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Small time periodic solutions of fully nonlinear telegraph equations in more spatial dimensions

by

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ABSTRACT. — The solution to the problem is expressed by means of a couple of new dependent variables—the time derivative of the desired solution and a function independent of time, which is the trace of the solution in a fixed time. This simple device makes it possible to prove the existence of a time-periodic solution without encountering the "loss of derivatives" phenomenon.

The result provides a small smooth solution, unique in a neighbourhood of zero, for any equation which is a perturbation of a suitable linear telegraph equation at least by quadratic terms which can involve any derivatives up to order two of the unknown function.

Key words : Hyperbolique equation with dissipation, time-periodic solutions.

RÉSUMÉ. — On démontre l'existence d'une solution périodique en temps pour une équation hyperbolique du deuxième ordre avec dissipation et une force périodique mais petite et avec une non-linéarité composée de dérivées de la variable dépendante dont l'ordre ne dépasse pas deux.

Le phénomène de « la perte des dérivées » est détourné par une simple transformation de la variable dépendante.

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1. INTRODUCTION

Denoting the number of spatial variables by n we set

$$\bar{n}=\left[\frac{n+1}{2}\right]+1 \quad \text{and} \quad M=\bar{n}+3,$$

where [ ] is used to denote the integer part of a real number. With the usual notation for derivatives we write

$$J(u)=\{\partial_x^\alpha \partial_t^p u; |\alpha|+p\leqq 2, \alpha=(\alpha_1, \ldots, \alpha_n)\}$$

to denote the jet of all derivatives up to order 2.

Let $G=G(\zeta)$; $\zeta=\{\zeta_\alpha^p\}_{|\alpha|+p\leqq 2}$ be a function on a neighbourhood $\mathcal{N}(\eta_0)=\{\zeta; |\zeta_\alpha^p|<\eta_0 \text{ for } |\alpha|+p\leqq 2\}$ of 0 in R^d, $d=\binom{n+3}{2}$. We shall suppose that $G\in C^{M+1}(\mathcal{N}(\eta_0))$, $G(0)=0$ and that the functions

$$g_\alpha^p(\zeta)=\frac{\partial G}{\partial \zeta_\alpha^p}(\zeta)$$

satisfy

$$g_\alpha^p(0)=0 \quad \text{for} \quad |\alpha|+p\leqq 2.$$

Throughout the paper Ω will denote an open and bounded domain in R^n of class C^{M+1}. We shall show the existence of a classical solution to the problem

$$Au+\partial_t u+\kappa\partial_t^2 u+\sum_{r=1}^n \kappa_r \partial_{x_r}\partial_t u+G(Ju)+f=0 \quad \text{in } \Omega\times R, \tag{1.1}$$

$$u(x,t)=0 \quad \text{for } x\in\partial\Omega, \quad t\in R, \tag{1.2}$$

$$u(x,t)=u(x,t+2\pi) \quad \text{for } x\in\Omega, \quad t\in R, \tag{1.3}$$

where κ and all κ_r are constants,

$$(Au)(x,t)=\sum_{|\alpha|,|\beta|\leqq 1}(-1)^{|\alpha|}\partial_x^\alpha(a_{\alpha\beta}(x)\partial_x^\beta u(x,t)), \tag{1.4}$$

and f is a smooth and small function. The detailed assumptions on A and f along with the main result will be given in the next section.

Let us note that those assumptions will be satisfied for $A=-\Delta$ and $\kappa_r=0$. Hence the equation

$$-\Delta u+\partial_t u+\kappa\partial_t^2 u+G(\partial_{x_i}\partial_{x_j}u, \partial_{x_i}\partial_t u, \partial_{x_i}u, \partial_t u, u)+f=0$$

has a unique small classical solution which is periodic in time with period 2π and which satisfies the Dirichlet boundary condition provided that f is 2π-periodic, sufficiently smooth and small.

In one spatial variable this problem was studied by P. H. Rabinowitz [7], with the help of a Moser theorem to overcome the "loss of derivatives" phenomenon, *see* [7], p. 16, encountered when the usual Newton iterative procedure is applied. In this connection *see* also [2], where even a bifurcation problem is treated.

The solution u to (1.1)-(1.3) will be written in the form

$$u(x,t)=\int_0^t v(x,\tau)\,d\tau+b(x),$$

where, obviously, $v=\partial_t u$ and, by virtue of 2π-periodicity of u in t, v satisfies $\int_0^{2\pi} v(x,t)\,dt=0$.

This device makes avoiding the occurence of the "loss of derivatives" possible. Hence, by using Schauder's theorem, we are able to extend the result of [7] to more spatial variables and also to suppress the explicit appearance of the small parameter and to improve the assumptions on regularity of f and G.

The idea of looking for $\partial_t u$ rather than for the solution u itself is not new. It has been used in time periodic problems at least in those having the form where it is possible to separate the determination of $\partial_t u$ and the time independent component of the solution, *see* [6]. As far as the initial-boundary value problems for (1.1) are concerned we refer to [8] and [9].

Equation (1.1) was taken in a form imitating the one-dimensional case of [3]. One of the crucial sufficient conditions for the existence, (2.2), has been taken over from [5].

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2. BASIC NOTATIONS AND THE MAIN RESULT

The notations are standard. All functions are real-valued. The spaces $L^2(\Omega)$ and $H^k(\Omega)$ will be, respectively, equipped with the norms

$$\|b\|_{0,\Omega}=\left(\int_\Omega |b(x)|^2\,dx\right)^{1/2},$$

$$\|b\|_{k,\Omega}=\max\{\|\partial_x^\alpha b\|_{0,\Omega};|\alpha|\leqq k\}.$$

$H_0^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$. Denoting

$$Q=\Omega\times(0,2\pi)$$

we equip the spaces $L^2(Q)$ and $H^k(Q)$, respectively, with the norms

$$\|u\|_{0,Q}=\left(\int_0^{2\pi}\int_\Omega |u(x,t)|^2\,dx\,dt\right)^{1/2},$$
$$\|u\|_{k,Q}=\max\{\|\partial_x^\alpha\partial_t^p u\|_{0,Q};|\alpha|+p\leqq k\}.$$

$C^\infty_{2\pi}(\Omega\times R)$ will be the space of C^∞ functions on $\Omega\times R$ which are 2π-periodic in t.

By $H^k(Q)$, the same symbol as above, we shall denote the completion of $C^\infty_{2\pi}(\Omega\times R)$ in the norm $\|.\|_{k,Q}$.

By $H_0^1(Q)$ we denote the closure of $\{u\in C^\infty_{2\pi}(\Omega\times R);\ u(.,t)\in C_0^\infty(\Omega)$ for all $t\in R\}$ in the norm $\|.\|_{1,Q}$.

In what follows, in agreement with the last two notations, all functions depending on t are supposed to be 2π-periodic in t without any particular reference to this fact; especially, there is no indication of the 2π-periodicity in t in the notation of the spaces. As usual, the inner product in $L^2(Q)$ is

$$(u,v)_0=\int_0^{2\pi}\int_\Omega u(x,t)\,v(x,t)\,dx\,dt.$$

For later use we also mention here the norm

$$|||u|||_k=\max(\|u\|_{k,Q},\|\partial_t^{k+1}u\|_{0,Q})$$

which was implicitely used in [7] and which is of a basic importance in the investigated problem.

Occasionally, the C^k norms, denoted by $\|.\|_{C^k(\bar Q)}$ and $\|.\|_{C^k(\bar\Omega)}$ will be used. With the above choice of $\bar n$ the Sobolev inequality takes up the form

$$\|u\|_{C(\bar Q)}\leqq C_S\|u\|_{\bar n,Q} \tag{2.1}$$

which, for simplicity, will be used with the same constants C_S and $\bar n$ also in cases, where the domain $\Omega\subset R^n$ appears in lieu of Q.

For A given by (1.4) with $a_{\alpha\beta}\in C^p_*(\Omega)$, $p=M-1+|\alpha|$, the class of functions which have continuous and bounded derivatives up to order p on Ω, we set

$$L_0 u=Au+\partial_t u+\kappa\,\partial_t^2 u+\sum_{r=1}^n \kappa_r\,\partial_{x_r}\partial_t u.$$

Throughout the paper we shall suppose that there exist positive λ and C_0 such that

$$(L_0 u,\Lambda u)_0\geqq C_0\|u\|_{1,Q}^2 \qquad \text{for}\quad u\in H^2(Q)\cap H_0^1(Q), \tag{2.2}$$

where

$$\Lambda u=\partial_t u+\lambda u. \tag{2.3}$$

In particular, when u is independent of t, (2.2) assumes the form, $u \in H_0^1(\Omega)$,

$$\sum_{|\alpha|, |\beta| \leqq 1} \int_\Omega a_{\alpha\beta}(x)\, \partial_x^\alpha u(x)\, \partial_x^\beta u(x)\, dx \geqq (C_0/\lambda)\, \|u\|_{1,\Omega}^2, \tag{2.4}$$

the condition of ellipticity of the bilinear form generated by A on $H_0^1(\Omega) \times H_0^1(\Omega)$.

With A, κ, κ_r and G satisfying the assumptions listed above we have

THEOREM 2.1. — *There are* $\varepsilon > 0$ *and* $\eta > 0$ *such that for every* f *with* $|||f|||_{M-1} < \varepsilon$, *there is a unique* $u \in H^{M+1}(Q)$, $\|u\|_{M,Q} < \eta$, *satisfying* (1.1)-(1.3).

The proof is given in the next four sections. Relying on an auxiliary result whose proof is put off to Section 4 we give in the next section, Section 3, the proof of the main part of the above theorem. The auxiliary result is based upon the study of a linear problem which is postponed to Section 5. In the last section, Section 6, the proof of Theorem 2.1 is completed by showing the uniqueness and improving the regularity result of Section 3.

3. PROOF OF THEOREM 2.1 (FIRST PART)

In this section we introduce various sets and mappings, whose properties will be studied later, and we prove the existence of $u \in H^M(Q)$ satisfying (1.1)-(1.3). Two minor parts of the proof which will require the application of techniques developed in the next sections are postponed to the closing section, Section 6.

We set

$$H_\#^k(Q) = \left\{ u \in H^k(Q);\ \int_0^{2\pi} u(x,t)\, dt = 0 \text{ for all } x \in \Omega \right\}$$

and

$$(I v)(x,t) = \int_0^t v(x,\tau)\, d\tau$$

for any $v \in H_\#^k(Q)$. Obviously, $I v$ is also 2π-periodic in t and, as we shall show; satisfies

$$\|I v\|_{0,Q} \leqq 2\, \|v\|_{0,Q}. \tag{3.1}$$

For any $z \in H^1(0, 2\pi)$, $z(0) = z(2\pi) = 0$ and almost all τ from $(0, 2\pi)$, we find easily after some simple arrangements that

$$2\frac{d}{d\tau}\left(z^2(\tau)\operatorname{cotg}\frac{\tau}{2}\right) = [2z'(\tau)]^2 - z^2(\tau) - \left[2z'(\tau) - z(\tau)\operatorname{cotg}\frac{\tau}{2}\right]^2.$$

Since $z^2(\tau)\operatorname{cotg}\frac{\tau}{2} \to 0$ as $\tau \to 0_+$ and $\tau \to 2\pi_-$, we have, by integrating over $(\varepsilon, 2\pi-\varepsilon)$ and letting $\varepsilon \searrow 0$,

$$\int_0^{2\pi} [(2z')^2 - z^2]\,d\tau = \int_0^{2\pi}\left[2z' - z\operatorname{cotg}\frac{\tau}{2}\right]^2 d\tau \geqq 0.$$

This gives

$$\int_0^{2\pi} z^2(\tau)\,d\tau \leqq 4\int_0^{2\pi} [z'(\tau)]^2\,d\tau,$$

from which, by setting $z(t) = \int_0^t y(\tau)\,d\tau$, it follows the following version of Wirtinger's inequality: for any $y \in L^2(0, 2\pi)$, $\int_0^{2\pi} y(\tau)\,d\tau = 0$ it holds

$$\int_0^{2\pi}\left(\int_0^t y(\tau)\,d\tau\right)^2 dt \leqq 4\int_0^{2\pi} y^2(t)\,dt.$$

From this inequality (3.1) is easy to obtain by integrating over Ω.

Let us denote the left-hand side of (1.1) by $F(u)$, *i. e.*

$$F(u) = L_0 u + G(J(u)) + f$$

and by $L(g(J(u)))v$ its linearization at u, *i. e.*

$$L(g(J(u)))v = L_0 v + \sum_{|\alpha|+p\leqq 2} g_\alpha^p(J(u))\,\partial_x^\alpha \partial_t^p v,$$

where $g(J(u))$ is short for $\{g_\alpha^p(J(u))\}_{|\alpha|+p\leqq 2}$.

For two positive ε_v and ε_b which are sufficiently small, and which will be fixed later, we set

$$V(\varepsilon_v) = \{v \in H_\#^M(Q) \cap H_0^1(Q);\ \|v\|_{M,Q} \leqq \varepsilon_v\},$$
$$B(\varepsilon_b) = \{b \in H^M(\Omega) \cap H_0^1(\Omega);\ \|b\|_{M,\Omega} \leqq \varepsilon_b\}.$$

For any $v \in V(\varepsilon_v)$ and $b \in B(\varepsilon_b)$ we have $\partial_t(Iv + b) = v$, since b is independent of t, and thus simple calculations give

$$\partial_t F(Iv+b) = L(g(J(Iv+b)))v + \partial_t f. \qquad (3.2)$$

We now express $F(Iv+b)(.,0)$, the trace of $F(Iv+b)$ on $\Omega \times \{0\}$, in a convenient form. This will be achieved in (3.5). We begin by arranging the components of $\mathrm{J}u$ like this

$$\mathrm{J}u=\{\partial_x^\beta u, \partial_x^\gamma \partial_t^p u\}, \qquad |\beta|\leqq 2, \qquad |\gamma|+p\leqq 2, \qquad p\geqq 1,$$

so that the derivatives with respect to x come first. Denoting

$$\mathrm{J}_x^2 u=\{\partial_x^\beta u\}_{|\beta|\leqq 2},$$
$$\mathrm{J}^1 u=\{\partial_x^\gamma \partial_t^{p-1} u\}_{|\gamma|+p\leqq 2,\, p\geqq 1},$$

so that $\mathrm{J}^1 u$ contains at most first derivatives of u, we have

$$\mathrm{J}u=\{\mathrm{J}_x^2 u, \mathrm{J}^1 \partial_t u\}.$$

Thus

$$\mathrm{J}(\mathrm{I}v+b)=\{\mathrm{J}_x^2(\mathrm{I}v+b), \mathrm{J}^1 v\}$$

and

$$\mathrm{J}(\mathrm{I}v+b)(x,0)=\{\mathrm{J}_x^2 b(x), (\mathrm{J}^1 v)(x,0)\} \tag{3.3}$$

since for $t=0$, as it is the case, all terms containing the integral disappear.

Owing to the assumptions on Ω, $a_{\alpha\beta}$ and (2.4), A can also be considered as a linear isomorphism of $\mathrm{H}^m(\Omega)\cap \mathrm{H}_0^1(\Omega)$ onto $\mathrm{H}^{m-2}(\Omega)$ for any m, $2\leqq m\leqq \mathrm{M}+1$, *see* Theorem 9.8 in [1]. We make use of it by referring to the inequality

$$\|\mathrm{A}^{-1}\varphi\|_{m,\Omega}\leqq \mathrm{C_A}\|\varphi\|_{m-2,\Omega}. \tag{3.4}$$

Eventually, as the last notation of this section, we put

$$\Phi(b,v)=-\mathrm{A}^{-1}(v(.,0)+\kappa\,\partial_t v(.,0)$$
$$+\sum_{r=1}^{n}\kappa_r\,\partial_{x_r} v(.,0)+\mathrm{G}(\mathrm{J}_x^2 b, (\mathrm{J}^1 v)(.,0))+f(.,0))$$

which satisfies

$$\mathrm{F}(\mathrm{I}v+b)(.,0)=\mathrm{A}(b-\Phi(b,v)) \tag{3.5}$$

as we immediately verify using (3.3).

The following lemma is a corner-stone of the proof of Theorem 2.1 and will be proved in the next section.

LEMMA 3.1. — *There exist positive* ε_b, ε_v *and* ε *such that for any* f *with* $|||f|||_{\mathrm{M}-1}<\varepsilon$ *it holds:*

(i) *For every* $b\in \mathrm{B}(\varepsilon_b)$ *there is a unique* $v\in \mathrm{V}(\varepsilon_v)$ *satisfying*

$$(\mathrm{L}(g(\mathrm{J}(\mathrm{I}v+b)))v+\partial_t f, \varphi)_0=0, \qquad \forall\varphi\in \mathrm{H}_\#^0(\mathrm{Q}); \tag{3.6}$$

when this v *is denoted by* $v(b)$, *then*

$$\|v(b_1)-v(b_2)\|_{\mathrm{M}-1,\mathrm{Q}}\leqq \|b_1-b_2\|_{\mathrm{M}-1,\Omega}. \tag{3.7}$$

(ii) *The mapping* $b\to\Phi(b,v(b))$ *has a fixed point in* $\mathrm{B}(\varepsilon_b)$.

Proof of Theorem 2.1. — By part (ii) of Lemma 3.1 there is $b \in B(\varepsilon_b)$ satisfying $b = \Phi(b, v(b))$, which, by (3.5) means

$$F(u)(.,0)=0 \tag{3.8}$$

for $u = I v(b) + b$. By (3.2) and part (i) of Lemma 3.1 this function u satisfies

$$(\partial_t F(u), \varphi)_0 = 0 \quad \text{for all } \varphi \in H^0_\# (Q)$$

which, since obviously $\partial_t F(u) \in H^0_\# (Q)$, implies $\partial_t F(u) = 0$. In virtue of (3.8) we have $F(u) = 0$; *i.e.* (1.1)-(1.3) are satisfied. Obviously, $u \in H^M(Q)$ and $u_t = v \in H^M(Q)$. Thus the proof of Theorem 2.1 will be complete if we show the uniqueness and prove also that $\partial_x^\alpha u \in L^2(Q)$ for $|\alpha| = M+1$. This is postponed to Section 6.

4. PROOF OF LEMMA 3.1

In what follows γ will stand for a collection of functions $\gamma_\alpha^p, |\alpha| + p \leqq 2$. For a given γ we set

$$L(\gamma) v = L_0 v + \sum_{|\alpha| + p \leqq 2} \gamma_\alpha^p \partial_x^\alpha \partial_t^p v,$$

and

$$|||\gamma|||_m = \max\{|||\gamma_\alpha^p|||_m; |\alpha| + p \leqq 2\}.$$

The proof of Lemma 3.1 relies on a proposition concerning the linear case.

LEMMA 4.1. — *There exists $\delta > 0$ such that for every γ, $|||\gamma|||_{M-2} < \delta$, and both $m = M$ and $m = M-1$ the following implication holds:*

When $|||h|||_{m-2} < \infty$, then there is a unique $v \in H^m_\# (Q) \cap H^1_0(Q)$ satisfying $(L(\gamma) v - h, \varphi)_0 = 0$ for all $\varphi \in H^0_\# (Q)$. Moreover,

$$\|v\|_{m, Q} \leqq K |||h|||_{m-2},$$

where K depends only on δ.

The proof of this lemma is postponed to the next section. A similar proposition could be stated about the solution of $L(\gamma) v = h$, *cf.* [7], Theorem 5, p. 33.

In the course of the proof we arrive at several inequalities involving ε_v, ε_b and some norms of f, which we shall suppose to be satisfied, *see* Remark 4.1. We have $\bar{n} + 2 < M$ and thus, by (2.1) and (3.1), for any $\bar{v} \in V(\varepsilon_v)$ and $b \in B(\varepsilon_b)$

$$\|\partial_x^\alpha \partial_t^p (I\bar{v} + b)\|_{C(\bar{Q})} \leqq C_S (\|I\bar{v}\|_{\bar{n}+2, Q} + \|b\|_{\bar{n}+2, \Omega})$$
$$\leqq C_S (2\|\bar{v}\|_{M, Q} + \|b\|_{M, \Omega}) < \eta_0$$

for all (α, p), $|\alpha|+p \leqq 2$, provided that

$$C_S(2\varepsilon_v+\varepsilon_b)<\eta_0. \tag{4.1}$$

As g_α^p are defined on $\mathcal{N}(\eta_0)$, we can set

$$\gamma_\alpha^p = g_\alpha^p(w), \qquad w=J(I\bar{v}+b) \tag{4.2}$$

and derive an estimate of $|||\gamma|||_{M-2}$. This is done *via* an auxiliary norm

$$|||\gamma|||_m^\sim = \max\{\|\gamma\|_{m,Q}, \|\partial_t\gamma\|_{m,Q}\},$$

where, as above, the same symbol is used to denote the norm of a scalar as well as vector function. Obviously

$$|||\gamma|||_m \leqq |||\gamma|||_m^\sim.$$

In obtaining various estimates the following two lemmas will often be used with θ equal either to $Q\subset R^{n+1}$ or $\Omega\subset R^n$. The first lemma is a particular case of Lemma 5.1 expressing the algebra character of the space $H^m(\theta)$.

LEMMA 4.2. – *Let integer m satisfy $m\geqq\bar{n}$. Then, for any a, $b\in H^m(\theta)$, it holds*

$$\|ab\|_{m,\theta}\leqq\sigma\|a\|_{m,\sigma}\|b\|_{m,\theta}.$$

LEMMA 4.3. – *Let $\hat{g}\in C^m(\mathcal{N})$, $\mathcal{N}=\{\zeta\in R^l;\ |\zeta_i|<\eta,\ i=1,\ldots,l\}$, $m\geqq\bar{n}$.*

(i) *There is a constant σ such that $\|\hat{g}(z)\|_{m,\theta}\leqq\sigma$ for any $z=(z_1,\ldots,z_l)$ satisfying $\|z_j\|_{m,\theta}<\eta/C_S$, $j=1,\ldots,l$.*

(ii) *If, moreover, $g\in C^{m+1}(\mathcal{N})$ and $\hat{g}(0)=0$, then*

$$\|\hat{g}(z)\|_{m,\theta}\leqq\sigma\max\{\|z_j\|_{m,\theta},\ j=1,\ldots,l\}.$$

(iii) *If $\hat{g}\in C^{m+2}(\mathcal{N})$, $\hat{g}(0)=0$ and also $(\partial\hat{g}/\partial\zeta_j)(0)=0$ for $j=1,\ldots,l$, then*

$$\|\hat{g}(z)\|_{m,\theta}\leqq\sigma\max\{\|z_j\|_{m,\theta}^2;\ j=1,\ldots,l\}.$$

Part (i) of this lemma can be proved along the same lines the proof of estimate (2.4) on page 273 of [4] has been achieved. If moreover, $\hat{g}(0)=0$, then

$$\hat{g}(z)=\sum_{i=1}^{l}\int_0^1\frac{\partial\hat{g}}{\partial\zeta_i}(\tau z)\,d\tau\, z_i.$$

By part (i) of the present lemma

$$\left\|\int_0^1\frac{\partial g}{\partial\zeta_i}(\tau z)\,d\tau\right\|_{m,\theta}\leqq\sigma,$$

and thus from Lemma 4.2 applied to the last equality, it follows

$$\|\hat{g}(z)\|_{m,\theta}\leqq\sigma\max\{\|z_j\|_{m,\theta},\ j=1,\ldots,l\},$$

i. e., part (ii) of the lemma. Similarly for part (iii).

In what follows σ (with index, bar or tilde) will denote the constants obtained as a result of estimates in which Lemmas 4.2, 4.3 or both of them have been applied.

For γ given by (4.2) with $\bar{v}\in V(\varepsilon_v)$ and $b\in B(\varepsilon_b)$ we have

$$\partial_t \gamma_\alpha^p = \sum_{|\beta|+q\leq 2} \frac{\partial g_\alpha^p}{\partial \zeta_\beta^q}(w)\,\partial_x^\beta \partial_t^q \bar{v}$$

since b is independent of t. Applying Lemma 4.2 and part (i) of Lemma 4.3 to any term behind the summation sign we find that the norm $\|\cdot\|_{M-2,Q}$ of any such term is estimated by

$$\bar{\sigma}_1 \|\partial_x^\beta \partial_t^q \bar{v}\|_{M-2,Q} \leq \bar{\sigma}_1 \|\bar{v}\|_{M,Q}.$$

Further, $\|\gamma_\alpha^p\|_{M-2,Q}$ is estimated by $\bar{\bar{\sigma}}_1\ (\|\bar{v}\|_{M,Q}+\|b\|_{M,\Omega})$ in virtue of part (ii) of Lemma 4.3 since $g_\alpha^p(0)=0$. Thus

$$|||\gamma|||_{\tilde{M}-2} \leq \sigma_1 (\|\bar{v}\|_{M,Q}+\|b\|_{M,\Omega})<\delta \tag{4.3}$$

as soon as

$$\sigma_1(\varepsilon_v+\varepsilon_b)<\delta. \tag{4.4}$$

Let us suppose

$$|||\partial_t f|||_{M-2}<\varepsilon_v/K \tag{4.5}$$

and fix any $b\in B(\varepsilon_b)$. Then, for any $\bar{v}\in V(\varepsilon_v)$, we get, by applying Lemma 4.1 with $m=M$, a unique $v\in H_\#^M(Q)\cap H_0^1(Q)$ satisfying

$$(L(g(J(I\bar{v}+b)))\,v+\partial_t f, \varphi)_0=0 \quad \text{for all } \varphi\in H_\#^0(Q) \tag{4.6}$$

and, by (4.5),

$$\|v\|_{M,Q}\leq K\,|||\partial_t f|||_{M-2,Q}\leq \varepsilon_v. \tag{4.7}$$

This means that the mapping $\bar{v}\to v$, which we shall denote by Ψ, maps $V(\varepsilon_v)$ into itself. We shall continue by showing

$$\|\Psi(\bar{v}_1)-\Psi(\bar{v}_2)\|_{M-1,Q}\leq \varepsilon_v\,\sigma_2\,K\,\|\bar{v}_1-\bar{v}_2\|_{M-1,Q} \tag{4.8}$$

for any $\bar{v}_i\in V(\varepsilon_v)$, $i=1, 2$. Setting $v_i=\Psi(\bar{v}_i)$, we immediately get

$$(L(g(w_1))(v_1-v_2)+h, \varphi)_0=0 \qquad \forall \varphi\in H_\#^0(Q) \tag{4.9}$$

where

$$h=L(g(w_1))\,v_2-L(g(w_2))\,v_2=\sum_{|\alpha|+p\leq 2} (g_\alpha^p(w_1)-g_\alpha^p(w_2))\,\partial_x^\alpha \partial_t^p v_2$$

and

$$w_i=J(I\bar{v}_i+b).$$

Lemma 4.2, applied to $\partial_t(ab)=(\partial_t a)b+a\,\partial_t b$, yields

$$|||ab|||_{\tilde{m}}\leq \tilde{\sigma}_m\,|||a|||_{\tilde{m}}\,|||b|||_{\tilde{m}}. \tag{4.10}$$

Hence,

$$|||h|||_{M-3}^{\sim} \leqq \tilde{\sigma}_{M-3} \sum_{|\alpha|+p \leqq 2} |||g_\alpha^p(w_1)-g_\alpha^p(w_2)|||_{M-3}^{\sim} |||\partial_x^\alpha \partial_t^p v_2|||_{M-3}^{\sim}.$$

Obviously, for $|\alpha|+p \leqq 2$,

$$|||\partial_x^\alpha \partial_t^p v_2|||_{M-3}^{\sim} \leqq \|\partial_x^\alpha \partial_t^p v_2\|_{M-2, Q} \leqq \|v_2\|_{M, Q} \leqq \varepsilon_v.$$

Here comes out the reason why we are not able to prove (4.8) with $\|.\|_{M, Q}$ norm: $|||\partial_x^\alpha \partial_t^p v_2|||_{M-2}^{\sim}$ cannot be estimated *via* $\|v_2\|_{M, Q}$. Next,

$$g_\alpha^p(w_1)-g_\alpha^p(w_2) = \sum_{|\beta|+q \leqq 2} \int_0^1 \frac{\partial g_\alpha^p}{\partial \zeta_\beta^q}(w(\tau))\, d\tau\, \partial_x^\beta \partial_t^q (I(\bar{v}_1-\bar{v}_2))$$

with $w(\tau)=\tau w_1+(1-\tau) w_2$. By (3.1) and by the definition of norms

$$|||\partial_x^\beta \partial_t^q (I(\bar{v}_1-\bar{v}_2))|||_{M-3}^{\sim} \leqq 2\|\bar{v}_1-\bar{v}_2\|_{M-1, Q}.$$

From part (i) of Lemma 4.3 it follows, $0 \leqq \tau \leqq 1$,

$$\left|\left|\left|\frac{\partial g_\alpha^p}{\partial \zeta_\beta^q}(w(\tau))\right|\right|\right|_{M-3}^{\sim} \leqq \left\|\frac{\partial g_\alpha^p}{\partial \zeta_\beta^q}(w(\tau))\right\|_{M-2, Q} \leqq \bar{\sigma}_2.$$

Hence,

$$|||h|||_{M-3}^{\sim} \leqq \varepsilon_v \sigma_2 \|\bar{v}_1-\bar{v}_2\|_{M-1, Q}. \tag{4.11}$$

When Lemma 4.1 with $m=M-1$ is applied to (4.9), (4.8) follows as a consequence of (4.11).

$V(\varepsilon_v)$ is a compact and convex subset of $H^{M-1}(Q)$ which Ψ maps into itself. Moreover, by (4.8), Ψ is continuous in $\|.\|_{M-1, Q}$. Hence, by Schauder's theorem, there is a fixed point of Ψ in $V(\varepsilon_v)$, say v, which, by virtue of (4.6), satisfies (3.6).

We now show (3.7). Let $b_i \in B(\varepsilon_b)$, $i=1, 2$. According to what we have proved, there are $v_i \in V(\varepsilon_v)$ satisfying, $i=1, 2$,

$$(L(g(J(I v_i+b_i)))\, v_i+\partial_t f, \varphi)_0=0 \qquad \forall \varphi \in H_\#^0 (Q).$$

Setting $\bar{w}_i=J(I v_i+b_i)$, we find

$$(L(g(\bar{w}_1))(v_1-v_2)+\bar{h}, \varphi)_0=0, \tag{4.12}$$

where

$$\bar{h}=L(g(\bar{w}_1))\, v_2-L(g(\bar{w}_2))\, v_2=\sum_{|\alpha|+p \leqq 2} (g_\alpha^p(\bar{w}_1)-g_\alpha^p(\bar{w}_2))\, \partial_x^\alpha \partial_t^p v_2.$$

Since

$$g_\alpha^p(\bar{w}_1)-g_\alpha^p(\bar{w}_2)=\sum_{|\beta|+q \leqq 2} \int_0^1 \frac{\partial g_\alpha^p}{\partial \zeta_\beta^q}(\ldots)\, d\sigma\, \partial_x^\beta \partial_t^q [I(v_1-v_2)+(b_1-b_2)]$$

we get by repeating arguments leading to (4.11)

$$|||\bar{h}|||_{M-3} \leqq \varepsilon_v \sigma_3 (\| v_1 - v_2 \|_{M-1, Q} + \| b_2 - b_1 \|_{M-1, \Omega}).$$

If

$$\varepsilon_v \sigma_3 K < 1/2, \tag{4.13}$$

Lemma 4.1 with $m = M-1$ applied to (4.12) yields

$$\| v_1 - v_2 \|_{M-1, Q} \leqq \frac{1}{2} (\| v_1 - v_2 \|_{M-1, Q} + \| b_1 - b_2 \|_{M-1, \Omega}),$$

an equivalent of (3.7). This completes the proof of part (i) of Lemma 3.1.

Part (ii) will be proved by using analogous techniques. We start by showing that $b \to \Phi(b, v(b))$ maps $B(\varepsilon_b)$ into itself. We shall use a very simple version of the trace theorem, namely,

$$\| w(.,0) \|_{m, \Omega} \leqq C_T |||w|||_{\tilde{m}, Q} \leqq C_T \| w \|_{m+1, Q}.$$

As above $J^1 = \{ \partial_x^\alpha \partial_t^p ; |\alpha| + p \leqq 1 \}$ and for simplicity we shall write for any norm $\|.\|$, $\| J^1 w \| = \max \{ \| \partial_x^\alpha \partial_t^p w \| ; |\alpha| + p \leqq 1 \}$. Hence,

$$\| J^1 v(.,0) \|_{M-2, \Omega} \leqq C_T \| v \|_{M, Q} \leqq C_T \varepsilon_v$$

for any $v \in V(\varepsilon_v)$. By this inequality and by part (iii) of Lemma 4.3 we have

$$\| G(J_x^2 b, (J^1 v)(.,0)) \|_{M-2, \Omega} \leqq \sigma_4 (\varepsilon_v + \varepsilon_b)^2.$$

With $\hat{\kappa} = 1 + |\kappa| + \sum_{r=1}^{n} |\kappa_r|$ and $v = v(b)$ we get by (3.4):

$$\begin{aligned} \| \Phi(b, v(b)) \|_{M, \Omega} \leqq C_A (\hat{\kappa} \| J^1 v(.,0) \|_{M-2, \Omega} & \\ + \| G(J_x^2 b, (J^1 v)(.,0)) \|_{M-2, \Omega} + \| f(.,0) \|_{M-2, \Omega}) & \\ \leqq C_A (\hat{\kappa} C_T \varepsilon_v + \sigma_4 (\varepsilon_v + \varepsilon_b)^2 + C_T |||f|||_{\tilde{M}-2}) \leqq \varepsilon_b & \end{aligned}$$

on condition that the next two inequalities hold

$$C_A (\hat{\kappa} C_T \varepsilon_v + \sigma_4 (\varepsilon_v + \varepsilon_b)^2) \leqq \varepsilon_b / 2, \tag{4.14}$$

$$|||f|||_{\tilde{M}-2} \leqq \varepsilon_b / (2 C_A C_T). \tag{4.15}$$

This implies $\Phi(b, v(b)) \in B(\varepsilon_b)$.

As a closing part of the proof we show

$$\| \Phi(b_1, v(b_1)) - \Phi(b_2, v(b_2)) \|_{M-1, \Omega} \leqq C_A \sigma_5 \| b_1 - b_2 \|_{M-1, \Omega} \tag{4.16}$$

for any $b_i \in B(\varepsilon_b)$, $i=1, 2$. Setting $d_i = J^1 v(b_i)(.,0)$, we get with the help of (3.7) in the last inequality that

$$\begin{aligned}
&\| G(J_x^2 b_1, d_1) - G(J_x^2 b_2, d_2) \|_{M-3,\Omega} \\
&\qquad \leqq \bar{\sigma}_5 (\| J_x^2 (b_1 - b_2) \|_{M-3,\Omega} + \| d_1 - d_2 \|_{M-3,\Omega}) \\
&\qquad \leqq \bar{\sigma}_5 (\| b_1 - b_2 \|_{M-1,\Omega} + C_T \| v(b_1) - v(b_2) \|_{M-1,Q}) \\
&\qquad\qquad \leqq \bar{\sigma}_5 (1 + C_T) \| b_1 - b_2 \|_{M-1,\Omega}.
\end{aligned}$$

This proves (4.16) since the estimates of the linear part of Φ are straightforward. Considering $b \to \Phi(b, v(b))$ as a mapping of $B(\varepsilon_b)$ into itself, we get a fixed point by Schauder's theorem. This completes the proof of Lemma 3.1.

Remark 4.1. — The proof has been carried out on condition that ε_b and ε_v satisfy (4.1), (4.4), (4.13) and (4.14) and f satisfies (4.5) and (4.15). It is possible to choose ε_v and ε_b so small that all four inequalities are satisfied. Since $\max(|||f_t|||_{M-2}, |||f|||_{\tilde{M}-2}) \leqq |||f|||_{M-1}$, (4.5) and (4.15) can be satisfied by taking ε sufficiently small.

5. LINEAR PROBLEM

In this section we give the proof of Lemma 4.1. The technique, a Galerkin approximation, is rather standard. Only $m=M$ is dealt with since the case $m=M-1$ is even simpler to treat. Thus, it is to be proved that for every γ with $|||\gamma|||_{M-2} < \delta$, δ sufficiently small, and any h, $|||h|||_{M-2} < \infty$, there exists a unique $v \in H_*^M(Q) \cap H_0^1(Q)$ satisfying

$$(L(\gamma) v, \varphi)_0 = (h, \varphi)_0 \quad \text{for all } \varphi \in H_*^0(Q). \tag{5.1}$$

Moreover,

$$\| v \|_{M,Q} \leqq K |||h|||_{M-2}, \tag{5.2}$$

where K depends only on δ.

We denote by ω_s and λ_s the sequence of all eigenfunctions and eigenvalues of the problem

$$A\omega = \lambda\omega, \qquad \omega \in H^2(\Omega) \cap H_0^1(\Omega).$$

[Due to the assumptions on Ω and A we even have $\omega_n \in H^{M+1}(\Omega)$.] Moreover, we shall suppose

$$\int_\Omega \omega_l(x)\, \omega_m(x)\, dx = \delta_{lm}.$$

Further, we set $e_j(t) = (\pi)^{-1/2} \sin jt$ for $j = 1, 2, \ldots$, and $e_j(t) = (\pi)^{-1/2} \cos jt$ for $j = -1, -2, \ldots$, so that the constant function is not considered.

For positive integers N and S we set

$$X^N = \{ v; v = \sum_{1 \leq |j| \leq N} v_j(x) e_j(t), v_j \in L^2(\Omega) \},$$
$$X_S^N = \{ v; v = \sum_{\substack{1 \leq |j| \leq N \\ s \leq S}} v_{sj} \omega_s(x) e_j(t) \}$$

and denote by Π^N (resp. Π_S^N) the orthogonal projectors on X^N (resp. X_S^N) in $L^2(Q)$ with the inner product $(\ ,\ )_0$ defined in Section 2.

The proof of Lemma 4.1 is divided into three steps. We start with an auxiliary estimate in step 1. In step 2 an approximating solution from X_S^N is found and its estimate in $H^2(Q)$ is obtained. Letting $S \to \infty$ we get an approximating solution which, as we show in step 3, is even an element of $H^M(Q)$. When an estimate of this solution in $\|.\|_{M,Q}$ is obtained, we get v satisfying (5.1) and (5.2) by letting $N \to \infty$.

STEP 1. — Let $1 \leq k \leq M$ and $v \in X^N \cap H^k(Q) \cap H_0^1(Q)$. Then

$$(L(\gamma) v, (-1)^{k-1} \Lambda \partial_t^{2(k-1)} v)_0 \geq (C_0 \| \partial_t^{k-1} v \|_{1,Q} - \delta K_k^* \| v \|_{k,Q}) \| \partial_t^{k-1} v \|_{1,Q} \quad (5.3)$$

where Λ is defined by (2.3) and K_k^* will denote a constant dependent only on k which can grow in each of the several stages of the proof. Integrating per partes (all functions are 2π-periodic in t and no boundary terms therefore appear) we get, by virtue of (2.2),

$$(L_0 v, (-1)^{k-1} \Lambda \partial_t^{2(k-1)} v)_0 = (L_0 \partial_t^{k-1} v, \Lambda \partial_t^{k-1} v)_0 \geq C_0 \| \partial_t^{k-1} v \|_{1,Q}^2.$$

Hence (5.3) follows as soon as we have shown that every

$$|(\gamma_\alpha^p \partial_x^\alpha \partial_t^p v, (-1)^{k-1} \Lambda \partial_t^{2(k-1)} v)_0|, \qquad |\alpha| + p \leq 2,$$

is estimated by

$$\delta K_k^* \| v \|_{k,Q} \| \partial_t^{k-1} v \|_{1,Q}. \quad (5.4)$$

To get this we begin by rearranging the scalar product.

We suppress the indexes of γ_α^p writing simply γ instead and, integrating per partes, we obtain

$$\begin{aligned}(\gamma \partial_x^\alpha \partial_t^p v, (-1)^{k-1} \Lambda \partial_t^{2(k-1)} v)_0 &= \sum_{s=0}^{k-1} \binom{k-1}{s} ((\partial_t^s \gamma) \partial_x^\alpha \partial_t^{k-1-s+p} v, \partial_t^k v)_0 \\ &\quad + \lambda (\gamma \partial_x^\alpha \partial_t^p v, (-1)^{k-1} \partial_t^{2(k-1)} v)_0 \\ &= \sum_{s=0}^{k-1} \binom{k-1}{s} I_s^{\alpha,p} + \lambda I_k^{\alpha,p}.\end{aligned}$$

At first, $I_0^{\alpha,p}$ will be dealt with. If $|\alpha| \leq 1$, then the desired estimate is easily achieved after simple computations. For example, when $|\alpha| = 1$ and

$p=1$, then

$$|\mathrm{I}_0^{\alpha,p}|=|(\gamma\,\partial_x^\alpha\,\partial_t^k v,\,\partial_t^k v)_0|=\frac{1}{2}|((\partial_x^\alpha\gamma)\,\partial_t^k v,\,\partial_t^k v)_0|$$
$$\leqq\frac{1}{2}\|\partial_x^\gamma\gamma\|_{C(\bar{Q})}\|v\|_{k,Q}\|\partial_t^{k-1}v\|_{1,Q},$$

since $v(x,t)=0$ for $x\in\partial\Omega$. As, by (2.1),

$$\|\partial_x^\alpha\gamma\|_{C(\bar{Q})}\leqq C_S\|\gamma\|_{\bar{n}+1,Q}\leqq C_S\|\gamma\|_{M-2,Q}\leqq C_S|||\gamma|||_{M-2,Q}\leqq C_S\,\delta,$$

the estimate of $\mathrm{I}_0^{\alpha,p}$ has the form (5.4).

For $|\alpha|=2$, *i.e.* $p=0$, we must be more careful.

Since $\partial_x^\alpha=\partial_{x_i}\partial_{x_j}$ for some i and j,

$$\begin{aligned}\mathrm{I}_0^{\alpha,p}&=(\gamma\partial_{x_i}\partial_{x_j}\partial_t^{k-1}v,\,\partial_t^k v)_0\\&=\frac{1}{2}(\gamma\,\partial_{x_i}\partial_{x_j}\partial_t^{k-1}v,\,\partial_t^k v)_0\\&+\frac{1}{2}(\gamma\,\partial_{x_j}\partial_{x_i}\partial_t^{k-1}v,\,\partial_t^k v)_0\\&=-\frac{1}{2}[((\partial_{x_i}\gamma)\,\partial_{x_j}\partial_t^{k-1}v,\,\partial_t^k v)_0\\&+((\partial_{x_j}\gamma)\,\partial_{x_i}\partial_t^{k-1},\,\partial_t^k v)_0]\\&-\frac{1}{2}\{(\gamma\,\partial_{x_j}\partial_t^{k-1}v,\,\partial_{x_i}\partial_t^k v)_0\\&+(\gamma\,\partial_{x_i}\partial_t^{k-1}v,\,\partial_{x_j}\partial_t^k v)_0\}.\end{aligned}$$

As above, the terms in [ ] can be directly estimated. The terms in { } equal

$$(\gamma,\,\partial_t((\partial_{x_j}\partial_t^{k-1}v)\,\partial_{x_i}\partial_t^{k-1}v))_0=-((\partial_t\gamma)\,\partial_{x_j}\partial_t^{k-1}v,\,\partial_{x_i}\partial_t^{k-1}v)_0$$

which is also easy to estimate. Thus $\mathrm{I}_0^{\alpha,p}$ is estimated in the form (5.4). On estimating $\mathrm{I}_s^{\alpha,p}$, $s\geqq 1$, the following lemma will be applied.

LEMMA 5.1. — *Let* $a\in\mathrm{H}^r(\mathrm{Q})$, *r integer,*

$$r>(n+1)/2\qquad[\text{i.e., } \mathrm{H}^r(\mathrm{Q})\subset C(\bar{\mathrm{Q}})].$$

Then

$$\|(\mathrm{D}^{l_1}a)(\mathrm{D}^{l_2}b)\|_{0,Q}\leqq\sigma_l\|a\|_{r,Q}\|b\|_{l,Q}$$

for any $b\in\mathrm{H}^l(\mathrm{Q})$ *with* $l_1+l_2=l\leqq r$. Here, D^l *denotes any derivative* $\partial_x^\alpha\partial_t^p$, $|\alpha|+p\leqq l$.

Remark 5.1. — In other words, $\mathrm{H}^l(\mathrm{Q})$ is a modul over $\mathrm{H}^r(\mathrm{Q})$. Only for completeness we present the proof. For simplicity we set $m=n+1$. We shall distinguish the following six cases:

(i) $l_1+m/2<r$,

(ii) $l_2+m/2<l$,

(iii) $l_1+m/2=r$, $l_2+m/2=l$,

(iv) $l_1+m/2>r$, $l_2+m/2=l$,

(v) $l_1+m/2=r$, $l_2+m/2>l$,

(vi) $l_1+m/2>r$, $l_2+m/2>l$.

Case (i). — By the Sobolev inequality $\| D^{l_1} a \|_{C(\bar{Q})} \leqq C_S \| a \|_{r, Q}$. Estimating the first factor in $(D^{l_1} a)(D^{l_2} b)$ with the help of this, we get the estimate in the lemma.

Case (ii). — Entirely analogous, since now

$$\| D^{l_2} b \|_{C(\bar{Q})} \leqq C_S \| b \|_{l, Q}.$$

Case (iii). — By embedding theorems,

$$(\alpha) \qquad \| D^{l_1} a \|_{L^p(Q)} \leqq \sigma \| a \|_{r, Q}$$

and

$$(\beta) \qquad \| D^{l_2} b \|_{L^q(Q)} \leqq \sigma \| b \|_{l, Q}$$

for any $p>0$ and $q<0$. Lemma follows by Hölder's inequality.

Case (iv). — In this case

$$(\gamma_1) \qquad \| D^{l_1} a \|_{L^p(Q)} \leqq \sigma \| a \|_{r, Q}$$

with

$$(\gamma_2) \qquad \frac{1}{p}=\frac{1}{2}-\frac{r-l_1}{m}.$$

From $l_1<r$ [$l_1=r$ implies $l_2=0$; thus $l=r$ and we are in case (ii)] it follows $p>2$ and by Hölder's inequality

$$\| (D^{l_1} a)(D^{l_2} b) \|_{0, Q} \leqq \sigma \| D^{l_1} a \|_{L^p(Q)} \| D^{l_2} b \|_{L^q(Q)}$$

with $q=2p'/(p'-1)$, where $p'=p/2$. Owing to (γ_1) and (β) we have the estimate.

Case (v). — Now

$$(\delta_1) \qquad \| D^{l_2} b \|_{L^q(Q)} \leqq \sigma \| b \|_{l, Q}$$

with

$$(\delta_2) \qquad \frac{1}{q}=\frac{1}{2}-\frac{l-l_2}{m}=\frac{1}{2}-\frac{l_1}{m}.$$

From $l_1>0$ [$l_1=0$ is covered by case (i)] it follows $q>2$ and the estimate is derived like in (iv).

Case (vi). — We get $m>r$, when summing up the inequalities in (vi). By assumption, $r>m/2$. Hence $\chi=1/\left(\frac{2r}{m}-1\right)>1$. Along with p from (γ_2) and q from (δ_2) this χ satisfies

$$\frac{1}{p/2}+\frac{1}{q/2}+\frac{1}{\chi}=1.$$

By Hölder's inequality,

$$\|(D^{l_1}a)(D^{l_2}b)\|_{0,Q}\leqq\|D^{l_1}a\|_{L^p(Q)}\|D^{l_2}b\|_{L^q(Q)}(\text{meas } Q)^{1/(2\chi)}$$

from which the inequality follows when (γ_1) and (δ_1) are taken into account. This completes the proof of Lemma 5.1.

We now return to estimating $I_s^{\alpha,p}$, $s\geqq 1$. The case $s=k$ we leave as the final stage of this step and we shall suppose $1\leqq s\leqq k-1$ at first. This implies that k must be at least 2. Obviously,

$$|I_s^{\alpha,p}|\leqq J_s^{\alpha,p}\|\partial_t^{k-1}v\|_{1,Q},$$

where

$$J_s^{\alpha,p}=\|(\partial_t^s\gamma)(\partial_x^\alpha\partial_t^{k-1-s+p}v)\|_{0,Q}.$$

We shall distinguish several cases:

(*a*) If $k\leqq M-1$, then

$$(\partial_t^s\gamma)\,\partial_x^\alpha\partial_t^{k-1-s+p}v=\partial_t^{s-1}(\partial_t\gamma)\,\partial_t^{k-1-s}(\partial_x^\alpha\partial_t^p v).$$

By Lemma 5.1 ($a=\partial_t\gamma$, $b=\partial_x^\alpha\partial_t^p v$, $D^{l_1}=\partial_t^{s-1}$, $D^{l_2}=\partial_t^{k-1-s}$, $r=M-3=\bar{n}$, so that $l_1+l_2=s-1+k-1-s=k-2\leqq M-3=r$),

$$J_s^{\alpha,p}\leqq\sigma_{k-2}\|\partial_t\gamma\|_{M-3,Q}\|\partial_x^\alpha\partial_t^p v\|_{k-2,Q}\leqq\sigma_{k-2}|||\gamma|||_{M-2}\|v\|_{k,Q},$$

which implies an estimate of type (5.4) for $I_s^{\alpha,p}$.

(*b*) Now let $k=M$ and let us treat $s=M-1$. Then,

$$J_s^{\alpha,p}=J_{M-1}^{\alpha,p}=\|\partial_t^{M-1}\gamma\|_{0,Q}\|\partial_x^\alpha\partial_t^p v\|_{C(\bar{Q})}\leqq C_S|||\gamma|||_{M-2}\|v\|_{M,Q}$$

which yields the right estimate of $I_{M-1}^{\alpha,p}$.

(*c*) Further, let $k=M$ and $1\leqq s\leqq M-2$. Then, as above,

$$(\partial_t^s\gamma)\,\partial_x^\alpha\partial_t^{M-1-s+p}v=\partial_t^{s-1}(\partial_t\gamma)\,\partial_t^{M-2-s}(\partial_x^\alpha\partial_t^{p+1}v).$$

By Lemma 5.1 ($a=\partial_t\gamma$, $b=\partial_x^\alpha\partial_t^{p+1}v$, $D^{l_1}=\partial_t^{s-1}$, $D^{l_2}=\partial_t^{M-2-s}$, $r=M-3=\bar{n}$, $l_1+l_2=s-1+M-2-s=M-3$),

$$J_s^{\alpha,p}\leqq\sigma_{M-3}\|\partial_t\gamma\|_{M-3,Q}\|\partial_x^\alpha\partial_t^{p+1}v\|_{M-3,Q}\leqq\sigma_{M-3}|||\gamma|||_{M-2}\|v\|_{M,Q}$$

and $I_s^{\alpha,p}$ again satisfies the estimate of type (5.4).

Finally, we estimate $I_k^{\alpha,p}$. For $k=1$, transferring, if necessary, one derivative from $\partial_x^\alpha\partial_t^p v$, we get $|(\gamma\,\partial_x^\alpha\partial_t^p v, v)_0|\leqq\delta K_1^*\|v\|_{1,Q}^2$, the estimate of form (5.4). For $k\geqq 2$,

$$|I_k^{\alpha,p}|=|(\partial_t^{k-2}(\gamma\,\partial_x^\alpha\partial_t^p v),\partial_t^k v)_0|\leqq\|\partial_t^{k-2}(\gamma\,\partial_x^\alpha\partial_t^p v)\|_{0,Q}\|\partial_t^{k-1}v\|_{1,Q}$$

and, by Lemma 5.1 with $r=\mathrm{M}-2$,

$$\| \partial_t^{k-2} (\gamma \, \partial_x^\alpha \partial_t^p v) \|_{0, Q} \leqq \sigma_{M-2} ||| \gamma |||_{M-2} \| \partial_x^\alpha \partial_t^p v \|_{k-2, Q},$$

which results in an estimate of type (5.4) for $I_k^{\alpha, p}$.

STEP 2. — Let N and S be positive integers. When $k=1$, then (5.3) takes up the form

$$(\mathrm{L}(\gamma) v, \Lambda v)_0 \geqq \mathrm{C}_1 \| v \|_{1, Q}^2, \tag{5.5}$$

with $\mathrm{C}_1 = \mathrm{C}_0 - \delta \mathrm{K}_1^*$ positive for δ sufficiently small and any $v \in \mathrm{X}_\mathrm{S}^\mathrm{N} \subset \mathrm{X}^\mathrm{N} \cap \mathrm{H}_0^1(\mathrm{Q})$. As $\Pi_\mathrm{S}^\mathrm{N} \mathrm{L}(\gamma)$ maps $\mathrm{X}_\mathrm{S}^\mathrm{N}$ into itself, (5.5) shows that there exists $v_\mathrm{S}^\mathrm{N} \in \mathrm{X}_\mathrm{S}^\mathrm{N}$ satisfying

$$(\mathrm{L}(\gamma) v_\mathrm{S}^\mathrm{N}, \varphi)_0 = (h, \varphi)_0, \qquad \forall \varphi \in \mathrm{X}_\mathrm{S}^\mathrm{N}. \tag{5.6}$$

Inserting $\varphi = -\Lambda \partial_t^2 v_\mathrm{S}^\mathrm{N}$ in (5.6) and $v = v_\mathrm{S}^\mathrm{N}$ in (5.3) with $k=2$, we have after simple calculations, for δ small,

$$\mathrm{C}_0 \| \partial_t v_\mathrm{S}^\mathrm{N} \|_{1, Q} \leqq (1+\lambda) \| \partial_t h \|_{0, Q} + \delta \mathrm{K}_2^* \| v_\mathrm{S}^\mathrm{N} \|_{2, Q}. \tag{5.7}$$

Since $\mathrm{A}\, \omega_s = \lambda_s \omega_s$, $\lambda_s > 0$, (5.6) is equivalent to

$$\mathrm{A}\, v_\mathrm{S}^\mathrm{N} = -g_\mathrm{S}^\mathrm{N} + \Pi_\mathrm{S}^\mathrm{N} h, \tag{5.8}$$

where

$$g_\mathrm{S}^\mathrm{N} = \partial_t v_\mathrm{S}^\mathrm{N} + \kappa \, \partial_t^2 v_\mathrm{S}^\mathrm{N} + \sum_{r=1}^{n} \kappa_r \Pi_\mathrm{S}^\mathrm{N} \partial_{x_r} \partial_t v_\mathrm{S}^\mathrm{N} + \sum_{|\alpha|+p \leqq 2} \Pi_\mathrm{S}^\mathrm{N} (\gamma_\alpha^p \partial_x^\alpha \partial_t^p v_\mathrm{S}^\mathrm{N}).$$

When, as above, $\hat{\kappa} = 1 + |\kappa| + \sum_{r=1}^{n} |\kappa_r|$, then

$$\| g_\mathrm{S}^\mathrm{N} \|_{0, Q} \leqq \hat{\kappa} \| \partial_t v_\mathrm{S}^\mathrm{N} \|_{1, Q} + \delta \sigma_6 \| v_\mathrm{S}^\mathrm{N} \|_{2, Q}, \tag{5.9}$$

since

$$\begin{aligned} \| \Pi_\mathrm{S}^\mathrm{N} (\gamma_\alpha^p \partial_x^\alpha \partial_t^p v) \|_{0, Q} &\leqq \| \gamma_\alpha^p \partial_x^\alpha \partial_t^p v \|_{0, Q} \\ &\leqq \| \gamma_\alpha^p \|_{C(\bar{Q})} \| \partial_x^\alpha \partial_t^p v \|_{0, Q} \leqq \mathrm{C}_\mathrm{S} ||| \gamma |||_{M-2} \| v \|_{2, Q}. \end{aligned}$$

Also here σ (indexed and/or barred) is used to denote various constants independent of δ and v.

The norm $\| \partial_t v_\mathrm{S}^\mathrm{N} \|_{1, Q}$ in (5.7) is close to $\| v_\mathrm{S}^\mathrm{N} \|_{2, Q}$ in the sense that only the L^2-norm of second order derivatives, *i.e.* $\| \partial_x^\beta v_\mathrm{S}^\mathrm{N} \|_{0, Q}$, $|\beta| = 2$, are missing. In obtaining these we shall apply the lemma, which will be useful also later.

LEMMA 5.2. — *Given* $g \in \mathrm{X}^\mathrm{N}$ *and a nonnegative integer* $s \leqq \mathrm{M}-1$, *let us suppose* $g \in \mathrm{H}^s(\mathrm{Q})$. *Then there is a unique* $v \in \mathrm{X}^\mathrm{N} \cap \mathrm{H}_0^1(\mathrm{Q})$ *with* $\partial_x^\beta v \in \mathrm{H}^s(\mathrm{Q})$, $|\beta| \leqq 2$, *satisfying* $\mathrm{A}\, v = g$. *Moreover,*

$$\max \{ \| \partial_x^\beta v \|_{s, Q} ; |\beta| \leqq 2 \} \leqq \hat{\sigma}_s \| g \|_{s, Q}, \tag{5.10}$$

$$\max \{ \| \partial_x^\beta v \|_{0, Q} ; |\beta| \leqq s+2 \} \leqq \tilde{\sigma}_s \max \{ \| \partial_x^\alpha g \|_{0, Q} ; |\alpha| \leqq s \}. \tag{5.11}$$

The proof is straightforward. For $g \in X^N$ we have

$$g = \sum_{1 \leqq |j| \leqq N} g_j(x) e_j(t)$$

and the corresponding $v \in X^N \cap H_0^1(Q)$ satisfies

$$v = \sum_{1 \leqq |j| \leqq N} v_j(x) e_j(t)$$

with $v_j = A^{-1} g_j$, where A^{-1} is taken in the sense of (3.4). Since $\|g\|_{0,Q}^2 = \sum_j \|g_j\|_{0,\Omega}^2$ and similarly for v, (5.10) follows.

Only for an easy reference in the next section we state:

LEMMA 5.3. — *The space* X^N *in the preceding lemma can be substituted by* $L^2(Q) = H^0(Q)$.

Really, for any $g \in L^2(Q)$ we can write

$$g(x, t) = \sum_{j \in Z} g_j(x) e_j(t),$$

where $e_0(t) = (2\pi)^{-1/2}$ and the proof, as above, follows from the properties of A^{-1} in $L^2(\Omega)$.

Coming back to (5.8) we begin by applying (5.10) with $s=0$ which, along with (5.9), implies

$$\max\{\|\partial_x^\beta v_S^N\|_{0,Q}; |\beta| \leqq 2\} \leqq \hat{\sigma}_0(\|g_S^N\|_{0,Q} + \|h\|_{0,Q})$$
$$\leqq \hat{\sigma}_0(\hat{\kappa}\|\partial_t v_S^N\|_{1,Q} + \delta\sigma_6\|v_S^N\|_{2,Q}) + \hat{\sigma}_0\|h\|_{0,Q}.$$

From this, in virtue of (5.7), it follows

$$C_0 \max\{\|\partial_x^\beta v_S^N\|_{0,Q}; |\beta| \leqq 2\}$$
$$\leqq \delta\hat{\sigma}_0(\hat{\kappa}K_2^* + C_0\sigma_6)\|v_S^N\|_{2,Q}$$
$$+ \hat{\sigma}_0(\hat{\kappa}(1+\lambda) + C_0)|||h|||_0.$$

By considering (5.7) and the least inequality we have

$$\|v_S^N\|_{2,Q} \leqq K_2 |||h|||_0,$$

for δ sufficiently small. Letting $S \to \infty$, we get $v^N \in X^N \cap H^2(Q) \cap H_0^1(Q)$ satisfying

$$(L(\gamma) v^N, \varphi)_0 = (h, \varphi)_0, \qquad \forall \varphi \in X^N, \tag{5.12}$$
$$\|v^N\|_{2,Q} \leqq K_2 |||h|||_0. \tag{5.13}$$

Also, (5.5) shows that v^N is unique. Besides, (5.12) is equivalent to

$$A v^N = -\sum_{|\alpha|=2} \Pi^N(\gamma_\alpha^0 \partial_x^\alpha v^N) - g^N, \tag{5.14}$$

$$g^N = \hat{c}_t v^N + \kappa \partial_t^2 v^N + \sum_{r=1}^n \kappa_r \partial_{x_r} \partial_t v^N + \sum_{\substack{|\alpha|+p=2 \\ p \geqq 1}} \Pi^N(\gamma_\alpha^p \partial_x^\alpha \partial_t^p v^N) + \Pi^N h. \tag{5.15}$$

STEP 3. — We shall denote by Z^k, $2 \leqq k$, a collection of functions for which

$$\| v \|_{Z^k} = \max \{ \| \partial_x^\alpha v \|_{k-2, Q}; |\alpha| \leqq 2 \} < \infty.$$

For any α, $|\alpha| \leqq 2$, ∂_x^α is a continuous map of Z^k into $H^{k-2}(Q)$. On the other hand (5.10) shows that A^{-1} is a continuous map of $H^{k-2}(Q) \cap X^N$ into $Z^k \cap X^N \cap H_0^1(Q)$. With the help of Lemma 5.1 is therefore easy to see that

$$v \to \sum_{|\alpha|=2} A^{-1} \Pi^N (\gamma_\alpha^0 \partial_x^\alpha v) \text{ is a contraction on any } Z^k \cap H_0^1(Q), \quad 0 \leqq k \leqq M, \tag{5.16}$$

provided that $\| \gamma \|_{M-2, Q}$ is sufficiently small.

Further we shall proceed by induction. Suppose

$$v^N \in H^{k+2}(Q) \cap H_0^1(Q) \cap X^N$$

for some k, $0 \leqq k \leqq M-3$, which is true for $k=0$ by the preceding step. Then just by looking on (5.15) we find that $g^N \in H^{k+1}(Q)$ (since $v^N \in X^N$, there is no worry about derivatives with respect to t) and therefore $A^{-1} g^N \in Z^{k+3} \cap X^N \cap H_0^1(Q)$. In virtue of (5.16) there is a unique $v \in Z^{k+3} \cap H_0^1(Q) \cap X^N$ satisfying

$$v = - \sum_{|\alpha|=2} A^{-1} \Pi^N (\gamma_\alpha^0 \partial_x^\alpha v) - A^{-1} g^N.$$

Obviously, $v = v^N$ and thus $v^N \in Z^{k+3} \cap X^N \cap H_0^1(Q)$ which means $v^N \in H^{k+3}(Q) \cap H_0^1(Q) \cap X^N$. This induction argument eventually gives $v^N \in H^M(Q)$.

Let L be any integer, $3 \leqq L \leqq M$. We shall suppose that

$$\| v^N \|_{L-1, Q} \leqq K_{L-1} ||| h |||_{L-3}, \tag{5.17}$$

which for $L=3$ is true by (5.13). Setting $\varphi = (-1)^{L-1} \Lambda \partial_t^{2(L-1)} v^N$ in (5.12) and applying (5.3), we get after simple calculations

$$C_0 \| \partial_t^{L-1} v^N \|_{1, Q} \leqq \delta K_L^* \| v^N \|_{L, Q} + (1+\lambda) ||| h |||_{L-2}.$$

This inequality says that

$$\max \{ \| \partial_x^\beta \partial_t^q v^N \|_{0, Q}; |\beta| + q = L, |\beta| \leqq l \} \leqq \delta K_{L, l}^* \| v^N \|_{L, Q} + \hat{K}_{L, l} ||| h |||_{L-2} \tag{5.18}$$

is valid for $l=1$. Constants $K_{L,l}^*$ and $\hat{K}_{L,l}$ are positive and independent of v^N, h, and γ. By induction we now show that (5.18) holds even for $l=L$. To this end we start by supposing (5.18) holds with some l, $1 \leqq l \leqq L-1$. Further, we give (5.12) the following form

$$A v^N = -g_1^N - g_2^N + h, \tag{5.19}$$

with

$$g_1^N = \partial_t v^N + \kappa \partial_t^2 v^N + \sum_{r=1}^{n} \kappa_r \partial_{x_r} \partial_t v^N.$$

and

$$g_2^N = \sum_{|\alpha|+p=2} \Pi^N (\gamma_\alpha^p \partial_x^\alpha \partial_t^p v^N),$$

applying ∂_t^{L-l-1} to (5.19), we have

$$A(\partial_t^{L-l-1} v^N) = -\partial_t^{L-l-1}(g_1^N + g_2^N - h). \tag{5.20}$$

It is easy to check that the inequality, $|\beta| \leqq l-1$,

$$\| \partial_x^\beta \partial_t^{L-l-1} g_1^N \|_{0,Q} \leqq \delta K_{L,l+1}^* \| v^N \|_{L,Q} + \hat{K}_{L,l+1} ||| h |||_{L-2}, \tag{5.21}$$

is a consequence of assumptions (5.17) and (5.18). Since $|\beta| + L - l - 1 \leqq L - 2$, we can apply Lemma 5.1 to obtain

$$\| \partial_x^\beta \partial_t^{L-l-1} g_2^N \|_{0,Q} \leqq \sigma_{L-2} \delta \| v^N \|_{L,Q}. \tag{5.22}$$

Really,

$$\begin{aligned} &\| \partial_x^\beta \partial_t^{L-l-1} \Pi^N (\gamma_\alpha^p \partial_x^\alpha \partial_t^p v^N) \|_{0,Q} \\ &\qquad = \| \Pi^N \partial_x^\beta \partial_t^{L-l-1} (\gamma_\alpha^p \partial_x^\alpha \partial_t^p v^N) \|_{0,Q} \\ &\qquad \leqq \sigma_{L-2} \| \gamma_\alpha^p \|_{L-2,Q} \| \partial_x^\alpha \partial_t^p v \|_{L-2,Q} \leqq \sigma_{L-2} \delta \| v^N \|_{L,Q}. \end{aligned}$$

Obviously, $\| \partial_t^{L-l-1} \partial_x^\beta h \|_{0,Q} \leqq \| h \|_{0,L-2}$. Thus (5.11) with $s = l-1$ applied to (5.20) along with (5.21) and (5.22) yield

$$\max \{ \| \partial_x^\beta \partial_t^{L-l-1} v^N \|_{0,Q} ; |\beta| \leqq l+1 \} \leqq \delta K_{L,l+1}^* \| v^N \|_{L,Q} + \hat{K}_{L,l+1} ||| h |||_{L-2}.$$

In particular, this shows that (5.18) holds also with $l+1$ in lieu of l. Hence, as said above, we can put $l = L$ in (5.18) and, for δ small, obtain $\| v^N \|_{L,Q} \leqq K_L ||| h |||_{L-2}$, which is (5.17) with L in place of L−1. Hence $\| v^N \|_M \leqq K_M ||| h |||_{M-2}$. Letting $N \to \infty$, we have v satisfying (5.1) and (5.2) whose uniqueness follows from (5.5). This completes the proof of Lemma 4.1.

6. CONCLUSION OF THE PROOF OF THEOREM 2.1

We shall prove the uniqueness of u found in Section 2. Let us suppose there are $u_1, u_2 \in H^M(Q) \cap H_0^1(Q)$, $\| u_i \|_{M,Q} < \eta$, satisfying $F(u_i) = 0$, $i = 1, 2$. Since

$$F(u_1) - F(u_2) = L_0 \bar{u} + \sum_{|\alpha|+p \leqq 2} \bar{\gamma}_\alpha^p \partial_x^\alpha \partial_t^p \bar{u}$$

with

$$\bar{u}=u_1-u_2, \qquad \bar{\gamma}_\alpha^p=\int_0^1 g_\alpha^p(J(u_1+\tau(u_2-u_1)))\,d\tau,$$

by (2.2) we obtain

$$0=(F(u_1)-F(u_2),\Lambda(u_1-u_2))_0\geqq C_0\|\bar{u}\|_{1,Q}^2+\sum_{|\alpha|+p\leqq 2}(\bar{\gamma}_\alpha^p\,\partial_x^\alpha\partial_t^p\bar{u},\Lambda\bar{u})_0.$$

Simple estimates, completely imitating those leading to (5.5), show that the right-hand side of the last inequality can be estimated from below by $C_1\|\bar{u}\|_{1,Q}^2$, $C_1>0$, provided that η is sufficiently small. This proves uniqueness.

It remains to show $u\in H^{M+1}(Q)$. As explained in Section 3 this means only to show $\partial_x^\alpha u\in L^2(Q)$ for $|\alpha|=M+1$. If $J_u=\{J_x^2 u, J^1\partial_t u\}$ as in Section 3, then u satisfies

$$A\,u=-G(J_x^2 u, w)+f^*, \tag{6.1}$$

where

$$-f^*=\partial_t u+\kappa\,\partial_t^2 u+\sum_{r=1}^n \kappa_r\,\partial_{x_r}\partial_t u+f$$

and

$$w=J^1\,\partial_t u.$$

Since $u_t=v$, $\|v\|_{M,Q}<\eta$ and $\|f\|_{M-1,Q}<\varepsilon$, we have

$$\|f^*\|_{M-1,Q}\leqq\hat{\kappa}\,\eta+\varepsilon \qquad \text{and} \qquad \|w\|_{M-1,Q}<\eta. \tag{6.2}$$

We shall be concerned with a mapping Ξ given by

$$\Xi(U)=A^{-1}(-G(J_x^2 U, w)+f^*)$$

on a ball $B=\{U\in Z^{M+1}\cap H_0^1(Q);\|U\|_{Z^{M+1}}<\rho\}$, where $\rho<\eta$ is a sufficiently small positive number subject to conditions specified later. By Lemma 4.3 (iii),

$$\|G(W,w)\|_{M-1,Q}\leqq\bar{\sigma}_1\max\{\|W\|_{M-1,Q}^2,\|w\|_{M-1,Q}^2\}. \tag{6.3}$$

By Lemma 4.2 (ii), for $j=1, 2$,

$$\begin{aligned}\|G(W_1,w)-G(W_2,w)\|_{M-j,Q}&\\ \leqq\bar{\sigma}_{1+j}\max\{&\|W_1\|_{M-j,Q},\\ &\|W_2\|_{M-j,Q},\|w\|_{M-j,Q}\}\,\|W_1-W_2\|_{M-j,Q}.\end{aligned} \tag{6.4}$$

In virtue of (5.10), we further have

$$\|A^{-1}g\|_{Z^{M+2-j}}\leqq\tilde{\sigma}_{M-j}\|g\|_{M-j,Q}, \qquad j=1,2. \tag{6.5}$$

By this inequality with $j=1$, (6.2), and (6.3), we get

$$\begin{aligned}\|\Xi(\mathrm{U})\|_{Z^{M+1}} &\leqq \tilde{\sigma}_{M-1}(\|G(J_x^2 U, w)\|_{M-1, Q}+\|f^*\|_{M-1, Q}) \\ &\leqq \tilde{\sigma}_{M-1}(\bar{\sigma}_1 \max\{\rho^2, \eta^2\}+\hat{\kappa}\eta+\varepsilon) \\ &\leqq \rho\end{aligned}$$

as soon as

$$\tilde{\sigma}_{M-1}(\bar{\sigma}_1 \max\{\rho^2, \eta^2\}+\hat{\kappa}\eta+\varepsilon) \leqq \rho. \tag{6.6}$$

By (6.4) with $j=1$, we have

$$\begin{aligned}&\|\Xi(U_1)-\Xi(U_2)\|_{Z^{M+1}} \\ &\qquad \leqq \tilde{\sigma}_{M-1}\|G(J_x^2 U_1, w)-G(J_x^2 U_2, w)\|_{M-1, Q} \\ &\qquad\qquad \leqq \tilde{\sigma}_{M-1}\bar{\sigma}_2 \max\{\rho, \eta\}\|U_1-U_2\|_{Z^{M+1}}\end{aligned}$$

for any U_1, $U_2 \in B$. If (6.6) and

$$\tilde{\sigma}_{M-1}\tilde{\sigma}_2 \max\{\rho, \eta\}<1, \tag{6.7}$$

are satisfied, then Ξ is a contraction on B and thus there is $U \in B$ satisfying $U=\Xi(U)$, *i.e.*

$$AU=-G(J_x^2 U, w)+f^*. \tag{6.8}$$

On the other hand $u \in H^M(Q) \subset Z^M$ satisfies (6.1). Subtracting (6.8) from (6.1) and using (6.5), and (6.4) both with $j=2$, we get

$$\begin{aligned}\|u-U\|_{Z^M} &\leqq \tilde{\sigma}_{M-2}\|G(J^2 u, w)-G(J^2 U, w)\|_{M-2, Q} \\ &\leqq \tilde{\sigma}_{M-2}\bar{\sigma}_3 \max\{\|u\|_{Z^M}, \|U\|_{Z^M}, \|w\|_{M-2, Q}\}\|u-U\|_{Z^M} \\ &\qquad \leqq \tilde{\sigma}_{M-2}\bar{\sigma}_3 \max\{\eta, \rho\}\|u-U\|_{Z^M}\end{aligned}$$

which in case

$$\tilde{\sigma}_{M-2}\bar{\sigma}_3 \max\{\eta, \rho\}<1 \tag{6.9}$$

implies $u=U$. Thus $u \in Z^{M+1}$ which, in particular, yields $\partial_x^\alpha u \in L^2(Q)$ for $|\alpha|=M+1$. Since η, ε and ρ can be taken so small to satisfy (6.6), (6.7), (6.9) and all the above inequalities they are subjected to the proof of Theorem 2.1 is complete.

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