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On the proof of Kuranishi's embedding theorem (*)

by

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ABSTRACT. — We prove a local holomorphic embedding theorem for a formally integrable, strictly pseudoconvex CR manifold M with dim $M=2n-1 \geqq 7$. This embedding is obtained as the limit of a sequence of approximate embeddings into complex n-space, which is constructed and shown to converge by the methods of Nash and Moser. The linearized problem is solved using the explicit integral operators constructed by Henkin. With estimates wich we have previously obtained for these operators, we show that if M is of class C^m, then it admits a C^k embedding provided $21 \leqq k$, $6k+5n-2 \leqq m$. Our argument is much shorter and simpler than previous arguments, which were based on the Neumann operator and carried out in the C^∞ category.

RÉSUMÉ. — Nous démontrons un théorème de plongement holomorphe local pour une variété CR M, intégrable et strictement pseudoconvexe, si $\dim M=2n-1 \geqq 7$. Ce plongement est obtenu comme limite d'une suite de plongements approximatifs dans l'espace $\mathbb{C}^n$. Nous construisons cette suite et démontrons sa convergence par les méthodes de Nash et Moser. Pour le problème linéarisé nous utilisons les opérateurs construits par Henkin et les bornes que nous avons obtenu auparavant. Si M est de classe C^m, le plongement est de classe C^k pourvu que $k \geqq 21$ et $m \geqq 6k+5n-2$.

Mots clés : Plongement CR, existence locale, méthodes de Nash-Moser.

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INTRODUCTION

We consider the local holomorphic embedding problem for a formally integrable CR structure of real hypersurface type with positive definite Levi form. It was first posed by Kohn [3] and has been solved in large part by Kuranishi [4] in the C^∞ case. His proof is rather long and technical, involving a delicate study in L^2-spaces of the Neumann operator for solving the tangential Cauchy-Riemann equations. This was used in conjunction with a Nash-Moser iteration scheme to produce an embedding. Due to its central importance and the difficult nature of the original proof, the problem merits further study and better understanding.

In broad outline our approch here is similar to Kuranishi's ([4], III), but differs significantly in several important details. We also set up a sequence of approximate holomorphic embeddings and show convergence using the methods of Nash and Moser [5]. But rather than using the Neumann operator to solve the "linearized problem", we use the totally explicit integral operators of Henkin [2] on approximating real hypersurfaces in C^n. The necessary estimates, which are given in [10], are much simpler than those in ([4], I, II). Working entirely in C^k-spaces, we are able to prove the following version of Kuranishi's theorem.

THEOREM. — *Let* M *be a* $(2n-1)$*-dimensional* CR *manifold of differentiability class* C^m. *Then* M *admits, locally near each point, a holomorphic embedding of class* C^k, *provided*

$$n \geqq 4, \qquad 2n-1 \geqq 7, \tag{0.1}$$

$$m \geqq 6k+5n-2, \qquad k \geqq 21. \tag{0.2}$$

Specifically, if the vector fields defining the structure locally have coefficients of class C^m, then they annihilate n independent complex valued functions of class C^k.

Kuranishi [4] requires $n \geqq 5$, $2n-1 \geqq 9$. Recently, Akahori [1] has given an argument assuming $n \geqq 4$. He also uses L^2-methods and the Neumann operator. In view of the examples of Nirenberg [8], only the case $n=3$ is still open. Aside from this the question of regularity remains to be settled. The inequalities (0.2) can probably be improved, even with the present methods. The derivative loss, $m-k$, comes mainly from small denominators occuring in the estimates of [10].

One aspect of our Nash-Moser argument should be pointed out. Typically in such an argument one controls derivatives up to some order k at the expense of allowing those of order $k+\mu$, say, to become unbounded. For $\mu > k$, some of the constants in section 5, e. g. in (5.3), must involve negative powers of the inner radius of the domain. As the domain shrinks at each stage of iteration, this would cause a problem. We are able to carry out our argument with $\mu < k$, thus avoiding this difficulty and greatly reducing the derivative loss. However, this requires the lower bound on k in (0.2).

In section 1 we give a simple Taylor series argument for finding an approximate holomorphic embedding. This follows from general principles and might have been omitted. However, it is an integral part of the argument and contributes to the greater part of the derivative loss. We set up the "homotopy formula" for the tangential CR complex in section 2, using the results of [2] and [10]. Section 3, which shows how to alter an embedding to make it more nearly holomorphic, contains the core of our argument. Sections 4 and 6 are dedicated to the technical details of controlling the embedding. Also in section 5 we state the results from [10] and make some minor modifications of them. Finally, in section 7 we present the convergence argument for the theorem. Aside from those considerations required by smoothing most of the ideas of our proof already occur in greatly simplified form in [9].

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1. INITIAL NORMALIZATION AND APPROXIMATE HOLOMORPHIC EMBEDDING

Let D be a neighborhood of 0 in $\mathbb{R}^{2n-1}$, with coordinates $z^\alpha = x^\alpha + iy^\alpha$, $1 \leqq \alpha \leqq n-1$, and $u = \bar{u}$; on which are given $n-1$ complex vector fields X_α of class C^m. We assume that the X_α together with their complex conjugates, $\bar{X}_\alpha \equiv X_{\bar{\alpha}}$, are pointwise linearly independent and satisfy the integrability condition, $[X_\alpha, X_\beta] = \Gamma^\gamma_{\alpha\beta} X_\gamma$. Where convenient we use the convention that greek indices run from 1 to $n-1$, latin ones from 1 to n, and repeated indices are summed. Our vector fields are determined up to a frame change $X_\alpha \to C^\beta_\alpha X_\beta$.

Initially we choose coordinates so that $X_\alpha u(0) = X_\alpha \bar{z}^\beta(0) = 0$, $X_\alpha z^\beta(0) = \delta^\beta_\alpha$, and adjust C so that ($\partial_\alpha = \partial/\partial z^\alpha$, etc.)

$$X_\alpha = \partial_\alpha + A^{\bar{\gamma}}_\alpha \partial_{\bar{\gamma}} + B_\alpha \partial_u, \qquad A(0) = 0, \quad B(0) = 0. \tag{1.1}$$

For such a frame the commutators vanish, $[X_\alpha, X_\beta] = 0$, giving

$$X_\alpha A^{\bar{\gamma}}_\beta = X_\beta A^{\bar{\gamma}}_\alpha, \qquad X_\alpha B_\beta = X_\beta B_\alpha. \tag{1.2}$$

The Levi form is the hermitian matrix g defined

$$\begin{aligned}[X_\alpha, X_{\bar{\beta}}] &= -ig_{\alpha\bar{\beta}} \partial_u, \qquad \bmod \{X_\gamma, X_{\bar{\gamma}}\},\\ g_{\alpha\bar{\beta}} &= i(X_\alpha B_{\bar{\beta}} - X_{\bar{\beta}} B_\alpha).\end{aligned} \tag{1.3}$$

We assume that g is positive definite.

We make a coordinate change F on $\mathbb{R}^{2n-1}$ to simplify A and B,

$$\mathrm{F}:\quad \begin{array}{ll} z'^{\beta}=\mathrm{F}^{\beta}(z, u)=z^{\beta}+f^{\beta}, & f^{\beta}=O(2), \\ u'=\mathrm{F}^{n}(z, u)=u+f^{n}, & f^{n}=\bar{f}^{n}=O(2), \end{array}$$

$$\mathrm{X}_{\alpha}=\mathrm{X}_{\alpha}\mathrm{F}^{\gamma}\,\partial'_{\gamma}+\mathrm{X}_{\alpha}\mathrm{F}^{\bar{\gamma}}\,\partial'_{\bar{\gamma}}+\mathrm{X}_{\alpha}\mathrm{F}^{n}\,\partial'_{u}\equiv \mathrm{C}^{\beta}_{\alpha}\mathrm{X}'_{\beta},$$

$$\mathrm{X}'_{\alpha}=\partial'_{\alpha}+\mathrm{A}'^{\bar{\gamma}}_{\alpha}\,\partial'_{\bar{\gamma}}+\mathrm{B}'_{\alpha}\,\partial'_{u}.$$

It follows that

$$\mathrm{X}_{\beta}\mathrm{F}^{\bar{\gamma}}=(\mathrm{X}_{\beta}\mathrm{F}^{\rho})(\mathrm{A}'^{\bar{\gamma}}_{\rho}\circ\mathrm{F}),\qquad \mathrm{X}_{\beta}\mathrm{F}^{n}=(\mathrm{X}_{\beta}\mathrm{F}^{\rho})(\mathrm{B}'_{\rho}\circ\mathrm{F}),$$

or more explicity,

$$\mathrm{A}^{\bar{\gamma}}_{\beta}+\mathrm{X}_{\beta}f^{\bar{\gamma}}=(\delta^{\rho}_{\beta}+\mathrm{X}_{\beta}f^{\rho})(\mathrm{A}'^{\bar{\gamma}}_{\rho}\circ\mathrm{F}), \tag{1.4}$$

$$\mathrm{B}_{\beta}+\mathrm{X}_{\beta}f^{n}=(\delta^{\rho}_{\beta}+\mathrm{X}_{\beta}f^{\rho})(\mathrm{B}'_{\rho}\circ\mathrm{F}). \tag{1.5}$$

We first take $f^{\gamma}=-z^{\bar{\sigma}}\mathrm{A}^{\gamma}_{\bar{\sigma}}(0, u)$, $f^{n}=0$ to get $\mathrm{A}'(0, u')=0$, and then $f^{\gamma}=0$, $f^{n}=-z^{\sigma}\mathrm{B}_{\sigma}(0, u)-z^{\bar{\sigma}}\mathrm{B}_{\bar{\sigma}}(0, u)$ to get $\mathrm{B}'(0, u)=0$. However, F would be of class C^{m} and A′, B′ only of class C^{m-1}. Therefore, we replace $\mathrm{A}(0, u)$, $\mathrm{B}(0, u)$, and other functions of u appearing in our transformation formulae by appropriate Taylor polynomials in u. This will result in normalization at $(z, u)=(0, 0)$. Next we take

$$f^{\gamma}=-z^{\rho}z^{\bar{\sigma}}\mathrm{X}_{\rho}\mathrm{A}^{\gamma}_{\bar{\sigma}}(0, u)-z^{\bar{\rho}}z^{\bar{\sigma}}\mathrm{X}_{\bar{\rho}}\mathrm{A}^{\gamma}_{\bar{\sigma}}(0, u),\qquad f^{n}=0.$$

By (1.2) the coefficient of $z^{\bar{\rho}}z^{\bar{\sigma}}$ is symmetric in ρ and σ, and hence may be removed by this change. If we differentiate (1.4) in the first instance by X_{α}, in the second by $\mathrm{X}_{\bar{\alpha}}$, and set $z=0$, we see that $\mathrm{A}'(z', u')=O(|z'|^{2})$. We next consider B_{α},

$$\mathrm{B}_{\alpha}=\mathrm{B}_{\alpha\beta}(u)\,z^{\beta}+\mathrm{B}_{\alpha\bar{\beta}}(u)\,z^{\bar{\beta}}+\mathrm{B}^{*}_{\alpha}(z, u),\qquad \mathrm{B}^{*}=O(|z|^{2}).$$

Again by (1.2), $\mathrm{B}_{\alpha\beta}=\mathrm{B}_{\beta\alpha}$, so that we may set

$$f^{\gamma}=0,\qquad f^{0}=-\mathrm{Re}(\mathrm{B}_{\alpha\beta}z^{\alpha}z^{\beta})-\frac{1}{2}(\mathrm{B}_{\alpha\bar{\beta}}+\bar{\mathrm{B}}_{\beta\bar{\alpha}})\,z^{\alpha}z^{\bar{\beta}}.$$

After substituting this into (1.5) and taking X_{α}-and $\mathrm{X}_{\bar{\alpha}}$-derivatives along $z=0$, we find that $\mathrm{B}'_{\alpha\beta}(u')=0$ and that $\mathrm{B}'_{\alpha\bar{\beta}}(u')$ is skew hermitian. Thus we may arrange that $\mathrm{B}_{\alpha}-ig_{\alpha\bar{\beta}}(u)\,z^{\bar{\beta}}=O(|z|^{2})$, where g is hermitian and by (1.3) positive definite. Finally, a change $z'^{\alpha}=z^{\beta}\mathrm{W}^{\alpha}_{\beta}(u)$, $u'=u$ makes $g_{\alpha\bar{\beta}}(u)=\delta_{\alpha\bar{\beta}}$.

This argument shows that with a polynomial change of coordinates we may achieve

$$\left.\begin{array}{l} \mathrm{A}^{\bar{\beta}}_{\alpha}(z, u)=O(2)\equiv O(|(z, u)|^{2}), \\ \mathrm{B}_{\alpha}(z, u)=i\,\delta_{\alpha\bar{\beta}}z^{\bar{\beta}}+\mathrm{B}^{*}_{\alpha}(z, u),\qquad \mathrm{B}^{*}=O(2). \end{array}\right\} \tag{1.6}$$

With $\mathrm{H}=\delta_{\alpha\bar{\beta}}z^{\alpha}z^{\bar{\beta}}\equiv z\,.\,\bar{z}$, we define

$$\mathrm{Z}(z, u)=(z, z^{n}),\qquad z^{n}=u+i\,\mathrm{H}(z, u), \tag{1.7}$$

which gives an approximate holomorphic embedding for our structure since $X_{\bar\alpha} z^{\beta}=O(2)$, $X_{\bar\alpha} z^{n}=O(2)$. We shall modify Z so that, in addition $X_{\bar\alpha} Z=O(m)$. Inductively, we assume $X_{\bar\alpha} Z=O(s)$, $2\leqq s\leqq m$, so that

$$X_{\bar\alpha} z^{j}=\sum E_{\bar\alpha}{}^{j}{}_{a\bar b i}\, z^{a}\, \bar z^{b}\, u^{i}+O(s+1),$$
$$z^{a}=z^{\alpha_1}\ldots z^{\alpha_p},\qquad \bar z^{b}=\bar z^{\beta_1}\ldots \bar z^{\beta_q}.$$

Here the coefficients E are symmetric separately in the multi-indices a and b, and the sum is over all indices with $p+q+i=s$. Since $X_{\bar\alpha} X_{\bar\beta} z^{j}=X_{\bar\beta} X_{\bar\alpha} z^{j}$, they are also symmetric in (αb). Therefore, we may make the change-of-embedding

$$z'^{j}=z^{j}-\sum E_{\bar\alpha}{}^{j}{}_{a\bar b i}\, \bar z^{\alpha}\, z^{a}\, \bar z^{b}\, u^{i}$$

to get $X_{\bar\alpha} Z=O(s+1)$. Since the term subtracted is $O(3)$, (1.6) is preserved.

We shall consider approximate holomorphic embeddings (1.7) of D onto a real hypersurface $M=Z(D)$ of the following more general form. With $z^{n}=u+iv$, M is given by

$$\begin{gathered} r\equiv -v+H(z, u)=0, \qquad H=b(z)+h(z, u), \\ h=\bar h\in C^{m}(D), \qquad h=O(3), \\ b=\bar b^{t}=I+\hat b, \qquad |\hat b(z)|\leqq \frac{1}{2}|z|^{2}. \end{gathered} \tag{1.8}$$

Following Kuranishi [4], for $\rho>0$ we define

$$\left.\begin{gathered} D_{\rho}\equiv D(\rho)\equiv D(\rho, H)=\{(z, u)\in D: u^{2}+H-H^{2}<\rho\}, \\ M_{\rho}=Z(D_{\rho}). \end{gathered}\right\} \tag{1.9}$$

As we shall see in section 4, D_{ρ} and M_{ρ} are smoothly bounded with compact closure if ρ is sufficiently small. We introduce locally defined vector fields on $\mathbb{C}^{n}$ by

$$\left.\begin{gathered} Y_{\alpha}=\partial_{\alpha}-(r_{\alpha}/r_{n})\,\partial_{n}, \qquad 1\leqq\alpha\leqq n-1 \quad (r_{j}=\partial_{j} r), \\ T=(1+ih_{u})\,\partial_{n}+(1-ih_{u})\,\partial_{\bar n}. \end{gathered}\right\} \tag{1.10}$$

They are tangent to M, and restricted to M, $T=\partial_{u}$, $Y_{\alpha}=\partial_{\alpha}-(r_{\alpha}/2 r_{n})\,\partial_{u}$. Also, $Y_{\alpha} r\equiv 0$, $T r\equiv 0$, $T z^{\alpha}\equiv 0$, $\operatorname{Re} T z^{n}\equiv 1$. *Changing our notation* from (1.1), we put

$$X_{\alpha}=Y_{\alpha}+A_{\alpha}{}^{\bar\beta}\, Y_{\bar\beta}+B_{\alpha} T \tag{1.11}$$

along M so that

$$A_{\bar\alpha}{}^{\beta}=X_{\bar\alpha} z^{\beta}, \qquad B_{\bar\alpha}=(1+ih_{u})^{-1}\{X_{\bar\alpha} z^{n}+X_{\bar\alpha} z^{\beta}(r_{\beta}/r_{n})\}. \tag{1.12}$$

In particular Z is a holomorphic embedding if and only if $A=0$, $B=0$. Vector fields X belonging to the original CR structure which satisfy (1.11) will be said to form a frame adapted to the embedding. They are uniquely determined by the condition

$$X_{\alpha} z^{\beta}=\delta_{\alpha}^{\beta}. \tag{1.13}$$

2. TANGENTIAL CAUCHY-RIEMANN EQUATIONS

For a function f on M as in (1.8) we define (summing greek indices from 1 to $n-1$)

$$\bar{\partial}_M f = Y_{\bar{\alpha}} f \, d\bar{z}^{\bar{\alpha}}. \tag{2.1}$$

For a "tangential" $(0, q)$-form φ on M,

$$\left.\begin{aligned} \varphi = \sum_{|C|=q} \varphi_C \, d\overline{z^C}, \qquad C = (\gamma_1, \ldots, \gamma_q), \quad 1 \leqq \gamma_j \leqq n-1, \\ \bar{\partial}_M \varphi = \sum \bar{\partial}_M \varphi_C \wedge d\overline{z^C}. \end{aligned}\right\} \tag{2.2}$$

Similarly, we define the operator

$$\bar{\partial}_X f = X_{\bar{\alpha}} f \, d\bar{z}^{\bar{\alpha}}, \qquad \bar{\partial}_X \varphi = \sum \bar{\partial}_X \varphi_C \wedge d\overline{z^C}. \tag{2.3}$$

Since both the Y_α's commute and the X_α's commute, we have

$$(\bar{\partial}_M)^2 = 0, \qquad (\bar{\partial}_X)^2 = 0. \tag{2.4}$$

The ultimate goal is to find n independent functions f^j satisfying $\bar{\partial}_X f^j = 0$.

We extend the fields Y_α in (1.10) to a (1,0)-frame on $\mathbb{C}^n$ by setting $Y_n = (i/r_n)\partial_n$. The dual coframe is dz^α, $\theta \equiv -i\,\partial r$, $\partial f = Y_\alpha f dz^\alpha + Y_n f \theta$. A $(0, q)$-form on $\mathbb{C}^n$ has the decomposition $\psi = \psi' + \psi'' \wedge \theta$, where ψ' and ψ'' are tangential $(0, q)$- and $(0, q-1)$-forms relative to M. Since $\bar{\partial}\bar{\theta} = 0$, we have

$$\bar{\partial}\psi = \bar{\partial}_M \psi' + (\bar{\partial}_v \psi' + \bar{\partial}_M \psi'') \wedge \bar{\theta},$$
$$\bar{\partial}_v \psi' \equiv (-1)^q \sum_C Y_{\bar{n}} \psi'_C \, d\overline{z^C}.$$

By definition $\bar{\partial}_b \varphi$, for a tangential $(0, q)$-form φ on M, is given by first extending the coefficients of $\varphi \equiv \varphi'$ to be independent of v, applying $\bar{\partial}$, and then restricting back to M. Thus,

$$\bar{\partial}_b \varphi = \bar{\partial}\varphi = \bar{\partial}_M \varphi + \bar{\partial}_v \varphi \wedge \bar{\theta}. \tag{2.5}$$

In [2] Henkin has constructed solution operators P, Q for $\bar{\partial}_b$ on the hypersurfaces M_ρ in (1.9). For a (tangential) $(0, q)$-form φ, $1 \leqq q \leqq n-3$, they satisfy

$$\begin{aligned} \varphi = \bar{\partial}_b P \varphi + Q \bar{\partial}_b \varphi &= (\bar{\partial}_b P \varphi)' + (Q \bar{\partial}_b \varphi)' \\ &= \bar{\partial}_M P' \varphi + Q' (\bar{\partial}_M \varphi + \bar{\partial}_v \varphi \wedge \bar{\theta}), \end{aligned} \tag{2.6}$$

where the prime denotes the tangential part. It is only at this point that we must restrict to $n \geqq 4$. As noted in [10], P and Q annihilate the ideal generated by $\bar{\theta}$. Thus, setting $P_M = P'$ and $Q_M = Q'$, we have the homotopy formula

$$\varphi = \bar{\partial}_M P_M \varphi + Q_M \bar{\partial}_M \varphi. \tag{2.7}$$

The operators $\bar{\partial}_M$, $\bar{\partial}_X$, P_M, Q_M will be applied component-wise when φ or f is a $\mathbb{C}^n$-valued or function on M_ρ.

3. ALTERATION OF THE EMBEDDING

We begin with Newton's method. Given an embedding Z as in (1.7), (1.8) with $\bar\partial_X Z$ sufficiently small, we set

$$Z_* = Z(z, u) + F(z, u), \tag{3.1}$$

and try to choose F so that $\bar\partial_X Z_*$ is even smaller. With $\varphi = \bar\partial_X Z$ in (2.7) we get

$$\begin{aligned}\bar\partial_X Z_* &= \bar\partial_X Z + \bar\partial_M F + (\bar\partial_X - \bar\partial_M) F\\ &= \bar\partial_M (P_M \bar\partial_X Z + F) + Q_M \bar\partial_M \bar\partial_X Z + (\bar\partial_X - \bar\partial_M) F. \end{aligned}\tag{3.2}$$

At first, we would choose $F = -P_M \bar\partial_X Z$. Then, *roughly* (suitable norms will be introduced later), $\| F \| = \| P_M \bar\partial_X Z \| \leqq \| \bar\partial_X Z \|$. From (1.11)

$$(\bar\partial_X - \bar\partial_M) F = (A_{\bar\alpha}^{\gamma} Y_\gamma + B_{\bar\alpha} T) F\, dz^{\bar\alpha}, \tag{3.3}$$

so that (1.12) gives $\| (\bar\partial_X - \bar\partial_M) F \| \leqq \| \bar\partial_X Z \|^2$. Furthermore,

$$\bar\partial_M \bar\partial_X Z = (\bar\partial_M - \bar\partial_X) \bar\partial_X Z = -(A_{\bar\alpha}^{\gamma} Y_\gamma + B_{\bar\alpha} T) X_{\bar\beta} Z\, dz^{\bar\alpha} \wedge dz^{\bar\beta}. \tag{3.4}$$

Thus, $\| Q_M \bar\partial_M \bar\partial_X Z \| \leqq \| \bar\partial_M \bar\partial_X Z \| \leqq \| \bar\partial_X Z \|^2$, and so $\| \bar\partial_X Z_* \| \leqq \| \bar\partial_X Z \|^2$.

To develope a rigorous argument, we first observe that the above choice of F will not work in an iteration scheme, since P_M does not regain fully the derivative lost in applying $\bar\partial_X$ to Z. Therefore, we set

$$F = -S_t P_M \bar\partial_X Z, \tag{3.5}$$

where S_t is a smoothing operator to be described later. From (3.2) we get

$$\left.\begin{aligned}\bar\partial_X Z_* &= I_1 + I_2 + I_3,\\ I_1 &= \bar\partial_M (I - S_t) P_M \bar\partial_X Z,\\ I_2 &= (\bar\partial_M - \bar\partial_X) S_t P_M \bar\partial_X Z,\\ I_3 &= Q_M \bar\partial_M \bar\partial_X Z.\end{aligned}\right. \tag{3.6}$$

While we may still expect F to be the size of $\bar\partial_X Z$, and $\bar\partial_X Z_*$ to be smaller, the pertubation (3.1), (3.5) destroys the form (1.8) of M, which is needed to establish (2.7) and the estimates of [10]. For certain constant vectors K we have

$$\begin{aligned}F(z, u) &= K_0 + z^\beta K_\beta + z^{\bar\beta} K_{\bar\beta} + u K_n + F_2(z, u)\\ &= K_0 + z^j K_j + z^{\bar\beta} K_{\bar\beta} - i \bar{H} K_n + F_2,\\ F_2 &= O(2).\end{aligned}\tag{3.7}$$

Since $\bar\partial_X Z(0) = 0$ initially, we have

$$\bar\partial_X Z_*(0) = K_{\bar\beta}\, dz^{\bar\beta},$$

so that the coefficients $K_{\bar\beta}$ are dominated by $\| \bar\partial_X Z_* \|$. Therefore, we may set

$$\left.\begin{aligned}Z' &= Z_* + E,\\ E(z, u) &= -K_0 - z^j K_j - z^{\bar\beta} K_{\bar\beta} + E_2(z, u),\end{aligned}\right\} \tag{3.8}$$

where $E_2 = O(2)$ is a term holomorphic in z^j, to be chosen shortly. We shall then have $\bar\partial_X Z'(0) = 0$ and (roughly)

$$\|\bar\partial_X Z'\| \leqq 2 \|\bar\partial_X Z_*\| \leqq 2 \|\bar\partial_X Z\|^2.$$

More precisely, we take $E_2^\alpha = 0$ so that

$$z'^\alpha = z^\alpha + f_2^\alpha, \qquad f_2^\alpha = F_2^\alpha - i\,HK_n^\alpha, \tag{3.9}$$

$$z'^n = z^n - i\,HK_n^n + F_2^n + E_2^n. \tag{3.10}$$

With $K_n^n = K_n'^n + i\,K_n''^n$, (3.10) gives

$$u' = u + f_2^n, \qquad f_2^n = \operatorname{Re}(F_2^n + E_2^n) + HK_n''^n, \tag{3.11}$$

$$v' = (1 - K_n'^n)\,H + \operatorname{Im}(F_2^n + E_2^n). \tag{3.12}$$

We may write

$$F_2^n = K_{\alpha\beta} z^\alpha z^\beta + K_{\bar\alpha\bar\beta} z^{\bar\alpha} z^{\bar\beta} + K_{\alpha\bar\beta} z^\alpha z^{\bar\beta} + K_{\alpha n} z^\alpha u + K_{\bar\alpha \bar n} z^{\bar\alpha} u + K_{nn} u^2 + F_3^n. \tag{3.13}$$

We substitute $u = z^n - i\,H$ in the fourth and sixth terms of (3.13), and $u = \bar z^n + i\,H$ in the fifth. Defining

$$E_2^n = -K_{\alpha j} z^\alpha z^j - K_{nn} (z^n)^2 + \bar K_{\bar\alpha \bar j} z^\alpha z^j \tag{3.14}$$

then gives

$$F_2^n + E_2^n = K_{\alpha\bar\beta} z^\alpha z^{\bar\beta} + 2\operatorname{Re}(K_{\bar\alpha \bar j} z^{\bar\alpha} z^{\bar j}) + F_3 - i(K_{\alpha n} z^\alpha - K_{\bar\alpha \bar n} z^{\bar\alpha})\,H - K_{nn}(2\,iz^n H + H^2). \tag{3.15}$$

Equation (3.12) may now be written as

$$v' = b^1_{\alpha\bar\beta} z^\alpha z^{\bar\beta} + h(z, u) + h^*(z, u), \tag{3.16}$$

$$b^1_{\alpha\bar\beta} = b_{\alpha\bar\beta} + \hat b_{\alpha\bar\beta}, \qquad \hat b_{\alpha\bar\beta} = -K_n'^n\, b_{\alpha\bar\beta} + \frac{1}{2i}(K_{\alpha\bar\beta} - \bar K_{\beta\bar\alpha}), \tag{3.17}$$

$$h^* = -K_n'^n h + \operatorname{Im}\{F_3^n - i(K_{\alpha n} z^\alpha - K_{\bar\alpha\bar n} z^{\bar\alpha})\,H - K_{nn}(2\,iu H - H^2)\}. \tag{3.18}$$

We denote by $f\colon D_\rho \to \mathbb{R}^{2n-1}$ the mapping given by (3.9), (3.11) and by g its (local) inverse:

$$f\colon \begin{array}{l} z'^\alpha = z^\alpha + f_2^\alpha(z, u) \\ u' = u + f_2^n(z, u) \end{array}; \qquad g\colon \begin{array}{l} z^\alpha = z'^\alpha + g_2^\alpha(z', u') \\ u = u' + g_2^n(z', u') \end{array}. \tag{3.19}$$

Then $Z_1 = Z' \circ g$, defined on an appropriate neighborhood of 0 in $\mathbb{R}^{2n-1}$, is the new embedding. By (3.16) and (3.19) the new hypersurface M^1 has the equation

$$\left.\begin{array}{l} r_1 \equiv -v + H_1 = 0, \qquad H_1 = b^1(z) + h_1(z, u), \\ h_1 = (h + h^*) \circ g + b^1_{\alpha\bar\beta}(z^\alpha g_2^{\bar\beta} + g_2^\alpha z^{\bar\beta} + g_2^\alpha g_2^{\bar\beta}). \end{array}\right\} \tag{3.20}$$

We also form the new fields Y^1_α, T^1 as in (1.10).

To define the new frame X^1_α adapted to Z_1, we first put

$$\left.\begin{array}{l} X'_\alpha = C^\beta_\alpha X_\beta, \qquad C = I + \hat C, \\ \delta^\beta_\alpha = X'_\alpha z'^\beta = C^\gamma_\alpha(\delta^\beta_\gamma + X_\gamma f^\beta_2). \end{array}\right\} \tag{3.21}$$

Then we set

$$X_\alpha^1 = df(X'_\alpha). \tag{3.22}$$

We may regard the X_α^1 as defining a new CR structure equivalent to the original via the map f. We have

$$\begin{aligned} X_\alpha^1 z_1^\beta &= X'_\alpha [z_1^\beta \circ f] \circ g = (X'_\alpha z'^\beta) \circ g = \delta_\alpha^\beta, \\ X_{\bar\alpha}^1 Z_1 &= X'_{\bar\alpha} [Z_1 \circ f] \circ g = (X'_{\bar\alpha} Z') \circ g, \\ \bar\partial_{X^1} Z_1 (0) &= 0. \end{aligned} \tag{3.23}$$

By repeating this construction we shall define a sequence of embeddings Z_j and diffeomorphisms f_j, $j = 0, 1, 2, \ldots$ We must then show that the sequences Z_j and $F_j = f_j \circ f_{j-1} \circ \ldots \circ f_0$ converge in a neighborhood of 0 in $\mathbb{R}^{2n-1}$ to Z_∞ and F_∞. Then Z_∞ will give a holomorphic embedding of the structure $dF_\infty(X_\alpha)$, or equivalently, $Z_\infty \circ F_\infty$ will embed our original structure.

4. GEOMETRIC PROPERTIES OF THE EMBEDDING

The approximating real hypersurface M given by (1.8) has an essentially nonlinear character, which is fully gauged only via the Cartan-Chern-Moser theory. We shall not require this theory, but we shall have to control the function h, which remains non-zero throughout our argument. Otherwise the domain D_ρ would tend to shrink too rapidly during iteration. We should point out that the domains $|z^n(z, u)| < \rho$ are in some ways more natural but need not be convex, a point which causes considerable difficulty. Thus, we have chosen D_ρ as in (1.9).

We set

$$x = (z, u), \qquad |x|^2 = |z|^2 + u^2, \quad |z|^2 = z.\bar z, \tag{4.1}$$

and assume that the domain D of h is initially a ball,

$$D = B(R) = \{ x : |x| < R \}, \qquad 0 < R \leqq 1.$$

Also, we define

$$\left.\begin{aligned} \psi(x) &= u^2 + H(x) - H(x)^2 = \operatorname{Re}(z^n(x)^2 - iz^n(x)), \\ \psi_0(x) &= u^2 + b(z) - b(z)^2 = \operatorname{Re}(z_0^n(x)^2 - iz_0^n(x)), \end{aligned}\right\} \tag{4.2}$$

where $z_0^n = u + ib$. Since $\frac{1}{2}|z|^2 \leqq b(z) \leqq \frac{3}{2}|z|^2$, taking $R^2 < 1/3$ gives

$$\psi_0(x) \leqq u^2 + b(z) \leqq \frac{3}{2}|x|^2,$$

$$\psi_0(x) \geqq u^2 + b(z)\left(1 - \frac{3}{2}R^2\right) \geqq \frac{1}{4}|x|^2.$$

If $|h(x)| \leqq c_h |x|^3$ on D, then $(H = b + h)$

$$|\psi - \psi_0| = |h(1 - 2b - h)| \leqq c_h |x|^3 (1 + 3|x|^2 + c_h |x|^3)$$
$$\leqq c_h |x|^3 \leqq c_h R |x|^2 \leqq c_h R \psi_0 \leqq \frac{1}{2}\psi_0$$

after changing c_h and shrinking R. Thus on D

$$\begin{gathered} \frac{1}{8}|x|^2 \leqq \psi(x) \leqq \frac{9}{4}|x|^2, \\ B\left(\frac{2}{3}\sqrt{\rho}\right) \subset D_\rho \subset B(\sqrt{8\rho}). \end{gathered} \tag{4.3}$$

We set

$$N \equiv N(h, \rho) = 1 + \sup\{|\partial^I h(x)| : |I| \leqq 3, x \in D_\rho\}. \tag{4.4}$$

In what follows we shall frequently have to assume ρ is so small that

$$c_0 N \sqrt{\rho} < 1, \tag{4.5}$$

for a sufficiently large absolute constant $c_0 \geqq 1$, in order that D_ρ have certain desired properties. In particular, we have just shown that D_ρ has compact closure and (4.3) holds on D_ρ if c_0 is sufficiently large. (Absolute constants, denoted by c, c', c_j, etc., are those independent of particular functions and of the number of derivatives taken in our argument.)

We claim that D_ρ is a smoothly bounded strictly convex neighborhood of 0 if c_0 in (4.5) is sufficiently large. Since $d\psi(0) = 0$, it will suffice to show that the hessian of ψ is positive definite on D_ρ. But

$$\begin{aligned} \psi_{ij}(x)\dot{x}^i\dot{x}^j = 2\dot{u}^2 + 2b(\dot{z}) + h_{ij}\dot{x}^i\dot{x}^j \\ - 2(H_i\dot{x}^i)^2 - 2HH_{ij}\dot{x}^i\dot{x}^j \\ \geqq |\dot{x}|^2 - c|\dot{x}|^2\{N|x| + (1 + N|x|)^2|x|^2\} \\ \geqq |\dot{x}|^2 - c|\dot{x}|^2\{c_0^{-1} + (1 + c_0^{-1})^2 c_0^{-2}\}. \end{aligned}$$

Hence, if c_0 is chosen large (relative to c), this is positive definite.

Next we estimate the distance between ∂D_ρ and $\partial D_{\rho(1-\sigma)}$ for $0 < \sigma < 1$. For this take $x_0 \in \partial D_{\rho(1-\sigma)}$, $x_1 \in \partial D_\rho$. By the mean value theorem on $x_0 x_1$,

$$\begin{aligned} \rho\sigma = \psi(x_1) - \psi(x_0) \leqq c|\partial_x \psi|_\rho |x_1 - x_0|, \\ |\partial_x \psi| \leqq c\{|x| + N|x|^2 + |x|^3(1 + N|x|)^2\} \\ \leqq c\sqrt{\rho}\{(1 + c_0^{-1}) + \rho(1 + c_0^{-1})^2\}. \end{aligned} \tag{4.6}$$

Thus for an absolute constant $c_1 \geqq 1$,

$$\operatorname{dist}(\partial D_\rho, \partial D_{\rho(1-\sigma)}) \geqq c_1^{-1}\rho\sigma. \tag{4.7}$$

Next we consider some properties of D_ρ which are necessary for the estimates of [10]. For this let $W=(w, w^n)$ be a second coordinate vector for $\mathbb{C}^n$ and put

$$S=i(r_{\bar{Z}}.\partial_Z-r_Z.\partial_{\bar{Z}}),$$
$$p(Z, W)=r_Z.(Z-W), \qquad q(Z, W)=-p(W, Z).$$

In [10] we required

$$|Sp|\geqq c_2^{-1}, \qquad |Sq|\geqq c_2^{-1}. \tag{4.8}$$
$$|p(Z, W)|\geqq c_2^{-1}|Z-W|^2, \tag{4.9}$$

for a constant $c_2>0$, and all W, Z in M_ρ.

First consider Sp (Sq is similar and simpler),

$$Sp=Sr_Z.(Z-W)+ir_Z.r_{\bar{Z}}=i(r_{\bar{Z}}.H_{ZZ}-r_Z.H_{Z\bar{Z}}).(Z-W)+\frac{i}{4}+iH_Z.H_{\bar{Z}}.$$

As

$$|r_{\bar{Z}}.H_{ZZ}|+|r_Z.H_{Z\bar{Z}}|\leqq c\{N|x|+|x|(1+N|x|)^2\},$$
$$H_Z.H_{\bar{Z}}\leqq c|x|^2(1+N|x|)^2,$$
$$|W-Z|\leqq c\sqrt{\rho},$$

it's clear that (4.8) will hold for an absolute constant c_2 if c_0 is sufficiently large.

For (4.9) we take a second order Taylor expansion of c about Z,

$$r(W)-r(Z)=2\,\mathrm{Re}\,p(Z, W)+2b(z-w)+2\int_0^1(1-t)\,\partial^2h(W-Z, W-Z)\,dt.$$

Here the hessian $\partial^2 h$ is evaluated at $Z_t=Z+t(W-Z)\in D_\rho\times\mathbb{R}$. Since $r(Z)=r(W)=0$,

$$|z-w|^2\leqq 2b(z-w)\leqq 2|p(Z, W)|+cN|x|\,.\,|Z-W|^2.$$

In the last term we substitute

$$z^n-w^n=(r_{z^n})^{-1}(p(Z, W)-H_z.(z-w))$$

to get

$$\begin{aligned}|z-w|^2\leqq{}&|p|(2+cN|x|.|r_n|^2|p|)\\&+|z-w|^2cN|x|(1+|r_n|^2(|x|+N|x|^2))\\&\leqq c'|p|+c''c_0^{-1}|z-w|^2.\end{aligned}$$

Here we have used a lower bound on $|r_n|$ and an upper bound on $|p|$ on D_ρ. By increasing c_0, (4.9) is attained with an absolute constant c_2.

5. C^k-NORM ESTIMATES

Using multi-index notation for derivatives, we define the usual C^k norm of a function f on a domain D in $\mathbb{R}^n$:

$$\left.\begin{aligned} |\partial^j f|_D &= \sup\{|\partial^J f(x)| : |J|=j,\ x\in D\}, \\ |f|_{D,k} &= \max\{|\partial^j f|_D : 0 \leqq j \leqq k\}. \end{aligned}\right\} \tag{5.1}$$

For vector-valued functions the max is taken over all components. We begin with some elementary properties of these norms relating to the product rule, chain rule, and inverse mapping lemma. With a fixed k in mind we shall consider differently derivatives of order b which are low $b<k$, intermediate $k<b<2k$, and high $b>2k$. High derivatives and the usual methods [7] for dealing with them will not enter into our arguments.

From the product rule we clearly have

$$|fg|_{D,k} \leqq c(k)\,|f|_{D,k}\,|g|_{D,k}, \tag{5.2}$$

with a constant $c(k)$ depending on k. If we take at most $2k$ derivatives, more than k can fall on only one function; thus

$$\left.\begin{aligned} |fg|_{D,k+\mu} \leqq c(k+\mu)\{|f|_{D,k}\,|g|_{D,k+\mu} + |f|_{D,k+\mu}\,|g|_{D,k}\}, \\ 0<\mu\leqq k. \end{aligned}\right\} \tag{5.3}$$

The estimate is linear in $(k+\mu)$-norms over k-norms.

For the chain rule let $G: U \to V$, $\varphi: V \to \mathbb{R}^p$, where $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$. For $j \geqq 1$, $|J|=j$, we have

$$\partial^J(\varphi\circ G) = \sum_{s=1}^{j} \varphi_{\alpha_1 \ldots \alpha_s} \sum_{I_1 \cup \ldots \cup I_s = J} \partial^{I_1} G^{\alpha_1} \ldots \partial^{I_s} G^{\alpha_s},$$

where the subscripts on φ denote single derivatives. Hence, for $1 \leqq j \leqq k$

$$|\partial^j(\varphi\circ G)|_U \leqq c(j)\,|\varphi|_{V,j}\,(|G|_{U,j})^j,$$

and by (5.1)

$$|\varphi\circ G|_{U,k} \leqq c(k)\,|\varphi|_{V,k}\,(1+|G|_{U,k})^k. \tag{5.4}$$

For $|J|=k+\mu$, $0<\mu\leqq k$, we write $\partial^J(\varphi\circ G) = S_1 + S_2$, where S_1 is the sum of the terms with $s \leqq k$, and S_2 the rest. If in S_1 one $|I_\nu|>k$, then the rest are $<k$. Thus,

$$|S_1|_U \leqq c(k+\mu)\,|\varphi|_{V,k}\{|G|_{U,k+\mu}(|G|_{U,k})^{k-1} + (|G|_{U,k})^k\}.$$

Since $s \geqq k+1$ and each $|I_\nu| \geqq 1$, each $|I_\nu| \leqq \mu \leqq k$ in S_2, so that

$$|S_2|_U \leqq c(k+\mu)\,|\varphi|_{V,k+\mu}(1+|G|_{U,k})^{k+\mu}.$$

Combining these we get

$$|\varphi\circ G|_{U,k+\mu} \leqq c(k+\mu)(1+|G|_{U,k})^{k+\mu}\{|\varphi|_{V,k+\mu} + |\varphi|_{V,k}\,|G|_{U,k+\mu}\}, \tag{5.5}$$

which again is linear in the $(k+\mu)$-norms.

For the inverse mapping lemma let D and D′ be domains containing 0 in $\mathbb{R}^n$. Suppose f maps D onto D′, $f(0)=0$, $f=I+f_2$, $f_2=O(|x|^2)$, and f has inverse $g=I+g_2$ mapping D′ into D, $f\circ g(x')\equiv x'$. With d denoting the Jacobian matrix and $k\geqq 1$, we also assume

$$|df_2|_D<\frac{1}{2},\qquad |f_2|_{D,k}<1. \tag{5.6}$$

Then

$$dg_2=\varphi\circ g,\qquad \varphi=\Phi(df_2)\,df_2,\qquad \Phi(X)=-(I+X)^{-1}. \tag{5.7}$$

By (5.4), (5.2), (5.6) and $V=\left\{|X|<\frac{1}{2}\right\}$,

$$\begin{aligned}|dg_2|_{D',k-1}&\leqq c(k-1)\,|\varphi|_{D,k-1}(1+|g_2|_{D',k-1})^{k-1},\\ |\varphi|_{D,k-1}&\leqq c(k-1)\,|df_2|_{D,k-1}\,|\Phi|_{V,k-1}(1+|df_2|_{D,k-1})^{k-1}\\ &\leqq c(k-1)\,|df_2|_{D,k-1}.\end{aligned} \tag{5.8}$$

Inductively, we see that

$$\left.\begin{aligned}|dg_2|_{D',k-1}&\leqq c(k-1)\,|df_2|_{D,k-1},\\ |g_2|_{D',k}&\leqq c(k)\,|f_2|_{D,k}.\end{aligned}\right\} \tag{5.9}$$

For the intermediate derivatives we assume $\mu\leqq k-1$. In (5.7) we use (5.5), (5.9), (5.6), (5.8) to get

$$|dg_2|_{D',k-1+\mu}\leqq c(k-1+\mu)\{|\varphi|_{D,k-1+\mu}+|g|_{D',k-1+\mu}\}.$$

From (5.7), (5.5), (5.6)

$$\begin{aligned}|\varphi|_{D,k-1+\mu}\leqq c(k-1+\mu)\{&|\Phi X|_{V,k-1+\mu}\\ &+|\Phi X|_{V,k-1}\,|df_2|_{D,k-1+\mu}\}\\ &\leqq c(k+\mu)(1+|f_2|_{D,k+\mu}).\end{aligned}$$

It follows by induction that

$$|g_2|_{D',k+\mu}\leqq c(k+\mu)(1+|f_2|_{D,k+\mu}),\qquad 0\leqq\mu\leqq k-1. \tag{5.10}$$

Next we recall the estimates for P (and Q) from [10] and make some adaptations. It's easy to see that inequalities (4.5) of [10] hold with $\delta=c\,\rho\sigma$ for M_ρ as in (1.9). Since $D_\rho\times\mathbb{R}$ is convex, the norms over $M_{\hat\rho}$ in (4.12) of [10] are bounded by the norms over D_ρ.

$$|\partial^j f|_\rho=|\partial^j f|_{D_\rho},\qquad |f|_{\rho,k}=|f|_{D_\rho,k}. \tag{5.11}$$

For $0<\rho$, $\sigma<1$ formula (4.9) of [10] gives, for a $(0,q)$-form φ on D_ρ,

$$|\partial^b P\varphi|_{\rho(1-\sigma)}\leqq c(b)(\rho\sigma)^{-2(n+b-1)}\sum_{j=0}^{b}|\partial^j\varphi|_\rho\,|w_0^j w_1^{b-j}A|_\rho, \tag{5.12}$$

where as explained there

$$w_0^j w_1^{b-j} A = \sum_{\alpha_1 + \ldots + \alpha_s = b-j} F_{\alpha_1 \ldots \alpha_s}(v)\, \partial^{\alpha_1} v \ldots \partial^{\alpha_s} v, \qquad (5.12\,a)$$
$$v = \partial^i r, \qquad 1 \leqq i \leqq 3.$$

The F's are certain rational functions whose denominators (4.8) are bounded away from 0 on D_ρ. With $b = k-3$, (5.2) and (1.8) readily give

$$|P\varphi|_{\rho(1-\sigma), k-3} \leqq K |\varphi|_{\rho, k-3}, \qquad (5.13)$$
$$K = c(k) N^{\gamma(k)} (\rho\sigma)^{-2(n+k)+8}. \qquad (5.14)$$

Here $\gamma(k)$ is a polynomial in k, and we have redefined N in (4.4) as

$$N = 1 + |h|_{\rho, k}, \qquad (5.15)$$

since we shall assume $k \geqq 3$.

For the intermediate derivative estimate we take $b = k-3+\mu$, $\mu \leqq k-3$, and write $|\partial^b P\varphi|_{\rho(1-\sigma)} \leqq S_1 + S_2$, where S_1 is the sum on j from 0 to $k-3$. In (5.12 a) for S_1, $\alpha_1 + \ldots + \alpha_s = k-3+\mu-j$, so that at most one $\alpha_i > k-3$. Thus

$$S_1 \leqq c(b) N^{\gamma(b)} N(\mu) (\rho\sigma)^{-2(n+k+\mu)+8} |\varphi|_{\rho, k-3},$$

where

$$N(\mu) = 1 + |h|_{\rho, k+\mu}. \qquad (5.16)$$

For S_2 we have $b-j < \mu - 1 < k-3$ in (5.12), (5.12 a) so that

$$S_2 \leqq c(b) N^{\gamma(b)} (\rho\sigma)^{-2(n+k+\mu)+8} |\varphi|_{\rho, k-3+\mu}.$$

Hence, for $0 < \mu \leqq k-3$

$$|P\varphi|_{\rho(1-\sigma), k-3+\mu} \leqq K^* \{ |\varphi|_{\rho, k-3+\mu} + N(\mu) |\varphi|_{\rho, k-3} \}, \qquad (5.17)$$
$$K^* = c(k-3+\mu) N^{\gamma(k+\mu)} (\rho\sigma)^{-2(n+k+\mu)+8}. \qquad (5.18)$$

Finally, we recall some standard results from [6] on smoothing operators. For each integer $k > 0$ and for $0 < t < \infty$ one may construct S_t, which is convolution with a smooth function supported in a ball of radius t in $\mathbb{R}^{2n-1}$. Thus, by (4.7)

$$S_t: \ C^0(D_\rho) \to C^\infty(D_{\rho(1-\sigma)}), \qquad 0 < t < c_1^{-1} \rho\sigma. \qquad (5.19)$$

For $0 \leqq a \leqq b \leqq k$, the following two properties also hold,

$$|S_t f|_{\rho(1-\sigma), b} \leqq c(k)\, t^{a-b} |f|_{\rho, a}, \qquad (5.20)$$
$$|(I - S_t) f|_{\rho(1-\sigma), a} \leqq c(k)\, t^{b-a} |f|_{\rho, b}. \qquad (5.21)$$

6. ESTIMATES FOR THE NEW EMBEDDING

We now assume that our vector fields X_α are of class C^m, $m \geqq k \geqq 3$, and that we have an approximate holomorphic embedding Z on D_ρ. Also, we assume inequality (4.5) with c_0 so large that all the results of section 4 hold. With $0<t<c_1^{-1}\rho\sigma$, $0<\sigma<1/2$, we make the perturbation (3.1), (3.5). Then for $\tilde{\rho}=\rho(1-2\sigma)$, (5.20) with $a=k-3$, $b=k$ and (5.13) give

$$|F|_{\tilde{\rho}, k} \leqq c(k)\, t^{-3}\, |P_M \bar{\partial}_X Z|_{\rho(1-\sigma), k-3}$$
$$\leqq c(k)\, t^{-3} K\, |\bar{\partial}_X Z|_{\rho, k-3}.$$

Absorbing $c(k)$ into K, we get

$$|F|_{\tilde{\rho}, k} \leqq \theta, \qquad \theta \equiv t^{-3} K \delta, \qquad \delta \equiv |\bar{\partial}_X Z|_{\rho, k-3}. \tag{6.1}$$

With $0<\mu \leqq k-3$, all the constants $c(k+\mu)$ in section 5 may be denoted $c(k)$. Then (5.20) $(a=k-3,\ b=k+\mu)$ and (5.13) give

$$|F|_{\tilde{\rho}, k+\mu} \leqq t^{-\mu}\theta, \qquad 0<\mu \leqq k-3, \tag{6.2}$$

where another $c(k)$ has been absorbed into K. With f_2 as in (3.19), (3.9), (3.11), we have, using (5.15), (3.14),

$$|f_2|_{\tilde{\rho}, k} \leqq c\{|F|_{\tilde{\rho}, k} + \theta N\} \leqq c\,\theta N, \tag{6.3}$$

whereas (6.2), (5.16) give

$$|f_2|_{\tilde{\rho}, k+\mu} \leqq c\,\theta\,(t^{-\mu} + N(\mu)), \qquad 0<\mu \leqq k-3. \tag{6.4}$$

By section 1 and (4.3) we may assume initially that for $\rho_0>0$

$$\delta_0 \leqq c_X\, \rho_0^{(1/2)(m-k+3)}, \tag{6.5}$$

where the constant c_X depends on the X_α. Then with $t_0=c_1^{-1}\rho_0\sigma_0$, σ_0 fixed, (6.1) and (5.14) give

$$\theta_0 \leqq c_X(k)\, \rho_0^{(1/2)(m-5k-4n+13)}. \tag{6.6}$$

Thus, if we require [to also get (7.21-22) with $s=1$, $j=0$]

$$m \geqq 5k + 4n - 10, \tag{6.7}$$

we can make δ_0 and θ_0 as small as we like by shrinking ρ_0. We therefore assume $0<t<1$, $0 \leqq \delta \leqq \theta < 1$ in the following.

Next we analyse the properties of the map f (3.19). We want to show that f maps $D_{\tilde{\rho}}$ onto $D_{\tilde{\rho}(1-\sigma)}$ and has inverse g mapping $D_{\tilde{\rho}(1-\sigma)}$ into $D_{\tilde{\rho}}$. For this we fix x' in $D_{\tilde{\rho}(1-\sigma)}$ and must show that the transformation $W(x)=x'-f_2(x)$ has a unique fixed point x in $D_{\tilde{\rho}}$. Since $f_2=O(2)$, (6.3) and (4.3) give, for $x \in D_{\tilde{\rho}}$,

$$|W(x)-x'| \leqq c\,|f_2|_{\tilde{\rho}, 2}\, |x|^2 \leqq c\,\theta N \tilde{\rho}.$$

Thus W will map $D_{\tilde{\rho}}$ into itself by (4.7), if

$$c_3\, \theta N < \sigma, \tag{6.8}$$

for an absolute constant $c_3>0$. If x_1, x_2 are in $D_{\tilde\rho}$, we apply the mean value theorem on the segment $x_1 x_2 \subset D_{\tilde\rho}$ and use (6.3) to get

$$\begin{aligned}|W(x_2)-W(x_1)| &= |f_2(x_1)-f_2(x_2)| \\ &\leqq c\,|f_2|_{\tilde\rho,1}\,|x_2-x_1| \\ &\leqq c\,\theta N\,|x_2-x_1|.\end{aligned}$$

Thus by (6.8), W is a contraction. We denote by $x=g(x')\in D_{\tilde\rho}$ the unique fixed point, $f(g(x'))\equiv x'$. It follows that $g=I+g_2$. By (6.3) and (6.8), (5.6) holds, so by (5.9) and (5.10)

$$|g_2|_{\tilde\rho(1-\sigma),k}\leqq c(k)\,|f_2|_{\tilde\rho,k}, \tag{6.9}$$

$$|g_2|_{\tilde\rho(1-\sigma),k+\mu}\leqq c(k)\,(1+|f_2|_{\tilde\rho,k+\mu}),\qquad 0<\mu\leqq k-3. \tag{6.10}$$

It furthermore c_3 in (6.8) is sufficiently large (i. e. θ sufficiently small), then $|f(x)-x|$ will be less than $c_1^{-1}\rho^*\sigma$ (4.7) on $D_{\rho^*(1-\sigma)}$, so that

$$f(D_{\rho^*(1-\sigma)})\subset D_{\rho^*},\qquad \text{for}\quad \rho^*\leqq\tilde\rho.$$

We have now established the new embedding $Z_1=Z'\circ g$ on $D_{\tilde\rho(1-\sigma)}$; however, its proper domain will be [*see* (1.9), (4.2)]

$$D^1_{\rho_1}=\{x:\psi_1(x)<\rho_1\},\qquad \psi_1=\operatorname{Re}\{(z_1^n)^2-iz_1^n\}, \tag{6.11}$$

for a suitable ρ_1. We also set $\psi'=\operatorname{Re}\{(z'^n)^2-iz'^n\}$, so that $\psi_1=\psi'\circ g$. For $x\in D_{\tilde\rho}$, (3.10) and (4.5) give

$$\begin{gathered}|\psi'-\psi|\leqq|z'^n-z^n|\,(1+2\,|z^n|+|z'^n-z^n|),\\ |z'^n-z^n|\leqq c\,(\theta\,|x|^2+\theta\,|H|)\leqq c\,\theta\,|x|^2(1+N\,|x|)\leqq c'\,\theta\psi(x),\\ |z^n|\leqq c\,|x|\,(1+|x|+N\,|x|^2)\leqq c''.\end{gathered}$$

Thus on $D_{\tilde\rho}$,

$$\left.\begin{aligned}|\psi'-\psi|\leqq\hat c\,\theta\,\psi,\\ \psi(1-\hat c\,\theta)\leqq\psi'\leqq\psi(1+\hat c\,\theta),\end{aligned}\right\} \tag{6.12}$$

so that $D'_{\rho^*(1-\hat c\theta)}\subset D_{\rho^*}$, if $\rho^*\leqq\tilde\rho$. Since

$$D^1_{\rho_1}=f(D'_{\rho_1})\subset f(D_{\rho_1(1-\hat c\theta)^{-1}})\subset D_{\rho_1(1-\hat c\theta)^{-1}(1-\sigma)^{-1}},$$

$\hat c\theta<\sigma$ by (6.8), and we need $D^1_{\rho_1}\subset D_{\tilde\rho(1-\sigma)}$, we require $\rho_1\leqq\tilde\rho(1-\sigma)^3$. Since $\rho(1-2\sigma)(1-\sigma)^2\geqq\rho(1-5\sigma)$, we take

$$\rho_1=\rho(1-5\sigma),\qquad \sigma<\frac{1}{5}. \tag{6.13}$$

On $D^1_{\rho_1}$ we may compare Z_1 and Z. By (1.7)

$$\begin{gathered}Z_1(z,u)-Z(z,u)=(0,\ i(H_1(z,u)-H(z,u))),\\ H_1-H=\hat b(z)+h_1(z,u)-h(z,u).\end{gathered}$$

By (3.17), since $|b_{\alpha\beta}|<2$,

$$|\hat b(z)|_{\rho_1,k}\leqq c\,\theta. \tag{6.14}$$

From (3.20), (6.9), (6.3), (6.8), (5.2), (5.4)

$$\begin{gathered} |h_1-h|_{\rho_1,k}\leqq J_1+J_2+J_3, \\ J_3\leqq c(k)\,|g_2|_{\rho_1,k}(1+|g_2|_{\rho_1,k})\leqq c(k)\,\theta\,N, \\ J_2=|h^*\circ g|_{\rho_1,k}\leqq c(k)\,|h^*|_{\tilde\rho,k}, \\ J_1=|h\circ g-h|_{\rho_1,k}. \end{gathered} \tag{6.15}$$

From (3.18)

$$|h^*|_{\tilde\rho,k}\leqq c(k)\{\theta|h|_{\rho,k}+\theta+\theta|H|_{\rho,k}+\theta|H^2|_{\rho,k}\}\leqq c(k)\,\theta\,N^2. \tag{6.16}$$

For J_1 we set $g_t(x)=x+tg_2(x)\in D_{\tilde\rho}$, then

$$h(g(x))-h(x)=\int_0^1 \nabla_x h(g_t(x)).g_2(x)\,dt,$$

$$|h\circ g-h|_{\rho_1,k}\leqq c(k)\,|h|_{\tilde\rho,k+1}(1+|g_2|_{\rho_1,k})^k|g_2|_{\rho_1,k}\leqq c(k)\,\theta\,NN(1),$$

where $N(1)$ is given by (5.16) with $\mu=1$. It follows that

$$|h_1-h|_{\rho_1,k}\leqq c(k)\theta\,NN(1), \tag{6.17}$$

and with $N_1=1+|h_1|_{\rho_1,k}$,

$$N_1\leqq N(1+c(k)\,\theta\,N(1)). \tag{6.18}$$

Next we consider the $(k-3)$-norm of $\bar\partial_{X^1}Z_1$, which we denote by δ_1. By (3.23), (3.22), (3.21), and (3.8)

$$\begin{aligned} |X^1_{\bar\alpha}Z_1|_{\rho_1,k-3}&=|C^{\bar\beta}_{\bar\alpha}X_{\bar\beta}[Z_*+E]\circ g|_{\rho_1,k-3} \\ &\leqq c(k)(1+|g_2|_{\rho_1,k-3})^{k-3}|C|_{\tilde\rho,k-3}|\bar\partial_X Z_*|_{\tilde\rho,k-3} \\ &\leqq c(k)\,|C|_{\tilde\rho,k-3}|\bar\partial_X Z_*|_{\tilde\rho,k-3}. \end{aligned} \tag{6.19}$$

By (3.21) $\hat C=(I+Xf_2)^{-1}Xf_2$. Hence, if $|Xf_2|_{\tilde\rho,0}<\dfrac{1}{2}$, then as in (5.7)

$$|\hat C|_{\tilde\rho,k-3}\leqq c(k-3)\,|Xf_2|_{\tilde\rho,k-3}(1+|Xf_2|_{\tilde\rho,k-3})^{k-3}.$$

For any function φ, (1.11) gives

$$|X_\alpha\varphi|_{\tilde\rho,k-3}\leqq c(k-3)\,|\varphi|_{\tilde\rho,k-2}\{(1+|r_\alpha/r_n|_{\tilde\rho,k-3})(1+|A|_{\tilde\rho,k-3})+|B|_{\tilde\rho,k-3}\}.$$

Since

$$|r_\alpha/r_n|_{\rho,k-3}\leqq c(k)\,N^{k-2},\qquad |(1+ih_u)^{-1}|_{\rho,k-3}\leqq c(k)\,N^{k-3},$$

we get from (1.12)

$$|B|_{\tilde\rho,k-3}\leqq c(k)\,N^{2k-5}|\bar\partial_X Z|_{\tilde\rho,k-3}. \tag{6.20}$$

Thus, as $\delta\leqq 1$,

$$|X_\alpha\varphi|_{\tilde\rho,k-3}\leqq c(k)\,N^{2k-5}|\varphi|_{\tilde\rho,k-3},$$

and from (6.3)

$$|X_\alpha f_2|_{\tilde{\rho}, k-3} \leqq c(k)\,\theta\,N^{2k-4},$$
$$|X_\alpha f_2|_{\tilde{\rho}, 0} \leqq c\,\theta\,N^2.$$

If we replace (6.8) by

$$c_3\,\theta\,N^{2k} < \sigma, \tag{6.21}$$

(*i. e.* decrease θ) we have $|C|_{\tilde{\rho}, k-3} \leqq c(k)$, so that (6.19) gives

$$\delta_1 \equiv |\bar{\partial}_{X^1} Z_1|_{\rho_1, k-3} \leqq c(k)\,|\bar{\partial}_X Z_*|_{\tilde{\rho}, k-3}. \tag{6.22}$$

From (3.6), (3.3), (5.20), (5.13) we get

$$|I_2|_{\tilde{\rho}, k-3} \leqq c(k)\,\{|A|(1+|r_\alpha/r_n|)+|B|\}\,|S_t P_M \bar{\partial}_X Z|_{\tilde{\rho}, k-2}$$
$$\leqq c(k)\,|\bar{\partial}_X Z|_{\rho, k-3}\,N^{2k-5}\,t^{-1}\,K\,|\bar{\partial}_X Z|_{\rho, k-3}.$$

By absorbing $c(k)\,N^{2k-5}$ into K, we have

$$|I_2|_{\tilde{\rho}, k-3} \leqq t^{-1}\,K\,\delta^2. \tag{6.23}$$

From (3.6), (5.21) with $a=k-2$, $b=k-3+\mu$, we get

$$|I_1|_{\tilde{\rho}, k-3} \leqq c(k-3)\,(1+N^{k-2})\,|(I-S_t)\,P_M \bar{\partial}_X Z|_{\tilde{\rho}, k-2}$$
$$\leqq c(k)\,N^{k-2}\,t^{-1+\mu}\,|P_M \bar{\partial}_X Z|_{\rho(1-\sigma), k-3+\mu}.$$

With $0<\mu\leqq k-3$, (5.17) gives

$$|I_1|_{\tilde{\rho}, k-3} \leqq t^{-1+\mu}\,K^*\,\{|\bar{\partial}_X Z|_{\rho, k-3+\mu}+N(\mu)\,\delta\},$$

where we have absorbed $c(k)\,N^{k-2}$ into K^*. Since $\delta\leqq 1$,

$$|I_1|_{\tilde{\rho}, k-3} \leqq t^{-1+\mu}\,K^* L(\mu), \tag{6.24}$$
$$L(\mu) = 1+|\bar{\partial}_X Z|_{\rho, k-3+\mu}+|h|_{\rho, k+\mu}. \tag{6.25}$$

Finally, we consider I_3 in (3.6). By (5.13) for Q, (3.4), and arguing as for I_2, we have

$$|I_3|_{\tilde{\rho}, k-3} \leqq K\,|\bar{\partial}_M \bar{\partial}_X Z|_{\rho(1-\sigma), k-3}$$
$$\leqq c(k)\,K\,|\bar{\partial}_X Z|_{\rho, k-3}\,N^{2k-5}\,|\bar{\partial}_X Z|_{\rho(1-\sigma), k-2}.$$

For the last factor we use

$$\bar{\partial}_X Z = S_t \bar{\partial}_X Z + (I-S_t)\,\bar{\partial}_X Z,$$
$$|S_t \bar{\partial}_X Z|_{\rho(1-\sigma), k-2} \leqq c(k)\,t^{-1}\,|\bar{\partial}_X Z|_{\rho, k-3},$$
$$|(I-S_t)\,\bar{\partial}_X Z|_{\rho(1-\sigma), k-2} \leqq c(k)\,t^{-1+\mu}\,|\bar{\partial}_X Z|_{\rho, k-3+\mu}.$$

Hence, we may write

$$|I_3|_{\tilde{\rho}, k-3} \leqq t^{-1}\,K\,\delta^2 + t^{-1+\mu}\,K\,\delta\,|\bar{\partial}_X Z|_{\rho, k-3+\mu}. \tag{6.26}$$

Combining the foregoing, we have, as $K\leqq K^*$,

$$\delta_1 \leqq t^{-1}\,K\,\delta^2 + t^{-1+\mu}\,K^*\,L(\mu). \tag{6.27}$$

The next step is to control the growth of $N(\mu)$ and $L(\mu)$. Arguing as in (6.15) but using the corresponding intermediate derivative estimates, we have

$$|h_1|_{\rho_1, k+\mu} \leqq J_1(\mu) + J_2(\mu),$$

$$J_2(\mu) \leqq c(k)(1+|g_2|_{\rho_1, k})|g_2|_{\rho_1, k+\mu}$$

$$\leqq c(k)(1+|f_2|_{\tilde{\rho}, k+\mu}) \leqq c(k)(1+\theta t^{-\mu}+\theta N(\mu)),$$

$$J_1(\mu) = |(h+h^*)\circ g|_{\rho_1, k+\mu} \leqq c(k)$$

$$\times \{ |h+h^*|_{\tilde{\rho}, k+\mu} + |h+h^*|_{\tilde{\rho}, k}(1+|g_2|_{\rho_1, k+\mu}) \}.$$

By (6.16), (6.21), (3.18), (6.4), we get

$$|h+h^*|_{\tilde{\rho}, k} \leqq c(k) N,$$

$$|h^*|_{\tilde{\rho}, k+\mu} \leqq c(k)\{\theta N + |F|_{\tilde{\rho}, k+\mu} + \theta N N(\mu)\}$$

$$\leqq c(k)\theta\{t^{-\mu} + N N(\mu)\}.$$

Thus, by (6.21), (6.10), and (6.4)

$$N_1(\mu) \leqq c(k)(N(\mu) + \theta t^{-\mu}). \tag{6.28}$$

For the term $\bar{\partial}_{X^1} Z_1$ we first note that by (3.21-23)

$$X^1_{\bar{\alpha}} Z_1 = \varphi \circ g, \qquad \varphi = \Phi(G),$$

$$G = (\bar{\partial}_X Z, \partial^1 F, \partial^1 h),$$

and apply (5.5) twice. This gives

$$|\bar{\partial}_{X^1} Z_1|_{\rho_1, k-3+\mu} \leqq c(k)\{ |\varphi|_{\tilde{\rho}, k-3+\mu} + |\varphi|_{\tilde{\rho}, k-3}(1+|g_2|_{\rho_1, k-3+\mu}) \},$$

$$|\varphi|_{\tilde{\rho}, k-3} \leqq c(k)(1+|G|_{\tilde{\rho}, k-3})^{k-3} \leqq c(k) N^{k-3},$$

$$|\varphi|_{\tilde{\rho}, k-3+\mu} \leqq c(k) N^{k-3+\mu}(1+|G|_{\tilde{\rho}, k-3+\mu}),$$

$$|G|_{\tilde{\rho}, k-3+\mu} \leqq |\bar{\partial}_X Z|_{\tilde{\rho}, k-3+\mu} + |h|_{\tilde{\rho}, k-2+\mu} + t^{2-\mu}\theta.$$

By (6.10) and (6.4)

$$1+|g_2|_{\rho_1, k-3+\mu} \leqq c(k)(1+\theta t^{3-\mu} + \theta N(\mu)).$$

Combining gives

$$|\bar{\partial}_{X^1} Z_1|_{\rho_1, k-3+\mu} \leqq c(k) N^{k-3+\mu}\{L(\mu) + \theta t^{2-\mu}\}.$$

Thus, using (6.28) we get

$$L_1(\mu) \leqq c(k) N^{k-3+\mu}\{L(\mu) + \theta t^{-\mu}\}. \tag{6.29}$$

7. THE SEQUENCE OF EMBEDDINGS

We must show that the foregoing process can be repeated an infinity of times and leads to a sequence of embeddings which converges to a holomorphic embedding. In view of (6.13) we first define for $j \geqq 0$

$$\rho_{j+1} = \rho_j (1 - 5\sigma_j), \qquad \sigma_j = 5^{-j-2}, \tag{7.1}$$

where $\rho_0 > 0$ is yet to be determined. By taking the logarithm of the infinite product, one sees that $\rho_* = \lim \rho_j > 0$,

$$\rho_* = \rho_0 \prod_{j=0}^{\infty} (1 - 5\sigma_j). \tag{7.2}$$

We first assume that we have constructed the $Z_j = (z, u + i H_j)$ on $D(\rho_j)$ with

$$c_0 N_j \sqrt{\rho_j} < 1, \tag{7.3}$$

$$0 < t_j \leqq c_1^{-1} \rho_j \sigma_j, \tag{7.4}$$

$$c(k) \theta_j N_j^{2k} < \sigma_j, \qquad \theta_j = t_j^{-3} K_j \delta_j, \tag{7.5}$$

where $N_j = 1 + |h_j|_{\rho j, k}$, etc, and determine what further conditions are need for convergence.

We define

$$F_l = f_l \circ f_{l-1} \circ \ldots \circ f_0 = f_l \circ F_{l-1} : U_l \to \mathbb{R}^{2n-1}, \tag{7.6}$$

$$G_l = g_0 \circ g_1 \circ \ldots \circ g_l = G_{l-1} \circ g_l : D(\rho_{l+1}) \subsetneq D(\rho_0). \tag{7.7}$$

Here, $U_l = G_l(D(\rho_{l+1})) \subset D_{\rho_0}$ contains 0 but is otherwise difficult to specify. However, since the $D(\rho_j) \supset B(2\sqrt{\rho_j}/3)$ (4.3), and they decrease, all the G_l are defined on

$$D^* \equiv \bigcap \{ D(\rho_j) : 0 \leqq j < \infty \},$$

which contains $B(2\sqrt{\rho^*}/3)$. By (6.14) and (6.17)

$$|b_{j+1} - b_j|_{\rho j+1, k} \leqq c\,\theta_j, \tag{7.8}$$

$$|h_{j+1} - h_j|_{\rho j+1, k} \leqq c(k) \theta_j N_j N_j(1). \tag{7.9}$$

Thus, if we can show

$$\sum \theta_j N_j N_j(1) < \infty, \tag{7.10}$$

then the Z_j will converge in C^k norm to $Z_* = (z, u + i H_*(z, u))$, an embedding defined on a neighborhood of 0 in $\mathbb{R}^{2n-1}$. The corresponding vector fields (1.10), Y_α^j, will converge in C^{k-1} to Y_α^*. Since, (1.11),

$$X_\alpha^j = Y_\alpha^j + A_\alpha^{j\bar\gamma} Y_{\bar\gamma} + B_\alpha^j \partial_u \tag{7.11}$$

and $(A^j, B^j) \to 0$ in C^{k-3} by (1.12), (6.20), and (7.5), X_α^j will converge to $X_\alpha^* \equiv Y_\alpha^*$ in C^{k-3}. Thus, we shall have produced an embedded real hypersurface of class C^k which is, in fact, equivalent to our original structure. However, we shall not yet have produced a C^k solution to $\bar\partial_X Z = 0$, for the original $X_\alpha = X_\alpha^0$. We must still analyse the maps F_l, G_l, as in [9].

By (6.3), (6.9)

$$|f_{2,j}|_{\tilde{\rho}_j,k} \leqq c\,\theta_j N_j, \qquad |g_{2,j}|_{\tilde{\rho}_j(1-\sigma_j),k} \leqq c(k)\,\theta_j N_j. \tag{7.12}$$

Letting d denote Jacobian matrix, and $\|\,.\,\|$ the operator norm on matrices, we have

$$\begin{aligned} dG_l &= (dG_{l-1} \circ g_l)\, dg_l = dg_0 \ldots dg_l, \\ \|dG_l\|_{\rho_{l+1}} &\leqq \prod_{j=0}^{l} (1 + \|dg_{2,j}\|_{\rho_{j+1}}) \\ &\leqq \prod_{j=0}^{\infty} (1 + c(k)\,\theta_j N_j). \end{aligned} \tag{7.13}$$

By (7.10) this is finite giving $|G_l|_{\rho_{l+1},1} \leqq b''$ for all l. From this follows the uniform convergence of the G_l to a continuous $G_*: D^* \to D(\rho_0)$. For, arguing as for J_1 in (6.15),

$$|G_l - G_{l-1}|_{\rho_{l+1},0} \leqq \|dG_{l-1} \circ g_l\|_{\rho_{l+1}} \|g_{2,l}\|_{\rho_{l+1},0} \leqq b''\,c(0)\,\theta_j N_j,$$

and we have convergence by (7.10). Similarly, the F_l are defined on the *set* $U_* = G_*(D^*)$, and we have a common bound

$$c^{-1}|F_l|_{U_l,1} \leqq \|dF_l\|_{U_l} \leqq b'.$$

Now,

$$\left.\begin{aligned} dF_j - dF_{l-1} &= (df_{2,l} \circ F_{l-1})\, dF_{l-1}, \\ \|dF_l - dF_{l-1}\|_{U_l} &\leqq \|df_{2,l}\|_{\tilde{\rho}_l} \|dF_{l-1}\|_{U_{l-1}} \leqq c\,\theta_l N_l\, b'. \end{aligned}\right\} \tag{7.14}$$

Thus, by (7.10), (7.14), and (7.15) with $s=3$, the $DG_l = (dF_l)^{-1} \circ G_l$ converge uniformly on D^*. Since the dG_l now have the positive lower bound $1/b'$ in norm, it follows that the G_l converge in C^1-norm to G_*, which is a diffeomorphism (after shrinking D_*). Thus, U_* is an open set and the F_j converge in C^1 to a diffeomorphism F_*. F_* is a CR equivalence between the structures X_α^0, X_α^*.

To show that the F_j converge in C^k to F_*, we fix s, $2 \leqq s \leqq k$ and assume that we have a bound

$$|F_l|_{U_l,s-1} \leqq b'_{s-1}, \tag{7.15}$$

where b'_{s-1} is a constant depending on our given structure. Then,

$$\begin{aligned} dF_l &= (df_l \circ F_{l-1})(dF_{l-1}), \\ |dF_l|_{U_l,s-1} &\leqq c(s)\,|df_l|_{\tilde{\rho}_l,s-1}(1 + |F_{l-1}|_{U_l,s-1})^{s-1}\,|dF_{l-1}|_{U_{l-1},s-1} \\ &\leqq c(s)\,(1 + b'_{s-1})^{s-1}\,|dF_{l-1}|_{U_{l-1},s-1}. \end{aligned} \tag{7.16}$$

Thus,

$$|F_l|_{U_l,s} \leqq b_s\,|F_{l-1}|_{U_{l-1},s} \leqq \ldots \leqq b_s^l\,a_s, \tag{7.17}$$

where b_s and a_s are constants depending on $s \leqq k$ and the X_α^0. By (7.14)

$$|dF_l - dF_{l-1}|_{U_l, s-1} \leqq c(s) |df_{2,l}|_{\tilde{\rho}_l, s-1} (1+b'_{s-1})^{s-1} b_s^l a_s \leqq c(s) \theta_l N_l (b_s^2)^l a_s. \quad (7.18)$$

By induction there is a constant $b = b(k, X_\alpha^0)$, such that, if

$$\sum_{j=0}^{\infty} \theta_j N_j N_j(1) b^j < \infty, \quad (7.19)$$

then (7.15) holds for $s = k+1$, and $F_l \to F_*$ in C^k.

If we can establish an *a priori* upper bound $N^* \geqq N_j$, and then shrink ρ_0 so that $c_0 N^* \sqrt{\rho_0} < 1$, (7.3) will hold for all j. From (6.18)

$$N_{j+1} \leqq N_j (1 + c(k) \theta_j N_j(1)) \leqq N_0 \prod_{i=0}^{j} (1 + c(k) \theta_i N_i(1)), \quad (7.20)$$

so that (7.19) will guarantee such a bound N^*. For a suitable $s > 0$ to be determined, we define

$$\zeta_j = t_j^{-s} \theta_j N_j N_j(1) b^j = t_j^{-s-3} \delta_j K_j N_j(1) b^j, \quad (7.21)$$

where for convenience we have absorbed another factor of N_j into K_j. If we can show $\zeta_j \leqq 1$, then by (7.4), (7.1)

$$\theta_j N_j N_j(1) b^j \leqq c_1^{-s} (\rho_j \sigma_j)^s \leqq \sigma_j^s.$$

We shall then have an upper bound for all partial sums in (7.19).

Now we assume that we have constructed Z_i, $i \leqq j$, satisfying (7.4), (7.5) and for a constant M

$$\zeta_j \leqq M, \qquad M < \frac{1}{2}. \quad (7.22)$$

Then with any $t_{j+1} \leqq c_1^{-1} \rho_{j+1} \sigma_{j+1}$, we may construct Z_{j+1} on $D_{\rho_{j+1}}$. By (6.18), (6.27), (6.28) and (6.29) we have (7.20) and

$$\delta_{j+1} \leqq t_j^{-1} K_j \delta_j^2 + t_j^{-1+\mu} K_j^* L_j(\mu), \quad (7.23)$$

$$N_{j+1}(\mu) \leqq c(k) (N_j(\mu) + \theta_j t_j^{-\mu}), \quad (7.24)$$

$$L_{j+1}(\mu) \leqq c(k) N_j^{k-3+\mu} (L_j(\mu) + \theta_j t_j^{-\mu}). \quad (7.25)$$

We investigate the growth of N_j, K_j, K_j^*, and $N_j(1)$ with j. By (7.20), (7.21), (7.22), (5.14), (5.18) we have, successively increasing $\hat{c}(k) \geqq 1$

each time,

$$\frac{N_{j+1}}{N_j} \leqq c(k)(1+t_j^s \zeta_j) \leqq \hat{c}(k),$$

$$\frac{K_{j+1}}{K_j} \leqq c(k)\hat{c}(k)^{\gamma(k)}\left(\frac{5}{1-5\sigma_j}\right)^{2(n+k)-8} \leqq \hat{c}(k), \tag{7.26}$$

$$\frac{K_{j+1}^*}{K_j^*} \leqq c(k)\hat{c}(k)^{\gamma(k+\mu)}\left(\frac{5}{1-5\sigma_j}\right)^{2(n+k+\mu)-8} \leqq \hat{c}(k).$$

From (7.24) and (7.21) and $\mu=1$

$$\frac{N_{j+1}(1)}{N_j(1)} \leqq c(k)(1+\zeta_j t_j^{s-1}) \leqq \hat{c}(k), \tag{7.27}$$

if we choose $s \geqq 1$.

Next we consider ζ_{j+1}. By (7.21), (7.23), (7.26), (7.27), we have

$$\begin{aligned}\zeta_{j+1} &\leqq t_{j+1}^{-s-3} K_{j+1} N_{j+1}(1) b^{j+1}\{t_j^{-1} K_j \delta_j^2 + t_j^{-1+\mu} K_j^* L_j(\mu)\} \\ &\leqq t_{j+1}^{-s-3} c_b \{t_j^{2s+5}\zeta_j^2 + t_j^{-1+\mu} b^j K_j K_j^* N_j(1) L_j(\mu)\}.\end{aligned}$$

$$c_b \equiv \hat{c}(k)^2 b. \tag{7.28}$$

If we choose t_{j+1} so that the coefficient of ζ_j^2 is 1;

$$t_{j+1} = c_b^{2-\kappa} t_j^{\kappa}, \qquad \kappa = \frac{2s+5}{s+3}, \qquad s = \frac{3\kappa-5}{2-\kappa}, \tag{7.29}$$

then

$$\zeta_{j+1} \leqq \zeta_j^2 + t_j^{\mu-2s-6} b^j K_j K_j^* N_j(1) L_j(\mu). \tag{7.30}$$

Therefore, we define the t_j inductively by $t_0 \leqq c_1^{-1}\rho_0\sigma_0$ and (7.29). If (7.4) holds for j then

$$t_{j+1} \leqq c_b^{2-\kappa}\left(\frac{\rho_j\sigma_j}{\rho_{j+1}\sigma_{j+1}}\right)^{\kappa}(\rho_{j+1}\sigma_{j+1})^{\kappa-1}\rho_{j+1}\sigma_{j+1} \leqq c(\rho_0\sigma_0)^{\kappa-1}\rho_{j+1}\sigma_{j+1}.$$

Once we have determined $\kappa>1$, we shrink ρ_0 so that $c(\rho_0\sigma_0)^{\kappa-1} \leqq C_1^{-1}$, then $(7.4)_{j+1}$ holds.

To (7.5) and (7.22) we add the inductive hypothesis

$$t_j^{\mu-2s-6} b^j K_j K_j^* N_j(1) L_j(\mu) < \frac{1}{2} M. \tag{7.31}$$

Clearly $(7.22)_j$ and $(7.31)_j$ imply $(7.22)_{j+1}$. By (7.21) and (7.22), $\theta_j N_j^{2k} \leqq t_j^s N_j^{2k-1}$, so that (7.5) may be replaced by the stronger

$$c(k)\, t_j^s N_j^{2k-1} \sigma_j^{-1} < 1. \tag{7.32}$$

But the ratio of the left hand side of (7.32) at $j+1$ to that at j does not exceed $(c_b^{2-\kappa} t_j^{\kappa-1})^s \hat{c}(k)^{2k-1} 5$. This will be less than one if ρ_0 is chosen sufficiently small, and $(7.32)_j$ will imply $(7.32)_{j+1}$.

Finally, consider $(7.31)_{j+1}$. If we set

$$Q_j \equiv b^j K_j K_j^* N_j(1) N_j^{k-3+\mu},$$

then by (7.28), (7.24-27), and $\theta_j \leqq M t_j^s$,

$$t_{j+1}^{\mu-2s-6} b^{j+1} K_{j+1} K_{j+1}^* N_{j+1}(1) L_{j+1}(\mu) \leqq c(k, b) M Q_j t_j^{\kappa(\mu-2s-6)} (t_j^{-\mu+2s+6} + t_j^{s-\mu}). \quad (7.33)$$

We require that both exponents of t_j be positive [note (7.29)]

$$\mu > 2s+6, \qquad (\kappa-1)\mu > 3s+10.$$

Then (7.33) is bounded by $c(k, b) M Q_j t_j^{2\alpha}$, $\alpha > 0$. But

$$\frac{Q_{j+1} t_{j+1}^{\alpha}}{Q_j t_j^{\alpha}} \leqq c(k, b) t_j^{(\kappa-1)\alpha},$$

which can be made arbitrarily small uniformly in j by shrinking ρ_0. Hence, if ρ_0 is taken sufficiently small $(7.31)_{j+1}$ will hold. This completes the induction step.

It remains to verify the above conditions for $j=0$. For this we take $t_0 = (c_1^{-1} \rho_0 \sigma_0)^{\beta}$, $\beta \geqq 1$. Then $(7.4)_0$ holds and $(7.32)_0$ will hold if ρ_0 is small enough. We can achieve $(7.31)_0$ by shrinking ρ_0 if [*see* (5.14) and (5.18)]

$$\beta(\mu - 2s - 6) > 2\mu + 4(n+k) - 16. \quad (7.34)$$

Likewise, by (6.5) and (7.21) we can get $(7.22)_0$ if

$$m - k + 3 > 2\beta(s+3) + 4(n+k) - 16. \quad (7.35)$$

Thus, if all parameters are chosen as indicated and $\rho_0 > 0$ is sufficiently small, the construction is possible for all j and yields a sequence of embeddings $Z_j \circ F_j$ of the original CR structure which converges in C^k-norm on a neighborhood of 0 in $\mathbb{R}^{2n-1}$ to a holomorphic embedding. We may take $s=1$, $\kappa = \frac{7}{4}$, and $\mu = 18$. Since $\mu \leqq k-3$, we need $k \geqq 21$. By the construction of section 1 we may take Z_0 to be a polynomial. For $j \geqq 1$ the Z_j as constructed in section 3 are C^∞. Hence $N_j(\mu)$ is finite, and by (1.11) $\bar{\partial}_{X^j} Z_j$ is as smooth as (A^j, B^j), *i. e.* of class C^m. By (6.25) $L_j(\mu)$ will be finite if $m \geqq k - 3 + \mu = k + 15$. If we choose $\beta = 2 + n + k$, then (7.34) holds and (7.35) becomes $m \geqq 6k + 5n - 2$. This implies $m \geqq k + 15$ since $k \geqq 21$, and the theorem is proved.

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