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# The stability of one dimensional stationary flows of compressible viscous fluids

by

**H. BEIRÃO DA VEIGA**

Istituto Matematiche Applicate « U. Dini »,
University of Pisa, I-56100 Pisa, Italy

ABSTRACT. — In [*1*], § 5 we gave a necessary and sufficient condition for the existence of stationary solutions for the system (1.1) in the presence of arbitrarily large external forces $f(x)$. Here we prove, see theorem 2.1, that every stationary solution is necessarily stable. We use Euler coordinates, since a proof in Lagrange coordinates (see for instance [*2*], [*3*], [*4*]) should present greater obstacles.

*Key-words:* Compressible fluids; stability; one dimensional flows; nonlinear P. D. E.

RÉSUMÉ. — On démontre que toute solution du système d'équations décrivant le mouvement unidimensionnel d'un fluide compressible et visqueux (avec des forces arbitraires et indépendantes du temps) est nécessairement stable.

## 1. REVIEW ON STATIONARY SOLUTIONS

We denote by $\| \ \|_k$ the norm in the Sobolev space $H^k(\Omega)$, by $\| \ \|$ the norm in $L^2(\Omega)$, and by $| \ |_q$ the norm in $L^q(\Omega)$, $q \in [1, +\infty]$, where $\Omega = ]0, 1[$. Moreover, $\Omega_\infty = [0, +\infty[ \times \Omega$. Other notations will be introduced when necessary. Denote by $u$ and $\rho$ the velocity and the density of the fluid, and by $\mu > 0$ the viscosity. Without loss of generality we assume that the total mass of the fluid is equal to 1, and that the bounded

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region $\Omega$ is the interval $]0, 1[$. The boundary is assumed to be impermeable. The equations of motion are

$$(1.1)\qquad \begin{cases} u_t + uu_x - \mu\rho^{-1}u_{xx} + \pi(\rho)_x = f(x), \\ \rho_t + (u\rho)_x = 0, \qquad \text{for} \quad (t, x) \in ]0, +\infty[ \times \Omega, \\ u(t, 0) = u(t, 1) = 0, \qquad \int_0^1 \rho(t, x)dx = 1. \end{cases}$$

Here $\pi(s)$, $s \in ]0, +\infty[$, is a real function defined by

$$\pi(s) = \int_1^s \tau^{-1}p'(\tau)d\tau .$$

The pressure $p(\cdot)$ is assumed to be a real valued continuously differentiable function defined on $]0, +\infty[$, and such that $p'(s) > 0$, $\forall s > 0$. For the convenience of the reader, we state here some results proved in [1], § 5.

Let F be a fixed primitive of $f$ in $\Omega$. The statements are independent of the particular choice of F. It is immediate to verify that a pair $(v(x), \eta(x))$ is a stationary solution of the system (1.1) if and only if $\eta$ satisfies the conditions

$$(1.2)\qquad p(\eta(x))_x = \eta(x)F'(x), \qquad \forall x \in \Omega,$$

$$(1.3)\qquad \int_0^1 \eta(x)dx = 1,$$

and $v \equiv 0$. The stationary solution $(0, \eta)$ will be denoted simply by $\eta$.

It is worth noting that we look for solutions that satisfy the condition

$$(1.4)\qquad 0 < \operatorname*{ess\,inf}_{x \in \Omega} \eta(x), \qquad \operatorname*{ess\,sup}_{x \in \Omega} \eta(x) < +\infty .$$

Denote by $]a, b[$ the image of the increasing function $\pi$. In other words, $a = \pi(0)$, $b = \pi(+\infty)$. Since $\pi(1) = 0$, one has $-\infty \leqq a < 0 < b \leqq +\infty$. Let $\Phi = \pi^{-1}$ be the inverse function of $\pi$. Clearly, $\Phi(]a, b[) = ]0, +\infty[$. We set $\Phi(a) = 0$, $\Phi(b) = +\infty$. Finally, we define

$$m_0 \equiv \operatorname*{ess\,inf}_{x \in \Omega} F(x), \qquad M_0 \equiv \operatorname*{ess\,sup}_{x \in \Omega} F(x).$$

In [1] we proved, in particular, the following result.

THEOREM 1.1. — *There is a stationary solution of problem (1.1) if and only if*

$$(1.5)\qquad a - m_0 < b - M_0,$$

*and*

$$(1.6)\qquad \int_0^1 \Phi(a + F(x) - m_0)dx < 1 < \int_0^1 \Phi(b + F(x) - M_0)dx.$$

*In this case the stationary solution is unique, moreover*

$$\eta(x) = \Phi(k + F(x)), \qquad \forall x \in \Omega, \tag{1.7}$$

*where k is the (unique) solution of the equation*

$$\int_0^1 \Phi(k + F(x))dx = 1 \tag{1.8}$$

*in the interval* $]a - m_0, b - M_0[$.

We also showed the following result in [*1*], § 5.

COROLLARY 1.2. — *The conditions (1.5) and* $(1.6)_1$ [*resp.* $(1.6)_2$] *hold for every* $F \in L^\infty(\Omega)$ *if* $a = -\infty$ [*resp.* $b = +\infty$]. *In particular, the stationary solution exists for every* $F \in L^\infty$, *if* $]a, b[ = ]-\infty, +\infty[$.

In particular ([*1*], corollary 5.5), the stationary solution exists if

$$M_0 - m_0 < \min\{-a, b\}, \tag{1.9}$$

hence if $|F|_\infty < (1/2)\min\{-a, b\}$. Note that this minimum is always strictly positive. Another sufficient condition for the existence of the stationary solution ([*1*], Eq. (5.13)) is

$$\left| \int_x^y f(\tau)d\tau \right| < \min\{-a, b\}, \qquad \forall x, y \in \Omega. \tag{1.10}$$

Note that the set of external forces $f$ in $L^\infty(\Omega)$ that satisfy condition (1.10) is unbounded, even when $\min\{-a, b\} < +\infty$.

Let us now consider the particular case $p(s) = Cs^\gamma$, where C and $\gamma$ are positive constants. If $\gamma = 1$ the corollary 1.2 shows that the stationary solution $\eta(x)$ exists, for every $F \in L^\infty(\Omega)$. On the contrary, if $\gamma > 1$, the stationary solution exists if and only if the condition $(1.6)_1$ is satisfied (since $a > -\infty$, $b = +\infty$). If condition $(1.6)_1$ is not satisfied, then vacuum occurs. Note that, for $\gamma > 1$, vacuum occurs even for a constant external force $f(x) \equiv \beta$, provided that $|\beta|$ is sufficiently large.

Finally, if $\gamma < 1$ one has $a = -\infty$, $b < +\infty$. Hence the stationary solution exists if and only if the condition $(1.6)_2$ is satisfied. If this condition is not satisfied infinite density occurs. Nevertheless, such a phenomena cannot occur if $f \in L^\infty(\Omega)$. Let us prove this last assertion.

PROPOSITION 1.3. — *Let* $p(s) = Cs^\gamma$, $0 < \gamma < 1$, *and let* $f \in L^\infty(\Omega)$. *Then, the stationary solution exists.*

*Proof.* — Set, for convenience, $C = 1$. One has $\pi(s) = \gamma(s^{\gamma-1} - 1)/(\gamma - 1)$. Hence $a = -\infty$, $b = \gamma/(1-\gamma)$. Since $a = -\infty$, conditions (1.5) and $(1.6)_1$ are satisfied. Let us show that $(1.6)_2$ holds if F is Lipschitz continuous.

The function $\Phi$ is given by $\Phi(y) = [1 - (1 - \gamma)y/\gamma]^{-1/(1-\gamma)}$. It readly follows that the condition $(1.6)_2$ becames, in the present case,

$$(1.11) \qquad \int_0^1 [M_0 - F(x)]^{-1/(1-\gamma)}dx > \left(\frac{1-\gamma}{\gamma}\right)^{-1/(1-\gamma)}.$$

Let $x_0 \in [0, 1]$ be such that $F(x_0) = M_0$, and let $\beta > 0$ be such that $F(x_0) - F(x) \leqq \beta \mid x - x_0 \mid$, $\forall x \in [0, 1]$. Hence,

$$(1.12) \quad \int_0^1 [M_0 - F(x)]^{-1/(1-\gamma)}dx \geqq \int_0^{x_0} [\beta(x_0 - x)]^{-1/(1-\gamma)}dx + \\ + \int_{x_0}^1 [\beta(x - x_0)]^{-1/(1-\gamma)}dx.$$

Since $(x - x_0)^{-1/(1-\gamma)}$ is not integrable near $x_0$, the right hand side of (1.12) is equal to $+\infty$. Hence, the left hand side satisfies (1.11).

The relationship between the regularity of $f$ and that of the stationary solution is very easy to establish. The particular case that follows will be useful in the next section.

PROPOSITION 1.4. — *Let $p \in C^2(]0, +\infty[)$, and assume that a stationary solution of the problem (1.1), (1.4) exists (or equivalently, assume that the conditions (1.5), (1.6) are satisfied). Then, $\eta$ belongs to $H^2(\Omega)$ if and only if $f$ belongs to $H^1(\Omega)$.*

*Proof.* — In fact, $f \in H^1(\Omega)$ if and only if $F \in H^2(\Omega)$. Equation (1.7) shows that the condition is sufficient. The condition is also necessary, since $F(x) = \pi(\eta(x)) - c$.

I want to quote here the recent paper [*4*], where the authors show that if a Lipschitz function $f(x)$ does not verify the assumptions of theorem 1.1 (i. e., if there is no stationary solution) then there are no solutions $(u, \rho)$ of the evolution problem which satisfy on $Q_\infty$ a estimate $k_1 \leqq \rho(t,x) \leqq k_2$, with some positive constants $k_1$, $k_2$.

## 2. STABILITY OF THE STATIONARY SOLUTION

In the following we denote by $c$ generic positive constants that depend at most on $\mu$, $p(\cdot)$, and $\eta$. Actually, the dependence of these constants on $\eta$ is only through the quantities $m$, M, $|\eta_x|_\infty$, and $\|\rho_{xx}\|$. Sometimes, we use symbols like $\bar{c}$, $c_0$, $c_1$, $c_2$, .... For convenience, we will use the abreviated notation $\int g = \int_0^1 g(x)dx$, for generic real functions on $\Omega$.

In this section we prove the stability of every stationary solution ([1]).

([1]) In higher dimensions stability is known only for small stationary solutions. See [*5*].

THEOREM 2.1. — *Assume that* $p \in C^2(]0, +\infty[)$ *satisfies the condition* $p'(s) > 0$, $\forall s > 0$. *Let* $f \in H^1(\Omega)$, *and let* $(0, \eta(x))$ *be a stationary solution of problem (1.1) in the class* $H^2(\Omega)$ ([2]). *Let* $(u_0, \rho_0) \in H_0^1(\Omega) \times H^1(\Omega)$, *and assume that* $\rho_0$ *satisfies the conditions (1.3), (1.4). Under the above assumptions, there is a positive constant* $\delta$ *such that if the initial data* $(u_0, \rho_0)$ *satisfies the condition*

$$\| u_0 \|_1^2 + \| \rho_0 - \eta \|_1^2 < \delta, \tag{2.1}$$

*the evolution problem (1.1), with initial data* $(u(0), \rho(0)) = (u_0, \rho_0)$, *has a (unique) global solution* $(u(t), \rho(t))$; *this solution satisfies the uniform estimate*

$$c_1 \leqq \rho(t, x) \leqq c_2, \qquad \forall (t, x) \in Q_\infty. \tag{2.2}$$

*Moreover,*

$$\| u(t) \|_1^2 + \| \rho(t) - \eta \|_1^2 \leqq c_3 ( \| u_0 \|_1^2 + \| \rho_0 - \eta \|_1^2 ) e^{-c_4 t}, \tag{2.3}$$

*for all* $t \in [0, +\infty[$. *Here,* $\delta$, $c_1$, $c_2$, $c_3$, $c_4$ *denote positive constants which depend only on* $\mu$, *on the particular function* $p(\cdot)$, *and on the stationary solution* $\eta(x)$.

*Proof of theorem* 2.1. — The proof of a local existence theorem for the solution of the problem (1.1) follows well known arguments. Hence we will concentrate our attention on the proof of the *a priori* estimates (2.2), (2.3) ([3]). Denote by $R(t, x)$ the perturbation of the stationary solution, i. e. set $\rho(t, x) = \eta(x) + R(t, x)$. Let $m$ and M be positive constants such that

$$4m \leqq \eta(x) \leqq M, \qquad \forall x \in \Omega. \tag{2.4}$$

We assume in the following that $R(t, x)$ satisfies the condition

$$| R(t, x) | \leqq 2m \tag{2.5}$$

and we show that (if $\delta$ is sufficiently small) we must have $| R(t, x) | \leqq m$. This shows, in particular, that (2.2) is satisfied.

Let us make some remarks on the function $\pi(\rho(t, x))$. For convenience, we denote here, by R, either a real number $R \in [-2m, 2m]$ or a function $R(t, x)$. Since $\pi \in C^2(]0, +\infty[)$, it follows that

$$\pi(\eta(x) + R) - \pi(\eta(x)) = \pi'(\eta(x))R + \omega_0(x, R)R, \tag{2.6}$$

for all $(x, R) \in \Omega x [-2m, 2m]$, where $|\omega_0(x, R)| \leqq c\,|R|$. We set

$$\omega_0(R)(t, x) = \omega_0(x, R(t, x)).$$

([2]) By the results of section 1, the stationary solution $\eta$ exists if and only if $f$ satisfies the conditions (1.5), (1.6). Moreover, the proposition 1.4 shows that $\eta \in H^2(\Omega)$.

([3]) These estimates, together with the local existence theorem, show that the solution is global.

Similarly, the function $\omega(x, \mathrm{R}) \equiv \pi'(\eta(x) + \mathrm{R}) - \pi'(\eta(x))$ satisfies on $\Omega \times [-2m, 2m]$ the estimate $|\omega(x, \mathrm{R})| \leqq c\mathrm{R}$. We set

$$\omega(\mathrm{R})(t, x) = \omega(x, \mathrm{R}(t, x)).$$

Since $\pi(\eta(x))_x = f(x)$, the perturbation $(u, \mathrm{R})$ satisfies the equations

$$(2.7) \qquad \begin{cases} u_t + uu_x - \dfrac{\mu}{\eta + \mathrm{R}} u_{xx} + \pi(\eta + \mathrm{R})_x - \pi(\eta)_x = 0, \\ \mathrm{R}_t + (\eta u)_x + (\mathrm{R}u)_x = 0, \end{cases}$$

the boundary condition $(1.1)_3$, and the constraint

$$(2.8) \qquad \int_0^1 \mathrm{R}(t, x)dx = 0 \qquad \forall t \geqq 0.$$

The equation $(2.7)_1$ will also be used in the equivalent forms

$$(2.9) \qquad u_t + uu_x - \frac{\mu}{\eta + \mathrm{R}} u_{xx} + (\pi'(\eta)\mathrm{R} + \omega_0(\mathrm{R})\mathrm{R})_x = 0,$$

and

$$(2.10) \quad u_t + uu_x - \frac{\mu}{\eta + \mathrm{R}} u_{xx} + \pi'(\eta)\mathrm{R}_x + \omega(\mathrm{R})\eta_x + \omega(\mathrm{R})\mathrm{R}_x = 0.$$

By multiplying both sides of the equation (2.9) by $\eta u$, and integrating on $\Omega$, one gets

$$\frac{1}{2}\frac{d}{dt}\int \eta u^2 + \int \eta u^2 u_x + \mu \int \eta(\eta + \mathrm{R})^{-1} u_x^2 + \mu \int (\eta_x \mathrm{R} - \mathrm{R}_x \eta)(\eta + \mathrm{R})^{-2} uu_x - \\ - \int (\eta u)_x \pi'(\eta)\mathrm{R} - \int (\eta u)_x \omega_0(\mathrm{R})\mathrm{R} = 0.$$

Since $2m \leqq \eta + \mathrm{R} \leqq \mathrm{M} + 2m$, it follows that

$$(2.12) \quad \frac{1}{2}\frac{d}{dt}\int \eta u^2 + c\,\|u_x\|^2 - \int (\eta u)_x \pi'(\eta)\mathrm{R} \leqq \\ \leqq c\,|\mathrm{R}|_\infty \|u\| \;\; \|u_x\| + c\,\|\mathrm{R}_x\| \;\; |u|_\infty \|u_x\| + c\,|u|_\infty \|\mathrm{R}\|^2 + c\,|u_x|_\infty \|\mathrm{R}\|^2 \\ + c\,|u|_\infty^2 \|u_x\| .$$

On the other hand, by multiplying $(2.7)_2$ by $\pi'(\eta)\mathrm{R}$, and integrating on $\Omega$, one gets

$$(2.13) \quad \frac{1}{2}\frac{d}{dt}\int \pi'(\eta)\mathrm{R}^2 + \int (\eta u)_x \pi'(\eta)\mathrm{R} \leqq c\,|u|_\infty \|\mathrm{R}_x\| \;\|\mathrm{R}\| + c\,|u_x|_\infty \|\mathrm{R}\|^2.$$

By adding (2.12) and (2.13) one obtains

$$(2.14)\quad \frac{1}{2}\frac{d}{dt}\int \eta u^2 + \frac{1}{2}\frac{d}{dt}\int \pi'(\eta)\mathrm{R}^2 + c\int u_x^2 \leqq c(\|\mathrm{R}_x\|^3 + \|u_x\|^3).$$

Note that $\|\mathrm{R}\| \leqq |\mathrm{R}|_\infty \leqq \|\mathrm{R}_x\|$, $\|u\| \leqq |u|_\infty \leqq \|u_x\|$.

Define

$$y(t, x) = \int_0^x \mathrm{R}(t, \xi)d\xi.$$

Clearly,

$$y(t, 0) = y(t, 1) = 0, y_x = \mathrm{R}, \|y_x\| = \|\mathrm{R}\|, \text{ and } \|y\| \leqq |y|_\infty \leqq |\mathrm{R}|_1 \leqq \|\mathrm{R}\|.$$

Equation $(2.7)_2$ shows that $y_t = -(\eta + \mathrm{R})u$. In particular, $u_t y = (uy)_t + (\eta + \mathrm{R})u^2$. Hence, by multiplying the equation $(2.9)_1$ by $y$ and integrating on $\Omega$, we easily obtain

$$\frac{d}{dt}\int uy + \int (\eta + \mathrm{R})u^2 - \frac{1}{2}\int u^2\mathrm{R} + \mu\int(\eta+\mathrm{R})^{-1}\mathrm{R}u_x$$
$$- \mu\int (\eta_x + \mathrm{R}_x)(\eta+\mathrm{R})^{-2}u_x y - \int \pi'(\eta)\mathrm{R}^2 - \int \omega_0(\mathrm{R})\mathrm{R}^2 = 0.$$

In particular,

$$(2.15)\quad -\frac{1}{2}\frac{d}{dt}\int uy + c\int \mathrm{R}^2 \leqq c(\|\mathrm{R}_x\|^3 + \|u_x\|^3) + c\|u\|^2 + + c\theta\|u_x\|^2 + c\theta^{-1}\|\mathrm{R}\|^2,$$

where the constant $\theta \geqq 1$ will be chosen later.

Multiplying the equation (2.10) by $-\eta u_{xx}$, integrating on $\Omega$, and taking

into account that $-\int u_t \eta u_{xx} = (1/2)\left(\int \eta u_x^2\right)_t + \int u_t \eta_x u_x$, and by using for $u_t$ the expression obtained from equation (2.10), it readily follows that

$$(2.16)\quad \frac{1}{2}\frac{d}{dt}\int \eta u_x^2 + c\|u_{xx}\|^2 - \int \pi'(\eta)\eta \mathrm{R}_x u_{xx} \leqq$$
$$\leqq c(\|u_x\|^3 + \|u_x\|^2 + \|\mathrm{R}_x\|\|u_x\| + \|\mathrm{R}_x\|^2\|u_x\| + \|u_x\|^4 + \|\mathrm{R}\|^2 + \|\mathrm{R}_x\|^4),$$

where Cauchy-Schwartz inequality was used in order to drop the term containing $\|u_{xx}\|$ from the right hand side of the inequality.

On the other hand, by differentiating with respect to $x$ both sides of equation $(2.7)_2$, then multiplying by $\pi'(\eta)R_x$, integrating on $\Omega$, and taking into account of the fact that

$$\int \pi'(\eta)uR_xR_{xx} = -\frac{1}{2}\int \pi'(\eta)u_xR_x^2 - \frac{1}{2}\int \pi''(\eta)\eta_x uR_x^2,$$

one gets

$$(2.17)\quad \frac{1}{2}\frac{d}{dt}\int \pi'(\eta)R_x^2 + \int \pi'(\eta)\eta R_x u_{xx} \leqq$$
$$\leqq c(\| u_x\| \; \| R_x\| + \| R_x\|^2 \| u_{xx}\| + \| R_x\|^2 | u_x|_\infty + \| u_x\| \; \| R_x\|^2).$$

By adding the equation (2.16) and (2.17), and by using Cauchy-Schwartz inequality, it readily follows that

$$(2.18)\quad \frac{1}{2}\frac{d}{dt}\int \eta u_x^2 + \frac{1}{2}\frac{d}{dt}\int \pi'(\eta)R_x^2 + c\| u_{xx}\|^2 \leqq$$
$$\leqq c(\| u_x\|^3 + \| u_x\|^4 + \| R_x\|^3 + \| R_x\|^4) + c(\| u_x\|^2 + \| R\|^2)$$
$$+ c(\theta^{-3}\| R_x\|^2 + \theta^3 \| u_x\|^2).$$

In proving (2.18), the term $\| R_x\|^2 | u_x|_\infty$, occurring on the right hand side of equation (2.17), was estimated as follows. Since $u_x$ has mean value zero on $\Omega$, one has $\| R_x\|^2 \; | u_x|_\infty \leqq \sqrt{2}\, \|R_x\|^2 \| u_x\|^{1/2} \| u_{xx}\|^{1/2}$. By using Young's inequality one shows that $\| R_x\|^2 \; | u_x|_\infty \leqq c \; \| R_x\|^{8/3} \; \| u_x\|^{2/3} + c_0 \| u_{xx}\|^2$. The term $c_0 \| u_{xx}\|^2$ was droped from the right hand side of equation (2.17), by chosing a sufficiently small value for the positive constant $c_0$. Finally the term $\| R_x\|^{8/3} \; \| u_x\|^{2/3}$ is bounded by $\| u_x\|^3 + \| u_x\|^4 + \| R_x\|^3 + \| R_x\|^4$.

Now, we multiply the equation (2.10) by $R_x$, we take into account of the fact that $u_tR_x = (uR_x)_t + u(\eta u)_{xx} + u(Ru)_{xx}$, and we integrate on $\Omega$. This leeds to the estimate

$$(2.19)\quad \int \pi'(\eta)R_x^2 - \frac{d}{dt}\int u_xR \leqq$$
$$\leqq c\| u_x\|^2 + c\| R_x\| \; \| u_x\|^2 + c\| R_x\|^3 + c(\theta^{-1}\| R_x\|^2 + \theta\| u_{xx}\|^2) +$$
$$+ c(\theta^{-1}\| R_x\|^2 + \theta\| R\|^2),$$

where $\theta$ is as above. Finally, we multiply the equation (2.14) by $\theta^4$, the equation (2.15) by $\theta^2$, the equation (2.19) by $\theta^{-2}$, and add these equations and also the equation (2.18). Denoting by $\tilde{c}_\theta$ a positive constant that depends only on $\mu$, $p(\cdot)$, $\eta(x)$, and on $\theta$, one gets

$$(2.20)\quad \frac{1}{2}\frac{d}{dt}\phi^2(t)+c[\theta^4\|u_x\|^2+\theta^2\|\mathrm{R}\|^2+\|u_{xx}\|^2+\theta^{-2}\|\mathrm{R}_x\|^2]\leqq$$
$$\leqq \tilde{c}_\theta(\|u_x\|^3+\|u_x\|^4+\|\mathrm{R}_x\|^3+\|\mathrm{R}_x\|^4)+c\theta^3\|u_x\|^2+$$
$$+c\theta\|\mathrm{R}\|^2+c\theta^{-3}\|\mathrm{R}_x\|^2+c\theta^{-1}\|u_{xx}\|^2.$$

In deducing (2.20) we use the fact that $\|u\| \leqq \|u_x\|$, and that $\theta \geqq 1$. By definition

$$(2.21)\quad \phi^2(t)=\int[\theta^4\eta u^2+\theta^4\pi'(\eta)\mathrm{R}^2-\theta^2 uy+\eta u_x^2+\pi'(\eta)\mathrm{R}_x^2-2\theta^{-2}u_x\mathrm{R}].$$

Since the functions $\eta(t, x)$ and $\pi'(\eta(t, x))$ are bounded away from zero and from infinity by positive constants of type $c$, it easily follows that

$$c\theta^4(\|u\|^2+\|\mathrm{R}\|^2)+c(\|u_x\|^2+\|\mathrm{R}_x\|^2)$$
$$\leqq \phi^2(t)\leqq c\theta^4(\|u\|^2+\|\mathrm{R}\|^2)+c(\|u_x\|^2+\|\mathrm{R}_x\|^2)$$

provided that $\theta \geqq \bar{c}$, for a suitable constant $\bar{c}$. On the other hand, the last four terms on the right hand side of the equation (2.20) are bounded by the second term on the left hand side of equation (2.20), provided that $\theta \geqq \bar{c}_1$, for a suitable $\bar{c}_1$. By choosing $\theta = \max\{\bar{c}, \bar{c}_1\}$, it follows that

$$(2.22)\quad \frac{1}{2}\frac{d}{dt}\phi^2(t)+c_5\phi^2(t)+c_6\|u_{xx}\|^2\leqq c_7[\phi^4(t)+\phi^3(t)],$$

and that

$$(2.23)\quad c_8(\|u\|_1^2+\|\mathrm{R}\|_1^2)\leqq\phi^2(t)\leqq c_9(\|u\|_1^2+\|\mathrm{R}\|_1^2).$$

In particular, $(\phi^2(t))_t + c_5\phi^2(t) \leqq 0$, $\forall t \geqq 0$, if $c_7(\phi^2(0)+\phi(0)) \leqq c_5/2$.

Hence, there exists a suitable constant $c_9$ such that $\phi^2(t)\leqq\phi^2(0)\exp[-c_5 t]$, if $\phi^2(0) \leqq c_9$. By using (2.23), it follows that there are constants $c_{10}$, $c_{11}$ such that

$$(2.24)\quad \|u(t)\|_1^2+\|\mathrm{R}(t)\|_1^2\leqq c_{10}(\|u(0)\|_1^2+\|\mathrm{R}(0)\|_1^2)e^{-c_5 t},$$
$$\text{if}\quad \|u(0)\|_1^2+\|\mathrm{R}(0)\|_1^2\leqq c_{11}.$$

Finally, we set (see (2.1)) $\delta=\min\{c_{11}, m^2/c_{10}\}$. Note that $u_0=u(0)$, $\rho(0)-\eta=\mathrm{R}(0)$. Since $\delta\leqq m^2/c_{10}$, the equation (2.26) shows that $|\mathrm{R}(t)|_\infty^2 \leqq \|\mathrm{R}_x\|^2\leqq m^2$, $\forall t\geqq 0$. Consequently, $|\mathrm{R}(t)|_\infty \leqq m$, for all $t \geqq 0$. The proof of theorem 2.1 is complete.

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