(g) - \|f_0\|_p \|g-h\|_q.$$

The same inequality is valid with g and h interchanged, hence (i).

(ii) If $g \in G$ then $k(g)$ is well defined, since $k(g)$ maximizes $\langle\,.\,, g\rangle$ relative to $\mathscr{F}$, and the maximizer is unique by Lemma 2.4. Fix $g \in G$ and let $\{g_n\}_{n=1}^{\infty}$ be a sequence in G such that $\|g_n - g\|_q \to 0$ as $n \to \infty$. Now

$$\begin{aligned}
\langle k(g_n), g\rangle &= \langle k(g_n), g_n\rangle + \langle k(g_n), g - g_n\rangle \\
&= \sigma(g_n) + \langle k(g_n), g - g_n\rangle \\
&\geqq \sigma(g_n) - \|f_0\|_p \|g - g_n\|_q \\
&\to \sigma(g)
\end{aligned}$$

as $n \to \infty$ by (i). Therefore $\{k(g_n)\}_{n=1}^{\infty}$ is a maximizing sequence for $\langle\,.\,, g\rangle$ relative to $\mathscr{F}$, so $\|k(g_n) - k(g)\|_p \to 0$ as $n \to \infty$, by Lemma 2.4. □

LEMMA 2.6. — *Let* (Ω, μ) *be a finite positive measure space, let* $1 \leqq p < \infty$, *let* $f_0 \in L^p(\mu)$ *and let* $\mathscr{F}$ *be the set of all rearrangements of* f_0 *on* Ω. *Then the relative weak and strong topologies on* $\mathscr{F}$ *coincide.*

Proof. — It will suffice to show that every strongly open subset of $\mathscr{F}$ is weakly open. Let q be the conjugate exponent of p and let U be a strongly open subset of $\mathscr{F}$. Consider $f_1 \in U$ and let $g = \tan^{-1} \circ f_1$, so $g \in L^q(\mu)$ and $f_1 = \tan \circ g$. It follows from Lemma 2.4 that if $\varepsilon > 0$ is sufficiently small.

then every $f \in \mathscr{F}$ satisfying $\langle f, g \rangle > \langle f_1, g \rangle - \varepsilon$ must lie in U. Then $V = \{ f \in \mathscr{F} \mid \langle f - f_1, g \rangle > -\varepsilon \}$ is weakly open and $f_1 \in V \subset U$. □

LEMMA 2.7. — *Let* (Ω, μ) *be a measure interval and let* $1 \leqq p < \infty$. *Then* $\| f^\Delta - g^\Delta \|_p \leqq \| f - g \|_p$ *for all* f *and* g *in* $L^p(\mu)$.

Remarks. — This result belongs to folklore. A neat proof in the case when f and g are non-negative follows from Corollary 1 of Crowe *et al.* [6], and the case when f and g are bounded below follows immediately from this. Now fix $f, g \in L^p(\mu)$ and for $m < 0$ define

$$f_m(x) = \max \{ m, f(x) \}$$
$$g_m(x) = \max \{ m, g(x) \}$$

for $x \in \Omega$. Then

$$\| f_m^\Delta - g_m^\Delta \|_p \leqq \| f_m - g_m \|_p.$$

As m tends to $-\infty$ the pointwise limits of f_m, g_m, f_m^Δ and g_m^Δ are f, g, f^Δ and g^Δ respectively, and the Dominated Convergence Theorem can be applied to deduce the general case.

LEMMA 2.8. — *Let* (Ω, μ) *be a measure interval, let* $1 \leqq p < \infty$, *let* $g : \Omega \to \mathbb{R}$ *be a measurable function and let* $f_1, f_2 \in L^p(\mu)$. *Suppose there exist increasing functions* φ_1 *and* φ_2 *such that* $f_1^* = \varphi_1 \circ g$ *and* $f_2^* = \varphi_2 \circ g$ *are rearrangements of* f_1 *and* f_2 *respectively. Then*

$$\| f_1^* - f_2^* \|_p \leqq \| f_1 - f_2 \|_p.$$

Proof. — Write $\omega = \mu(\Omega)$. If $i = 1$ or 2 then $\varphi_i \circ g^\Delta$ is decreasing and is a rearrangement of $\varphi_i \circ g = f_i^*$, so $\varphi_i \circ g^\Delta = f_i^{*\Delta} = f_i^\Delta$. Therefore

$$\begin{aligned} \| f_1^* - f_2^* \|_p &= \left(\int_\Omega | \varphi_1 - \varphi_2 |^p \circ g \, d\mu \right)^{1/p} \\ &= \left(\int_0^\omega | \varphi_1 - \varphi_2 |^p \circ g^\Delta \right)^{1/p} \\ &= \| f_1^\Delta - f_2^\Delta \|_p \leqq \| f_1 - f_2 \|_p \end{aligned}$$

by Lemma 2.7. □

LEMMA 2.9. — *Let* (Ω, μ) *be a finite positive measure space, let* $f : \Omega \to \mathbb{R}$ *and* $g : \Omega \to \mathbb{R}$ *be measurable functions, and suppose that every level set of* g *has zero measure. Then there is an increasing function* φ *such that* $\varphi \circ g$ *is a rearrangement of* f. *In particular,*

$$\varphi(s) = f^\Delta(\mu(\{ x \in \Omega \mid g(x) \geqq s \})).$$

Proof. — Since the level sets of g have zero measure, g^Δ is strictly decreasing and therefore injective. A right inverse for g^Δ is defined by the decreasing function

$$\psi(s)=\mu(\{x\in\Omega \mid g(x) \geqq s\}),$$

that is, $\psi(g^{\Delta}(t))=t$ for $0 < t < \omega$. Define $\varphi=f^{\Delta}\circ\psi$, which is an increasing function. Then $\varphi\circ g^{\Delta}=f^{\Delta}$, hence $\varphi\circ g$ is a rearrangement of f. □

LEMMA 2.10. — *Let* (Ω, μ) *be a finite positive measure space, let* $1 \leqq p < \infty$, *let* $1 \leqq r < \infty$, *let* $J \subset \mathbb{R}$ *be an interval, let* $f_0 \in L^p(\mu)$, *let* $\tau: J \to \mathbb{R}$ *be a continuous function such that* $\tau\circ f_0 \in L^r(\mu)$ *and let* $\mathscr{F}$ *be the set of rearrangements of* f_0 *on* Ω. *Then* $f \mapsto \tau\circ f$ *defines a continuous map from* $\mathscr{F}$ *to* $L^r(\mu)$.

Proof. — Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in $\mathscr{F}$ converging to a limit $f\in\mathscr{F}$ and let $\varepsilon > 0$. For $M > 0$ define

$$\tau_M(s)=\begin{cases} M & \text{if } \tau(s)\geqq M \\ \tau(s) & \text{if } -M<\tau(s)<M \\ -M & \text{if } \tau(s)\leqq -M \end{cases}$$

$$\sigma_M(s)=\tau(s)-\tau_M(s)$$

when $s\in J$. Fix M so large that $\|\sigma_M\circ f\|_r < \varepsilon$, so $\|\sigma_M\circ f_n\|_r < \varepsilon$ for $n=1, 2, \ldots$ by Lemma 2.1. Then

$$\|\tau\circ f_n-\tau\circ f\|_r \leqq \|\tau_M\circ f_n-\tau_M\circ f\|_r+\|\sigma_M\circ f_n\|_r+\|\sigma_M\circ f\|_r < \|\tau_M\circ f_n-\tau_M\circ f\|_r+2\varepsilon.$$

Since τ_M is bounded and continuous it follows from the Dominated Convergence Theorem that $\|\tau_M\circ f_n-\tau_M\circ f\|_r\to 0$ as $n\to\infty$, hence $\|\tau\circ f_n-\tau\circ f\|_r < 3\varepsilon$ for all sufficiently large n. □

LEMMA 2.11. — *Let* (Ω, μ) *be a measure interval, let* $1 \leqq p < \infty$, *let* $f_0\in L^p(\mu)$ *and let* $\mathscr{F}$ *be the set of rearrangements of* f_0 *on* Ω. *Then* $\mathscr{F}$ *is path-connected.*

Proof. — It is sufficient to consider the case when (Ω, μ) is an interval $[0, \omega]$. Let $f\in\mathscr{F}$. We construct a path from f to f_0^{Δ} as follows. Let $\tau: \mathbb{R}\to(0, \pi)$ be defined by $\tau(s)=\pi/2+\tan^{-1}s$. Let $F=\tau\circ f$, so $F\in L^p[0, \omega]$ also, and for $s, t\in[0, \omega]$ define

$$F_t(s)=\begin{cases} F(s) & \text{for } 0\leqq s\leqq t \\ 0 & \text{for } t<s\leqq\omega \end{cases}$$

$$G_t(s)=F(s)-F_t(s)$$

$$F_t^*(s)=F_t^{\Delta}(s)+G_t(s).$$

Then F_t^* is a rearrangement of F for each $t\in[0, \omega]$ and $t\mapsto F_t^*$ is a continuous map from $[0, \omega]$ to $L^p[0, \omega]$ by Lemma 2.7. Define $f_t^*=\tau^{-1}\circ F_t^*$ for $0\leqq t\leqq\omega$, so $f_t^*\in\mathscr{F}$. Then $f_0^*=f$ and $f_\omega^*=f^{\Delta}=f_0^{\Delta}$, and the continuity of the map $t\mapsto f_t^*$ follows from Lemma 2.10. It now follows that any two points of $\mathscr{F}$ can be connected by a path through f_0^{Δ}. □

LEMMA 2.12. — *Let* (Ω, μ) *be a measure interval, let* $1 \leqq p < \infty$, *let* q *be the conjugate exponent of* p, *let* $f_0 \in L^p(\mu)$, *let* $\mathcal{F}$ *be the set of rearrangements of* f_0 *on* Ω, *let* $g \in L^p(\mu)$, *suppose every level set of* g *has zero measure, and let* φ *be an increasing function such that* $f^* = \varphi \circ g \in \mathcal{F}$. *Let* $f \in \mathcal{F}$. *Then there is a continuous path from* f *to* f^* *in* $\mathcal{F}$, *along which* $\langle \cdot, g \rangle$ *is increasing.*

Proof. — The existence of φ follows from Lemma 2.9. Define $\tau: \mathbb{R} \to (0, \pi)$ by $\tau(s) = \pi/2 + \tan^{-1} s$, let $F = \tau \circ f$ and let $F^* = \tau \circ f^*$, so F, $F^* \in L^p(\mu)$. For $-\infty \leqq s \leqq \infty$ and $x \in \Omega$ define

$$\Gamma(s) = \{y \in \Omega \mid g(y) \geqq s\}$$
$$F_s(x) = \begin{cases} F(x) & \text{for } x \in \Gamma(s) \\ 0 & \text{for } x \in \Omega \setminus \Gamma(s) \end{cases}$$
$$H_s(x) = F(x) - F_s(x).$$

By Lemma 2.9 for each $-\infty \leqq s \leqq \infty$ we can choose an increasing function φ_s such that $\varphi_s \circ g \mid \Gamma(s)$ is a rearrangement of $F \mid \Gamma(s)$. We extend φ_s so that $\varphi_s(t) = 0$ for $t < s$, define $F_s^\wedge = \varphi_s \circ g$ on Ω, and define $F_s^* = F_s^\wedge + H_s$. Observe that $F_{-\infty}^* = F^*$, that $F_\infty^* = F$, and that F_s^* is a rearrangement of F for each s. Consider $-\infty \leqq s \leqq t \leqq \infty$. Then $\Gamma(t) \subset \Gamma(s)$ and by Lemma 2.8 we have

$$\| F_t^* - F_s^* \|_p \leqq \| F_t^\wedge - F_s^\wedge \|_p + \| H_t - H_s \|_p$$
$$\leqq \| F_t - F_s \|_p + \| H_t - H_s \|_p = 2 \| F_t - F_s \|_p.$$

Therefore $t \mapsto F_t^*$ is continuous from $[-\infty, \infty]$ to $L^p(\mu)$.

Define $f_t^* = \tau^{-1} \circ F_t^*$ for $-\infty \leqq t \leqq \infty$. Then f_t^* defines a continuous path in $\mathcal{F}$ from f^* to f, by Lemma 2.10. It remains to check that $\langle f_t^*, g \rangle$ is a decreasing function of t. Let $-\infty \leqq s \leqq t \leqq \infty$. Then

$$\langle f_s^* - f_t^*, g \rangle = \int_{\Gamma(s)} f_s^* g \, d\mu - \left(\int_{\Gamma(t)} f_t^* g \, d\mu + \int_{\Gamma(s) \setminus \Gamma(t)} fg \, d\mu \right).$$

Relative to $\Gamma(s)$ we have $f_s^* = \tau^{-1} \circ \varphi_s \circ g$ and that f_s^* is a rearrangement of $f_t^* + 1_{\Gamma(s) \setminus \Gamma(t)} f$, so it follows from Lemma 2.4 that

$$\langle f_s^* - f_t^*, g \rangle \geqq 0$$

as required. □

Lemmas 2.13 and 2.15 are steps in the proof of Theorems 8 and 9 of [5], but the proofs given here are clearer.

LEMMA 2.13. — *Let* C *be a convex set in a real vector space* X, *let* x^* *and* y^* *be linear functionals on* X, *let* I *be a real number and suppose there exist* x_1 *and* x_2 *in* C *such that* $x^*(x_1) < I < x^*(x_2)$. *Suppose* $x_0 \in C$ *is such that* $y^*(x) \leqq y^*(x_0)$ *for all* $x \in C$ *satisfying* $x^*(x) = I$. *Then there is a real number* λ *such that* x_0 *maximizes* $y^* + \lambda x^*$ *relative to* C.

Proof. — Let J be the range of x^* on C, and for $\alpha \in J$ define

$$V(\alpha) = \inf\{-y^*(x) \mid x \in C,\ x^*(x) = \alpha\} < \infty.$$

Then $J \subset \mathbb{R}$ is an interval, V is a convex function, I is interior to J and $V(I) = -y^*(x_0) > -\infty$. It now follows that V is everywhere finite-valued, and that V is subdifferentiable on the interior of J. Let $\lambda \in \partial V(I)$. Then for all $x \in C$ we have

$$-y^*(x) \geqq V(x^*(x)) \geqq V(I) + \lambda(x^*(x) - I) = -y^*(x_0) + \lambda x^*(x) - \lambda x^*(x_0),$$

therefore

$$(y^* + \lambda x^*)(x) \leqq (y^* + \lambda x^*)(x_0). \quad \square$$

LEMMA 2.14. — *Let* (Ω, μ) *be a finite positive measure space, let* $1 \leqq p < \infty$, *let* $f_0 \in L^p(\mu)$ *be non-negative and let* $\mathscr{F}$ *be the set of rearrangements of* f_0 *on* Ω. *Let* $f \in \overline{\overline{\mathscr{F}}}$. *Then* $f \geqq 0$ *and*

$$\mu\{x \in \Omega \mid f(x) > 0\} \geqq \mu\{x \in \Omega \mid f_0(x) > 0\}.$$

Remarks. — Since $f \in \overline{\operatorname{conv} \mathscr{F}}$ it is clear that $f \geqq 0$, and the remaining assertion follows from Lemma 5 of [5].

LEMMA 2.15. — *Let* Ω *be an open set in* $\mathbb{R}^N$, μ *a finite positive measure on* Ω *equivalent to* N*-dimensional Lebesgue measure, and let*

$$\mathscr{L} u = \sum_{1 \leqq |\alpha| \leqq m} a^\alpha(x) D^\alpha u$$

define a linear partial differential operator in Ω, *where the* a^α *are measurable and there is no* 0-th *order term. Let* $1 \leqq p < \infty$, *let* q *be the conjugate exponent of* p, *let* $f_0 \in L^p(\mu)$ *be non-negative, let* $\mathscr{F}$ *be the set of rearrangements of* f_0 *on* Ω *and let* $g \in L^q(\mu) \cap W^m(\Omega)$. *Suppose that* $f^* \in \overline{\overline{\mathscr{F}}}$ *maximizes* $\langle\, .\,, g \rangle$ *relative to* $\overline{\overline{\mathscr{F}}}$ *and that* $\mathscr{L} g \geqq f^*$ *almost everywhere in* Ω. *Then* $f^* \in \mathscr{F}$ *and there is an increasing function* φ *such that* $f^* = \varphi \circ g$ *almost everywhere in* Ω.

Proof. — Let

$$\begin{aligned} &S = \{x \in \Omega \mid f^*(x) > 0\} \\ &s = \mu(S) \\ &\sigma = \mu\{x \in \Omega \mid f_0(x) > 0\} \\ &\gamma = \operatorname{ess\,inf} g(S) \\ &\omega = \mu(\Omega). \end{aligned}$$

We first show that $g(x) \leqq \gamma$ for almost all $x \in \Omega \setminus S$. Suppose this is false. Then for some $\beta > \gamma$ and for some set $A \subset \Omega \setminus S$ having positive measure we have $g(x) > \beta$ for all $x \in A$. Let $\beta > \delta > \gamma$. Then by the definition of γ there is a subset $B \subset S$ having positive measure such that $g(x) < \delta$ for all

$x \in B$. Since A and B are measure intervals, we can replace A and B by subsets of themselves to ensure further that $0 < \mu(A) = \mu(B)$. There is then a measure preserving bijection $\pi: A \to B$. For $v \in L^p(\mu)$ define

$$T v(x) = \begin{cases} v(\pi(x)) & \text{if } x \in A \\ v(\pi^{-1}(x) & \text{if } x \in B \\ v(x) & \text{if } x \in \Omega \setminus (A \cup B). \end{cases}$$

Then $T: L^p(\mu) \to L^p(\mu)$ is bounded and $T(\mathscr{F}) \subset \mathscr{F}$ so $T(\bar{\mathscr{F}}) \subset \bar{\mathscr{F}}$. Write $f_1 = Tf^*$, so $f_1 \in \bar{\mathscr{F}}$. We have

$$\begin{aligned} \langle f_1, g \rangle - \langle f^*, g \rangle &= \int_{A \cup B} (f_1 - f^*) g \, d\mu \\ &= \int_A (f^* \circ \pi - f^*) g \, d\mu + \int_B (f^* \circ \pi^{-1} - f^*) g \, d\mu \\ &= \int_A (f^* \circ \pi) g \, d\mu - \int_B f^* g \, d\mu \\ &= \int_B f^* (g \circ \pi^{-1} - g) \, d\mu \geqq \int_B f^* (\beta - \delta) \, d\mu > 0 \end{aligned}$$

which is impossible since f^* maximizes $\langle \cdot, g \rangle$ relative to $\bar{\mathscr{F}}$. Thus $g(x) \leqq \gamma$ for almost every $x \in \Omega \setminus S$ as claimed.

We next show that every level set of $g \mid S$ has zero measure. Suppose $\alpha \in \mathbb{R}$ and let $H = \{x \in \Omega \mid g(x) = \alpha\}$. Then $\mathscr{L} g = 0$ almost everywhere in H, by for example Lemma 7.7 of [7]. But $\mathscr{L} g \geqq f^* > 0$ in S, so $\mu(H \cap S) = 0$. In particular, we now have $g(x) > \gamma$ for almost all $x \in S$. It follows from Lemma 2.9 that there exists an increasing function φ such that $\varphi \circ (g \mid S)$ is a rearrangement of $f_0^\Delta \mid [0, s]$. Since $g > \gamma$ on S and $f_0 \geqq 0$ we can suppose that $\varphi \geqq 0$ and φ is undefined on $(-\infty, \gamma]$. It follows from Lemma 2.14 that $s \geqq \sigma$, hence $f_0^\Delta = 0$ on $[s, \omega]$; if we now define $\varphi(t) = 0$ for $t \leqq \gamma$, the function φ is increasing and $\varphi \circ g$ is a rearrangement of f_0^Δ, so $\varphi \circ g \in \mathscr{F}$. It follows from Lemma 2.4 that $\varphi \circ g$ is the unique maximizer for $\langle \cdot, g \rangle$ relative to $\bar{\mathscr{F}}$, so $f^* = \varphi \circ g$ and $f^* \in \mathscr{F}$. □

3. VARIATIONAL PRINCIPLES

DEFINITIONS. — Suppose (Ω, μ) is a positive measure space, that $1 \leqq p \leqq \infty$, that q is the conjugate exponent of p and that $K: L^p(\mu) \to L^q(\mu)$ is a bounded linear operator. We say K is *symmetric* if $\langle v, K w \rangle = \langle w, K v \rangle$ for all $v, w \in L^p(\mu)$. We say K is *positive* if $\langle v, K v \rangle \geqq 0$ for all $v \in L^p(\mu)$, and we say K is *strictly positive* if in addition, equality only holds for $v = 0$.

We present three abstract variational principles, each of which ensures the existence of a stationary point of a functional Ψ relative to the weak closure $\bar{\mathscr{F}}$ of the class $\mathscr{F}$ of rearrangements of a function f_0. We give three Corollaries in which the stationary point is shown to lie in $\mathscr{F}$ in the case of certain boundary value problems. The trick used there is Lemma 2.15, and it requires $f_0 \geqq 0$. Our first result is an analogue for the present situation of Theorem 2.1 of Ambrosetti and Rabinowitz [1], which applied to C^1 functionals on a Banach space, without constraints. The terms and notation used in Theorem 3.1 have been defined at the beginning of Section 2.

THEOREM 3.1 (Mountain Pass Lemma for rearrangements). — *Let* (Ω, μ) *be a measure interval, let* $1 < p < \infty$, *let* $\Psi : L^p(\mu) \to \mathbb{R}$ *be a continuously differentiable convex functional, let* $f_0 \in L^p(\mu)$, *let* $\mathscr{F}$ *be the set of rearrangements of* f_0 *on* Ω, *let* $e_0, e_1 \in \mathscr{F}$ *and define*

$$\mathscr{C} = \{h \in C([0, 1], \mathscr{F}) \mid h(0) = e_0 \text{ and } h(1) = e_1\}$$
$$c = \sup_{h \in \mathscr{C}} \inf_{0 \leqq t \leqq 1} \Psi(h(t)).$$

Suppose

$$\inf \Psi(L^p(\mu)) < c < \min\{\Psi(e_0), \Psi(e_1)\}.$$

Then there exists a sequence $\{v_n\}_{n=1}^{\infty}$ in $\mathscr{F}$ satisfying

$$\Psi(v_n) \to c$$
$$\sup_{\mathscr{F}} \langle \,.\,, \Psi'(v_n)\rangle - \langle v_n, \Psi'(v_n)\rangle \to 0$$

as $n \to \infty$.

Proof. — There is no loss of generality in supposing (Ω, μ) to be the interval $[0, \omega]$ with Lebesgue measure. Introduce the notation

$$\sigma(g) = \sup_{\mathscr{F}} \langle \,.\,, g\rangle$$

for $g \in L^q[0, \omega]$ where q is the conjugate exponent of p. Let $\varepsilon > 0$ satisfy

$$c + 2\varepsilon < \min\{\Psi(e_0), \Psi(e_1)\} \tag{3.1}$$
$$c - 2\varepsilon > \inf \Psi(L^p[0, \omega]). \tag{3.2}$$

We will prove the existence of a point $v \in \mathscr{F}$ satisfying both

$$c - 2\varepsilon < \Psi(v) < c + 2\varepsilon \tag{3.3}$$
$$\sigma(v) - \langle v, \Psi'(v)\rangle < 2\varepsilon. \tag{3.4}$$

Suppose, to seek a contradiction, that no point v of $\mathscr{F}$ satisfies both (3.3) and (3.4). Consider any $h \in \mathscr{C}$ satisfying

$$\inf \Psi(h[0, 1]) > c - \varepsilon. \tag{3.5}$$

Define

$$\gamma = \inf\{\| \Psi'(h(t)) \| \mid 0 \leqq t \leqq 1\},$$

so $\gamma > 0$ in view of (3.2), (3.5) and the convexity of Ψ.

Write V_N for the linear subspace of $L^q[0, \omega]$ spanned by the functions $\sin(\pi x/\omega), \ldots, \sin(N\pi x/\omega)$, whenever N is a natural number, and let Λ_N denote the nearest-point map from $L^q[0, \omega]$ onto V_N. Then Λ_N is strongly continuous. Let us choose N such that

$$\| \Psi'(h(t)) - \Lambda_N \Psi'(h(t)) \|_q < \delta = \min\{\gamma, \varepsilon/(2\| f_0 \|_p)\} \tag{3.6}$$

for all $t \in [0, 1]$, so that in particular $\Lambda_N \Psi'(h(t)) \neq o$. Each level set of each nonzero element of V_N is finite so by Lemmas 2.9 and 2.4, for each $t \in [0, 1]$ there is a unique maximizer $k(t)$ for the functional $\langle ., \Lambda_N \Psi'(h(t)) \rangle$ relative to $\mathscr{F}$, and $k(t) = \varphi_t \circ (\Lambda_N \Psi'(h(t)))$ almost everywhere for some increasing function φ_t. It now follows from Lemma 2.5 that $k: [0, 1] \to \mathscr{F}$ is continuous.

Let $0 \leqq t \leqq 1$. We now show that

$$\Psi(k(t)) > c. \tag{3.7}$$

Consider first the case when $\Psi(h(t)) \geqq c + 2\varepsilon$. Then by convexity we have

$$\begin{aligned} \Psi(k(t)) &\geqq \Psi(h(t)) + \langle k(t) - h(t), \Psi'(h(t)) \rangle \\ &\geqq \Psi(h(t)) + \langle k(t) - h(t), \Lambda_N \Psi'(h(t)) \rangle - 2\| f_0 \|_p \delta \\ &\geqq \Psi(h(t)) - 2\| f_0 \|_p \delta \\ &\geqq c + 2\varepsilon - \varepsilon = c + \varepsilon, \end{aligned}$$

where we have used (3.6) and the maximizing property of $k(t)$. Thus (3.7) holds in this case. Now consider the other case, when $\Psi(h(t)) < c + 2\varepsilon$. Then by (3.5) we have (3.3) for $v = h(t)$ so (3.4) must fail. Thus

$$\sigma(\Psi'(h(t))) \geqq \langle h(t), \Psi'(h(t)) \rangle + 2\varepsilon. \tag{3.8}$$

By convexity, (3.6), (3.8), Lemma 2.5 and (3.5) we have

$$\begin{aligned} \Psi(k(t)) &\geqq \Psi(h(t)) + \langle k(t) - h(t), \Psi'(h(t)) \rangle \\ &\geqq \Psi(h(t)) + \langle k(t), \Lambda_N \Psi'(h(t)) \rangle - \delta \| f_0 \|_p - \langle h(t), \Psi'(h(t)) \rangle \\ &\geqq \Psi(h(t)) + \sigma(\Lambda_N \Psi'(h(t))) - \sigma(\Psi'(h(t))) - \delta \| f_0 \|_p + 2\varepsilon \\ &\geqq \Psi(h(t)) - 2\delta \| f_0 \|_p + 2\varepsilon \geqq c + \varepsilon. \end{aligned}$$

This completes the proof of (3.7).

By Lemma 2.12 we can choose k_0 and k_1 in $C([0, 1], \mathscr{F})$ such that for $i = 0, 1$ we have $k_i(0) = e_i$, $k_i(1) = k(i)$ and $\langle k_i(t), \Lambda_N \Psi'(e_i) \rangle$ is an increasing function of $t \in [0, 1]$. Thus

$$\begin{aligned} \Psi(k_i(t)) &\geqq \Psi(e_i) + \langle k_i(t) - e_i, \Psi'(e_i) \rangle \\ &\geqq \Psi(e_i) + \langle k_i(t) - e_i, \Lambda_N \Psi'(e_i) \rangle - 2\delta \| f_0 \|_p \\ &\geqq (c + 2\varepsilon) + 0 - \varepsilon = c + \varepsilon. \end{aligned}$$

Define

$$h_0(t)=\begin{cases} k_0(3t), & 0 \leqq t \leqq 1/3 \\ k(3t-1), & 1/3 \leqq t \leqq 2/3 \\ k_1(3-3t), & 2/3 \leqq t \leqq 1. \end{cases}$$

Then $h_0 \in \mathscr{C}$ and $\Psi(h(t)) > c$ for $0 \leqq t \leqq 1$ contrary to the definition of c. So for all sufficiently small $\varepsilon > 0$ we can choose $v \in \mathscr{F}$ satisfying (3.3) and (3.4) as required. □

COROLLARY 3.2. — *Let Ω be an open set in $\mathbb{R}^N$, let μ be a finite positive measure on Ω equivalent to* N*-dimensional Lebesgue measure, let $1 < p < \infty$, let $p^{-1}+q^{-1}=1$, let*

$$\mathscr{L} u = \sum_{1 \leqq |\alpha| \leqq m} a^\alpha(x) D^\alpha$$

define a linear partial differential operator on Ω, where the a^α are measurable functions for $1 \leqq |\alpha| \leqq m$ and there is no 0*-th order term, let $K: L^p(\mu) \to L^q(\mu)$ be a compact, symmetric, positive linear operator, suppose $Kv \in W^m(\Omega)$ and $\mathscr{L} Kv = v$ almost everywhere in Ω for all $v \in L^p(\mu)$ and let $w \in L^q(\mu) \cap W^m(\Omega)$ satisfy $\mathscr{L} w = 0$ almost everywhere in Ω. Let*

$$\Psi(v) = \frac{1}{2}\int_\Omega v K v \, d\mu + \int_\Omega v w \, d\mu$$

for all $v \in L^p(\mu)$, let $f_0 \in L^p(\mu)$ be non-negative, let $\mathscr{F}$ be the set of rearrangements of f_0 on Ω, let $e_0, e_1 \in \mathscr{F}$ and define

$$\mathscr{C} = \{h \in C([0, 1], \mathscr{F}) \mid h(0)=e_0 \text{ and } h(1)=e_1\}$$
$$c = \sup_{h \in \mathscr{C}} \inf_{0 \leqq t \leqq 1} \Psi(h(t)).$$

Suppose

$$\inf \Psi(L^p(\mu)) < c < \min\{\Psi(e_0), \Psi(e_1)\}.$$

Then there exists $v \in \mathscr{F}$ and $u = Kv + w$ such that

$$\Psi(v) = c$$
$$\mathscr{L} u = \varphi \circ u$$

almost everywhere in Ω, for some increasing function φ.

Proof. — Write

$$\sigma(g) = \sup_{\mathscr{F}} \langle \,.\,, g \rangle$$

for $g \in L^q(\mu)$. By Theorem 3.1 we can choose a sequence $\{v_n\}_{n=1}^\infty$ in $\mathscr{F}$ such that

$$\Psi(v_n) \to c \tag{3.9}$$
$$\sigma(K v_n + w) - \langle v_n, K v_n + w \rangle \to 0 \tag{3.10}$$

as $n \to \infty$. Passing to a subsequence we can suppose $v_n \to v$ weakly in $L^p(\mu)$ as $n \to \infty$, for some $v \in \bar{\mathcal{F}}$. Then $Kv_n \to Kv$ in the q-norm, and Lemma 2.5 shows that $\sigma(Kv_n+w) \to \sigma(Kv+w)$. It now follows from (3.10) that

$$\langle v, Kv+w \rangle = \sigma(Kv+w).$$

We also have $\mathcal{L}(Kv+w)=v$ almost everywhere. It now follows from Lemma 2.15 with $g=Kv+w$ that $v \in \mathcal{F}$ and $v=\varphi \circ (Kv+w)$ almost everywhere, for some increasing function φ. Hence $u=Kv+w$ has the required properties. □

Remark. — In applications of Corollaries 3.2 and 3.4 we anticipate that K will be defined by inverting $\mathcal{L}$ with homogeneous boundary conditions. Non-homogeneous boundary conditions may then be accommodated by an appropriate choice of w.

In [5] an abstract result concerning the existence of maximizers for a convex functional relative to a set of rearrangements was proved. Here we remove the convexity assumption. The results obtained are weaker, but they are adequate for applications to boundary value problems, and they apply equally to minimization problems, in contrast to the results of [5]. We also study the variational conditions satisfied by local maximizers and local minimizers.

THEOREM 3.3. — *Let (Ω, μ) be a measure interval, let $1 \leq p < \infty$, let $f_0 \in L^p(\mu)$ and let $\mathcal{F}$ be the set of rearrangements of f_0 on Ω. Let $\Psi: L^p(\mu) \to \mathbb{R}$ be a weakly sequentially continuous Gâteaux differentiable functional. Then*

(i) *There exists a maximizer (resp. minimizer) for Ψ relative to $\bar{\mathcal{F}}$.*

(ii) *If v^* is a maximizer (resp. minimizer) for Ψ relative to $\bar{\mathcal{F}}$ then v^* maximizes (resp. minimizes) $\langle .,\Psi'(v^*) \rangle$ relative to $\bar{\mathcal{F}}$.*

(iii) *If $v^* \in \mathcal{F}$, if U is a strong neighbourhood of v^* relative to $\mathcal{F}$, and if $\Psi(v) \leqq \Psi(v^*)$ (resp. $\geqq$) for all $v \in U$, then v^* maximizes (resp. minimizes) $\langle ., \Psi'(v^*) \rangle$ relative to $\bar{\mathcal{F}}$.*

Proof. — We first consider the case of maximization. Since $\bar{\mathcal{F}}$ is weakly sequentially compact the existence of a maximizer relative to $\bar{\mathcal{F}}$ is immediate. Suppose v^* maximizes Ψ relative to $\bar{\mathcal{F}}$. The variational condition is now proved by a standard argument as follows. Consider $v \in \bar{\mathcal{F}}$ and $0 < t \leqq 1$. Then $(1-t)v^*+tv \in \bar{\mathcal{F}}$ by convexity (Lemma 2.2), so we have

$$\Psi(v^*) \geqq \Psi((1-t)v^*+tv)=\Psi(v^*)+t\langle v-v^*, \Psi'(v^*)\rangle+o(t)$$

as t tends to zero, whence

$$\langle v-v^*, \Psi'(v^*)\rangle \leqq 0.$$

This proves (ii).

Now suppose the assumptions of (iii) apply. By Lemma 2.6 the weak and strong topologies agree on $\mathscr{F}$, so we can choose a weakly open set $W \subset L^p(\mu)$ such that $W \cap \mathscr{F} = U$. Consider any $w \in W \cap \bar{\mathscr{F}}$. Since $\bar{\mathscr{F}}$ with the weak topology is metrizable, w is the weak limit of a sequence $\{w_n\}_{n=1}^{\infty}$ in $\mathscr{F}$. For all sufficiently large n we have $w_n \in W$ and therefore $\Psi(w_n) \leqq \Psi(v^*)$. By weak continuity it follows that $\Psi(w) \leqq \Psi(v^*)$, and this holds for every $w \in W \cap \bar{\mathscr{F}}$. Now let $v \in \bar{\mathscr{F}}$ and $0 < t \leqq 1$, so $(1-t)v^* + tv \in \bar{\mathscr{F}}$ by convexity. If t is sufficiently small we also have $(1-t)v^* + tv \in W$, and then

$$\Psi(v^*) \geqq \Psi((1-t)v^* + tv) = \Psi(v^*) + t \langle v - v^*, \Psi'(v^*) \rangle + o(t)$$

as t tends to zero, whence

$$\langle v - v^*, \Psi'(v^*) \rangle \leqq 0.$$

This proves (iii).

The corresponding results for minimization follow by considering $-\Psi$. □

COROLLARY 3.4. — *Let Ω be an open set in $\mathbb{R}^N$, let μ be a finite positive measure on Ω equivalent to N-dimensional Lebesgue, let $1 \leqq p < \infty$, let q be the conjugate exponent of p, let*

$$\mathscr{L} u = \sum_{1 \leqq |\alpha| \leqq m} a^\alpha(x) D^\alpha u$$

define a linear partial differential operator on Ω, where the a^α are measurable functions for $1 \leqq |\alpha| \leqq m$ and there is no 0-th order term, let $K: L^p(\mu) \to L^q(\mu)$ be a compact, symmetric, linear operator, suppose $Kv \in W^m(\Omega)$ and $\mathscr{L} K v = v$ almost everywhere in Ω for all $v \in L^p(\mu)$, and let $w \in L^q(\mu) \cap W^m(\Omega)$ satisfy $\mathscr{L} w = 0$ almost everywhere in Ω. Let

$$\Psi(v) = \frac{1}{2} \int_\Omega v K v \, d\mu + \int_\Omega v w \, d\mu$$

for all $v \in L^p(\mu)$, let $f_0 \in L^p(\mu)$ be non-negative, and let $\mathscr{F}$ be the set of rearrangements of f_0 on Ω. Then

(i) *There exists a maximizer (resp. minimizer) for Ψ relative to $\mathscr{F}$. If f^* is any maximizer (resp. minimizer) of Ψ relative to $\bar{\mathscr{F}}$ then $f^* \in \mathscr{F}$ and if $u = K f^* + w$ then $\mathscr{L} u = \varphi \circ u$ almost everywhere in Ω, for some increasing (resp. decreasing) function φ.*

(ii) *If $f^* \in \mathscr{F}$, if U is a strong neighbourhood of f^* relative to $\mathscr{F}$ such that $\Psi(f) \leqq \Psi(f^*)$ (resp. $\geqq$) for all $f \in U$, and if $u = K f^* + w$ then $\mathscr{L} u = \varphi \circ u$ almost everywhere in Ω, for some increasing (resp. decreasing) function φ.*

Proof. — The existence of a maximizer relative to $\bar{\mathscr{F}}$ follows from Theorem 3.3 (i). If f^* is any maximizer for Ψ relative to $\bar{\mathscr{F}}$ and

$u=Kf^*+w$ then $\mathscr{L}u=f^*$ almost everywhere, and f^* maximizes $\langle ., u \rangle$ relative to $\bar{\bar{\mathscr{F}}}$ by Theorem 3.3 (ii). It now follows from Lemma 2.15 that $f^* \in \mathscr{F}$ and $f^*=\varphi \circ u$ almost everywhere, for some increasing function φ.

Now suppose the assumptions of (ii) are satisfied. Then with $u=Kf^*+w$ it follows that f^* maximizes $\langle ., u \rangle$ relative to $\mathscr{F}$ by Theorem 3.3 (iii). By Lemma 2.15 there is an increasing function φ such that $f^*=\varphi \circ u$ almost everywhere.

The results for minimizers follow by considering $-\Psi$. □

Remarks. — In some situations one can prove uniqueness of the minimizer as follows. Let (Ω, μ) be a measure interval, $1 \leqq p < \infty$, and let $\Psi: L^p(\mu) \to \mathbb{R}$ be strictly convex. Then, since $\bar{\bar{\mathscr{F}}}$ is convex, Ψ can have at most one minimizer relative to $\bar{\bar{\mathscr{F}}}$; if Ψ is also weakly sequentially continuous then any minimizer of Ψ relative to $\mathscr{F}$ is also a minimizer relative to $\bar{\bar{\mathscr{F}}}$, hence Ψ has at most one minimizer relative to $\mathscr{F}$.

McLeod [10] has considered an example, that satisfies the assumptions of Corollary 3.4 except that f_0 takes positive and negative values, and where furthermore Ψ is convex, and has shown that there is no minimizer relative to $\mathscr{F}$. This contrasts with the maximization of the same Ψ, where McLeod has proved the existence of a maximizer. Theorem 7 of the author [5] proves the existence of a maximizer for a convex Ψ in quite a general context.

We now turn our attention to constrained maximization and minimization. Problems of the type considered below have been proposed by Benjamin [3], although we shall not apply our results to any examples here.

THEOREM 3.5. — *Let (Ω, μ) be a measure interval, let $1 \leqq p < \infty$, let q be the conjugate exponent of p, let $f_0 \in L^p(\mu)$, let $\mathscr{F}$ be the set of rearrangements of f_0 on Ω, let $w \in L^q(\mu)$, let I be real and let $\Lambda=\{v \in L^p(\mu) \mid \langle v, w \rangle = I\}$. Suppose there exist f_1 and $f_2 \in \mathscr{F}$ with $\langle f_1, w \rangle < I < \langle f_2, w \rangle$. Let $\Psi: L^p(\mu) \to \mathbb{R}$ be a weakly sequentially continuous Gâteaux differentiable functional. Then*

(i) *There exists a maximizer (resp. minimizer) for Ψ relative to $\bar{\bar{\mathscr{F}}} \cap \Lambda$.*

(ii) *If v^* is a maximizer (resp. minimizer) for Ψ relative to $\bar{\bar{\mathscr{F}}} \cap \Lambda$ then v^* maximizes (minimizes) $\langle ., \Psi'(v^*)+\lambda w \rangle$ relative to $\bar{\bar{\mathscr{F}}}$, for some real λ.*

Proof. — (i) The convexity of $\bar{\bar{\mathscr{F}}}$ and the existence of f_1 and f_2 show that $\bar{\bar{\mathscr{F}}} \cap \Lambda \neq \varnothing$. Since $\bar{\bar{\mathscr{F}}}$ is weakly sequentially compact it follows that Ψ has a maximizer relative to $\bar{\bar{\mathscr{F}}} \cap \Lambda$.

(ii) Let v^* maximize Ψ relative to $\bar{\bar{\mathscr{F}}} \cap \Lambda$. Consider $v \in \bar{\bar{\mathscr{F}}} \cap \Lambda$ and $0 < t \leqq 1$. Then $(1-t)v^*+tv \in \bar{\bar{\mathscr{F}}} \cap \Lambda$ by convexity, so we have

$$\Psi(v^*) \geqq \Psi((1-t)v^*+tv) = \Psi(v^*)+t\langle v-v^*, \Psi'(v^*)\rangle + o(t)$$

as t tends to zero, whence

$$\langle v - v^*, \Psi'(v^*) \rangle \leqq 0.$$

This holds for every $v \in \bar{\mathscr{F}} \cap \Lambda$. It now follows from Lemma 2.13 that there is a real λ for which v^* maximizes $\langle \,.\,, \Psi'(v^*) + \lambda w \rangle$ relative to $\bar{\mathscr{F}}$.

The case of minimization follows by considering $-\Psi$. □

COROLLARY 3.6. — *Let Ω be an open set in $\mathbb{R}^N$, let μ be a finite positive measure on Ω equivalent to N-dimensional Lebesgue measure, let $1 \leqq p < \infty$, let q be the conjugate exponent of p, let*

$$\mathscr{L} u = \sum_{1 \leqq |\alpha| \leqq m} a^\alpha(x) D^\alpha u$$

define a linear partial differential operator on Ω, where the a^α are measurable functions for $1 \leqq |\alpha| \leqq m$ and there is no 0-th order term, let $K : L^p(\mu) \to L^q(\mu)$ be a compact symmetric linear operator, suppose $Kv \in W^m(\Omega)$ and $\mathscr{L} K v = v$ almost everywhere in Ω for all $v \in L^p(\mu)$, and let $w \in L^q(\mu) \cap W^m(\Omega)$ satisfy $\mathscr{L} w = 0$ almost everywhere in Ω. Let

$$\Psi(v) = \frac{1}{2} \int_\Omega v K v \, d\mu$$

for all $v \in L^p(\mu)$, let $f_0 \in L^p(\mu)$ be non-negative and let $\mathscr{F}$ be the set of rearrangements of f_0 on Ω. Let I be a real number such that there exist f_1 and f_2 in $\mathscr{F}$ satisfying

$$\int_\Omega f_1 w \, d\mu < I < \int_\Omega f_2 w \, d\mu.$$

Then there exists a maximizer (resp. minimizer) for $\Psi(f)$ relative to

$$f \in \bar{\mathscr{F}} \quad \text{and} \quad \int_\Omega f w \, d\mu = I,$$

and if f^* is any such maximizer (resp. minimizer) then $f^* \in \mathscr{F}$ and there exists a real λ and an increasing (resp. decreasing) function φ such that $u = K f^* + \lambda w$ satisfies $\mathscr{L} u = \varphi \circ u$ almost everywhere in Ω.

Proof. — From Theorem 3.5 and Lemma 2.15. □

Remarks. — A special case of Corollary 3.6 was given in [5] as Theorem 9. Strict convexity of Ψ is sufficient for uniqueness of the minimizer in Theorem 3.5, and therefore strict positivity of K is sufficient for the uniqueness of the minimizer in Corollary 3.6.

4. MULTIPLE CONFIGURATIONS FOR STEADY VORTICES

NOTATION. – Let $r > R_1 \geqq R_2 > \varepsilon > 0$, let $\Omega(1)$ and $\Omega(2)$ be open discs in the $x_1 x_2$ plane having radii R_1 and R_2 whose centres are $(-r, 0)$ and $(r, 0)$ respectively, and let Ω be the union of $\Omega(1)$ and $\Omega(2)$ with the rectangle $|x_1| < r$, $|x_2| < \varepsilon$. Thus Ω is a dumb-bell-shaped region with ends $\Omega(1)$ and $\Omega(2)$ connected by a channel that forms part of the rectangle.

Let μ denote 2-dimensional Lebesgue measure, let $K: L^2(\Omega) \to L^2(\Omega)$ be the inverse of $-\Delta$ in Ω with zero Dirichlet boundary conditions on $\partial\Omega$, and for $v \in L^2(\Omega)$ let

$$\Psi(v) = \frac{1}{2}\int_\Omega v K v \, d\mu.$$

THEOREM 4.1. – *Let $0 < a < \infty$, let $2 < p < \infty$ and let $f_0 \in L^p[0, a]$ be a positive function. Let $\sqrt{(a/\pi)} < R_2 \leqq R_1 < \infty$. Then there exist $r \in (R_1, \infty)$ and $\varepsilon \in (0, a/r)$ such that when the region Ω has dimensions R_1, R_2, r, ε, the following holds:*

Let f_0 be extended by defining $f_0(s) = 0$ for $a < s \leqq \mu(\Omega)$, let $\mathscr{F}$ be the set of rearrangements of f_0 on Ω, and for $i = 1, 2$ let

$$\mathscr{F}_i = \{ f \in \mathscr{F} \mid f(x) = 0 \text{ for all } x \in \Omega(3-i) \}.$$

Then

(i) *For $i = 1, 2$ there exists a maximizer v_i for Ψ relative to $\mathscr{F}_i$, and v_i is a local maximizer for Ψ relative to $\mathscr{F}$. If $u_i = K v_i$ then $-\Delta u_i = \varphi_i \circ u_i$ almost everywhere in Ω, for some increasing function φ_i.*

(ii) *There is exactly one minimizer v_3 for Ψ relative to $\mathscr{F}$. If $u_3 = K v_3$ then $-\Delta u_3 = \varphi_3 \circ v_3$ almost everywhere in Ω, for some decreasing function φ_3.*

(iii) *Define*

$$\mathscr{C} = \{ h \in C([0, 1], \mathscr{F}) \mid h(0) = v_1 \text{ and } h(1) = v_2 \}$$
$$c = \sup_{h \in \mathscr{C}} \inf_{0 \leqq t \leqq 1} \Psi(h(t)).$$

Then

$$\Psi(v_3) < c < \min\{\Psi(v_1), \Psi(v_2)\}$$

and there exists a $v_4 \in \mathscr{F}$ such that $\Psi(v_4) = c$ and $u_4 = K v_4$ satisfies $-\Delta u_4 = \varphi_4 \circ u_4$ almost everywhere in Ω, for some increasing function φ_4.

Before giving the proof, let us interpret this result physically. Any function u that satisfies $-\Delta u = \varphi(u)$ in Ω for some sufficiently smooth φ and that satisfies $u = 0$ on $\partial\Omega$, represents the stream function for the steady flow of an ideal fluid in two dimensions, confined by a solid wall in the shape of $\partial\Omega$. The velocity field is given by $(\partial u/\partial x_2, -\partial u/\partial x_1)$, and the vorticity, which is given by the curl of the velocity, has magnitude $-\Delta u$. In Theorem 4.1, flows are sought in which the vorticity is a rearrangement of a prescribed function f_0, and it is shown that at least four solutions exist. These solutions represent different configurations of a region of vorticity in an otherwise irrotational flow.

Integrating by parts in the formula for $\Psi(v)$ yields

$$\Psi(v)=\frac{1}{2}\int_{\Omega}|\nabla u|^2\,d\mu$$

so $\Psi(v)$ represents the kinetic energy of the fluid. Our variational methods are based on the principle that steady flows correspond to stationary points of the kinetic energy relative to rearrangement of the vorticity.

We do not investigate the regularity of the solution or the smoothness of φ_i here; some results on this subject would be required before it could be asserted that our velocity fields satisfied the hydrodynamic equations.

We now derive some estimates that are needed for the proof of Theorem 4.1.

LEMMA 4.2. — *Let $0<\alpha<1$ and $0<\beta\leqq\sqrt{\pi}$. Then there exists $k=k(\alpha,\beta)>0$ with the following property: If $R>0$, and D_R denotes a disc of radius R in the plane, if $u\in W^{2,2}(D_R)$ and $v=-\Delta u$, and if u and v are non-negative and $\|v\|_1\geqq\beta R\|v\|_2$, then $u(x)\geqq k\|v\|_1$ for all $x\in D_{\alpha R}$, the concentric disc of radius αR.*

Proof. — First consider the case $R=1$. Define

$$S=\{v\in L^2(D_1)\mid v\geqq 0,\ \|v\|_1=1 \text{ and } \|v\|_2\leqq 1/\beta\}.$$

Then S is closed, bounded and convex, since $\|\ \|_1$ is additive over non-negative functions. Hence S is weakly compact, and $S\neq\varnothing$ since $1/\pi\in S$. For $v\in L^2(D_1)$ let Tv be the solution of $-\Delta u=v$ with $u=0$ on ∂D_1. Then $T: L^2(D_1)\to W^{2,2}(D_1)$ is a bounded linear operator, and the embedding $W^{2,2}(D_1)\to C(\bar{D}_1)$ is compact, hence the function $m: L^2(D_1)\to\mathbb{R}$ defined by

$$m(v)=\inf\{Tv(x)\mid x\in D_\alpha\}$$

is weakly sequentially continuous. Write

$$k=\inf\{m(v)\mid v\in S\}.$$

Then $k=m(v_0)$ for some $v_0\in S$, and $m(v_0)>0$ by the Maximum principle, so $k>0$. It now follows that if $v\in L^2(D_1)$ is non-negative and $\|v\|_1\geqq\beta\|v\|_2$ then $Tv(x)\geqq k\|v\|_1$ for all $x\in D_\alpha$; any non-negative $u\in W^{2,2}(D_1)$ such that $-\Delta u=v$ satisfies $u\geqq Tv$ by the Maximum principle. The result is now established when $R=1$.

Now suppose $u \in W^{2,2}(D_R)$, $-\Delta u = v$, $u \geqq 0$, $v \geqq 0$ and $\|v\|_1 \geqq \beta R \|v\|_2$. Taking all discs to have centre o, define

$$\bar{u}(x) = u(R\,x)$$
$$\bar{v}(x) = R^2 v(R\,x)$$

for $x \in D_1$. Then $\bar{u}$ satisfies $-\Delta \bar{u} = \bar{v}$, and

$$\|\bar{v}\|_1 = \|v\|_1 \geqq \beta R \|v\|_2 = \beta \|\bar{v}\|_2$$

so for $x \in D_{\alpha R}$ we have

$$u(x) = \bar{u}(x/R) \geqq k \|\bar{v}\|_1 = k \|v\|_1$$

as required.

LEMMA 4.3. – *Let* $R_1 \geqq R_2 > 0$ *and* $c > 0$ *be given. Then there exist* $r_0 > R_1$ *with the following property: If* $r \geqq r_0$ *and* $0 < \varepsilon \leqq R_2$, *if* $i = 1$ *or* 2, *and* $v \in L^2(\Omega)$ *satisfies* $v = 0$ *almost everywhere in* $\Omega \setminus \Omega(i)$, *then* $\sup\{|K v(x)| \mid x \in \Omega(3-i)\} \leqq c \|v\|_1$.

Proof. – We lose no generality by taking $i = 1$, since the inequality $R_1 \geqq R_2$ plays no part in the ensuing argument. Assume $r \geqq R_1 + R_2$ and let $G(x, y)$ be the Green's function for $-\Delta$ in the half-plane $x_1 > -2R_1 - r$ so

$$G(x, y) = \frac{1}{4\pi} \log \frac{(X_1 + Y_1^2) + (X_2 - Y_2)^2}{(X_1 - Y_1)^2 + (X_2 - Y_2)^2} = \frac{1}{4\pi} \log\left(1 + \frac{4 X_1 Y_1}{\rho^2}\right)$$

where (X_1, X_2) and (Y_1, Y_2) are the coordinates of x and y referred to $(-2R_1 - r, 0)$ as origin and $\rho = |x - y|$. Let $v \in L^2(\Omega)$ satisfy $v = 0$ almost everywhere in $\Omega \setminus \Omega(1)$, and for $x \in \bar{\Omega}$ write

$$u(x) = \int_\Omega G(x, y)\, |v(y)|\, d\mu(y).$$

Then $u \in W^{2,2}(\Omega)$ is a strong solution of $-\Delta u = |v|$ in Ω, and u is positive on $\partial\Omega$, hence by the Maximum principle $|K v(x)| \leqq u(x)$ for all $x \in \Omega$.

Now suppose $x \in \Omega(2)$. Since the contribution to the above integral from $\Omega \setminus \Omega(1)$ is zero, we need only consider $y \in \Omega(1)$. We then have $X_1 \leqq 2R_1 + 2r + R_2 \leqq 4r$, $Y_1 \leqq 3R_1$ and $\rho \geqq 2r - R_1 - R_2 \geqq r$, hence

$$G(x, y) \leqq \frac{1}{\pi} \frac{X_1 Y_1}{\rho^2} \leqq \frac{12 R_1}{\pi r}.$$

Then $u(x) \leqq (12 R_1/\pi r) \|v\|_1$. We obtain the result by taking $r_0 = \max\{R_1 + R_2,\ 12 R_1/\pi c\}$.

Proof of Theorem 4.1. – Let $k = k(\alpha, \beta)$ be the positive number provided by Lemma 4.2 with α, β chosen such that $\pi \alpha^2 R_2^2 = a$ and $2\beta R_1 \|f_0\|_2 = \|f_0\|_1$. Let r be the number r_0 provided by Lemma 4.3 with R_1 and R_2 as in the statement of the Theorem and $c = k/4$. Let

$\varepsilon_0 = a/(4r) < R_2$, and let $M \in (0, \infty)$ be such that for all $0 < \varepsilon < \varepsilon_0$, if Ω has dimensions R_1, R_2, r, ε then

$$\|Kv\|_\infty \leqq M \|v\|_2 \tag{4.1}$$

for all $v \in L^2(\Omega)$. The existence of M may be established by using the Maximum principle to compare Kv with the function u introduced in the proof of Lemma 4.3. Fix $\varepsilon \in (0, \varepsilon_0)$ satisfying

$$\left(\int_0^{4r\varepsilon} (f_0^\Delta)^2\right)^{1/2} < (k/4M) \|f_0\|_1. \tag{4.2}$$

Let $\delta = (p-2)/(2p-2)$, so $0 < \delta < 1$ and the interpolation inequality

$$\|v\|_2 \leqq \|v\|_1^\delta \|v\|_p^{1-\delta} \tag{4.3}$$

holds for all $v \in L^p(\Omega)$. Write $P = \|f_0\|_p^{1-\delta}$. Fix $i = 1$ or 2, let $j = 3 - i$ and let $\Omega(i, \alpha)$ be the open disc of radius αR_i concentric with $\Omega(i)$.

Consider $v \in \mathscr{F}$ and write $v(j) = 1_{\Omega(j)} v$ and $w = v - v(j)$. The subset V of $\Omega(i, \alpha)$ where v vanishes has at least as great measure as the set where $v(j)$ is positive; since both of these sets are measure intervals, there is a rearrangement $v(i)$ of $v(j)$ that is defined on Ω and vanishes outside V. Let $\bar{v} = w + v(i) \in \mathscr{F}$. Also let $w(i) = 1_{\Omega(i)} w$ and $w_0 = w - w(i)$, which vanishes in $\Omega(i)$ and $\Omega(j)$. Then by (4.2) we have

$$\|w_0\|_2 \leqq (k/(4M)) \|f_0\|_1. \tag{4.4}$$

We have

$$\begin{aligned}\Psi(\bar{v}) - \Psi(v) &= \frac{1}{2}\int_\Omega ((w + v(i)) K(w + v(i)) - (w + v(j)) K(w + v(j))) \, d\mu \\ &= \int_\Omega w K(v(i) - v(j)) \, d\mu + \frac{1}{2}\int_\Omega v(i) K v(i) \, d\mu - \frac{1}{2}\int_\Omega v(j) K v(j) \, d\mu \\ &\geqq \int_\Omega v(i) K w \, d\mu - \int_\Omega v(j) K w(i) \, d\mu \\ &\quad - \int_\Omega v(j) K w_0 \, d\mu - \frac{1}{2}\int_\Omega v(j) K v(j) \, d\mu. \end{aligned} \tag{4.5}$$

In view of the definition of β we can choose a positive $\theta < \|f_0\|_2$ such that

$$\beta R_1 \|u\|_2 \leqq \|u\|_1$$

holds for all $u \in L^2(\Omega)$ satisfying $\|u - f\|_2 < \theta$ for some $f \in \mathscr{F}$. If $\|v(j)\|_2 < \theta$ we can now apply Lemmas 4.2 and 4.3, and relations (4.1) and (4.5), to obtain

$$\begin{aligned}\Psi(\bar{v}) - \Psi(v) \geqq k \|w\|_1 \|v(i)\|_1 &- (k/4) \|w(i)\|_1 \|v(j)\|_1 \\ &- M \|w_0\|_2 \|v(j)\|_1 - \frac{1}{2} M \|v(j)\|_1 \|v(j)\|_2.\end{aligned}$$

From this, (4.2) and (4.3) it follows that

$$\begin{aligned}\Psi(\bar{v})-\Psi(v) &\geqq k\|w\|_1\|v(i)\|_1-(k/4)\|v(j)\|_1\|w(i)\|_1 \\ &\quad -(k/4)\|v(j)\|_1\|f_0\|_1-\frac{1}{2}\mathrm{MP}\|v(j)\|_1^{1+\delta} \\ &\geqq \|v(j)\|_1\left(\frac{3}{4}k\|w\|_1-\frac{k}{4}\|f_0\|_1-\frac{1}{2}\mathrm{MP}\|v(j)\|_1^{\delta}\right) \\ &\geqq \|v(j)\|_1\left(\frac{k}{2}\|f_0\|_1-\frac{3}{4}k\|v(j)\|_1-\frac{1}{2}\mathrm{MP}\|v(j)\|_1^{\delta}\right).\end{aligned}$$

We can choose $\nu > 0$ and $\eta \in (0, \theta)$ such that

$$\frac{k}{2}\|f_0\|_1-\frac{3}{4}k\|u\|_1-\frac{1}{2}\mathrm{MP}\|u\|_1^{\delta} > \nu\,\mathrm{P}^{1/\delta}$$

for all $u \in \mathrm{L}^2(\Omega)$ satisfying $\|u\|_2 \leqq 2\eta$. Then

$$\Psi(\bar{v})-\Psi(v) \geqq \nu\|v(j)\|_2^{1/\delta} \tag{4.6}$$

provided that $\|v(j)\|_2 \leqq 2\eta$.

Let $\mathscr{E}_i$ be the set of maximizers of Ψ relative to $\mathscr{F}_i$. It follows from Theorem 3.3 (i) that $\mathscr{E}_i$ is nonempty. Let

$$\mathrm{U}=\{v \in \mathscr{F} \mid \operatorname{dist}(v, \mathrm{F}_i) < \eta\}$$

where the distance is calculated in the 2-norm. Then U is strongly open relative to $\mathscr{F}$. Consider $v \in \mathrm{U}$. Then there exists $v^* \in \mathscr{F}_i$ with $\|v-v^*\|_2 < \eta$, so if $v(j)=1_{\Omega(j)}v$ we have

$$\|v(j)\|_2=\left(\int_{\Omega(j)} v^2\,d\mu\right)^{1/2}=\left(\int_{\Omega(j)}(v-v^*)^2\right)^{1/2} \leqq \eta.$$

Hence, if $\bar{v} \in \mathscr{F}_i$ is formed from v as above, we have $\|v-\bar{v}\|_2 \leqq 2\eta$ and so

$$\begin{aligned}\Psi(v) &\leqq \Psi(\bar{v})-\nu\|v(j)\|_2^{1/\delta} \\ &\leqq \Psi(\bar{v}) \leqq \sup\Psi(\mathscr{F}_i);\end{aligned}$$

if $\Psi(v)=\sup\Psi(\mathscr{F}_i)$ then we must have $v_j=0$ and $\bar{v} \in \mathscr{E}_i$. Thus

$$\Psi(v) < \sup\Psi(\mathscr{F}_i)$$

if $v \in \mathrm{U}\setminus\mathscr{E}_i$. It now follows from Corollary 3.4 (ii) that if $v_i \in \mathscr{E}_i$ and $u_i=\mathrm{K}v_i$ then $-\Delta u_i=\varphi_i \circ u_i$ almost everywhere in Ω, for some increasing function φ_i.

The existence of a minimizer v_3 for Ψ relative to $\mathscr{F}$ follows from Corollary 3.4 (i), and by Corollary 3.4 (ii) we have $-\Delta u_3=\varphi_3 \circ u_3$ almost everywhere in Ω, where $u_3=\mathrm{K}v_3$ and φ_3 is some decreasing function. The uniqueness of v_3 follows from the Remarks after Corollary 3.4.

We now turn to the existence of the fourth solution. Choose any $v_i \in \mathscr{E}_i$ for $i=1$, 2 and let $\mathscr{C}$ and c be as in the statement of the Theorem. The unique minimizer of Ψ relative to $L^2(\Omega)$ is the zero function, which does not lie in $\mathscr{F}$, hence

$$c \geqq \Psi(v_3) > \inf \Psi(L^2(\Omega)).$$

Let $i=1$ or 2 be such that $\Psi(v_i) \leqq \Psi(v_j)$ where $j=3-i$, let η, ν, U be defined as above and let ∂U denote the boundary of U relative to $\mathscr{F}$. Consider $v \in \partial U$. Then dist $(v, F_i)=\eta$. As above let us write $v(j)=1_{\Omega(j)} v$ and form $\bar{v} \in \mathscr{F}_i$ by replacing $v(j)$ with a rearrangement of itself vanishing outside $\Omega(i, \alpha)$. Then as above we have

$$\| v(j) \|_2 \leqq \eta$$

and further

$$2 \| v(j) \|_2 \geqq \| v-\bar{v} \|_2 \geqq \eta,$$

hence from (4.6) it follows that

$$\Psi(v) \leqq \Psi(\bar{v}) - \nu \| v(j) \|_2^{1/\delta} \leqq \Psi(v_i) - \nu (\eta/2)^{1/\delta}.$$

Thus

$$\sup \Psi(\partial U) < \Psi(v_i) \leqq \Psi(v_j).$$

Now $v_j \notin U$, so if $h \in \mathscr{C}$ we have $h(t) \in \partial U$ for some $t \in [0, 1]$, therefore

$$\inf_{0 \leqq t \leqq 1} \Psi(h(t)) \leqq \sup \Psi(\partial U)$$

and therefore in view of Lemma 2.11 we have

$$c \leqq \sup \Psi(\partial U).$$

Corollary 3.2 now applies to prove the existence of a $v_4 \in \mathscr{F}$ and an increasing function φ_4 such that $\Psi(v_4)=c$, and if $u_4=K v_4$ then $-\Delta u_4 = \varphi_4 \circ u_4$ almost everywhere in Ω.

Finally let us observe that $c > \Psi(v_3)$. For otherwise, by uniqueness of the minimizer, we would have $v_3=\varphi_3 \circ u_3=\varphi_4 \circ u_3$ almost everywhere, where φ_3 is decreasing and φ_4 is increasing, so v_3 would be constant, which it is not.

Remarks. — The functional Ψ does of course possess at least one global maximizer relative to $\mathscr{F}$, but we believe it coincides with v_1 or v_2. In the case when $R_1=R_2$, one can construct a fourth solution by maximizing Ψ over the elements of $\mathscr{F}$ that are even in x_1, without using the Mountain Pass Lemma.

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