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Minimax characterization of solutions for a semi-linear elliptic equation with lack of compactness

by

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ABSTRACT. — In this paper conditions ensuring bifurcation from any boundary point of the spectrum is studied for a class of nonlinear operators. We give a general minimax result which allows an enlargement of the class of non-linearities which has been studied up to now. The general result is applied to study the existence of solutions $(u, \lambda) \in H^1(\mathbb{R}^N) \times \mathbb{R}$ for the equation

$$-\Delta u + pu - N(u) = \lambda u, \; u \neq 0,$$

where λ is located in a prescribed gap of the spectrum of $-\Delta u + pu$. The function p is periodic and the superlinear term N derives from a potential but is not assumed to be compact.

Key words : Elliptic equation, bifurcation, minimax method, loss of compactness.

RÉSUMÉ. — Dans cet article, nous donnons des conditions assurant la bifurcation pour tous les points au bord du spectre pour une classe d'opérateurs non linéaires. A l'aide d'un résultat de type minimax, nous élargissons la classe des non-linéarités étudiées jusqu'à présent. La théorie

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est appliquée à l'étude des solutions $(u, \lambda) \in H^1(\mathbb{R}^N) \times \mathbb{R}$ de l'équation

$$-\Delta u + pu - N(u) = \lambda u, \; u \neq 0,$$

où λ se situe dans une lacune donnée du spectre de $-\Delta u + pu$. La fonction p est périodique et le terme surlinéaire N dérive d'un potentiel mais n'est pas nécessairement compact.

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1. INTRODUCTION

In this paper we consider the equation

$$\left.\begin{array}{c} -\Delta u(x) + p(x) u(x) - r(x) |u(x)|^\sigma u(x) = \lambda u(x), \\ x \in \mathbb{R}^N, \qquad \lambda \in \mathbb{R}, \end{array}\right\} \tag{1.1}$$

where it is assumed that

(B1) $p \in L^\infty(\mathbb{R}^N)$, $p(x + a_i) = p(x)$ a.e. on $\mathbb{R}^N$, $1 \leqq i \leqq N$,
span $\{a_i, 1 \leqq i \leqq N\} = \mathbb{R}^N$ (p is periodic),

(B2) $r \in L^\infty(\mathbb{R}^N)$, $r \geqq 0$ a.e. on $\mathbb{R}^N$,

(B3) $0 < \sigma < \dfrac{4}{N-2}$ for $N \geqq 3$ and $\sigma > 0$ for $N = 1, 2$.

We seek weak solutions of (1.1) in $H^1(\mathbb{R}^N)$.

The linearisation of (1.1) is the periodic Schrödinger equation

$$-\Delta u(x) + p(x) u(x) = \lambda u(x), \qquad x \in \mathbb{R}^N.$$

It is therefore natural to introduce the self-adjoint operator

$$S: \; D(S) \subset L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$$

defined by

$$D(S) = H^2(\mathbb{R}^N) \qquad \text{and} \qquad Su = -\Delta u + pu.$$

The spectrum of S, $\sigma(S)$, is bounded from below and consists of a union of closed intervals. Let $l \in \sigma(S)$ be such that $[l - \varepsilon, l[\cap \sigma(S) = \emptyset$ for some $\varepsilon > 0$. We want to derive conditions on the nonlinearity which insure that l is a bifurcation point. More precisely we require this bifurcation to occur by regular values. This is the case if there exists a sequence $\{(u_n, \lambda_n)\} \subset H^1(\mathbb{R}^N) \times [l - \varepsilon, l[$ such that

1. $\forall n$, $\forall v \in H^1(\mathbb{R}^N)$:

$$\int_{\mathbb{R}^N} (\nabla u_n \nabla v + p u_n v - r |u_n|^\sigma u_n v) \, dx = \lambda_n \int_{\mathbb{R}^N} u_n v dx,$$

2. $\forall n: u_n \not\equiv 0$,
3. $(u_n, \lambda_n) \to (0, l)$.

In the following, l will always refer to a point of $\sigma(S)$ such that $[l-\varepsilon, l[\cap \sigma(S)=\varnothing$ for some $\varepsilon>0$.

Clearly (1.1) can be viewed as a generalization of the equation

$$-\Delta u(x)-r(x)\,|u(x)|^{\sigma} u(x)=\lambda u(x), \qquad x \in \mathbb{R}^N, \quad \lambda \in \mathbb{R}, \tag{1.2}$$

where (B2), (B3) hold. This equation has raised a lot of interest in recent years. One of the reasons is that bifurcation can occur at the infimum of the spectrum associated with the linearized equation

$$-\Delta u(x)=\lambda u(x)$$

(*see* [19-22] and the references there). It is interesting since the bifurcation point belongs to the essential spectrum and hence the methods of classical Fredholm theory do not work. Multiplicity results has also been proved *i.e.* the existence of infinitely many bifurcating branches [17, 18].

In view of the results obtained on (1.2), one is led to inquire if points of type l can be bifurcation points for equation (1.1) and under which conditions. It was only very recently that the first results in this direction were obtained [7-13]. In [11], under appropriate assumptions on the nonlinearity, it is shown that for each point l there exists a bifurcating sequence $\{(u_n, \lambda_n)\} \subset H^1(\mathbb{R}^N) \times [l-\varepsilon, l[$. Moreover each u_n has a defined (small) norm in $L^2(\mathbb{R}^N)$. Another way of attacking the problem was put forth by Heinz [7] and Heinz-Stuart [10]. Let $a, b \in \sigma(S)$ with $]a, b[\cap \sigma(S)=\varnothing$. In [7] conditions are given on the nonlinearity which insure that for every $\lambda \in]a, b[$ there exists a solution u_λ of (1.1). Under stronger assumptions, it is also proved that $\|u_\lambda\|_{H^1(\mathbb{R}^N)} \to 0$ as $\lambda \to b$. Thus bifurcation from the point b is obtained. In this approach there also exist multiplicity and regularity results [8]. The method used by Heinz and Stuart is based on abstract critical point theorems for strongly indefinite functionals developed by Benci and Rabinowitz [3, 4, 16]. In order to use these theorems it is necessary to assume that the nonlinear term in (1.1) is compact. This amounts to requiring the decay property $\lim_{|x| \to \infty} r(x)=0$. This (strong) restriction is also necessary in [11, 12]. A first step towards a release of the compactness condition was made independently by Alama and Li [1] and the authors [5]. In [1], the existence of a solution for every $\lambda \in \mathbb{R} \setminus \sigma(S)$ is proved in the autonomous case (*i.e.* $r(x)=r>0$, a.e. on $\mathbb{R}^N$). Under the same assumptions on $r(x)$ we obtained the existence of a bifurcating sequence of solutions for each given point l.

In this paper, an original variational approach to the existence of non trivial solutions is developed. In particular we obtain a minimax characterization of critical points. This allows us to treat a larger class of

non-compact nonlinearities. Let us introduce the condition (B4):
(B4) one of the following conditions hold:

1. $N=1$ and $0<\sigma<2$,
2. $N\geqq 1$, there exists $L>0$ and $I\geqq 0$ such that $r(x)\geqq L|x|^{-I}$ a.e. on C, where $C=\{tx,\ t\geqq 1 \text{ and } x\in B(x_0, d)\}$ for some $x_0\in\mathbb{R}^N$ and $0<d<|x_0|$. Moreover $0<\sigma<\dfrac{4-2I}{N}$.

Our main result is:

THEOREM 1.1. – *Let (B1) to (B4) be satisfied and assume that there exists a periodic function $r^P(x)$ of period comparable to the one of $p(x)$ such that*

1. $\lim\limits_{|x|\to+\infty}\{r(x)-r^P(x)\}=0$,
2. $r(x)\geqq r^P(x)$ a.e. *on* $\mathbb{R}^N$,
3. $r^P(x)>0$ a.e. *or* $r^P(x)=0$ *a.e.*

Then l is a bifurcation point towards regular values for equation (1.1).

Note that compact and non-compact cases are simultaneously treated here.

In order to avoid to use too specific properties of equation (1.1) (in particular the fact that the nonlinearity is homogeneous), we develop the variational method in an abstract setting. This also enables us as we believe to present in the clearest way the essential features of the method. This setting is introduced at the beginning of Section 2 where existence and bifurcation theorem are obtained for an equation set in an abstract Hilbert space. As the reader will see, the abstract results presented there go beyond the scope of equation (1.1). Application to (1.1) and extensions to other partial differential equations are presented at the end of Section 3. In particular we rederive in a simple way some of the results previously obtained on equation (1.2).

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2. THE THEORETICAL SETTING

Instead of working directly with equation (1.1), we shall consider first a general equation, set in an abstract Hilbert space, which contains (1.1) as a special case. Let $H^k(\mathbb{R}^N)$ denote the usual Sobolev space of functions with square integrable derivatives of order up to k. In discussing (1.1), we can assume without loss of generality that $p(x) \geqq 1$ a.e. on $\mathbb{R}^N$. Consequently the bilinear form:

$$\langle \cdot, \cdot \rangle: \quad H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \to \mathbb{R}$$

defined by

$$\langle u, v \rangle = \int_{\mathbb{R}^N} (\nabla u \nabla v + puv)\, dx$$

is a continuous scalar product. The corresponding norm $\|.\|$ is equivalent to the usual norm on $H^1(\mathbb{R}^N)$. In the following it is assumed that $H^1(\mathbb{R}^N)$ is equipped with this scalar product. Since the form

$$u, v \in H^1(\mathbb{R}^N) \to \int_{\mathbb{R}^N} uv\, dx$$

is continuous bilinear symmetric, we know that there exists a unique bounded selfadjoint operator

$$A: \quad H^1(\mathbb{R}^N) \to H^1(\mathbb{R}^N)$$

such that for $v \in H^1(\mathbb{R}^N)$

$$\langle A u, v \rangle = \int_{\mathbb{R}^N} uv\, dx.$$

Trivially $\langle A u, u \rangle > 0$ for $u \neq 0$. Concentrating on the nonlinearity we see that under assumptions (B1) to (B3) the function

$$v \in H^1(\mathbb{R}^N) \to \int_{\mathbb{R}^N} r |u|^\sigma uv\, dx$$

is continuous for all $u \in H^1(\mathbb{R}^N)$. Consequently there exists $N(u) \in H^1(\mathbb{R}^N)$ such that $\forall u, v \in H^1(\mathbb{R}^N)$:

$$\int_{\mathbb{R}^N} r |u|^\sigma uv\, dx = \langle N(u), v \rangle.$$

Recall that solutions of (1.1) are defined as couples $(u, \lambda) \in H^1(\mathbb{R}^N) \times \mathbb{R}$ satisfying for all $v \in H^1(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} (\nabla u \nabla v + puv - r |u|^\sigma uv)\, dx = \lambda \int_{\mathbb{R}^N} uv\, dx.$$

Using the definitions introduced above, this can be written:

$$\forall v \in H: \quad \langle u, v \rangle - \langle N(u), v \rangle = \lambda \langle A u, v \rangle$$

or equivalently

$$I u - N(u) = \lambda A u.$$

The preceding observations justify the introduction of the following abstract problem.

Let H be a real Hilbert space with scalar product $\langle .,. \rangle$ and norm $\|.\|$ and A and B two continuous selfadjoint operators satisfying

$$AB=BA \quad \text{and} \quad \forall u \neq 0: \ \langle Au, u \rangle > 0.$$

The resolvant set of the pair (B, A) is defined by

$$\rho(B, A) = \{ \lambda \in \mathbb{R} : B - \lambda A \text{ has a bounded inverse} \}$$

and the spectrum of (B, A) by $\sigma(B, A) = \mathbb{R} - \rho(B, A)$. We introduce an auxiliary scalar product:

$$\langle .,. \rangle_A : \quad H \times H \to \mathbb{R}$$

defined by

$$\langle u, v \rangle_A = \langle Au, v \rangle$$

and $\|.\|_A$ is the associated norm (which in general is not equivalent to $\|.\|$)

Let $\varphi \in C^2(B_r, \mathbb{R})$, where $r \in]0, +\infty[$ and $B_r = \{ u \in H : \|u\| < r \}$. We denote by $N : B_r \to H$ the derivative of φ and assume that $N(0)=0$.

Finally we consider $l \in \sigma(B, A)$ and $\varepsilon > 0$ such that $[l-\varepsilon, l[\subset \rho(A, B)$. We are interested in the (non-linear) eigenvalue problem:

To find couples $(u, \lambda) \in B_r \times [l-\varepsilon, l[$ verifying

$$Bu - N(u) = \lambda A u, \qquad u \neq 0.$$

For convenience, we state now all the conditions which will be required on φ and N:

(H1) $\varphi(0)=0$, $N(0)=0$ and $\exists\, C, \sigma > 0$ such that

$$\forall u \in B_r: \quad \|N'(u)\| \leqq C \|u\|^{\sigma},$$

(H2) $N(u_n) \rightharpoonup N(u)$ whenever $u_n \rightharpoonup u$ (weakly in H),

(H3) $\forall u \in B_r : \langle N(u), u \rangle \geqq 2\varphi(u)$,

(H4) φ is even and convex, and $\exists \gamma > 1$ such that

$$\forall u \in B_r: \quad \langle N(u), u \rangle \leqq (1+\gamma)\varphi(u),$$

(H5) φ is defined on all H and there exists a sequence $\{v_n\} \subset H$ and a constant $M > 0$ such that:

- $\forall n : \|v_n\|_A = 1$,
- $\forall n : \varphi(v_n) > 0$ and $\{\varphi(v_n)\}$ is bounded,
- $\forall n : \|(B - lA)v_n\|^2 \leqq M |\langle (B - lA)v_n, v_n \rangle|$,
- $\displaystyle \lim_{n \to \infty} \frac{\langle (B - lA)v_n, v_n \rangle}{\varphi(v_n)} = 0.$

Clearly if for $u \in H^1(\mathbb{R}^N)$ we set

$$\varphi(u) = \frac{1}{\sigma+2} \int_{\mathbb{R}^N} r(x) |u(x)|^{\sigma+2} \, dx,$$

by standard arguments as in [19, 20], we see that φ satisfies the conditions (H1) to (H4) with $H = H^1(\mathbb{R}^N)$, $\varphi'(u)v = \langle N(u), v \rangle$ and $\eta = \gamma = \sigma + 1$. The fact that (B4) implies that the condition (H5) is satisfied has been established by Heinz-Kupper-Stuart [9].

This proves that (1.1) is of the form

$$\mathrm{B}\,u-\mathrm{N}(u)=\lambda\,\mathrm{A}\,u.$$

In the rest of this section we will work on this abstract equation.

LEMMA 2.1. – *If condition* (H1) *is verified, then*

$$\forall\, u\in \mathrm{B}_r : \|\mathrm{N}(u)\| \leqq \frac{\mathrm{C}}{1+\sigma}\|u\|^{1+\sigma}$$

and

$$|\varphi(u)| \leqq \frac{\mathrm{C}}{(1+\sigma)(2+\sigma)}\|u\|^{2+\sigma}.$$

Proof. – For $u\in \mathrm{B}_r$ and $v\in \mathrm{H}$,

$$\begin{aligned}|\langle \mathrm{N}(u), v\rangle| &\leqq \left|\int_0^1 \langle \mathrm{N}'(tu)\,u, v\rangle\, dt\right| \\ &\leqq \int_0^1 \mathrm{C}\, t^{\sigma}\|u\|^{1+\sigma}\|v\|\, dt \\ &= \frac{\mathrm{C}}{1+\sigma}\|u\|^{1+\sigma}\|v\|\end{aligned}$$

and

$$\begin{aligned}|\varphi(u)| &\leqq \left|\int_0^1 \langle \mathrm{N}(tu), u\rangle\, dt\right| \\ &\leqq \int_0^1 \frac{\mathrm{C}}{1+\sigma} t^{1+\sigma}\|u\|^{2+\sigma}\, dt \\ &= \frac{\mathrm{C}}{(1+\sigma)(2+\sigma)}\|u\|^{2+\sigma}. \quad \blacksquare\end{aligned}$$

It is well known that if $u_0\in \mathrm{B}_r$ is a critical point of the functional

$$\mathrm{J}(u)=\frac{1}{2}\langle \mathrm{B}\,u, u\rangle-\varphi(u), \qquad u\in \mathrm{B}_r,$$

under the condition $\|u\|_{\mathrm{A}}=c$, where $c>0$ is constant, then there exists a Lagrange multiplier λ_0 such that $\mathrm{B}\,u_0-\mathrm{N}(u_0)=\lambda_0\,\mathrm{A}\,u_0$. Clearly all the critical points are not interesting since we require $\lambda_0\in[l-\varepsilon, l[$. In order to introduce a variational method which allows us to find appropriate critical points we need the following definitions:

DEFINITIONS:

1. For $c>0$, $S(c)=\{u\in H: \|u\|_A=c\}$ is the sphere of radius c (defined with the norm $\|.\|_A$).

2. P is the orthogonal projection on the eigenspace of the operator $B-(l-\varepsilon)A$ associated with the positive part of its spectrum. (Some properties of P are derived in Lemma 4.1.)

3. For $v\in PH$ with $v\neq 0$,

$$M(v)=\left\{u\in H: \|u\|_A=\|v\|_A,\ Pu=\frac{\|Pu\|_A}{\|v\|_A}v \text{ and } \|(I-P)u\|\leq\frac{\|v\|_A}{2\sqrt{\|A\|}}\right\}$$

can be considered as a piece of the "meridian" on $S(\|v\|_A)$ of direction v.

The fact that the norm $\|.\|_A$ and $\|.\|$ are not equivalent on PH presents a difficulty when we try to develop our variational procedure. In order to overcome this difficulty we introduce the open set:

$$V=\{v\in PH: \langle Bv, v\rangle<(l+1)\langle Av, v\rangle\}.$$

As we see in Lemma 4.2 the equivalence of norms is satisfied on V and we will show that our search of critical points can be confined into this set (in a sense which will be precised later).

The idea of the method is to study

$$m(c)\equiv \inf_{v\in V\cap S(c)}\ \sup_{u\in M(v)} J(u), \qquad c>0.$$

The next result says that *sup* can be replaced by *max*. Its proof and the proofs of the main theorems stated below will be given in the last section.

THEOREM 2.1. – *If condition* (H1) *holds, then for all* $v\in V$ *with* $\|v\|_A$ *small enough,*

1. $M(v)\subset B_r$,

2. *there exists an unique* $G(v)\in M(v)$ *satisfying*

$$(J\circ G)(v)=\max\{J(u): u\in M(v)\},$$

3. G *is continuously differentiable*, $PG(v)\neq 0$ *and* $(F\circ G)(v)=0$, *where*

$$F(u)=(I-P)\{Bu-N(u)-q(u)Au\},$$
$$q(u)=\frac{\langle Bu-N(u), Pu\rangle}{\|Pu\|_A^2}$$
$$(u\in B_r \textit{ with } Pu\neq 0).$$

4. *there exists* K *independent of* v *such that* $\|(I-P)G(v)\|\leq K\|v\|^{1+\sigma}$.

Clearly if $(I-P)H=\{0\}$ then $M(v)=\{v\}$ and the theorem is obvious (set $G(v)=v$). This situation happen when l is the infimum of $\sigma(B, A)$. From Theorem 2.1 we deduce that

$$m(c)=\inf\{(J\circ G)(v): v\in V\cap S(c)\}.$$

THEOREM 2.2. – *Under conditions* (H1) *and* (H2) *and for* $c>0$ *small enough, if* $\{v_n\} \subset V \cap S(c)$ *and* $v_c \in PH$ *satisfy*

1. $(J \circ G)(v_n) \to m(c)$,
2. $v_n \rightharpoonup v_c$,

then there exist u_c, λ_c *and a constant* $K>0$ *such that*

1. $B u_c - N(u_c) = \lambda_c A u_c$,
2. $u_c = 0 \Leftrightarrow v_c = 0$,
3. $\|u_c\| \leqq K c$,
4. $|\lambda_c - l| \leqq K c^{\sigma/2}$.

Moreover $\lambda_c \leqq \dfrac{2m(c)}{c^2}$ *if* (H3) *is also satisfied.*

As one clearly sees in the proof given in the last section, this result is valid because without loss of generality each minimising sequence of $(J \circ G)$ on $V \cap S(c)$ stays uniformly bounded away from the frontier of V. This fact, which was first pointed out in [5], occurs because the nonlinearity is of order higher than linear at the origin.

Since $V \cap S(c)$ is bounded, all minimising sequences of $(J \circ G)$ admit a subsequence converging weakly, but the weak limit can be null. We must now find a sequence $\{v_n\} \subset V \cap S(c)$ and $v_c \in PH$ such that $(J \circ G)(v_n) \to m(c)$, $v_n \rightharpoonup v_c$ and $v_c \neq 0$. With this aim, we introduce $\tilde{\varphi}$ and $\tilde{N}$ satisfying (H1) and (H2) with the same constants r, σ, K as φ and N. We define $\tilde{J}$, $\tilde{G}$, $\tilde{m}(c)$ as before.

THEOREM 2.3. – *We suppose that* φ *and* $\tilde{\varphi}$ *verify* (H1) *and* (H2) *and that* $c>0$ *is small enough.*

1. *If* $\varphi - \tilde{\varphi} \geqq 0$, *then* $m(c) \leqq \tilde{m}(c)$.
2. *If* $\{v_n\} \subset V \cap S(c)$ *satisfies* $(J \circ G)(v_n) \to m(c)$, $v_n \rightharpoonup 0$, *and if* $\varphi - \tilde{\varphi}$ *is weakly sequentially continuous at* 0, *then* $\tilde{m}(c) \leqq m(c)$.
3. *If* $\varphi - \tilde{\varphi} \geqq 0$ *and if* $\{v_n\} \subset V \cap S(c)$ *satisfies* $(\tilde{J} \circ \tilde{G})(v_n) \to m(c)$, *then* $(J \circ G)(v_n) \to m(c)$.

COROLLARY 2.1. – *Under the same conditions as the preceding theorem, if* $\varphi - \tilde{\varphi}$ *is non-negative and weakly sequentially continuous at* 0, *and if there exists a sequence* $\{\tilde{v}_n\} \subset V \cap S(c)$ *such that*

- $(\tilde{J} \circ \tilde{G})(\tilde{v}_n) \to \tilde{m}(c)$,
- $\tilde{v}_n \rightharpoonup \tilde{v}_c \neq 0$,

then there exists a sequence $\{v_n\} \subset V \cap S(c)$ *such that*

- $(J \circ G)(v_n) \to m(c)$,
- $v_n \rightharpoonup v_c \neq 0$.

Proof. – By Theorem 2.3, Part 1, $m(c) \leqq \tilde{m}(c)$. First, suppose that $m(c) = \tilde{m}(c)$ and set $v_n = \tilde{v}_n$. Part 3 shows that $\{v_n\}$ has the desired properties. Now suppose that $m(c) < \tilde{m}(c)$. Since $V \cap S(c)$ is bounded, we can always find a minimising sequence of $J \circ G$ in $V \cap S(c)$ converging weakly. By Part 2, the weak limit cannot be null because $m(c) < \tilde{m}(c)$. ■

THEOREM 2.4. – *Under conditions* (H1) *to* (H5), *there exists a sequence* $\{c_n\} \subset]0, \infty[$ *such that*

1. $c_n \to 0$,
2. $\dfrac{2m(c_n)}{c_n^2} < l$.

The proof is given in the last section. We can state now our first bifurcation result:

THEOREM 2.5. – *If conditions* (H1) *to* (H5) *hold and if* φ *is weakly sequentially continuous at* 0, *then* l *is a bifurcation point towards regular values*, i.e. *there exists a sequence* $\{(u_n, \lambda_n)\} \subset H \times [l-\varepsilon, l[$ *such that*

1. $\forall n: B u_n - N(u_n) = \lambda_n A u_n$,
2. $\forall n: u_n \neq 0$,
3. $(u_n, \lambda_n) \to (0, l)$.

Proof. – Since $\varphi(0)=0$ and φ is convex, we see that φ is not negative. By Theorem 2.4, there exists a sequence $\{c_n\} \subset]0, \infty[$ such that

1. $c_n \to 0$,
2. $\dfrac{2m(c_n)}{c_n^2} < l$.

For each n, we can find $\{v_k^n\} \subset V \cap S(c_n)$ satisfying

$$(J \circ G)(v_k^n) \to m(c_n),\ v_k^n \rightharpoonup v_n.$$

Set $\tilde{\varphi} \equiv 0$ and $\tilde{N} \equiv 0$. It is easy to check that $\forall c>0: \tilde{m}(c) = \frac{1}{2} l c^2$. Theorem 2.3 (Part 2) then shows that $v_n \neq 0$. We conclude by Theorem 2.2. ■

In the next section, we shall apply this theory in a case where φ is not weakly sequentially continuous at 0 and therefore Theorem 2.5 is no more applicable. The last theorem of this section emphasizes the importance of the sign of the non-linear term:

THEOREM 2.6. – *If the condition* (H1) *holds and if* φ *is convex, then there exists no sequence* $\{(u_n, \lambda_n)\} \subset H \times [l-\varepsilon, l[$ *such that*

1. $\forall n: B u_n + N(u_n) = \lambda_n A u_n$,
2. $\forall n: u_n \neq 0$,
3. $(u_n, \lambda_n) \to (0, l)$.

3. APPLICATION

We now turn back to equation (1.1) and prove Theorem 1.1. In order to use the general theory obtained in Section 2 we need to check that $\sigma(S)$ is equal to the cojointed spectrum $\sigma(I, A)$. Since this demonstration is rather lenghty we admit the result here and delay the proof (*see* Lemma 4.5 in the last section).

Theorem 1.1 in the case $r^P \equiv 0$ a.e. is a direct consequence of Theorem 2.5, because φ is then weakly sequentially continuous (*see* [19-22] for such results). Now suppose that $r(x) \geqq r^P(x) > 0$ a.e. on $\mathbb{R}^N$. We set

$$\tilde{\varphi}(u) = \frac{1}{\sigma+2} \int_{\mathbb{R}^N} r^P(x) \, | u(x) |^{\sigma+2} \, dx.$$

Since (B1) to (B4) are still satisfied when $r(x)$ is replaced by $r^P(x)$, $\tilde{\varphi}$ satisfies conditions (H1) to (H5). Moreover $\varphi \geqq \tilde{\varphi}$ because $r(x) \geqq r^P(x)$ and $\varphi - \tilde{\varphi}$ is weakly sequentially continuous since $\lim_{|x| \to \infty} \{ r(x) - r^P(x) \} = 0$. Applying Theorem 2.4 to $\tilde{\varphi}$ and then Theorem 2.3 (Part 1), we deduce that there exists a sequence $\{ c_n \} \subset]0, \infty[$ such that

1. $c_n \to 0$,
2. $\dfrac{2\, m(c_n)}{c_n^2} \leqq \dfrac{2\, \tilde{m}(c_n)}{c_n^2} < l.$

Now assume that the following result holds:

LEMMA 3.1. – *There exists a sequence* $\{ \tilde{v}_k^n \} \subset \mathrm{V} \cap \mathrm{S}(c_n)$ *such that*
- $(\tilde{\mathrm{J}} \circ \tilde{\mathrm{G}})(\tilde{v}_k^n) \to \tilde{m}(c_n)$,
- $\tilde{v}_k^n \rightharpoonup \tilde{v}^n \neq 0$.

We deduce by Corollary 2.1 that there exists a sequence $\{ v_k^n \} \subset \mathrm{V} \cap \mathrm{S}(c_n)$ such that
- $(\mathrm{J} \circ \mathrm{G})(v_k^n) \to m(c_n)$,
- $v_k^n \rightharpoonup v^n \neq 0$.

Theorem 1.1 is now a direct consequence of Theorem 2.2. It remains to prove the lemma. Let $\{ \tilde{v}_k^n \} \subset \mathrm{V} \cap \mathrm{S}(c_n)$ satisfies

$$(\tilde{\mathrm{J}} \circ \tilde{\mathrm{G}})(\tilde{v}_k^n) \to \tilde{m}(c_n).$$

Since $\mathrm{V} \cap \mathrm{S}(c_n)$ is bounded, we can assume that $\tilde{v}_k^n \rightharpoonup \tilde{v}^n$ (passing to a subsequence). We shall prove that the minimising sequence can be chosen such that $\tilde{v}^n \neq 0$. For convenience, we shall from now on omit the indice n.

Clearly there exists a subsequence (still denoted by $\{ \tilde{v}_k \}$) satisfying one of the two following possibilities

1. (vanishing)

$$\lim_{k \to +\infty} \sup_{y \in \mathbb{R}^N} \int_{y + \mathrm{B_R}} \tilde{v}_k^2(x)\, dx = 0, \qquad \forall \mathrm{R} < +\infty,$$

2. (non-vanishing)

$$\exists \alpha > 0, \qquad \mathrm{R} < +\infty \qquad \text{and} \qquad \{ y_k \} \subset \mathbb{R}^N$$

such that

$$\liminf_{k\to+\infty}\int_{y_k+B_R}\tilde{v}_k^2(x)\,dx\geqq\alpha>0.$$

We shall prove that only the case of non-vanishing can occur.

Since

$$\lim_{k\to+\infty}(\tilde{J}\circ\tilde{G})(\tilde{v}_k)=\tilde{m}(c)<\frac{1}{2}lc^2$$

and $(\tilde{J}\circ\tilde{G})(\tilde{v}_k)\geqq\tilde{J}(\tilde{v}_k)$, we can deduce

$$\limsup_{k\to+\infty}\tilde{J}(\tilde{v}_k)<\frac{1}{2}lc^2\quad(*).$$

This implies $\exists\beta>0$ such that

$$\liminf_{k\to+\infty}\tilde{\varphi}(\tilde{v}_k)\geqq\beta>0.$$

Indeed since $\{\tilde{v}_k\}\subset V\cap S(c)$, we get $\frac{1}{2}\langle B\tilde{v}_k,\tilde{v}_k\rangle\geqq\frac{1}{2}lc^2$. The result follows from (*) and the definition of $\tilde{J}$.

On the other hand:

$$\begin{aligned}\tilde{\varphi}(\tilde{v}_k)&=\frac{1}{\sigma+2}\int_{\mathbb{R}^N}r^P(x)\,|\tilde{v}_k(x)|^{\sigma+2}\,dx\\&\leqq\frac{|r^P|_\infty}{\sigma+2}\int_{\mathbb{R}^N}|\tilde{v}_k(x)|^{\sigma+2}\,dx\\&\leqq K\,\|\tilde{v}_k\|_{L^{\sigma+2}}^{\sigma+2}.\end{aligned}$$

Using Lemma I.1 of [15] and setting in this lemma $p=2$ and $q=2$, we see that in the case of a vanishing sequence

$$\limsup_{n\to+\infty}\tilde{\varphi}(\tilde{v}_k)=0.$$

This is a contradiction. So only case (2) can occur.

Since we are considering periodic nonlinearities, we can take advantage of the invariance under translation to prove the existence of a minimising sequence $\{\tilde{v}_k\}$ for which the sequence $\{y_k\}\subset\mathbb{R}^N$ appearing in case (2) stays bounded.

The basic idea is to consider an arbitrary minimising sequence $\{\tilde{v}_k\}$ and to pull it back if the sequence $\{y_k\}$ goes to infinity. A rigourous proof of the fact that the sequence so obtained stays a minimising sequence of $(\tilde{J}\circ\tilde{G})$ on $V\cap S(c)$ can be found in [5]. We do not repeat it here.

Now let Ω be an arbitrary bounded domain in $\mathbb{R}^N$ such that

$$\{y_k\}+B_R\subset\Omega.$$

Since $\tilde{v}_k \rightharpoonup \tilde{v}$ weakly in $H^1(\mathbb{R}^N)$, by the compactness of the Sobolev embeddings $H^1(\Omega) \subset L^2(\Omega)$ on bounded domains we get:

$$\|\tilde{v}\|_{L^2}^2 \geqq \int_\Omega \tilde{v}^2(x)\,dx \geqq \liminf_{n\to+\infty} \int_\Omega \tilde{v}_k^2(x)\,dx \geqq \alpha > 0.$$

We conclude that $\tilde{v} \neq 0$ and this ends the proof of Theorem 1.1.

Remarks:

1. As announced in the introduction, our general theory can be used to improve the previous results concerning the bifurcation properties of the equation

$$-\Delta u(x) - r(x)\,|u(x)|^\sigma u(x) = \lambda u(x), \qquad x \in \mathbb{R}^N. \tag{1.2}$$

For the sake of simplicity, we directly work with a non-compact nonlinearity and assume that the following conditions hold:

(*a*) $0 < \sigma < \dfrac{4}{N}$ for $N \geqq 1$,

(*b*) $\lim_{|x|\to+\infty} r(x) = r > 0$,

(*c*) $\exists x_0 \in \mathbb{R}^N$ such that $\forall c \in \mathbb{R}$

$$\int_{|x-x_0|=c} r(x)\,dx \geqq \int_{|x-x_0|=c} r\,dx.$$

This equation has been studied by Stuart [22] and Zhu-Zhou [23] under analogous conditions. Clearly (1.2) is obtained from (1.1) when we set $p(x) \equiv 0$. The spectrum associated with this equation is purely continuous, $\sigma(S) = [0, +\infty[$ and consequently 0 is the only point were bifurcation towards regular values can occur. Since there is no spectrum on the left of 0, $G(v) = v$. Setting

$$\tilde{\varphi}(u) = \frac{1}{\sigma+2} \int_{\mathbb{R}^N} r\,|u(x)|^{\sigma+2}\,dx.$$

we then see clearly that we can replace in Corollary 2.1 the condition $(\varphi - \tilde{\varphi}) \geqq 0$ by the weaker requirement $\forall k : (\varphi - \tilde{\varphi})(\tilde{v}_k) \geqq 0$. [$\tilde{v}_k$ denote a minimising sequence of $(\tilde{J} \circ \tilde{G}) = \tilde{J}$ on $S(c) \cap V$.]

Now if $\{\tilde{v}_k\}$ is a minimising sequence of $\tilde{J}$ on $S(c) \cap V$, the sequence $\{\tilde{v}_k^*\}$ of Schartz-symmetrized (about x_0) functions of $\tilde{v}_k$ is also a minimising sequence of $\tilde{J}$ on $S(c) \cap V$. We recall that these functions are decreasing spherically symmetric. Since $\tilde{m}(c) < 0$, (*see* [19, 22] for such results), the vanishing does not occur and automatically the minimising sequence $\{\tilde{v}_k^*\}$ has a non-null weak limit.

The third condition on $r(x)$ implies that $\forall k : (\varphi - \tilde{\varphi})(v_k^*) \geqq 0$. Therefore we can apply the modified version of Corollary 2.1 and conclude as precedingly.

2. In the examples we have presented, the nonlinearity is homogeneous. It should nevertheless be made clear that such assumption is nowhere needed in the abstract part (Section 2) or when working directly on the equation (Section 3). In particular we can also treated equations where $r(x)\,|u(x)|^{\sigma}u(x)$ is replaced by a finite sum

$$\sum_{i=1}^{n} r_i(x)\,|u(x)|^{\sigma_i}u(x)$$

with $r_i(x)$ and σ_i satisfying the hypotheses (B2) and (B3).

4. PROOFS OF THE MAIN THEOREMS

In this last section we give the proofs of the theorems stated in Section 2. From now on, the letter K will denote various constants whose exact value may change from line to line, but are not essential to the analysis of the problem.

LEMMA 4.1. – *Set* $\delta=\|\{B-(l-\varepsilon)A\}^{-1}\|^{-1}$. *We have*
1. $PB=BP$ *and* $PA=AP$,
2. $PH\neq\{0\}$,

$$l=\inf\left\{\frac{\langle Bu,u\rangle}{\langle Au,u\rangle}: u\in PH,\ u\neq 0\right\}$$

and $\forall u\in(I-P)H:\langle Bu,u\rangle\leqq(l-\varepsilon)\langle Au,u\rangle$.

3. $\forall\lambda\in[l-\varepsilon, l[, \forall u\in PH:\langle(B-\lambda A)u,u\rangle\geqq\frac{\delta}{\varepsilon}(l-\lambda)\|u\|^2$ *and*

$$\|(B-\lambda A)u\|\geqq\frac{\delta}{\varepsilon}(l-\lambda)\|u\|,$$

4. $\forall\lambda\geqq l-\varepsilon, \forall u\in(I-P)H:\langle(\lambda A-B)u,u\rangle\geqq\delta\|u\|^2$ *and*

$$\|(B-\lambda A)u\|\geqq\delta\|u\|,$$

5. $\forall\lambda\in[l-\varepsilon, l[,\forall u\in H:\|(B-\lambda A)u\|\geqq\frac{\delta}{\varepsilon}(l-\lambda)\|u\|.$

Proof. – Since A and B commute with $B-(l-\varepsilon)A$, we deduce that $PA=AP$ and $PB=BP$. Moreover

$$\forall u\in PH:\langle\{B-(l-\varepsilon)A\}u,u\rangle\geqq\delta\|u\|^2,$$
$$\forall u\in(I-P)H:\langle\{(l-\varepsilon)A-B\}u,u\rangle\geqq\delta\|u\|^2.$$

If $PH=\{0\}$, then

$$\forall u\in H:\langle(lA-B)u,u\rangle\geqq\langle\{(l-\varepsilon)A-B\}u,u\rangle\geqq\delta\|u\|^2$$

and so $B-lA$ has a bounded inverse and $l\in\rho(B, A)$. Since we suppose $l\in\sigma(B, A)$, we deduce $PH\neq\{0\}$. Set

$$\bar{l}=\inf\left\{\frac{\langle Bu, u\rangle}{\langle Au, u\rangle}: u\in PH,\ u\neq 0\right\}.$$

Clearly $\bar{l}\geqq l-\varepsilon$. Since $B-\bar{l}A$ and P commute,

$$\sigma(B-\bar{l}A)=\sigma(B-\bar{l}A|_{PH})\cup\sigma(B-\bar{l}A|_{(I-P)H}).$$

Since $0\in\sigma(B-\bar{l}A|_{PH})$, we see that $\bar{l}\in\sigma(B, A)$. For $\lambda\in[l-\varepsilon, \bar{l}[$ and $u\in PH$:

$$\begin{aligned}\langle(B-\lambda A)u, u\rangle&=\langle\{B-(l-\varepsilon)A\}u, u\rangle-\{\lambda-(l-\varepsilon)\}\langle Au, u\rangle\\ &\geqq\langle\{B-(l-\varepsilon)A\}u, u\rangle-\frac{\lambda-(l-\varepsilon)}{\bar{l}-(l-\varepsilon)}\langle\{B-(l-\varepsilon)A\}u, u\rangle\\ &=\frac{\bar{l}-\lambda}{\bar{l}-(l-\varepsilon)}\langle\{B-(l-\varepsilon)A\}u, u\rangle\\ &\geqq\delta\frac{\bar{l}-\lambda}{\bar{l}-(l-\varepsilon)}\|u\|^2\end{aligned}$$

and

$$\|(B-\lambda A)u\|\geqq\delta\frac{\bar{l}-\lambda}{\bar{l}-(l-\varepsilon)}\|u\|.$$

For $\lambda\geqq l-\varepsilon$ and $u\in(I-P)H$:

$$\langle(\lambda A-B)u, u\rangle\geqq\langle\{(l-\varepsilon)A-B\}u, u\rangle\geqq\delta\|u\|^2$$

and $\|(\lambda A-B)u\|\geqq\delta\|u\|$. Finally, for $\lambda\in[l-\varepsilon, \bar{l}[$ and $u\in H$:

$$\begin{aligned}\|(B-\lambda A)u\|^2&=\|(B-\lambda A)Pu\|^2+\|(B-\lambda A)(I-P)u\|^2\\ &\geqq\left\{\delta\frac{\bar{l}-\lambda}{\bar{l}-(l-\varepsilon)}\right\}^2(\|Pu\|^2+\|(I-P)u\|^2)\\ &=\left\{\delta\frac{\bar{l}-\lambda}{\bar{l}-(l-\varepsilon)}\right\}^2\|u\|^2\end{aligned}$$

and

$$\|(B-\lambda A)u\|\geqq\delta\frac{\bar{l}-\lambda}{\bar{l}-(l-\varepsilon)}\|u\|.$$

This shows that $[l-\varepsilon, \bar{l}[\subset\rho(B, A)$ and then $\bar{l}=l$. ■

LEMMA 4.2. — V *is a non empty open set and*

$$\forall v\in V: \|v\|^2<\frac{1+\varepsilon}{\delta}\langle Av, v\rangle.$$

Proof. — The relation $\langle Bv, v\rangle<(l+1)\langle Av, v\rangle$ implies

$$\delta\|v\|^2\leqq\langle\{B-(l-\varepsilon)A\}v, v\rangle<(1+\varepsilon)\langle Av, v\rangle.\quad ■$$

Proof of Theorem 2.1. — For $u \in \mathrm{M}(v)$,

$$\|u\| \leqq \|v\| + \|(\mathrm{I}-\mathrm{P})u\| \leqq \sqrt{\frac{1+\varepsilon}{\delta}}\|v\|_{\mathrm{A}} + \frac{\|v\|_{\mathrm{A}}}{2\sqrt{\|\mathrm{A}\|}} < r$$

if $\|v\|_{\mathrm{A}}$ is small enough and therefore $\mathrm{M}(v) \subset \mathrm{B}_r$. We define the function h by

$$h: \begin{cases} \mathrm{D}(h) = \left\{ (v, w) \in \mathrm{V} \times (\mathrm{I}-\mathrm{P})\mathrm{H} : \|w\| \leqq \dfrac{\|v\|_{\mathrm{A}}}{2\sqrt{\|\mathrm{A}\|}} \right\} \to \mathrm{H}, \\ h(v, w) = \sqrt{\|v\|_{\mathrm{A}}^2 - \|w\|_{\mathrm{A}}^2}\, \dfrac{v}{\|v\|_{\mathrm{A}}} + w. \end{cases}$$

Note that $\|w\|_{\mathrm{A}} \leqq \sqrt{\|\mathrm{A}\|}\, \|w\| \leqq \frac{1}{2}\|v\|_{\mathrm{A}}$ if $(v, w) \in \mathrm{D}(h)$ and therefore h is well defined and admits a twice continuously differentiable extension defined on an open set containing $\mathrm{D}(h)$. Also $\|h(v, w)\|_{\mathrm{A}} = \|v\|_{\mathrm{A}}$ and for all fixed $v \in \mathrm{V}$, $\mathrm{R}(h(v, .)) = \mathrm{M}(v)$. Let us show that $(\mathrm{J} \circ h)(v, .)$ is strictly concave in w and admits an unique maximum as a function of w. For $z \in (\mathrm{I}-\mathrm{P})\mathrm{H}$,

$$(\partial/\partial w)\, h(v, w)\, z = -\frac{\langle \mathrm{A} w, z \rangle}{\|v\|_{\mathrm{A}} \sqrt{\|v\|_{\mathrm{A}}^2 - \|w\|_{\mathrm{A}}^2}}\, v + z,$$

hence

$$\begin{aligned} &\langle (\partial/\partial w)(\mathrm{J} \circ h)(v, w), z \rangle \\ &\quad = \langle \mathrm{B} h(v, w) - \mathrm{N}(h(v, w)), (\partial/\partial w)\, h(v, w)\, z \rangle \\ &= \langle \mathrm{B} h(v, w) - \mathrm{N}(h(v, w)), z \rangle \\ &\quad - \frac{\langle \mathrm{A} w, z \rangle}{\|v\|_{\mathrm{A}} \sqrt{\|v\|_{\mathrm{A}}^2 - \|w\|_{\mathrm{A}}^2}} \langle \mathrm{B} h(v, w) - \mathrm{N}(h(v, w)), v \rangle \\ &= \langle (\mathrm{F} \circ h)(v, w), z \rangle. \end{aligned}$$

Now using the definition of F and the orthogonality between PH and $(\mathrm{I}-\mathrm{P})\mathrm{H}$ we get

$$\begin{aligned} \langle (\partial/\partial w)(\mathrm{F} \circ h)(v, w)\, z, z \rangle &= \langle \mathrm{B} z - q(h(v, w))\, \mathrm{A} z, z \rangle \\ &- \langle (\partial/\partial w)(\mathrm{N} \circ h)(v, w)\, z, z \rangle - \langle (\partial/\partial w)(q \circ h)(v, w), z \rangle \langle \mathrm{A} w, z \rangle. \end{aligned}$$

Since

$$q(h(v, w)) = \frac{\langle \mathrm{B} v, v \rangle}{\|v\|_{\mathrm{A}}^2} - \frac{\langle \mathrm{N}(h(v, w)), v \rangle}{\|v\|_{\mathrm{A}} \|\mathrm{P} h(v, w)\|_{\mathrm{A}}},$$

$$(\partial/\partial w)\, \|(\mathrm{P} \circ h)(v, w)\|_{\mathrm{A}}\, z = -\frac{\langle \mathrm{A} w, z \rangle}{\|\mathrm{P} h(v, w)\|_{\mathrm{A}}},$$

we have

$$\langle (\partial/\partial w)(q \circ h)(v, w) z \rangle = - \frac{\langle N'(h(v, w))(\partial/\partial w) h(v, w) z, v \rangle}{\|v\|_A \|P h(v, w)\|_A} - \frac{\langle N(h(v, w)), v \rangle \langle A w, z \rangle}{\|v\|_A \|P h(v, w)\|_A}.$$

The following inequalities are easily obtained using the definition of h and the equivalence of norms on V:

$$\|h(v, w)\| \leqq K \|v\|_A,$$
$$\|(\partial/\partial w) h(v, w) z\| \leqq K \|z\|,$$
$$\|P h(v, w)\|_A \geqq K \|v\|_A.$$

We then obtain for $\|v\|_A$ small enough

$$q(h(v, w)) \geqq l - K \|v\|_A^\sigma \geqq l - \varepsilon,$$
$$\langle B z - q(h(v, w)) A z, z \rangle \leqq -\delta \|z\|^2,$$
$$\langle (\partial/\partial w)(N \circ h)(v, w) z, z \rangle \leqq K \|v\|_A^\sigma \|z\|^2 \leqq \frac{\delta}{4} \|z\|^2$$

and

$$\langle (\partial/\partial w)(q \circ h)(v, w) z \rangle \langle A w, z \rangle \leqq K \|v\|_A^\sigma \|z\|^2 \leqq \frac{\delta}{4} \|z\|^2.$$

Therefore

$$\langle (\partial/\partial w)(F \circ h)(v, w) z, z \rangle \leqq -\frac{\delta}{2} \|z\|^2$$

and so $(J \circ h)(v, w)$ is strictly concave in w and

$$(\partial/\partial w) F(h(v, w)) : \quad (I-P) H \to (I-P) H$$

has a bounded inverse. Now if we assume $v \in V$, $\|w\| = \dfrac{\|v\|_A}{2\sqrt{\|A\|}}$, we have

$$\begin{aligned}
&(J \circ h)(v, 0) - (J \circ h)(v, w) \\
&\geqq \frac{1}{2} \langle B v, v \rangle - \frac{\|v\|_A^2 - \|w\|_A^2}{2\|v\|_A^2} \langle B v, v \rangle \\
&\quad - \frac{1}{2} \langle B w, w \rangle - K \|v\|_A^{2+\sigma} \\
&= \frac{\|w\|_A^2}{2\|v\|_A^2} \langle B v, v \rangle - \frac{1}{2} \langle B w, w \rangle - K \|v\|_A^{2+\sigma} \\
&\geqq \frac{1}{2} (l \langle A w, w \rangle - \langle B w, w \rangle) - K \|v\|_A^{2+\sigma}
\end{aligned}$$

$$\geqq \frac{\delta}{2}\|w\|^2 - K\|v\|_A^{2+\sigma}$$
$$\geqq K\|v\|_A^2.$$

As a consequence, there exists an unique $u(v) \in M(v)$ which maximises J on $M(v)$. Moreover $u(v)$ can be written in the form $h(v, w(v))$ and $(F \circ h)(v, w(v)) = 0$. By the implicit function theorem, $w(v)$ is continuously differentiable and to obtain Parts 2 and 3 of Theorem 2.1 it suffices to set $G(v) = h(v, w(v))$.

Finally, using $(F \circ h)(v, w(v)) = 0$, we get

$$\{B - q(G(v))A\} w(v) = (I - P) N(G(v))$$

and

$$\|w(v)\| \leqq \frac{1}{\delta}\|\{B - q(G(v))A\} w(v)\|$$
$$\leqq \frac{1}{\delta}\|N(G(v))\| \leqq K\|G(v)\|^{1+\sigma}$$
$$\leqq K\{\|v\| + \|w(v)\|\}^{1+\sigma}.$$

Hence

$$\|w(v)\|^{1/(1+\sigma)} \leqq K^{1/(1+\sigma)}\|v\| + \frac{1}{2}\|w(v)\|^{1/(1+\sigma)}$$

if $\|v\|_A$ is small, and

$$\|w(v)\| \leqq K\|v\|_A^{1+\sigma}. \quad \blacksquare$$

LEMMA 4.3. – *Under conditions* (H1) *and* (H2) *and for* $c > 0$ *small enough, if* $\{v_n\} \subset V$ *verifies* $\forall n : \|v_n\|_A \leqq c$ *and* $v_n \rightharpoonup 0$, *then there exists a subsequence such that* $G(v_{n_i}) \rightharpoonup 0$.

Proof. – There exists a subsequence v_{n_i} such that

$$(I - P) G(v_{n_i}) \rightharpoonup w$$

and

$$q(G(v_{n_i})) \to \lambda \geqq l - \varepsilon.$$

Passing to the limit in $(F \circ G)(v_{n_i}) = 0$, we obtain

$$(B - \lambda A) w = (I - P) N(w).$$

Consequently

$$\|w\| \leqq \frac{1}{\delta}\|(B - \lambda A) w\|$$
$$= \frac{1}{\delta}\|(I - P) N(w)\| \leqq K\|w\|^{1+\sigma}.$$

Since

$$\| w \| \leqq \liminf \| (I-P)\, G(v_{n_i}) \| \leqq K \| v \|_A \leqq K c,$$

we deduce that $\| w \| = 0$ (for c small enough). ■

PROOF OF THEOREM 2.2. – For $0 < c \leqq 1$ and $v, u \in V \cap S(c)$ such that

$$\langle B v, v \rangle \geqq \left(l + \frac{1}{2} c^{\sigma/2} \right) c^2$$

and

$$\langle B u, u \rangle \leqq \left(l + \frac{1}{4} c^{\sigma/2} \right) c^2,$$

we have

$$\begin{aligned}
&(J \circ G)(v) - (J \circ G)(u) \\
&\quad = \frac{1}{2} \langle BG(v), G(v) \rangle - \frac{1}{2} \langle BG(u), G(u) \rangle - \varphi(G(v)) + \varphi(G(u)) \\
&\quad \geqq \frac{1}{2} (1 - K c^{2\sigma}) \langle Bv, v \rangle - \frac{1}{2} \langle Bu, u \rangle - K c^{2+\sigma} \\
&\quad \geqq \frac{1}{2} (1 - K c^{2\sigma}) \left(l + \frac{1}{2} c^{\sigma/2} \right) c^2 - \frac{1}{2} \left(l + \frac{1}{4} c^{\sigma/2} \right) c^2 - K c^{2+\sigma} \\
&\quad > 0
\end{aligned}$$

if c is small enough. Consequently taking a subsequence, we can suppose that for all n:

$$\langle B v_n, v_n \rangle \leqq \left(1 + \frac{1}{2} c^{\sigma/2} \right) c^2$$

and

$$(J \circ G)(v_n) \leqq m(c) + \frac{1}{n^2}.$$

We apply the Ekeland's ε-variational principle [2] on the closed metric space

$$E = \{ v \in PH : \| v \|_A = c \text{ and } \langle B v, v \rangle \leqq (l + c^{\sigma/2}) c^2 \} \subset V \cap S(c).$$

For all $n \geqq 1$, there exists $v_n^* \in E$ with

1. $(J \circ G)(v_n^*) \leqq (J \circ G)(v_n)$,
2. $\| v_n - v_n^* \| \leqq \dfrac{1}{n}$,
3. $\forall v \in E : (J \circ G)(v_n^*) \leqq (J \circ G)(v) + \dfrac{1}{n} \| v - v_n^* \|$.

Clearly $(J\circ G)(v_n^*)\to m(c)$ and $v_n^*\rightharpoonup v_c$. Taking a subsequence we also have

$$\forall n:\langle B v_n^*, v_n^*\rangle\leqq\left(l+\frac{3}{4}c^{\sigma/2}\right)c^2.$$

For $v\in PH$, $v\neq 0$, we set

$$T_v=\{z\in PH:\langle A v, z\rangle=0\}.$$

For $z\in T_{v_n^*}$ with $z\neq 0$, we define

$$h:\left\{\begin{array}{l}\left]-\frac{c}{\|z\|_A},\frac{c}{\|z\|_A}\right[\to S(c)\cap PH,\\ h(t)=\sqrt{1-t^2\frac{\|z\|_A^2}{c^2}}\,v_n^*+tz.\end{array}\right.$$

This function satisfies $h(0)=v_n^*$, $h'(0)=z$ and

$$(J\circ G)(h(0))\leqq(J\circ G)(h(t))+\frac{1}{n}\|h(t)-h(0)\|$$

if $|t|$ is small enough. Hence

$$\begin{aligned}\langle (J\circ G)'(v_n^*), z\rangle&=\langle (J\circ G)'(h(0)), h'(0)\rangle\\ &=\lim_{t\to 0}\frac{(J\circ G)(h(t))-(J\circ G)(h(0))}{t}\\ &\geqq-\frac{1}{n}\|h'(0)\|\\ &=-\frac{1}{n}\|z\|.\end{aligned}$$

Replacing z by $-z$, we obtain

$$|\langle (J\circ G)'(v_n^*), z\rangle|\leqq\frac{1}{n}\|z\|,\qquad\forall z\in T_{v_n^*}.$$

For $v\in E$ and $z\in T_v$, let us develop $\langle (J\circ G)'(v), z\rangle$.

Since $\langle AG(v), G(v)\rangle=\langle A v, v\rangle$ we have

$$\langle AG(v), G'(v)z\rangle=\langle A v, z\rangle=0.$$

We deduce

$$\begin{aligned}&\langle (J\circ G)'(v), z\rangle\\ &\quad=\langle BG(v)-N(G(v))-q(G(v))AG(v), G'(v)z\rangle\\ &\quad=\langle BG(v)-N(G(v))-q(G(v))AG(v), PG'(v)z\rangle\end{aligned}$$

[using the fact that $F(G(v))=0$].

The relations

$$\begin{aligned} PG'(v)z &= \frac{d}{dv}\{PG(v)\}z \\ &= \frac{1}{c}\frac{d}{dv}(\|PG(v)\|_A v)z \\ &= \frac{1}{c}\|PG(v)\|_A z + \frac{\langle APG(v), PG'(v)z\rangle}{c\|PG(v)\|_A} v, \end{aligned}$$

and

$$\langle BG(v) - N(G(v)) - q(G(v))AG(v), v\rangle = 0$$

be definition of q, imply that

$$\langle (J \circ G)'(v), z\rangle = \frac{\|PG(v)\|_A}{c}\langle BG(v) - N(G(v)) - q(G(v))AG(v), z\rangle, \quad \forall z \in T_v.$$

For $z \in PH$, we have

$$\begin{aligned} &|\langle BG(v_n^*) - N(G(v_n^*)) - q(G(v_n^*))AG(v_n^*), z\rangle| \\ &= \left|\left\langle BG(v_n^*) - N(G(v_n^*)) - q(G(v_n^*))AG(v_n^*), z - \frac{\langle A v_n^*, z\rangle}{c^2} v_n^* \right\rangle\right| \\ &= \frac{c}{\|PG(v_n^*)\|_A}\left|\left\langle (J \circ G)'(v_n^*), z - \frac{\langle A v_n^*, z\rangle}{c^2} v_n^* \right\rangle\right| \\ &\leqq \frac{c}{n\|PG(v_n^*)\|_A}\left\| z - \frac{\langle A v_n^*, z\rangle}{c^2} v_n^* \right\| \\ &\leqq \frac{c}{n\|PG(v_n^*)\|_A}\left(1 + \frac{\|A\|\,\|v_n^*\|^2}{c^2}\right)\|z\| \\ &\leqq \frac{K}{n}\|z\|. \end{aligned}$$

Therefore

$$P\{BG(v_n^*) - N(G(v_n^*)) - q(G(v_n^*))AG(v_n^*)\} \to 0$$

and

$$BG(v_n^*) - N(G(v_n^*)) - q(G(v_n^*))AG(v_n^*) \to 0$$

since $F(G(v_n^*)) = 0$. Taking a subsequence, we can suppose

$$G(v_n^*) \rightharpoonup u_c \quad \text{and} \quad q(G(v_n^*)) \to \lambda_c$$

and we get

$$\mathrm{B}\,u_c-\mathrm{N}(u_c)=\lambda_c\,\mathrm{A}\,u_c,$$

$$\begin{aligned}\|u_c\| &\leqq \liminf_{n\to\infty}\|\mathrm{G}(v_n^*)\|\\ &\leqq \liminf_{n\to\infty}(\|\mathrm{PG}(v_n^*)\|+\|(\mathrm{I}-\mathrm{P})\,\mathrm{G}(v_n^*)\|)\\ &\leqq \mathrm{K}\liminf_{n\to\infty}\|v_n^*\|\leqq \mathrm{K}\,c,\end{aligned}$$

$$\begin{aligned}\lambda_c &= \lim_{n\to\infty} q(\mathrm{G}(v_n^*))\\ &= \lim_{n\to\infty}\frac{\langle \mathrm{BG}(v_n^*)-\mathrm{N}(\mathrm{G}(v_n^*)),\,\mathrm{PG}(v_n^*)\rangle}{\|\mathrm{PG}(v_n^*)\|_{\mathrm{A}}^2}\\ &= \lim_{n\to\infty}\left\{\frac{1}{c^2}\langle \mathrm{B}\,v_n^*,\,v_n^*\rangle-\frac{\langle \mathrm{N}(\mathrm{G}(v_n^*)),\,\mathrm{PG}(v_n^*)\rangle}{\|\mathrm{PG}(v_n^*)\|_{\mathrm{A}}^2}\right\}.\end{aligned}$$

Since

$$\forall n:\quad l\leqq\frac{\langle \mathrm{B}\,v_n^*,\,v_n^*\rangle}{c^2}\leqq l+\frac{1}{4}c^{\sigma/2},$$

we obtain after calculation $|\lambda_c-l|\leqq \mathrm{K}\,c^{\sigma/2}$. Finally, if condition (H3) holds,

$$\begin{aligned}\lambda_c &= \lim_{n\to\infty} q(\mathrm{G}(v_n^*))\\ &= \lim_{n\to\infty}\frac{\langle \mathrm{BG}(v_n^*)-\mathrm{N}(\mathrm{G}(v_n^*)),\,\mathrm{G}(v_n^*)\rangle}{\|\mathrm{G}(v_n^*)\|_{\mathrm{A}}^2}\\ &= \frac{1}{c^2}\lim_{n\to\infty}\{2\,\mathrm{J}(\mathrm{G}(v_n^*))+2\,\varphi(\mathrm{G}(v_n^*))-\langle \mathrm{N}(\mathrm{G}(v_n^*)),\,\mathrm{G}(v_n^*)\rangle\}\\ &\leqq \frac{1}{c^2}\lim_{n\to\infty} 2\,\mathrm{J}(\mathrm{G}(v_n^*))\\ &= \frac{2\,m(c)}{c^2}.\end{aligned}$$

The second equality is obtained by combining the expression of q coming from the definition with the one derived from $(\mathrm{F}\circ\mathrm{G})(v_n^*)=0$. ■

Proof of Theorem 2.3:

1. Consider $\{v_n\}\subset \mathrm{V}\cap \mathrm{S}(c)$ with $(\tilde{\mathrm{J}}\circ\tilde{\mathrm{G}})(v_n)\to\tilde{m}(c)$. Since

$$\begin{aligned}(\tilde{\mathrm{J}}\circ\tilde{\mathrm{G}})(v_n) &\geqq (\tilde{\mathrm{J}}\circ\mathrm{G})(v_n)\\ &= (\mathrm{J}\circ\mathrm{G})(v_n)+(\varphi-\tilde{\varphi})(\mathrm{G}(v_n))\\ &\geqq (\mathrm{J}\circ\mathrm{G})(v_n),\end{aligned}$$

we have

$$m(c)\leqq\liminf_{n\to\infty}(\mathrm{J}\circ\mathrm{G})(v_n)\leqq\liminf_{n\to\infty}(\tilde{\mathrm{J}}\circ\tilde{\mathrm{G}})(v_n)=\tilde{m}(c).$$

2. Passing to a subsequence, we can suppose $\tilde{G}(v_n) \rightharpoonup 0$. The relation

$$(J \circ G)(v_n) \geqq (J \circ \tilde{G})(v_n) = (\tilde{J} \circ \tilde{G})(v_n) - (\varphi - \tilde{\varphi})(\tilde{G}(v_n))$$

implies

$$\tilde{m}(c) \leqq \liminf_{n \to \infty} (\tilde{J} \circ \tilde{G})(v_n) \leqq \liminf_{n \to \infty} (J \circ G)(v_n) = m(c).$$

3. Since

$$\begin{aligned}(\tilde{J} \circ \tilde{G})(v_n) &\geqq (\tilde{J} \circ G)(v_n) \\ &= (J \circ G)(v_n) + (\varphi - \tilde{\varphi})(G(v_n)) \\ &\geqq (J \circ G)(v_n),\end{aligned}$$

we get

$$\begin{aligned}m(c) &\leqq \liminf_{n \to \infty} (J \circ G)(v_n) \leqq \limsup_{n \to \infty} (J \circ G)(v_n) \\ &\leqq \lim_{n \to \infty} (\tilde{J} \circ \tilde{G})(v_n) \\ &= m(c). \quad \blacksquare\end{aligned}$$

In order to prove Theorem 2.4, we need the following lemma:

LEMMA 4.4. – *Under conditions* (H1) *to* (H4), *if there exists a sequence* $\{z_n\} \subset V$ *such that*

- $\|z_n\|_A = 1$,
- $\varphi(z_n) > 0$,
- $\lim_{n \to \infty} \dfrac{\langle (B - lA) z_n, z_n \rangle}{\varphi(z_n)} = 0.$

then there exists a sequence $\{c_n\} \subset]0, \infty[$ *such that*

1. $c_n \to 0$,
2. $\dfrac{2m(c_n)}{c_n^2} < l.$

Proof. – Let us show that we can find a sequence $\{t_n\} \subset \mathbb{R}^+$ such that

1. $t_n \to 0$ as $n \to +\infty$,
2. $2(J \circ G)(v_n) < l \|G(v_n)\|_A^2$, where $v_n = t_n z_n$.

If it is the case, the lemma will be proved since we can set

$$c_n \equiv \|G(v_n)\|_A.$$

We have

$$\begin{aligned}2(J \circ G)(v_n) &= \langle BG(v_n), G(v_n) \rangle - 2\varphi(G(v_n)) \\ &= \langle BPG(v_n), PG(v_n) \rangle \\ &\quad + \langle B(I-P)G(v_n), (I-P)G(v_n) \rangle - 2\varphi(G(v_n)) \\ &= \langle (B - lA)PG(v_n), PG(v_n) \rangle \\ &\quad + \langle (B - lA)(I-P)G(v_n), (I-P)G(v_n) \rangle \\ &\quad + l\langle AG(v_n), G(v_n) \rangle - 2\varphi(G(v_n))\end{aligned}$$

$$
\begin{aligned}
&\leqq \langle (\mathrm{B}-l\mathrm{A})\,\mathrm{PG}(v_n),\ \mathrm{PG}(v_n)\rangle \\
&\quad +\langle (\mathrm{B}-l\mathrm{A})(\mathrm{I}-\mathrm{P})\,\mathrm{G}(v_n),\ (\mathrm{I}-\mathrm{P})\,\mathrm{G}(v_n)\rangle \\
&\quad +l\langle \mathrm{AG}(v_n),\ \mathrm{G}(v_n)\rangle \\
&\quad -2^{1-\gamma}\varphi(\mathrm{PG}(v_n))+2\varphi((\mathrm{I}-\mathrm{P})\,\mathrm{G}(v_n)) \\
&\leqq \langle (\mathrm{B}-l\mathrm{A})\,\mathrm{PG}(v_n),\ \mathrm{PG}(v_n)\rangle-\delta\,\|(\mathrm{I}-\mathrm{P})\,\mathrm{G}(v_n)\|^2 \\
&\quad +l\langle \mathrm{AG}(v_n),\ \mathrm{G}(v_n)\rangle \\
&\quad -2^{1-\gamma}\varphi(\mathrm{PG}(v_n))+\mathrm{K}\,\|(\mathrm{I}-\mathrm{P})\,\mathrm{G}(v_n)\|^{2+\sigma} \\
&\leqq \langle (\mathrm{B}-l\mathrm{A})\,\mathrm{PG}(v_n),\ \mathrm{PG}(v_n)\rangle \\
&\quad +l\langle \mathrm{AG}(v_n),\ \mathrm{G}(v_n)\rangle-2^{1-\gamma}\varphi(\mathrm{PG}(v_n)) \\
&\leqq \langle (\mathrm{B}-l\mathrm{A})\,v_n,\ v_n\rangle \\
&\quad +l\langle \mathrm{AG}(v_n),\ \mathrm{G}(v_n)\rangle-2^{1-\gamma}\varphi(\mathrm{PG}(v_n)) \\
&\leqq t_n^2\langle (\mathrm{B}-l\mathrm{A})\,z_n,\ z_n\rangle+l\langle \mathrm{AG}(v_n),\ \mathrm{G}(v_n)\rangle \\
&\quad -t_n^{1+\gamma}2^{-\gamma}\varphi(z_n)
\end{aligned}
$$

if $\|v_n\|$ is small enough, *i.e.* t_n is small enough. In order to satisfy the condition (2), we require

$$
t_n^2\langle (\mathrm{B}-l\mathrm{A})\,z_n,\ z_n\rangle<t_n^{1+\gamma}2^{-\gamma}\varphi(z_n) \quad\Leftrightarrow\quad t_n^{\gamma-1}>2^{\gamma}\frac{\langle (\mathrm{B}-l\mathrm{A})\,z_n,\ z_n\rangle}{\varphi(z_n)}
$$

Since the right member tends toward zero as $n\to+\infty$, the t_n can be chosen such that $\lim_{n\to+\infty} t_n=0$. This ends the proof. ■

Proof of Theorem 2.4. – The demonstration is very close to the one given by Heinz and Stuart in [10]. Starting from the sequence $\{v_n\}\subset \mathrm{H}$, which appears in the condition (H5), we show that we can construct a sequence $\{z_n\}\subset \mathrm{V}$ as required in the preceding lemma. Since $\{\varphi(v_n)\}$ is bounded, we deduce that $\lim_{n\to+\infty}\langle (\mathrm{B}-l\mathrm{A})\,v_n,\ v_n\rangle=0$, which implies $\lim_{n\to+\infty}\|(\mathrm{B}-l\mathrm{A})\,v_n\|=0$. Moreover

$$
\begin{aligned}
\|(\mathrm{B}-l\mathrm{A})\,v_n\| &\geqq \|(\mathrm{B}-l\mathrm{A})(\mathrm{I}-\mathrm{P})\,v_n\| \\
&\geqq \delta\,\|(\mathrm{I}-\mathrm{P})\,v_n\| \\
&\geqq \frac{\delta}{\sqrt{\|\mathrm{A}\|}}\,\|(\mathrm{I}-\mathrm{P})\,v_n\|_{\mathrm{A}},
\end{aligned}
$$

which shows that $\|(\mathrm{I}-\mathrm{P})\,v_n\|\to 0$ and $\|(\mathrm{I}-\mathrm{P})\,v_n\|_{\mathrm{A}}\to 0$. We can then assume without loss of generality that

$$
\forall n:\quad \|\mathrm{P}\,v_n\|_{\mathrm{A}}\geqq\frac{1}{2}\qquad\text{and}\qquad \|(\mathrm{I}-\mathrm{P})\,v_n\|\leqq 1.
$$

Now

$$
\begin{aligned}
&|\langle (\mathrm{B}-l\mathrm{A})\,\mathrm{P}\,v_n,\ \mathrm{P}\,v_n\rangle| \\
&\qquad\leqq|\langle (\mathrm{B}-l\mathrm{A})\,v_n,\ v_n\rangle|+|\langle (\mathrm{B}-l\mathrm{A})(\mathrm{I}-\mathrm{P})\,v_n,\ (\mathrm{I}-\mathrm{P})\,v_n\rangle|,
\end{aligned}
$$

with

$$\begin{aligned}
&|\langle (B-lA)(I-P)v_n, (I-P)v_n\rangle| \\
&\leqq \|(B-lA)(I-P)v_n\| \, \|(I-P)v_n\| \\
&\leqq \frac{1}{\delta}\|(B-lA)(I-P)v_n\|^2 \\
&\leqq \frac{1}{\delta}\|(B-lA)v_n\|^2 \\
&\leqq \frac{M}{\delta}|\langle (B-lA)v_n, v_n\rangle|
\end{aligned}$$

and then

$$|\langle (B-lA)Pv_n, Pv_n\rangle| \leqq K|\langle (B-lA)v_n, v_n\rangle|.$$

In particular $\langle (B-lA)Pv_n, Pv_n\rangle \to 0$. Since $\forall n: \|Pv_n\|_A \geqq 1/2$, we can then assume that $\{Pv_n\} \subset V$. Now we set $z_n = Pv_n/\|Pv_n\|_A$ and show that $\{z_n\}$ has all the desired properties. Clearly $\forall n: \|z_n\|_A = 1$ and $\{z_n\} \subset V$. Now

$$\begin{aligned}
\varphi(z_n) = \varphi\left(\frac{Pv_n}{\|Pv_n\|_A}\right) &\geqq \frac{1}{\|Pv_n\|_A^2}\varphi(Pv_n) \\
&\geqq \varphi(Pv_n) \geqq 2^{-\gamma}\varphi(v_n) - \varphi((I-P)v_n) \\
&\geqq 2^{-\gamma}\varphi(v_n) - K\|(I-P)v_n\|^{2+\sigma} \\
&\geqq 2^{-\gamma}\varphi(v_n) - K\|(B-lA)v_n\|^{2+\sigma} \\
&\geqq 2^{-\gamma}\varphi(v_n) - K|\langle (B-lA)v_n, v_n\rangle|^{(2+\sigma)/2} \\
&\geqq 2^{-(\gamma+1)}\varphi(v_n)
\end{aligned}$$

if

$$K|\langle (B-lA)v_n, v_n\rangle|^{(2+\sigma)/2} \leqq 2^{-(\gamma+1)}\varphi(v_n).$$

But this is true for n large enough since

$$\lim_{n\to+\infty} \frac{|\langle (B-lA)v_n, v_n\rangle|^{(2+\sigma)/2}}{\varphi(v_n)} = 0.$$

We can then assume that $\forall n \in \mathbb{N}$, $\varphi(z_n) > 0$. Finally,

$$\begin{aligned}
\langle (B-lA)z_n, z_n\rangle &\leqq 4\langle (B-lA)Pv_n, Pv_n\rangle \\
&\leqq K|\langle (B-lA)v_n, v_n\rangle|.
\end{aligned}$$

We can conclude that

$$\lim_{n\to+\infty} \frac{\langle (B-lA)z_n, z_n\rangle}{\varphi(z_n)} = 0$$

and the proof is complete. ■

Proof of Theorem 2.6. – Set $v_n = P u_n$ and $w_n = (I-P) u_n$. We have

$$\begin{aligned}
&\delta \| w_n \|^2 + \frac{\delta}{\varepsilon}(l-\lambda_n) \| v_n \|^2 \\
&\leqq \langle (B-\lambda_n A) v_n, v_n \rangle - \langle (B-\lambda_n A) w_n, w_n \rangle \\
&= \langle B(v_n+w_n), v_n - w_n \rangle + \lambda_n (\| w_n \|_A - \| v_n \|_A) \\
&= \langle \lambda_n A(v_n+w_n) - N(v_n+w_n), v_n - w_n \rangle + \lambda_n (\| w_n \|_A - \| v_n \|_A) \\
&= \langle N(v_n+w_n), w_n - v_n \rangle = \langle N(v_n+w_n), 2 w_n - (v_n+w_n) \rangle \\
&\leqq \varphi(2 w_n) \leqq K \| w_n \|^{2+\sigma}.
\end{aligned}$$

For n large enough,

$$\frac{\delta}{2} \| w_n \|^2 + \frac{\delta}{\varepsilon}(l-\lambda_n) \| v_n \|^2 \leqq 0$$

and therefore $w_n = v_n = u_n = 0$, which is a contradiction. ■

Proof of Lemma 4.5. – Now we establish the equivalence $\sigma(S) = \sigma(I, A)$. We abbreviate $H^1(\mathbb{R}^N)$, $L^2(\mathbb{R}^N)$ and $H^2(\mathbb{R}^N)$ by H, L^2 and H^2. Moreover $\langle .,. \rangle$ denotes the scalar product on $H^1(\mathbb{R}^N)$ introduced at the beginning of Section 2. We start with the inclusion $\rho(I, A) \subset \rho(S)$. For fix $\lambda \in \rho(I, A)$, we know by Lemma 4.1 that there exist $\delta > 0$ and a projector operator $P: H \to H$ such that

1. $AP = PA$
2. $\forall u \in PH: \langle (I-\lambda A) u, u \rangle \geqq \delta \| u \|^2$
3. $\forall u \in (I-P) H: \langle (I-\lambda A) u, u \rangle \leqq -\delta \| u \|^2$.

Setting $H_+ = adh_{L^2} PH$ and $H_- = adh_{L^2}(I-P) H$, we see that $(L_2, \| . \|_{L^2})$ can be decomposed into $L^2 = H_+ \oplus H_-$. Indeed

$$\forall u \in PH, \quad \forall v \in (I-P) H: \quad \langle u, v \rangle_{L^2} = \langle A u, v \rangle = 0$$

and, using the dense inclusion $H \subset L^2$, we obtain

$$\forall u \in H_+, \quad \forall v \in H_-: \quad \langle u, v \rangle_{L^2} = 0.$$

Moreover

$$\forall u \in H_+ \cap H^2: \quad \langle (S-\lambda I) u, u \rangle_{L^2} = \langle (I-\lambda A) u, u \rangle \geqq \delta \| u \|^2 \geqq \delta \| u \|_{L^2}^2$$

and

$$\forall u \in H_- \cap H^2: \quad \langle (S-\lambda I) u, u \rangle_{L^2} \leqq -\delta \| u \|_{L^2}^2.$$

We denote by Q the projector defined in L^2 with range H_+. It is easy to show that Q and S commute. Consequently $\sigma(S) = \sigma(S|_{H^+}) \cup \sigma(S|_{H^-})$ and we deduce that $\lambda \in \rho(S)$.

Let us now prove that $\rho(S) \subset \rho(I-A)$. For $\lambda \leqq 0$,

$$\forall u \in H^2, \langle (I-\lambda A) u, u \rangle \geqq \langle u, u \rangle = \| u \|^2.$$

Using the dense inclusion $H^2 \subset (H, \|.\|)$, we conclude that

$$\forall u \in H : \langle (I - \lambda A) u, u \rangle \geqq \| u \|^2.$$

This proves $\lambda \in \rho(I, A)$. For $\lambda > 0$, we introduce the projector operator $P : L^2 \to L^2$ defined by $P = I - E_S(\lambda)$. Setting $\mu = \inf\{ t \in \sigma(S)$ such that $t > \lambda \}$, we have for all $u \in PL^2 \cap H^2$:

$$\begin{aligned} \langle (S - \lambda I) u, u \rangle_{L^2} &= \langle \{ (S - \mu I) + (\mu - \lambda) I \} u, u \rangle_{L^2} \\ &= \frac{\mu - \lambda}{\mu} \left\langle \left\{ \frac{\mu}{\mu - \lambda} (S - \mu I) + \mu I \right\} u, u \right\rangle_{L^2} \\ &\geqq \frac{\mu - \lambda}{\mu} \langle \{ (S - \mu I) + \mu I \} u, u \rangle_{L^2} \\ &= \frac{\mu - \lambda}{\mu} \| u \|^2. \end{aligned}$$

Moreover there exists $\delta > 0$ such that for all $u \in (I - P) L^2 \cap H^2$,

$$\langle (S - \lambda I) u, u \rangle_{L^2} \leqq -\delta \| u \|_{L^2}^2.$$

Since $\| u \|^2 = \langle S u, u \rangle_{L^2} \leqq \lambda \| u \|_{L^2}^2$, we get:

$$\forall u \in (I - P) L^2 \cap H^2 : \langle (S - \lambda I) u, u \rangle_{L^2} \leqq -\frac{\delta}{\lambda} \| u \|^2.$$

Using the dense inclusion $H^2 \subset (H, \|.\|)$, we conclude that

$$\forall u \in PH \cap H^2 : \quad \langle (B - \lambda) u, u \rangle \geqq \frac{\mu - \lambda}{\mu} \| u \|^2$$

and

$$\forall u \in (I - P) H \cap H^2 : \quad \langle (B - \lambda) u, u \rangle \leqq -\frac{\delta}{\lambda} \| u \|^2.$$

Since $P|_H$ is an orthogonal projection, we deduce that $\lambda \in \rho(I, A)$. ■

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