

On Tartar's conjecture

by

Vladimír ŠVERÁK

Department of Mathematics,
Heriot-Watt University, Edinburgh EH14 4AS, UK.
(On leave from MFF UK, Sokolovská 83, Praha)

ABSTRACT. — We prove that the only probability measures supported at connected subsets of 2×2 matrices without rank-one connections and commuting with the determinant are Dirac masses. We also prove some regularity results for fully nonlinear 2×2 elliptic systems of the first order.

Key words : Young measures, compactness, regularity.

RÉSUMÉ. — Soit K un sous-ensemble connexe de matrices deux par deux sans connexion de rang un et soit ν une mesure de probabilité concentrée sur K qui commute avec le déterminant. On démontre que ν est une masse de Dirac. On démontre aussi quelques résultats de régularité pour des systèmes elliptiques deux par deux du premier ordre.

1. INTRODUCTION

Let $\Omega \subset \mathbf{R}^2$ be open and bounded. For functions $v : \Omega \rightarrow \mathbf{R}^2$ we consider nonlinear systems given by $Dv(x) \in K$, where K is a submanifold of the

Classification A.M.S. : 35 B.

set $M^{2 \times 2}$ of all 2×2 matrices. We shall be interested in regularity of solutions of these systems and also in the following question: if $v_j: \Omega \rightarrow \mathbf{R}^2$ is a sequence of functions such that $|Dv_j| \leq c$ and $\text{dist}(Dv_j(\cdot), K) \rightarrow 0$ in L^p , what can be said about compactness of the sequence Dv_j in L^p ? Since for every $A, B \in M^{2 \times 2}$ with $\text{rank}(A - B) = 1$ we can construct a sequence of piecewise linear functions whose gradients oscillate between A and B , a necessary condition to get some positive results is that $\text{rank}(A - B) \geq 2$ for any two distinct matrices $A, B \in K$. Tartar's conjecture (see [14]) in our special situation says that this condition should be also sufficient for the compactness of the sequences above. Here we prove that this holds true under the additional assumption that K is connected. (Without additional assumptions the conjecture fails. For a counterexample with K consisting of four matrices see [7]. Counterexamples in higher dimensions can be found in [2].) We also give a simple proof of the fact that if K is connected, $\text{rank}(A - B) \geq 2$ for each $A, B \in K$ distinct, and the system $Dv(x) \in K$ is elliptic (i. e. planes tangent to K do not contain rank-one directions), then the solutions which are Lipschitzian belong to $C^{1, \alpha}$ for some $\alpha > 0$. If, moreover, K is smooth, then the solutions are smooth. *A priori* estimates for the $C^{1, \alpha}$ -norm of twice differentiable solutions of the systems considered here are well-known. (See, for example, [8], Chapter 12.) I am not aware of any previous regularity results for Lipschitzian solutions, with the exception of the Monge-Ampère equation, which, of course, can be considered as a first-order elliptic system. In general, if K is two dimensional and is contained in symmetric matrices, then the equation $Dv(x) \in K$ can be viewed as a fully nonlinear scalar equation of the second order for the potential of the vector field v . *A priori* estimates for solutions of such equations in arbitrary dimensions have been obtained in [5]. See also [8], Chapter 17.

2. PRELIMINARIES

Throughout this paper Ω denotes a nonempty, bounded, open subset of \mathbf{R}^2 . The Lebesgue spaces L^p , the Sobolev spaces $W^{k, p}$ and the spaces $C^{k, \alpha}$ of Hölder continuous functions are defined in the usual way.

Let us briefly recall basic facts concerning Young measures. (We refer the reader to [1] or [14] for more details.) Let $z_j: \Omega \rightarrow \mathbf{R}^n$ be a sequence of functions bounded in $L^\infty(\Omega)$. It is possible to prove that there exists a subsequence z_{j_μ} of z_j such that for any continuous function $f: \mathbf{R}^n \rightarrow \mathbf{R}$ the sequence $f \circ z_{j_\mu}$ converges weakly* in $L^\infty(\Omega)$ to some function h_f . Moreover, it is also possible to prove that there is a subset S of Ω of measure zero and a family $\{\nu_x, x \in \Omega \setminus S\}$ of probability measures on \mathbf{R}^n such that for

each continuous $f: \mathbf{R}^n \rightarrow \mathbf{R}$ we have $h_f(x) = \int_{\mathbf{R}^n} f(\lambda) dv_x(\lambda)$ for almost every $x \in \Omega$. We shall use the notation $\int_{\mathbf{R}^n} f(\lambda) dv_x(\lambda) = \langle v_x, f \rangle$. If almost all of the measures v_x are Dirac masses, then the sequence z_μ is compact in $L'(\Omega)$ for any $r < \infty$ and *vice versa*. The measures v_x are called Young measures.

We shall use the following lemma.

LEMMA 1. — *Let K be a connected topological space and let $g: K \times K \rightarrow \mathbf{R}$ be a continuous function such that $g(x, y) = g(y, x) \neq 0$ for every $x, y \in K$, $x \neq y$ and $g(x, x) = 0$ for every $x \in K$. Then either $g(x, y) \geq 0$ for every $x, y \in K$ or $g(x, y) \leq 0$ for every $x, y \in K$.*

Proof. — We notice that if g changes sign on $K \times K$, then there exists $y \in K$ such that $g(\cdot, y)$ changes sign on K . Indeed, supposing this is not the case, we consider the sets $K^+ = \{y \in K, g(\cdot, y) \geq 0 \text{ on } K\}$ and $K^- = \{y \in K, g(\cdot, y) \leq 0 \text{ on } K\}$. These sets are clearly closed and $K^+ \cap K^- = \emptyset$. Since K is connected, we cannot have $K^+ \cup K^- = K$. Therefore the lemma will be proved if we show that under our assumptions the function $g(\cdot, y)$ does not change sign for any $y \in K$. Suppose this is not true and let $y_0 \in K$ be such that $g(\cdot, y_0)$ changes sign. Let $K_+ = \{x \in K, g(x, y_0) \geq 0\}$ and $K_- = \{x \in K, g(x, y_0) \leq 0\}$. We claim that K_+ and K_- are connected. To see this, suppose that $K_+ = U \cup V$, where U, V are nonempty disjoint closed subsets of K_+ . We can suppose $y_0 \notin V$. We now consider the sets $\tilde{U} = K_- \cup U$ and $\tilde{V} = V$. These are closed sets covering K , i. e. $\tilde{U} \cup \tilde{V} = K$. We have

$$\tilde{U} \cap \tilde{V} = (K_- \cap V) \cup (U \cap V) \subset (K_- \cap K_+) \setminus \{y_0\}.$$

Since g does not vanish outside the diagonal, the last set is empty. Since K is connected and \tilde{U} is nonempty, the set $V = \tilde{V}$ must be empty. This shows that K_+ is connected. The proof for K_- is the same. Let $x_+ \in K_+ \setminus \{y_0\}$ and $x_- \in K_- \setminus \{y_0\}$. The function $g(x_+, \cdot)$ is positive at y_0 and does not vanish on the connected set K_- containing y_0 . Therefore it is positive on K_- and in particular $g(x_+, x_-) > 0$. On the other hand, the function $g(x_-, \cdot)$ is negative at y_0 and does not vanish on the connected set K_+ containing y_0 and therefore $g(x_-, x_+) = g(x_+, x_-) < 0$, a contradiction. The proof is finished.

3. COMPACTNESS

LEMMA 2. — Let K be a connected subset of $M^{2 \times 2}$ and suppose that $\text{rank}(X - Y) \geq 2$ for every two distinct matrices $X, Y \in K$. Then either $\det(X - Y) \geq 0$ for all $X, Y \in K$ or $\det(X - Y) \leq 0$ for all $X, Y \in K$.

Proof. — This is an obvious consequence of Lemma 1.

LEMMA 3. — Let K be a bounded Borel measurable subset of $M^{2 \times 2}$ such that $\text{rank}(X - Y) \geq 2$ for any two distinct $X, Y \in K$ and suppose that $\det(X - Y)$ does not change sign on $K \times K$. Let ν be a probability measure on $M^{2 \times 2}$ carried by K (i.e. $\nu(M^{2 \times 2} \setminus K) = 0$) and satisfying $\langle \nu, \det \rangle = \det \langle \nu, \text{identity} \rangle$. Then ν is a Dirac mass, i.e. $\nu = \delta_A$ for some $A \in K$.

Proof. — Let $A = \langle \nu, \text{identity} \rangle$ be the centre of mass of ν . Let b be the symmetric bilinear form on $M^{2 \times 2}$ determined by $\det X = \frac{1}{2}b(X, X)$. We can write

$$\begin{aligned} & \int_{M^{2 \times 2}} d\nu(X) \int_{M^{2 \times 2}} d\nu(Y) \det(X - Y) \\ &= \int_{M^{2 \times 2}} d\nu(X) \int_{M^{2 \times 2}} d\nu(Y) (\det X + \det Y - b(X, Y)) \\ &= \int_{M^{2 \times 2}} d\nu(X) (\det X + \det A - b(X, A)) \\ &= \det A + \det A - b(A, A) = 0. \end{aligned}$$

Since $\det(X - Y)$ does not change sign and vanishes only at the diagonal of $K \times K$, we see that the measure $\nu \otimes \nu$ is supported at the diagonal of $K \times K$ and therefore it must be a Dirac mass. The proof is finished.

THEOREM 1. — Let $U^{(j)} = \begin{pmatrix} u_1^{(j)} & u_2^{(j)} \\ v_1^{(j)} & v_2^{(j)} \end{pmatrix}$ be a uniformly bounded sequence of matrix-valued functions on Ω and suppose that the sequences $\text{curl } u^{(j)}$ and $\text{curl } v^{(j)}$ are compact in $H^{-1}(\Omega)$. Let K be a closed connected subset of $M^{2 \times 2}$ such that $\text{rank}(X - Y) \geq 2$ for any two distinct $X, Y \in K$ and suppose that $\text{dist}(U^{(j)}(x), K) \rightarrow 0$ for a.e. $x \in \Omega$. Then the sequence $U^{(j)}$ is compact in $L^p(\Omega)$ for every $1 \leq p < \infty$.

Proof. — Following L. Tartar [14] we consider a family of Young measures ν_x associated to a subsequence of the sequence $U^{(j)}$ we and use the div-curl lemma (see [14]) to infer that $\langle \nu_x, \det \rangle = \det \langle \nu_x, \text{identity} \rangle$ for almost every $x \in \Omega$. Our assumptions clearly imply that ν_x is supported

on a bounded subset of K for a.e. $x \in \Omega$. From Lemma 2 and Lemma 3 we see that ν_x is a Dirac mass for almost every $x \in \Omega$. The proof is finished.

4. RANK-ONE CONNECTIONS IN SETS OF GRADIENTS

The results of Section 3 can be used to generalize some results of [2], Section 5.

THEOREM 2. — *Let $u: \Omega \rightarrow \mathbf{R}^2$ be a Lipschitzian function which coincides with an affine function A at the boundary of Ω and suppose that Du is continuous in Ω . If u is not affine, then there exist $x, y \in \Omega$ such that $\text{rank}(Du(x) - Du(y)) = 1$.*

Proof. — Let us first assume that Ω is connected and $A = 0$. Let $K = \{ Du(x), x \in \Omega \}$ and let ν be the probability measure on $M^{2 \times 2}$ given by

$$\langle \nu, f \rangle = \frac{1}{\text{meas } \Omega} \int_{\Omega} f(Du(x)) dx$$

for every continuous function $f: M^{2 \times 2} \rightarrow \mathbf{R}$. Under our assumptions the set K is bounded and connected. The measure ν is carried by K . We claim that $\langle \nu, \det \rangle = \det \langle \nu, \text{identity} \rangle$. For this it is enough to prove that under our assumptions we have $\int_{\Omega} Du(x) dx = 0$ and $\int_{\Omega} \det Du(x) dx = 0$. This is well known if u is Lipschitzian and compactly supported in Ω . (See, for example, [11].) The general case can be brought to this case by extending u by 0 outside Ω and integrating over a sufficiently large ball in which Ω is compactly contained. We can now apply Lemma 2 and Lemma 3 and we see that if Du is not constant, then there must be rank-one connections in K . The proof in the case when Ω is connected and $A = 0$ is finished. The general case follows easily, since clearly $u = A$ on the boundary of every connected component of Ω and since we can replace u by $u - A$, if necessary.

Remarks. — 1. For any open set $\Omega \subset \mathbf{R}^2$ it is possible to construct a Lipschitzian function $u: \Omega \rightarrow \mathbf{R}^2$ vanishing at the boundary of Ω and a bounded countable set $S \subset M^{2 \times 2}$ such that there are no rank-one connections in the closure K of S , $0 \notin K$, and $Du \in S$ a.e. in Ω . See [13].

2. For examples showing that Theorem 2 fails in higher dimensions (except, perhaps, for mappings from $\Omega \subset \mathbf{R}^2$ to \mathbf{R}^3) see [2].

5. REGULARITY

THEOREM 3. — *Let K be a bounded subset of $M^{2 \times 2}$ and suppose that there is $\lambda > 0$ such that either $\det(X - Y) \geq \lambda |X - Y|^2$ for each $X, Y \in K$ or $\det(X - Y) \leq -\lambda |X - Y|^2$ for each $X, Y \in K$. Let $v: \Omega \rightarrow \mathbf{R}^2$ be a Lipschitzian function satisfying $Dv(x) \in K$ for almost every $x \in \Omega$. Then there is $p > 2$ such that v belongs to $W_{loc}^{2,p}(\Omega)$. In particular, the gradient Dv of v is Hölder continuous.*

Proof. — We will consider only the case $\det(X - Y) \geq \lambda |X - Y|^2$. For the proof in the case $\det(X - Y) \leq -\lambda |X - Y|^2$ it is enough to replace \det by $-\det$ in the formulae below. Let $a \in \mathbf{R}^2$ and for $h > 0$ let $v_h(x) = (v(x + ha) - v(x))/h$. (We can extend v by zero outside Ω , for example.) Let η be a smooth nonnegative function compactly supported in Ω . Let $b \in \mathbf{R}^2$. For sufficiently small h we have

$$\begin{aligned} 0 &= \int_{\Omega} \det D(\eta(v_h - b)) \, dx \\ &\geq \int_{\Omega} (-\eta |Dv_h| |D\eta| |v_h - b| + \eta^2 \det Dv_h) \, dx \\ &\geq \int_{\Omega} (-\eta |Dv_h| |D\eta| |v_h - b| + \lambda \eta^2 |Dv_h|^2) \, dx \\ &\geq -\frac{1}{2\lambda} \int_{\Omega} |D\eta|^2 |v_h - b|^2 \, dx + \frac{\lambda}{2} \int_{\Omega} \eta^2 |Dv_h|^2 \, dx. \end{aligned}$$

We see that the L^2 -norm of Dv_h on compact subsets of Ω is estimated by the L^2 -norm of v_h . We can now use the well-known Nirenberg's Lemma to infer that $Dv \in W_{loc}^{1,2}(\Omega)$. It is well-known that if there exists $C > 0$ such that

$$\int_{\Omega} \eta^2 |Dv_h|^2 \, dx \leq C \int_{\Omega} |D\eta|^2 |v_h - b|^2 \, dx$$

for every η as above and every $b \in \mathbf{R}^2$, or in another words, if v_h satisfies the Caccioppoli's inequality, then there exists a $p > 2$ such that the L^p -norm of Dv_h on every set $\tilde{\Omega}$ compactly contained in Ω is bounded by $C_1 \|v_h\|_{L^2(\tilde{\Omega})}$, where C_1 depends only on C , p , $\tilde{\Omega}$ and Ω . (For a proof of this which is based on the technique of reverse Hölder inequalities see [6].) Using Nirenberg's Lemma again, we see that Dv is bounded in $W_{loc}^{1,p}(\Omega)$. The Hölder continuity of Dv follows from the Sobolev Imbedding Theorem. The proof is finished.

COROLLARY. — *Let K be a closed connected smooth submanifold of $M^{2 \times 2}$ such that $\text{rank}(X - Y) \geq 2$ for any two distinct $X, Y \in K$. Suppose moreover*

that K is “elliptic”, or in other words, that for any $X \in K$ the tangent space to K passing through X does not contain rank-one directions. Then every Lipschitz function $v : \Omega \rightarrow \mathbb{R}^2$ satisfying $Dv(x) \in K$ for a.e. $x \in K$ is smooth.

Proof. — We notice that the ellipticity condition together with Lemma 1 implies that for each bounded subset K_1 of K there exists $\lambda > 0$ such that either

$$\det(X - Y) \geq \lambda |X - Y|^2 \quad \text{for every } X, Y \in K_1$$

or

$$\det(X - Y) \leq -\lambda |X - Y|^2 \quad \text{for every } X, Y \in K_1.$$

We can use Theorem 2 to infer that Dv is Hölder continuous and that v belongs to the space $W_{loc}^{2,2}(\Omega)$. Since $Dv(x) \in K$ in Ω , the derivatives $\frac{\partial}{\partial x_i} Dv(x)$ belong to the tangent space of K at $Dv(x)$ for a.e. $x \in \Omega$. Since

Dv is Hölder continuous and K is elliptic, we see that $\frac{\partial}{\partial x_i} v(x)$ can be viewed as solutions of a certain linear first order elliptic system with Hölder continuous coefficients. Therefore $D^2 v$ is Hölder continuous. (See, for example, [11].) Applying the usual procedure of improving regularity we see that v must be smooth. The proof is finished.

6. EXAMPLES

Classical examples of K 's which are elliptic in the above sense are

$$K_0 = \{ X \in M^{2 \times 2}, X \text{ is symmetric and Trace } X = 0 \}$$

and

$$K_1 = \{ X \in M^{2 \times 2}, X \text{ is symmetric, positive definite, and } \det X = 1 \}.$$

Clearly K_0 can be viewed as the tangent space to K_1 at the unit matrix.

The following examples arise in connection with problems concerning invariant “wells” which appear in the theory of microstructures. (See, for example, [3], [4], [9], and [10] for motivation). Let $A_1, \dots, A_m \in M^{2 \times 2}$ with $\det A_k > 0$ for each $k = 1, \dots, m$ and let

$$K_w = SO(2) \cdot A_1 \cup \dots \cup SO(2) \cdot A_m.$$

It is easy to check that if K_w does not contain rank-one connections (*i.e.* rank $(X - Y) \geq 2$ for any two distinct $X, Y \in K_w$), then there exists $\nu > 0$ such that $\det(X - Y) \geq \nu |X - Y|^2$ for each $X, Y \in K_w$. We see that in this case Lemma 3 and Theorem 3 can be applied to K_w . This shows, for example, that if K_w does not contain rank-one connections, then the deformations $\varphi : \Omega \rightarrow \mathbb{R}^2$ satisfying $D\varphi \in K_w$ a.e. in Ω belong to $C_{loc}^{1,\alpha}(\Omega)$

for some $\alpha > 0$. Using this it is not difficult to see that if K_w does not contain rank-one connections, then $D\phi \in K_w$ a.e. in Ω implies that in fact $D\phi$ is locally constant in Ω .

We can also consider continuous families of invariant wells. A simple example is the following: let $\mu: [0, 1] \rightarrow \mathbf{R}$ and $\lambda: [0, 1] \rightarrow \mathbf{R}$ be smooth strictly positive functions with $\mu'(t) > 0$ and $\lambda'(t) > 0$ for all $t \in [0, 1]$ and let $K_c = \bigcup_{t \in [0, 1]} \text{SO}(2) \cdot \begin{pmatrix} \lambda(t) & 0 \\ 0 & \mu(t) \end{pmatrix}$. It is easy to check that K_c satisfies the assumptions of Theorem 1 and Theorem 3.

ACKNOWLEDGEMENTS

I thank John Ball for several useful suggestions. I also thank the referee for useful remarks.

REFERENCES

- [1] J. M. BALL, A Version of the Fundamental Theorem for Young Measures, in *Partial Differential Equations and Continuum Models of Phase Transitions*, M. RASCLE, D. SERRE and M. SLEMROD Eds., pp. 107-215, Springer-Verlag.
- [2] J. M. BALL, Sets of Gradients with No Rank-One Connections, *J. Math. pures et appl.*, Vol. **69**, 1990, pp. 241-159.
- [3] J. M. BALL and R. D. JAMES, Fine Phase Mixtures as Minimizers of Energy, *Arch. Rat. Mech. Anal.*, Vol. **100**, 1987, pp. 13-52.
- [4] J. M. BALL and R. D. JAMES, *Proposed Experimental Tests of a Theory of Fine Microstructures and the Two-Well Problem*, preprint, 1990.
- [5] L. C. EVANS, Classical Solutions of Fully Nonlinear Second Order Elliptic Equations, *Comm. Pure Appl. Math.*, Vol. **25**, 1982, pp. 333-363.
- [6] M. GIAQUINTA, *Multiple Integrals in the Calculus of Variations*, Princeton University Press, 1983.
- [7] N. FIROOZY, R. D. JAMES and R. KOHN (to appear).
- [8] D. GILBARG and N. S. TRUDINGER, *Elliptic Partial Differential Equations of Second Order*, Second Edition, Springer, 1983.
- [9] D. KINDERLEHRER, Remarks about equilibrium configurations of crystals, in *Symp. Material Instabilities in Continuum Mechanics*, pp. 217-242, J. M. BALL Ed., Heriot-Watt, Oxford University Press, 1988.
- [10] J. P. MATOS, *Young Measures and the Absence of Fine Microstructures in the α - β Quartz Phase Transition*, preprint, 1991.
- [11] Ch. B. MORREY, *Multiple Integrals in the Calculus of Variations*, Springer, 1966.
- [12] V. ŠVERÁK, *On the Problem of Two Wells* (to appear).
- [13] V. ŠVERÁK (to appear).
- [14] L. TARTAR, The Compensated Compactness Method Applied to Systems of Conservation Laws, in *Systems of Nonlinear Partial Differential Equations*, J. M. BALL Ed., NATO ASI Series, Vol. C111, Reidel, 1982.

(Manuscript received May 21, 1991;
revised August 21, 1991.)