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# On solutions of the exterior Dirichlet problem for the minimal surface equation

by

**Ernst KUWERT**

Mathematisches Institut, Universität Bonn,
Wegelerstrasse 10, D-53115 Bonn,
Germany

ABSTRACT. — Uniqueness and existence results for boundary value problems for the minimal surface equation on exterior domains obtained by Langévin-Rosenberg and Krust in dimension two are generalized to arbitrary dimensions. A suitable $n$-dimensional version of the maximum principle at infinity is given.

*Key words :* Minimal surface equation, exterior domain problems, maximum principle at infinity.

RÉSUMÉ. — On présente des résultats d'unicité et d'existence pour l'équation des surfaces minimales sur un domaine extérieur de $\mathbb{R}^n$. On donne une généralisation du principe de maximum à l'infini, valable quel que soit $n$.

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# 1. INTRODUCTION

Let $U \subset \mathbb{R}^n$ be a domain such that $K = \mathbb{R}^n \setminus U$ is compact. In this paper we consider solutions $u \in C^2(U)$ of the minimal surface equation

$$\text{(E)} \qquad \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0 \quad \text{in } U$$

which are *regular at infinity* in the sense that their graph has a welldefined asymptotic normal $\nu_\infty \in \{\nu \in S^n : \nu^{n+1} > 0\}$. Given $\varphi \in C^0(\partial U)$ a function $u \in C^0(\bar{U})$ is called a *solution of the exterior Dirichlet problem* if $u$ satisfies (E) – in particular $u \in C^2(U)$ – and $u | \partial U = \varphi$.

In case of a *bounded* domain U the solvability of the corresponding boundary value problem for *all* $\varphi \in C^0(\partial U)$ is equivalent to the mean curvature of $\partial U$ being nonnegative [3]. Since this condition is necessarily violated for an exterior region, the existence problem is quite difficult. In [11] Osserman presented smooth functions on the unit circle which do not admit a bounded solution. Recently Krust [5] showed that the boundary data in Osserman's examples would not even admit solutions having a vertical normal at infinity. Krust's main result says that for $n=2$ all solutions having the same $\nu_\infty$ form a *foliation*. From this he could derive the nonexistence statement using a symmetry argument. We will prove the above foliation property in arbitrary dimensions. We shall also give a simple proof different from [7] of the so-called *maximum principle at infinity*. Our argument is similar to the one given in [8] and suitably generalizes to the $n$-dimensional case.

Let us finally mention that $\Gamma = \operatorname{graph} \varphi$ always bounds a minimal surface having a planar end by the work of Tomi and Ye [13] and the author [6]; here "minimal surface" refers either to a parametric solution ($n=2$) or to an embedded surface (possibly with singularities if $n \geqq 7$).

# 2. ASYMPTOTIC EXPANSIONS AND MAXIMUM PRINCIPLE AT INFINITY

We will use the following notation:

$\omega_n = \mathcal{H}^n(S^n)$

$px = \xi$ for $x = (\xi, x^{n+1}) \in \mathbb{R}^{n+1}$

$r = r(\xi) = |\xi|$, $e_r = e_r(\xi) = \dfrac{\xi}{r}$ for $\xi \in \mathbb{R}^n \setminus \{0\}$

$A(r, R) = \{\xi \in \mathbb{R}^n : r < |\xi| < R\}$, $A(r) = A(r, \infty)$

U will always denote an open neighbourhood of infinity in $\mathbb{R}^n$. If $V \subset\subset U$ and $\partial V$ is of class $C^1$, $u$ is a solution of (E) in U and $\varphi$ is locally

Lipschitz continuous in U, then

$$\int_{V} \langle T(\nabla u), \nabla\varphi \rangle \, d\mathscr{L}^n = \int_{\partial V} \varphi \langle T(\nabla u), N \rangle \, d\mathscr{H}^{n-1} \tag{1}$$

where as usual $T(p)=(1+|p|^2)^{-1/2}p$ for $p\in\mathbb{R}^n$ and N is the exterior unit normal along $\partial V$. Setting $w(p)=(1+|p|^2)^{1/2}$, the ellipticity of (E) can be stated as follows:

$$\langle T(p_1)-T(p_2), p_1-p_2 \rangle \geqq (\max_{i=1,2} w(p_i))^{-3} |p_1-p_2|^2 \qquad \forall p_{1,2}\in\mathbb{R}^n \tag{2}$$

A connected, oriented and embedded minimal surface $M^n \subset \mathbb{R}^{n+1}$ will be called *simple at infinity* if M has a welldefined normal $\nu_\infty \in S^n$ at infinity and M can be written as a graph over its asymptotic tangent plane outside some compact set. Assuming $\nu_\infty = e_{n+1}$ is the vertical direction, it is shown in [12] that the corresponding graph function has a *twice differentiable expansion*

$$u(\xi)=h+\alpha g(r)+O(r^{1-n}) \tag{3}$$

where $h\in\mathbb{R}$, $\alpha\in\mathbb{R}$ and $g$ is the Newtonian potential in $\mathbb{R}^n$:

$$g(r)= \begin{cases} \log r & (n=2) \\ \dfrac{r^{2-n}}{2-n} & (n\geqq 3) \end{cases}$$

For example the graph function of an $n$-dimensional catenoid is given by

$$|x^{n+1}|=c_a(r)=a\int_1^{r/a} (s^{2(n-1)}-1)^{-1/2}\, ds \qquad (a>0) \tag{4}$$

and satisfies (3) with $\alpha=a^{n-1}$. If $\nu_\infty$ is fixed, we will refer to $h$ as the *height* and $\alpha$ as the *growth rate* (at infinity). The following result is due to Langévin and Rosenberg [7] in case $n=2$.

THEOREM 1 (MAXIMUM PRINCIPLE AT INFINITY). — *Suppose* $M_i\,(i=1, 2)$ *are minimal surfaces which are simple and disjoint at infinity. If the* $M_i$ *are at distance zero at infinity, then* $n\geqq 3$ *and their growth rates are different.*

*Proof.* — We may assume that $M_i=\operatorname{graph} u_i$ where $u_i\in C^0(\overline{A(R)})$, $u_1<u_2$ and $u_i=h_i+\alpha_i g(r)+O(r^{1-n})$. The assumptions imply $h_1=h_2$ and $u_2-u_1\to 0$ uniformly as $\xi\to\infty$; for $n=2$ we also had $\alpha_1=\alpha_2$. The expansions yield

$$\langle T(\nabla u_i), e_r \rangle = \alpha_i r^{1-n}+O(r^{-n}).$$

Setting $d=\inf\{u_2(\xi)-u_1(\xi): |\xi|=R\}>0$ we consider for any $\varepsilon\in(0, d)$ the test function

$$\varphi_\varepsilon := \begin{cases} 0 & \text{if } u_2-u_1\geqq d \\ u_2-u_1-d & \text{if } \varepsilon<u_2-u_1<d \\ \varepsilon-d & \text{if } 0<u_2-u_1<\varepsilon \end{cases}$$

Using $\varphi_\varepsilon$ in (1) on $V=A(R, \rho)$ and letting $\rho \to \infty$ we obtain

$$(d-\varepsilon)\,\omega_{n-1}\,\alpha_i = \int_{\{\varepsilon < u_2 - u_1 < d\}} \langle T(\nabla u_i), \nabla u_1 - \nabla u_2 \rangle \, d\mathscr{L}^n \ (i=1, 2).$$

Now subtract these two identifies, apply (2) and let $\varepsilon \to 0$:

$$d\omega_{n-1}(\alpha_1 - \alpha_2) \geqq \int_{\{u_2 - u_1 < d\}} (\max_{i=1,2} w(\nabla u_i))^{-3} |\nabla u_1 - \nabla u_2|^2 \, d\mathscr{L}^n.$$

Hence $\alpha_1 \leqq \alpha_2$ is impossible. □

COROLLARY 1. – *Let* $u_i \in C^0(\bar{U})\,(i=1, 2)$ *be two solutions of* (E) *having the same asymptotic normal and* $u_1 | \partial U = u_2 | \partial U$. *Let* $h_i$ *and* $\alpha_i$ *be their heights and growth rates respectively. Then*

(i) $\alpha_1 \geqq \alpha_2 \Leftrightarrow u_1 \geqq u_2$.

(ii) *If* $n \geqq 3$, *we also have:* $h_1 \geqq h_2 \Leftrightarrow u_1 \geqq u_2$.

Corollary 1 follows easily by looking at vertical translates of the graph of $u_1$. Now let M be simple at infinity such that $\nu_\infty = e_{n+1}$ and suppose M is of class $C^1$ up to the boundary. Let $\nu$ be the continuous unit normal on M determined by $\nu_\infty$, $M_R = \{x \in M : |px| < R\}$ and denote by $\eta$ the exterior unit normal along $\partial M_R$ in M. For sufficiently large $|\xi| = R$ we have

$$\eta(\xi) = \frac{e_r - \langle e_r, \nu \rangle \nu}{\sqrt{1 - \langle e_r, \nu \rangle^2}} = e_r + \alpha r^{1-n} e_{n+1} + O(r^{-n}),$$
$$(1 + |\nabla u|^2 - (\partial_r u)^2)^{1/2} = 1 + O(r^{2(1-n)}).$$

Applying the divergence theorem on $M_R$ to the tangential component of a constant vector $v \in \mathbb{R}^{n+1}$ and letting $R \to \infty$ we obtain the "balancing formula" (compare [12], [4])

$$\alpha \nu_\infty = -\frac{1}{\omega_{n-1}} \int_{\partial M} \eta(x) \, d\mathscr{H}^{n-1}(x). \tag{5}$$

COROLLARY 2. – *Let* $U \subset \mathbb{R}^n$ *be an exterior domain with* $\partial U \in C^1$. *If* $u_i \in C^1(\bar{U})\,(i=1, 2)$ *are solutions of* (E) *having the same asymptotic normal and satisfying*

$$\langle T(\nabla u_1), N \rangle = \langle T(\nabla u_2), N \rangle \ \textit{along } \partial U,$$

*then the difference* $u_1 - u_2$ *is a constant.*

*Proof.* – Writing down (5) in terms of the graph functions, we obtain

$$\alpha_1 = \alpha_2 = -\frac{1}{\omega_{n-1} \langle \nu_\infty, e_{n+1} \rangle} \int_{\partial U} \langle T(\nabla u_i), N \rangle \, d\mathscr{H}^{n-1}.$$

Let $t_0=\inf\{t: u_2+t\geqq u_1\}$. Then we have either $u_1-u_2\equiv t_0$ or $u_2+t_0>u_1$ in all of U. But the second case is impossible because of Thm 1 and the maximum principle at the boundary. □

## 3. FOLIATION PROPERTY OF THE SOLUTIONS

The following result is a consequence of the interior maximum principle (*see* [11]).

LEMMA 1. — *If* $M\subset\mathbb{R}^{n+1}$ *is a compact minimal surface such that*

$$(\partial M\cap(A(\rho)\times\mathbb{R}))\subset\{x:|px|=R,\ |x^{n+1}|\leqq h_0\}\ \textit{for some}\ \rho\in(0,R),$$

*then*

$$(M\cap(A(\rho)\times\mathbb{R}))\subset\{x:|x^{n+1}|\leqq h_0+c_\rho(R)-c_\rho(r)\}.$$

*Remark.* — Let $n\geqq 3$ and $u\in C^0(\bar U)$ be a solution of (E) with $\nu_\infty=e_{n+1}$. Then if $B\subset\mathbb{R}^n$ is a closed ball of radius $\rho>0$ containing $\partial U$, we conclude from the above lemma that

$$|u(\xi)-u(\infty)|\leqq c_\rho(\infty)-c_\rho(r)\quad\text{in } U\setminus\operatorname{int}B.$$

Setting $A=\partial B\cap\partial U$, we have in particular $\operatorname*{osc}_A(u)\leqq 2\rho\, c_1(\infty)$. For example there is no solution of the exterior Dirichlet problem having a vertical normal at infinity if $\varphi\in C^0(S^{n-1})$ is given with $\operatorname{osc}(\varphi)>2c_1(\infty)$.

Now let $U\subset\mathbb{R}^n$ be an exterior region, $K=\mathbb{R}^n\setminus U$, and suppose that $u^\pm\in C^0(\bar U)$ are two solutions of (E) satisfying $u^-<u^+$ in U and $u^\pm\,|\,\partial U=\varphi$.

LEMMA 2. — *Let* $K\subset\subset B_R(0)=B\subset\mathbb{R}^n$ *and suppose* $\psi\in C^2(\partial B)$ *satisfies* $u^-<\psi<u^+$ *on* $\partial B$. *Setting* $U_R=U\cap B$, *there is a unique solution* $u\in C^0(\bar U_R)$ *of* (E) *satisfying* $u\,|\,\partial U=\varphi$, $u\,|\,\partial B=\psi$. *Moreover* $u^-<u<u^+$ *in* $U_R$.

*Proof.* — We refer to Haar's solution of the nonparametric Plateau problem [2] which is described in the book of Giusti [1]. Let us first consider the case that U has Lipschitz continuous boundary and $\max\{\operatorname{Lip}(u^\pm, U_R)\}=l<\infty$. Then for any sufficiently large $k>l$, we can take $u^k\in C^0((\bar U_R)$ minimizing the area functional in the class $\{u\in C^0((\bar U_R):\operatorname{Lip}(u)\leqq k,\ u\,|\,\partial U=\varphi,\ u\,|\,\partial B=\psi\}$. The weak maximum principle [1], 12.5, yields $u^-\leqq u^k\leqq u^+$ in $U_R$. Now because of [1], 12.7, we know that

$$\operatorname{Lip}(u^k)=\sup\left\{\frac{|u^k(\xi)-u^k(\eta)|}{|\xi-\eta|}:\xi\in U_R,\ \eta\in\partial U_R\right\}.$$

Since $u^-\leqq u^k\leqq u^+$, for any $\eta\in\partial U$ and any $\xi\in U_R$ we have

$$|u^k(\xi)-u^k(\eta)|\leqq l\,|\xi-\eta|.$$

On the other hand, it is easy to construct barriers in a neighbourhood of $\partial B$ (*see* [1], pp. 142-144). Hence $\operatorname{Lip}(u^k)<k$ for sufficiently large $k$, which means that $u^k$ is a weak solution of (E) in $U_R$; in fact because of the regularity theory ([1], 12.11) $u^k$ is smooth. To treat the general case, we choose a regular value $\varepsilon>0$ of $u^+ - u^-$. Letting

$$V_\varepsilon=\{\xi\in U_R : u^+(\xi)-u^-(\xi)>\varepsilon\}, \qquad \varphi_\varepsilon=u^+ \quad \text{on } \partial V_\varepsilon\setminus\partial B,$$

we can apply the argument above to obtain a solution $v_\varepsilon\in C^0(\bar{V}_\varepsilon)$ of (E) which coincides with $u^+$ on $\partial V_\varepsilon\setminus B$, and with $\psi$ on $\partial B$. Since $u^-<v_\varepsilon<u^+$ in $V_\varepsilon$, the *a priori* estimates in [1] imply that $v_\varepsilon\to u$ locally uniformly in $C^2(U_R)$ as $\varepsilon\to 0$. Clearly $u$ must attain the boundary values on $\partial U$. But on $\partial B$ the same barriers apply to all the $v_\varepsilon$ and hence $u\in C^0((\bar{U}_R)$ is a solution of our problem. Uniqueness follows easily from the interior maximum principle. □

The following result is due to Krust [5] in the two dimensional case. In order to obtain the approximating solutions, he solved the parametric Plateau problem for a minimal annulus and referred to an embeddedness result of Meeks and Yau [9] together with the well-known argument of Kneser-Radó to show the graph property.

THEOREM 2. – *Let* $U\subset\mathbb{R}^n$ *be an exterior region and* $\varphi\in C^0(\partial U)$. *The set of solutions of the exterior Dirichlet problem with boundary data* $\varphi$ *having the same asymptotic normal forms a (possibly empty) foliation.*

*Proof.* – Let us first consider the case $n\geqq 3$. Suppose $u^\pm\in C^0(\bar{U})$ are two solutions with asymptotic normal $\nu_\infty$. Because of corollary 1, we may assume that $h^+>h^-$ and $u^+>u^-$ in U. Given any $h\in(h^-, h^+)$, we let

$$H=\{x\in\mathbb{R}^{n+1} : \langle x, \nu_\infty\rangle=h\},$$
$$\Gamma_R=\{x\in H : |px|=R\}.$$

For any sufficiently large R there is a minimal graph $u_R\in C^0((\bar{U}_R)$ such that $u_R\,|\,\partial U=\varphi$ and graph $(u_R\,|\,\partial B_R(0))=\Gamma_R$; moreover $u^-<u_R<u^+$ in $U_R$. As in lemma 2, we can let $R\to\infty$ to obtain a solution $u\in C^0(\bar{U})$ of the exterior Dirichlet problem with boundary data $\varphi$ satisfying $u^-\leqq u\leqq u^+$ in U. Let $\pi$ be the orthogonal projection onto H and let $B\subset H$ be an $n$-dimensional ball of radius $\rho>0$ containing $\pi(\operatorname{graph}\varphi)$. Applying lemma 1 to graph $u_R$ and then letting $R\to\infty$ we infer that

$$|\langle x, \nu_\infty\rangle-h|\leqq c_\rho(\infty)-c_\rho(|\pi x|)$$

for any $x\in\operatorname{graph} u$ satisfying $\pi x\notin\bar{B}$. In particular graph $u$ is at a height $h$ at infinity. Now the gradient of $u$ is bounded ([1], 13.6) and in fact converges to a limit (*see* [10], thm 6). This means that $u$ is regular at infinity in the sense of the introduction and has asymptotic normal $\nu_\infty$. Thus we have shown that for any $h\in(h^-, h^+)$ there is a solution $u_h$ with asymptotic height equal to $h$ and moreover $u_h<u_{h'}$ for $h<h'$. Now let

$x=(\xi, x^{n+1})\in U\times\mathbb{R}$ be given such that $u^-(\xi)<x^{n+1}<u^+(\xi)$. Then we let $h_1=\sup\{h: u_h(\xi)<x^{n+1}\}$, $h_2=\inf\{h: u_h(\xi)>x^{n+1}\}$. We see that $h_1<h_2$ is impossible because otherwise we would have $u_{h_1}(\xi)<u_h(\xi)<u_{h_2}(\xi)$ for any $h\in(h_1, h_2)$. This proves the theorem if $n\geqq 3$.

The case $n=2$ was treated in [5]; the main difference in this case is that one has to replace the parameter $h$ by the growth rate $\alpha$. Taking as $\Gamma_R$ the intersection of the cylinder $\{x:|px|=R\}$ with a half catenoid of the desired growth rate centered around the axis $\mathbb{R}\nu_\infty$ one proceeds essentially in the same way as above. □

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