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Harmonic gauges on Riemann surfaces and stable bundles

by

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ABSTRACT. — We consider, on a compact Riemann surface M, a set of equations, generalizing harmonic maps from M into the unitary group.

Using a 0-curvature representation, we describe every solution on S^2 in terms of appropriate holomorphic vector bundles (called unitons), extending some previous results by Uhlenbeck and the author; and we make some remarks on the Morse and algebro-geometric stability of the solutions.

Key words : Harmonic maps, stability, gauge theories.

RÉSUMÉ. — Nous considérons, sur une surface de Riemann compacte M, un système d'équations, généralisant les fonctions harmoniques de M dans le groupe unitaire.

En employant une représentation de Zakharov-Shabat, nous décrivons toutes les solutions sur S^2 en termes d'appropriés fibrés vectoriels holomorphes (appelés unitons), en étendant quelques résultats précédents de Uhlenbeck et de l'auteur; nous faisons des observations sur la stabilité de Morse et algébro-géométrique des solutions.

Classification A.M.S. : 32 G, 53 C.

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0. INTRODUCTION

Let $P \to M^2$ be a principal $U(N)$-bundle over a compact Riemann surface M^2.

We study the equations:

$$\left.\begin{array}{c} F(A)+1/2[\Phi, \Phi] = -2i\pi * \mu(P) \\ d_A \Phi = 0, \qquad d_A * \Phi = 0 \end{array}\right\} \qquad (\bigstar)$$

where: A is a unitary connection on $P \to M^2$, of curvature $F(A)$; and Φ is a section of $T^*(M) \otimes \mathrm{ad}\, P$ [$\mathrm{ad}\, P \to M^2$ is the "adjoint bundle" associated to $P \to M^2$ *via* the adjoint representation of $U(N)$], and μ is the normalised 1st Chern class of P.

Equations ($\bigstar$) generalise the harmonicity equations for maps $M^2 \to U(N)$, and they still maintain a variational origin (*cf.* § 3 and [V 2]).

Here we generalise some previous work in this subject by Uhlenbeck [U] and Valli [V 1], to this twisted situation: we show the existence of a recursive procedure (called "addition of a uniton" or "flag transform" in the literature *cf.* [U], [B-R]), which generates solutions of ($\bigstar$) by means of choices of appropriate holomorphic vector subbundles.

We prove that, on the Riemann sphere, this procedure generates all solutions of ($\bigstar$), starting from one with $\Phi = 0$; while, on general surfaces M^2's, it fails to work, as long as we reach a pair (d_A, Φ), which is semistable in the sense of Hitchin (*cf.* [H 2]).

Our proof is based, as in [V 1], on progressive reduction of the energy $2|\Phi|^2$ of a solution (A, Φ); and on a topological expression for the decrease of energy in a flag transform.

Finally, we apply the same formula to show that the Morse (semi)-stability of a solution (A, Φ) of ($\bigstar$) implies the (semi)-stability of the holomorphic structure defined by the $\bar{\partial}$-operator $\bar{\partial}_A$; to do that, we split the energy Hessian, restricted to a special class of variations, into sum of two pieces, the difference of which is a topological term.

We regret that so much of the paper consists of preliminaries; our notation is anyway the "most standard" in current literature (in particular, *cf.* [A-B], [D], [H] and § 1).

Finally, we wish to thank J. C. Wood and N. Hitchin for having informed us about their work.

1. DICTIONARY

Let M^2 be a compact Riemann surface, and $P \to M^2$ be a given smooth principal U(N)-bundle. *Equivalently*, we may consider the associated complex hermitian vector bundle $V \to M^2$, associated to $P \to M^2$ *via* the standard representation of U(N) in $\mathbb{C}^N$ (we will generally prefer the vector bundle terminology throughout the following).

The adjoint bundle $\mathrm{ad}\, P \to M^2$ is the Lie algebra bundle associated to $P \to M^2$ via the adjoint representation of U(N) in $\mathfrak{u}(N)$; equivalently, it consists of the skew-hermitian elements of End(V).

Let $\mathscr{A} = \mathscr{A}(P)$ be the space of unitary connections on $P \to M^2$. Each connection $A \in \mathscr{A}(P)$ defines exterior differential operators d_A, defined on the space of sections of each of the bundles above.

Let $\mathscr{G}$ be the "gauge group" of smooth automorphisms of the principal bundle $P \to M^2$. Its "Lie algebra" $\mathfrak{g}$ consists of smooth sections of the adjoint bundle $\mathrm{ad}\, P \to M^2$.

The action of $\mathscr{G}$ on P induces an action on the space of connections:

$$\mathscr{G} \times \mathscr{A} \to \mathscr{A}, \qquad (g, A) \mapsto g_* A \tag{1}$$

defined by:

$$d_{g_* A} V = g^{-1} d_A (gvg^{-1}) g \tag{2}$$

for each v in $\mathfrak{g}$.

Finally, given a connection $A \in \mathscr{A}(P)$, we may split:

$$d_A = \partial_A + \bar{\partial}_A$$

using the complex structure on M^2.

2. STABLE BUNDLES AND STABLE PAIRS

From a topological point of view, complex vector bundles $V \to M^2$ are classified by their rank, and by their 1st Chern class:

$$c_1(V) \in H^2(M^2, \mathbb{Z}) \cong \mathbb{Z}$$

(the last isomorphism being evaluation on the fundamental 2-cycle).

We define the normalized 1st Chern class of $V \to M^2$ to be

$$\mu(V) = c_1(V)/rk\, V \tag{3}$$

Moreover, if $\underline{\mathbf{p}} \subseteq V$ is a complex vector subbundle, we then define, for later typographical convenience:

$$\sigma(\underline{\mathbf{p}}) = rk\, \underline{\mathbf{p}} (\mu(V) - \mu(\underline{\mathbf{p}})) \tag{4}$$

Let now $V \to M^2$ be a given complex vector bundle.

By a theorem of Koszul and Malgrange (*cf.* [A-B]), each unitary connection A on V defines a unique holomorphic structure on V, such that, for any local section s, we have:

$$s \text{ is holomorphic} \iff \bar{\partial}_A s = 0$$

More precisely, holomorphic structures on $V \to M^2$ are in $1-1$ correspondance with $\bar{\partial}$-operators $\bar{\partial}_A$ on it.

In the following, we shall indicate the complex vector bundle $V \to M^2$, equipped with the holomorphic structure induced by $\bar{\partial}_A$, with $(V, \bar{\partial}_A)$.

We recall now some standard definitions.

DEFINITION 1. — A holomorphic vector bundle $V \to M^2$ is called *stable* (resp. *semistable*), if, for any proper holomorphic subbundle $\underline{p} \subseteq V$ we have: $\sigma(\underline{p}) > 0$ (resp. $\geqq 0$).

DEFINITION 2 (*cf.* [H 2]). — Let $V \to M^2$ be a holomorphic vector bundle, and let Φ_z be a holomorphic section of $\text{End}(V) \otimes K$ [where K is the canonical bundle of (1,0) forms]. We say that (V, Φ_z) is a *stable pair* (resp. a *semistable pair*) if for each proper holomorphic subbundle $\underline{p} \subset V$, which is Φ_z-invariant, we have $\sigma(\underline{p}) > 0$ (resp. $\geqq 0$).

We consider now the case $M^2 = \mathbb{CP}^1$. We then have a complete classification of holomorphic vector bundles.

THEOREM (*Birkhoff-Grothendieck*).

(i) *Each holomorphic vector bundle over* $\mathbb{CP}^1$ *splits as direct sum of holomorphic line bundles.*

(ii) *For each integer* k, *there exists one and only one holomorphic line bundle* L^k *over* $\mathbb{CP}^1$ *of* 1*st Chern class* k (*up to isomorphism*).

Using this well known classification theorem, it's easy to prove the following.

LEMMA 1. — *Let* $V \to \mathbb{CP}^1$ *be a holomorphic vector bundle, and let* Φ_z *be a holomorphic section of* $\text{End}(V) \otimes K$. *Suppose* (V, Φ_z) *is a semistable pair. Then:*

(i) $\Phi_z = 0$;

(ii) $\mu(V)$ *is an integer*;

(iii) V *splits as direct sum of* $N = rk\, V$ *copies of the line bundle* $L^{\mu(V)}$.

Proof. — By the Birkhoff-Grothendieck theorem, we can split V into the direct sum of N line bundles. Let $\underline{p} \subseteq V$ be the holomorphic subbundle generated by the line bundles of highest 1st Chern class; then we have $\mu(\underline{p}) > \mu(V)$, unless $\underline{p} = V$.

By standard arguments involving the positivity of the 1st Chern class of the tangent bundle of S^2 (*cf.* [V 1]), $\underline{p}$ must lie in the kernel of Φ_z. In particular it is Φ_z-invariant, so that we must have, by the semistability assumption, $\mu(\underline{\mathbf{p}}) \leqq \mu(V)$; therefore $\underline{\mathbf{p}} = V$. But this in turn implies $\Phi_z = 0$. □

More generally, on Riemann surfaces of greater genus, stable pairs form a moduli space (*cf.* [H 2]).

3. THE EQUATIONS

Let $V \to M^2$ be a complex hermitian vector bundle, $A \in \mathscr{A}$ a unitary connection on V, and Φ a skew-hermitian section of $\mathrm{End}(V) \otimes T^* M^2$.

[Equivalently, $P \to M^2$ is the associated principal $U(N)$-bundle, $A \in \mathscr{A}(P)$, and Φ is a section of $\mathrm{ad}\, P \otimes T^* M^2$.]

We want to study the following system of equations:

$$\left.\begin{array}{c} F(A) + 1/2\,[\Phi, \Phi] = -2\pi i * \mu(V) \\ d_A \Phi = 0, \qquad d_A * \Phi = 0 \end{array}\right\} \qquad (\bigstar)$$

Equivalently, if we decompose $\Phi = \Phi_z + \Phi_{\bar{z}}$ using the complex structure on M^2, then Φ_z is a section of $\mathrm{End}(V) \otimes K$, and ($\bigstar$) is equivalent to ($\bigstar\bigstar$):

$$\left.\begin{array}{c} F(A) + [\Phi_z, \Phi_{\bar{z}}] = -2\pi i * \mu(V) \\ \bar{\partial}_A \Phi_z = 0 \end{array}\right\} \qquad (\bigstar\bigstar)$$

We remark that equations ($\bigstar$) are *not* conformally invariant, because of the term $* \mu(V)$, which needs a volume form on M^2 in order to be defined. Alternatively, let $\mu^{\sim}$ be *any* 2-form on M^2, such that $\mu . rk\, V$ is integral, and represents the 1st Chern class $c_1(V)$ (this constraint is due to Proposition 2 below, and to the Chern-Weil theory of characteristic classes). Then we can replace $* \mu(V)$ with $\mu^{\sim}$, in ($\bigstar$), and throughout the following. And, up to a choice of $\mu^{\sim}$, equations ($\bigstar$) *are* conformally invariant.

REMARK. — Equations ($\bigstar$) are "gauge" invariant under the $\mathscr{G}$-action:

$$A \mapsto g_* A, \qquad \Phi \mapsto g_* \Phi = g^{-1} \Phi g \qquad (5)$$

The simplest possible example of solutions to ($\bigstar$) is the case when $\Phi = 0$. Then ($\bigstar$) becomes:

$$F(A) = -2\pi i * \mu(V)$$

i.e. A is a connection with constant central curvature $-2\pi i * \mu(V)$. We can generalize this fact, as follows.

Let (A, Φ) be as above. We consider the loop of unitary connections on $V \to M^2$:

$$A_t = A + \cos t\, \Phi + \mathrm{sen}\, t * \Phi, \qquad \forall t \in [0, 2\pi] \qquad (6)$$

Equivalently:

$$\bar{\partial}_{A_t}=\bar{\partial}_A+\lambda^{-1}\,\mathrm{ad}\,\Phi_{\bar{z}}, \qquad \lambda=e^{it}\in S^1 \tag{7}$$

PROPOSITION 2. — *The following statements are equivalent.*

(i) (A, Φ) *is a solution of* (★).

(ii) A_t *has constant central curvature* $F(A_t)=-2\pi i*\mu(V)$, $\forall t\in[0, 2\pi]$.

Proof.

$$F(A_t)=F(A)+1/2\,[\Phi, \Phi]+\cos t\, d_A\Phi+\mathrm{sen}\, t\, d_A*\Phi. \quad \square$$

We call a circle of unitary connections, with constant central curvature, of the form (6), an "Uhlenbeck loop".

By proposition 2, Uhlenbeck loops are in natural bijective correspondance with the space of solutions of (★), *modulo the* S^1*-action* $\Phi_z \mapsto e^{it}\Phi_z$.

The system of equations (★) has a simple variational origin. Suppose we take two unitary connections on $V\to M^2$; we call them B, C (we should call them A_π, A_0, because of Proposition 2, but we prefer to avoid proliferation of indexes).

We define B, C to be harmonic one with respect to the other (*cf.* [V 2]) if fixing one of them (say C) then the other (say B) is critical, with respect to the $\mathcal{G}$-action, with respect to the natural, conformally invariant, norm (Energy) on the space of connections $\mathcal{A}$:

$$E_C(B)=E_B(C)=1/2\,\|C-B\|^2=1/2\int \mathrm{Tr}(C-B)\wedge *(C-B) \tag{8}$$

$$d/dt\,\|g_{t*}B-C\|^2_{t=0}=0$$

for each variation g_t:

$$\forall g_t:(-\varepsilon, \varepsilon)\to\mathcal{G}, \qquad g_0=I.$$

PROPOSITION 3. — *Let* B, C *be unitary connections with constant central curvature:*

$$F(B)=F(C)=-2\pi i*\mu(V)$$

Then the following statements are equivalent.

(i) *B and C are harmonic one with respect to each other.*

(ii) $A=1/2(B+C)$ $\Phi=1/2\,(B-C)$ *are a solution of* (★).

Proof. — B and C are harmonic one with respect to each other if and only if $d_A*\Phi=0$.

And

$$F(B)=F(C)=-2\pi i*\mu(V) \quad \text{iff} \quad \begin{cases} F(A)+1/2\,[\Phi, \Phi]=-2\pi i*\mu(V) \\ d_A\Phi=0 \quad \square \end{cases}$$

Remark. — Let us consider the special case $V = M^2 \times \mathbb{C}^N$; then the gauge group $\mathcal{G}$ is $\mathcal{G} = \{\text{smooth maps } M^2 \to U(N)\}$. Set $C = 0$, the zero connection; and, for $f \in \mathcal{G}$ set $B = f_* C = f^{-1} df$; and A, Φ, as above. Then (A, Φ) is a solution of ($\bigstar$) if and only if f is an harmonic map $f: M^2 \to U(N)$ (*cf.* [H 1], [U], [V 1]).

PROPOSITION 4. — *Let* (A, Φ) *be a solution of* ($\bigstar$).
The second variation of energy is then:

$$H(u, u) = |d_A u|^2 - |[\Phi, u]|^2, \qquad u \in \mathcal{G} \tag{9}$$

Proof. — Let $B = A + \Phi$, $C = A - \Phi$, as above.
We consider a variation $g_t: (-\varepsilon, \varepsilon) \to \mathcal{G}$, $g_0 = I$; and let $B_t = g_{t*} B$. Then we have:

$$d/dt\, |B_t - C|^2 = 2 \langle B_t', B_t - C \rangle$$

$$d/dt^2\, |B_t - C|^2_{t=0} = 2 \{ |d_{B_t}(g^{-1} g')|^2 + \langle B_t - C, d_{B_t}(g^{-1} g')' + [d_{B_t}(g^{-1} g'), g^{-1} g'] \rangle \} |_{t=0}.$$

Set $u = g^{-1} g'_{|t=0}$. Then, because of the harmonicity equations, we have:

$$H(u, u) = 1/2\, d/dt^2\, |B_t - C|^2 |_{t=0} = |d_B u|^2 + \langle B - C, [d_B u, u] \rangle = \langle d_B u, d_C u \rangle = |d_A u|^2 - |[\Phi, u]|^2. \quad \square$$

4. ADDING UNITONS

Let (A, Φ) be a solution of ($\bigstar$) on $V \to M^2$; and let A_t be the associated Uhlenbeck circle of connections (6).

For each complex subbundle $\underline{p} \subseteq V$, let p be the associated hermitian projection operator $p: V \to \underline{p}$, $p^2 = p$. Let us associate to $\underline{p}$ the closed 1-parameter subgroup of $\mathcal{G}$:

$$g_t = \exp(itp), \qquad t \in [0, 2\pi]$$

Let $A_t^\sim = g_{t*} A_t$, $\forall t \in [0, 2\pi]$.

PROPOSITION 5. — *The following statements are equivalent:*
(i) $A_t^\sim$ *is an Uhlenbeck loop;*
(ii) $\underline{p}$ *is a* $\bar{\partial}_A$*-holomorphic subbundle of* V, *and it is* Φ_z*-invariant.*

Proof. — $A_t^\sim$ is certainly a loop of unitary connections with constant central curvature. We have to check when it is of the form (6), for appropriate $\Phi^\sim$, $A^\sim$. Now, if $\lambda = e^{it}$, then we have:

$$g_t = (p^\perp + \lambda p),$$

[where $p^\perp = I - p$ is the hermitian projection operator onto $(\underline{p})^\perp \subseteq V$]; and:

$$A_{\bar{z}, t} = A_{\bar{z}} + \lambda^{-1} \Phi_{\bar{z}};$$

so that we have:

$$A^{\sim}_{\bar z,\,t}=(p^{\perp}+\lambda p)_{*}A_{\bar z}+\lambda^{-1}(p^{\perp}+\lambda^{-1}p)\Phi_{\bar z}(p^{\perp}+\lambda p)$$

And, using local coordinates it' s easy to show that:

$$A^{\sim}_{\bar z,\,t}=\lambda(p^{\perp}\bar\partial_{A}p)+(p^{\perp}\bar\partial_{A}p^{\perp}+p\bar\partial_{A}p+p^{\perp}\Phi_{\bar z}p) \\ +\lambda^{-1}(p\bar\partial_{A}p^{\perp}+p\Phi_{\bar z}p+p^{\perp}\Phi_{\bar z}p^{\perp})+\lambda^{-2}(p\Phi_{\bar z}p^{\perp}) \quad (10)$$

But $A^{\sim}_t$ is of the form (6) if and only if $A^{\sim}_{\bar z,\,t}$ does not contain terms with λ, λ^{-2}.

$$\Leftrightarrow \quad \begin{array}{l} p^{\perp}\bar\partial_{A}p=0 \\ p\Phi_{\bar z}p^{\perp}=0, \end{array}$$

$$\Leftrightarrow \quad \begin{cases} p^{\perp}\bar\partial_{A}p=0 & (11\,a) \\ p^{\perp}\Phi_{z}p=0 & (11\,b) \end{cases}$$

But (11) are the equations expressing (ii). □

We call equations (11) "uniton equations" (*cf.* [U]). We call a subbundle $\underline{p}\subseteq V$, satisfying equations (11) (*i. e.* $\bar\partial_A$-holomorphic and Φ_z-invariant) a "uniton"; and we say that the new Uhlenbeck loop $A^{\sim}_t$ has been obtained from A_t by addition of the uniton $\underline{p}$ (*cf.* [U]), or by "flag transform" (*cf.* [B-R 1], [B-R 2]).

Consistently with paragraph 3, given a solution (A, Φ) of (★), we call $E=2|\Phi|^2$ the *energy* of the solutions (A, Φ).

PROPOSITION 5. — *Let* (A, Φ) *be a solution of* (★) *on* $V\to M^2$; *and let* $(A^{\sim}, \Phi^{\sim})$ *be obtained from* (A, Φ) *by addition of the uniton* $\underline{p}\subseteq V$. *Then we have:*

$$1/2\,\Delta E=|\Phi^{\sim}|^2-|\Phi|^2=2\pi\sigma(\underline{p}) \quad (12)$$

Proof. — From (10) we get:

$$\Phi^{\sim}_{z}=-(p^{\perp}\partial_{A}p)+(p\Phi_{z}p)+(p^{\perp}\Phi_{z}p^{\perp}) \quad (13)$$

Therefore:

$$|\Phi^{\sim}_{z}|^2=|p^{\perp}\partial_{A}p|^2+|p\Phi_{z}p|^2+|p^{\perp}\Phi_{z}p^{\perp}|^2$$

and:

$$1/2(|\Phi^{\sim}|^2-|\Phi|^2)=|\Phi^{\sim}_{z}|^2-|\Phi_{z}|^2=|p^{\perp}\partial_{A}p|^2+|p\Phi_{z}p|^2+|p^{\perp}\Phi_{z}p^{\perp}|^2-|\Phi_z|^2 \\ =|p^{\perp}\partial_{A}p|^2+|p\Phi_{z}p|^2-|\Phi_{z}|^2+|\Phi_{z}p^{\perp}|^2-|p\Phi_{z}p^{\perp}|^2 \\ =|p^{\perp}\partial_{A}p|^2-|p^{\perp}\Phi_{z}p|^2-|p\Phi_{z}p^{\perp}|^2 \\ =(|p^{\perp}\partial_{A}p|^2-|p\Phi_{z}p^{\perp}|^2)-(|p^{\perp}\bar\partial_{A}p|^2-|p^{\perp}\Phi_{z}p|^2).$$

[The last passage being possible because of the uniton equations (11).]

Proposition 5 follows then from Lemma 6, in next paragraph. □

5. A FORMULA

Let $V \to M^2$ be a complex hermitian vector bundle over the compact Riemann surface M^2, equipped with a hermitian metric; let A be a unitary connection on V, and $\Phi = \Phi_z + \Phi_{\bar{z}}$ a skew-hermitian section of $\mathrm{End}(V) \otimes T^*(M^2)$.

LEMMA 6. — *Suppose* (A, Φ) *satisfy the* 1*st eq. of* (★):

$$F(A) + 1/2\,[\Phi, \Phi] = -2\pi i * \mu(V)$$

Let $\underline{p} \subseteq V$ be any complex subbundle of V, and let $p : V \to \underline{p}$ be the associated hermitian projection operator. Then we have:

$$(|p^\perp \partial_A p|^2 - |p^\perp \Phi_{\bar{z}} p|^2) - (|p^\perp \bar{\partial}_A p|^2 - |p^\perp \Phi_z p|^2) = 2\pi\sigma(\underline{p}) \quad (14)$$

Proof. — Let us equipp $\underline{p}$ with the connection induced from A. If ∇ is the associated exterior differential, then we have, for each section v of p:

$$\nabla v = p\, d_A v = d_A v - d_A p v$$

And the curvature $F(\nabla)$ is:

$$F(\nabla) = p(F(A) + d_A p \wedge d_A p)p$$

By the Chern-Weil formulas for characteristic classes, we have:

$$-2i\pi c_1(\underline{p}) = \int \mathrm{Tr}\, F(\nabla) = \int \mathrm{Tr}\,(p\, F(A)) + \int \mathrm{Tr}\,(p\, d_A p \wedge d_A p)$$

$$= -2i\pi \int \mathrm{Tr}(p) * \mu(V) - 1/2 \int \mathrm{Tr}\,[\Phi, \Phi] p + \int \mathrm{Tr}\,(p\, d_A p \wedge d_A p) \quad (15)$$

(using the hypotheses of the Lemma).

But we have:

$$\int \mathrm{Tr}\,(p\, d_A p \wedge d_A p) = -i(|p^\perp \bar{\partial}_A p|^2 - |p\, \bar{\partial}_A p|^2)$$
$$= -i(|p^\perp \bar{\partial}_A p|^2 - |p^\perp \partial_A p|^2) \quad (16)$$

(because $p\, d_A p = d_A p p^\perp$ and $p^\perp d_A p = d_A p p$); and:

$$1/2 \int \mathrm{Tr}\,[\Phi, \Phi] p = \int \mathrm{Tr}\,[\Phi_z, \Phi_{\bar{z}}] p = -i(|\Phi_z p|^2 - |p\, \Phi_z|^2)$$
$$= -i(|p^\perp \Phi_z p|^2 - |p\, \Phi_z p^\perp|^2) \quad (17)$$

$$\int \mathrm{Tr}\,(p) * \mu(V) = rk(\underline{p})\, \mu(V) \quad (18)$$

Substituting (16), (17) and (18) in (15), we get:

$$-2i\pi c_1(\underline{p}) = (-2i\pi)\, rk(\mathbf{p})\, \mu(V) + i(|p^\perp \Phi_z p|^2 - |p\, \Phi_z p^\perp|^2) + \ldots$$
$$- i(|p^\perp \bar{\partial}_A p|^2 - |p^\perp \partial_A p|^2)$$

which is (15). □

6. MAIN RESULTS

THEOREM 7. — *Let* (A, Φ) *be a solution of* (★) *on* $V \to M^2$. *Then there exists a solution of* (★) (A^0, Φ^0) *such that:*

(i) $((V, \bar\partial_{A^0}), \Phi^0_z)$ *is a semistable pair;*

(ii) (A, Φ) *is obtained from* (A^0, Φ^0) *by a finite number of flag transforms, each one making the energy* $E^i = 2|\Phi^i|^2$ *increse by a positive integral multiple of* $8\pi/rk(V)$.

Proof. — If $((V, \bar\partial_A), \Phi_z)$ is not a semistable pair, then there exists a $\bar\partial_A$-holomorphic subbundle $\underline{p} \subseteqq V$, Φ_z-invariant, and with $\sigma(\underline{p}) < 0$.

Therefore $\underline{p}$ is a uniton, and we may add it to (A, Φ), to produce a new solution $(A^\sim, \Phi^\sim)$ of (★), such that:

$$\Delta E = 2(|\Phi^\sim|^2 - |\Phi|^2) = 4(|\Phi_z^\sim|^2 - |\Phi_z|^2) = 8\pi\sigma(\underline{p}) < 0$$

(using Proposition 5).

Moreover

$$\begin{aligned}\sigma(\underline{p}) = rk(\underline{p})(\mu(V) - \mu(\underline{p})) = rk(\underline{p})\,\mu(V) - c_1(\underline{p}) \\ = 1/rk(V)\{rk(\underline{p})\,c_1(V) - rk(V)\,c_1(\underline{p})\}\end{aligned}$$

Repeating this procedure, if necessary, we must eventually come to a stop, when we reach a semistable pair (A^0, Φ^0). □

THEOREM 8. — *Let* (A, Φ) *be a solution of* (★) *on* $V \to \mathbb{CP}^1$. *Then:*

(i) $\mu(V)$ *is an integer;*

(ii) (A, Φ) *is obtained from a solution* (A^0, Φ^0), *with:*

$$\begin{cases} \Phi^0 = 0 \\ F(A^0) = -2\pi i * \mu(V) \end{cases}$$

by a finite number of flag transforms, each one making the energy $E^i = 2|\Phi^i|^2$ *increase by an integral multiple of* $8\pi/rk(V)$.

(iii) V, *with the* $\bar\partial_{A^0}$-*holomorphic structure, is a direct sum of* N *copies of the line bundle* $L^{\mu(V)} \to \mathbb{CP}^1$.

(iv) $E = 2|\Phi|^2 = 1/8\pi\{a\mu(V) + b\}$ *where* $a \in \mathbb{N}$, $b \in \mathbb{Z}$.

Proof. — It easily follows from Theorem 7, Lemma 1, and Proposition 5. □

Remarks. — The flag transforms in Theorems 7, 8 may be chosen to be "canonical" in some sense. For example, we can choose at each step the most energy-decreasing uniton, which is the one generated by all the unitons $\underline{p}' \subseteqq V$, with $\sigma(\underline{p}') > 0$. Another possible "canonical" choice, when $M^2 = \mathbb{CP}^1$, is to choose the image bundle (or the kernel bundle) of Φ_z at each step as uniton: it is not necessarily energy decreasing, but it arrives to the 0-energy solution, after a finite number of steps, as in Theorem 8.

For a more detailed analysis, *cf.* [V 3]. For a more explicit description of the factorization, when $M^2 = \mathbb{CP}^1$, with a unicity result, *cf.* [W].

Let (A, Φ) be a solution of ($\bigstar$) on $V \to M^2$. We want to use Lemma 6 in order to study the energy hessian:

$$H(u, u) = |d_A u|^2 - |[\Phi, u]|^2 \tag{9}$$

where $u \in \mathfrak{g}$ is a smooth skew-symmetric section of $\mathrm{End}(V) \to M^2$.

We consider infinitesinal variations of the form $u = ip$, where $p : V \to \underline{\mathbf{p}}$ is the hermitian projection operator onto a complex subbundle $\underline{\mathbf{p}} \subseteq V$; we may call this kind of variations *Grassmannian variations.*

PROPOSITION 9. — *There exist two quadratic functionals* $H^1(p, p)$, $H^2(p,p)$, *on the space of Grassmannian variations, such that we have:*

$$\begin{gathered} H^1(p, p) + H^2(p, p) = H(ip, ip) \\ H^1(p, p) - H^2(p, p) = 4\pi\sigma(\underline{\mathbf{p}}), \qquad \forall \underline{\mathbf{p}} \subseteq V \end{gathered} \tag{19}$$

Proof. — Define:

$$\begin{aligned} H^1(p, p) &= 2(|p^\perp \partial_A p|^2 - |p\Phi_z p^\perp|^2) \\ H^2(p, p) &= 2(|p^\perp \bar{\partial}_A p|^2 - |p^\perp \Phi_z p|^2) \end{aligned}$$

and apply Lemma 6. □

COROLLARY 10. — *Suppose the energy hessian* (9) *of a given solution* (A, Φ) *of* ($\bigstar$) *is positive definite* (*resp. semipositive*).

Then the bundle (V, $\bar{\partial}_A$) *is stable* (*resp. semistable*).

Proof. — If (V, $\bar{\partial}_A$) is not (semi)-stable, then $\exists\, \underline{\mathbf{p}} \subseteq V$ $\bar{\partial}_A$-holomorphic subbundle, with $\sigma(\underline{\mathbf{p}}) \leqq 0$ (resp. < 0). Let us take a variation $u = ip \in \mathfrak{g}$, with p projection onto $\underline{\mathbf{p}}$. Then we have:

$H^2(p, p) \leqq 0$, so that:

$$H(ip, ip) = 2H^2(p, p) + 4\pi\sigma(p) \leqq 0 \quad (\text{resp. } < 0). \quad \square$$

COROLLARY 11. — *Let* (A, Φ) *be a solution of* ($\bigstar$) *on* $V \to M^2$; *let* $\underline{\mathbf{p}} \subseteq V$ *be a* $\bar{\partial}_A$*-holomorphic,* Φ_z*-invariant subbundle* (i. e. *a uniton*), *and let* p : $V \to \underline{\mathbf{p}}$ *be the associated projection.*

Then $u = ip$ *is critical point for the functional* (9):

$$H(ip, ip) = |d_A(p)|^2 - |[\Phi, p]|^2$$

restricted to the space of Grassmannian variations.

Proof. — Because of Proposition 9, it's sufficient to show that p is a critical point of:

$$H^2(p, p) = 2(|p^\perp \bar{\partial}_A p|^2 - |p^\perp \Phi_z p|^2)$$

on the space of p's.

The Euler-Lagrange equations for p [we allow variations of the form $p \mapsto g_t^{-1} p g_t$, with $g_t : (-\varepsilon, \varepsilon) \to \mathcal{G}$] are of the form:

$$D'(p) + D''(p) = 0 \tag{20}$$

where $D'(p)$, $D''(p)$, are differential expressions in p, coming out from the 1st and 2nd term in $H^2(p, p)$.

But $D'(p) = 0$, because $p^\perp \bar{\partial}_A p = 0$, so p is a minimum of $|p^\perp \bar{\partial}_A p|^2$, and therefore a critical point for $|p^\perp \bar{\partial}_A p|^2$, in the space of $\mathbf{p}$'s.

Similarly, $D''(p) = 0$, because $\underline{\mathbf{p}}$ is $\Phi_{\bar{z}}$-invariant. Therefore, if $\underline{\mathbf{p}}$ is a uniton, it satisfies the Euler-Lagrange equations (20). □

Remark. — Corollary 11 still holds if $\underline{\mathbf{p}}$ is an "antiuniton", *i.e.* a ∂_A-antiholomorphic, $\Phi_{\bar{z}}$-invariant subbundle of V. Indeed, just repeat the proof above, considering now H^1; or observe that $\mathbf{p}$ is a antiuniton if and only if $\underline{\mathbf{p}}^\perp$ is a uniton; and that $H(ip, ip) = H(ip^\perp, ip^\perp)$ for each p. □

7. SOME OPEN QUESTIONS

Question 1. — *Is the converse of Corollary* 10 *true?*

This question is closely related to the following:

Question 2. — *Is the converse of Corollary* 11 *true; in other words: is every critical point of the functional* (9) *on the space of complex subbundles of* V *either a uniton or antiuniton?*

Question 3. — *Is it possible to generalize some of the constructions and of the results in this paper to the case when* M *is a Kähler manifold?*

For a partial answer to Question 3, *see* [O-V], where factorization theorems for pluriharmonic maps (in the sense of Ohnita [O]) into Lie groups are obtained.

Question 4. — Motivated by a theorem of Gaveau (*cf.* [G]), we ask if the following is true.

Let $V \to M^2$ *be a complex vector bundle on a compact Riemann surface* M^2; *and let* (B, C) *be two unitary connections on* V, *qith constant central curvature* $F(B) = F(C) = -2\pi i * \mu(V)$.

Then there exists a unitary connection $B^\sim$, *gauge equivalent to* B, *such that* $B^\sim$, C, *are harmonic one with respect to the other.*

This would give information on the moduli space of solutions of (★).

REFERENCES

[A-B] M. F. Atiyah and R. Bott, The Yang-Mills Equations over Riemann Surfaces, *Phil. Trans. R. Soc. London*, Vol. **A308**, pp. 523-615.

[B-R 1] F. Burstall and J. Rawnsley, Sphères harmoniques, etc., *C. R. Acad. Sci. Paris*, T. 302, Series I, 1986, pp. 709-712.

[B-R 2] F. Burstall and J. Rawnsley, *Twistors, Harmonic Maps, and Homogeneous Geometry* (in preparation).

[D 1] S. K. Donaldson, A New Proof of a Theorem of Narasimhan and Seshadri, *J. Diff. Geometry*, Vol. **18**, 1983, pp. 269-277.

[D 2] S. K. Donaldson, Anti-Self-Dual Yang-Mills Connections over Complex Algebraic Surfaces and Stable Vector Bundles, *Proc. London Math. Soc.*, Vol. **50**, 1985, pp. 1-26.

[G] B. Gaveau, Intégrales harmoniques non-abéliennes, *Bull. Sc. Math.*, 2ᵉ série, T. **106**, 1982, pp. 113-169.

[H 1] N. Hitchin, *Harmonic Maps from a 2-Torus to the 3-Sphere*, Preprint, 1987.

[H 2] N. Hitchin, Self-Duality Equations on a Riemann Surface, *Proc. London Math. Soc.*, Vol. **55**, 1987, pp. 59-126.

[H 3] N. Hitchin, Stable Bundles and Integrable Systems, *Duke Math. J.*, Vol. **54**, 1987, pp. 91-114.

[O] Y. Ohnita, On Pluriharmonicity of Stable Harmonic Maps, *J. London Math. Soc.*, Vol. **35**, 1987, pp. 563-568.

[O-V] Y. Ohnita and G. Valli, *Pluriharmonic Maps into Lie Groups*, Max-Plank-Institut, preprint, 1989.

[U] K. Uhlenbeck, *Harmonic Maps into Lie Groups (Classical Solutions of the Chiral Model)*, Preprint, University of Chicago, 1985.

[V 1] G. Valli, On the Energy Spectrum of Harmonic 2-Spheres in Unitary Groups. *Topology* (to appear).

[V 2] G. Valli, *Some Remarks on Geodesics in Gauge Groups and Harmonic Maps*, Preprint, International Centre of Theoretical Physics, Trieste, 1987.

[V 3] G. Valli, *Ph. D. Thesis*, University of Warwick (G. B.) (in preparation).

[W] J. C. Wood, *Explicit Construction and Parametrization of Harmonic 2-Spheres in the Unitary Group*, Preprint, University of Leeds, 1987.

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