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An approach of deterministic control problems with unbounded data

by

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ABSTRACT. — We prove that the value function of a deterministic unbounded control problem is a viscosity solution and the maximum viscosity subsolution of a family of Bellman Equations; in particular, the one given by the hamiltonian, generally discontinuous, associated formally to the problem by analogy with the bounded case. In some cases, we show that this equation is equivalent to a first-order Hamilton-Jacobi Equation with gradient constraints for which we give several existence and uniqueness results. Finally, we indicate other applications of these results to first-order H. J. Equations, to some cheap control problems and to uniqueness results in the nonconvex Calculus of Variations.

Key-words: Deterministic unbounded control problems, Bellman Equations, Hamilton-Jacobi Equations, gradient-constraints, comparison results.

RÉSUMÉ. — Nous prouvons que la fonction valeur d'un problème de contrôle déterministe non borné est une solution de viscosité et la sous-solution de viscosité maximale d'une famille d'équations de Bellman; en particulier, celle donnée par l'hamiltonien, généralement discontinu, associé formellement au problème par analogie avec le cas borné. Dans certains cas, nous montrons que cette équation est équivalente à une équation de Hamilton-Jacobi du premier ordre avec contraintes sur le gradient pour laquelle nous donnons des résultats d'existence et d'unicité variés. Enfin, nous indiquons d'autres applications de ces résultats à des équations de H. J. du premier ordre, à certains problèmes de contrôle impulsionnel ainsi qu'à des résultats d'unicité dans des problèmes non convexes du Calcul des Variations.

Mots-clés : Contrôle déterministe non borné, équations de Bellman, équations de Hamilton-Jacobi, contraintes sur le gradient, résultats de comparaison.

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INTRODUCTION

The starting point of this work is the study of deterministic unbounded control problems in $\mathbb{R}^N$. « Unbounded » means both that the space of controls is not compact and that the given functions of the problems (i. e. the field in the Dynamics, the running cost and the discount factor) may have unbounded $W^{1,\infty}$ norms. Our aim is to show how to get a Bellman Equations for the value function of the control problem and to discuss uniqueness properties for this equation. The difficulty is that, since the dynamic is unbounded, it is, in general, impossible to derive directly the Bellman Equation from the Dynamic Programming Principle as in the classical case of bounded control (cf. W. H. Fleming and R. W. Rishel [*14*], P. L. Lions [*22*]). The other difficulty, closely connected to the preceding one, is that the Hamiltonian (which we obtain at least formally by analogy with the classical case) may be discontinuous and, in particular, may have a domain—in the sense of convex analysis—which is not all the space. Before giving details on our results and our methods, let us precise that our approach is based on the notion of viscosity solution introduced by M. G. Crandall and P. L. Lions [*10*] (see also M. G. Crandall, L. C. Evans and P. L. Lions [*9*], and P. L. Lions [*22*]) and extended to discontinuous Hamiltonians by H. Ishii [*16*].

In the first part, we consider a very general—but necessarily coarse—approach of the problem: we approximate the control problem by problems set on compact subsets of the control space; by classical results of P. L. Lions [*22*], we get a Bellman Equation for the value function of the approximated problem and we pass to the limit by the stability result of G. Barles and B. Perthame [*5*], extended by H. Ishii [*16*]. The obtained Bellman Equation is associated to the formal Hamiltonian obtained by analogy with the bounded case and we prove that *the value function of the unbounded control problem is the maximum viscosity subsolution (and solution) of this Equation.* Another approach consists in dealing with rescaled Hamiltonians: although the obtained equation is not, in general, equivalent to the preceding one, the above result remain valid for this equation. Then, we are interested in further investigations on the uniqueness properties of the Bellman Equation. In order to motivate the following results and to show the typical phenomena due to the unboundedness of the control, let us give an example in dimension 1. We consider the value function

$$u(x)=\operatorname{Inf}\left\{\int_0^\infty [f(y)\,|\,\dot{y}\,|^2+g(y)\,|\,\dot{y}\,|+h(y)]e^{-t}dt\,;\,y(\cdot)\in W^{1,\infty}((0,\infty);\mathbb{R})\right\}$$

for $x\in\mathbb{R}$; f, g, h are, say, bounded and lipschitz functions.

The formal Hamiltonian associated to the problem is

$$H(x, t, p) = \sup_{q\in\mathbb{R}} \{ q\cdot p - f(x) \mid q \mid^2 - g(x) \mid q \mid - h(x) \} + t$$

Three typical cases can be considered

a) $f(x) \geqslant \alpha > 0$ in $\mathbb{R}$.

This corresponds to a coercivity assumption on the running cost. In this case, H is continuous and the existence, uniqueness and regularity properties of viscosity solutions of

$$\text{(B)} \qquad H(x, u, Du) = 0 \quad \text{in} \quad \mathbb{R}^N$$

has been studied by many authors (cf. references in the bibliography).

b) $f \equiv 0, \quad g(x) \geqslant \alpha > 0$ in $\mathbb{R}$.

In this case, H is discontinuous and it is easy to see that the Bellman Equation (B) is, at least formally, equivalent to

$$\begin{cases} \mid u' \mid \leqslant g(x) & \text{and} \quad u(x) \leqslant h(x) \quad \text{in } \mathbb{R}, \\ u(x) \geqslant h(x) & \text{if} \quad \mid u' \mid < g(x) \quad \text{in } \mathbb{R}. \end{cases}$$

Of course, all the inequalities have to be understood in the viscosity sense. There exists, at least, two ways to see these equations. The first one is as a *gradient constraint* i. e. as a problem like

$$(\mathcal{P}) \qquad \begin{cases} Du \in C(x) & \text{and} \quad H(x, u\, Du) \leqslant 0 \quad \text{in } \mathbb{R}^{N.} \\ H(x, u, Du) \geqslant 0 & \text{if} \quad Du \in \text{Int}\, C(x) \quad \text{in } \mathbb{R}^{N.} \end{cases}$$

(In our example, $C(x) = \{ p \in \mathbb{R} / \mid p \mid \leqslant g(x) \}$, $H(x, t, p) = t - h(x)$). The second one is a classical first-order equation

$$(1) \qquad \max (H(x, u, Du);\ \varphi(x, Du)) = 0 \quad \text{in} \quad \mathbb{R}^N$$

(In our example, $\varphi(x, p) = \mid p \mid - g(x)$ and $H(x, t, p) = t - h(x)$).

A priori, the first way is more general since H may be defined only on $C(x)$. This type of situations will motivate the study of existence and uniqueness result for $(\mathcal{P})$, done at the end of the first part and in the second section.

c) $g \equiv 0$ in $\mathbb{R}$, $\quad f(x) > 0$ for $x \neq 0$, $\quad f(0) = 0$.

In this case, the Bellman Equation is (again formally) equivalent to

$$\begin{cases} \dfrac{\mid u' \mid^2}{4f(x)} + u = h \quad \text{in } \mathbb{R} - \{0\}, \\ u(0) \leqslant h(0), \qquad \mid u'(0) \mid = 0. \end{cases}$$

Then, there is typically a non-uniqueness feature: indeed, for all $\alpha \leqslant \operatorname{Inf} h$, the unique viscosity solution u_α in $W^{1,\infty}(\mathbb{R})$ of

$$\begin{cases} \dfrac{|v'|^2}{4} + f(x)v = h(x)f(x) & \text{in } \mathbb{R} - \{0\}, \\ v(0) = \alpha, \end{cases}$$

is also a viscosity solution of the above Bellman Equation. In fact, the value of u at 0 is not determined: this is a consequence of the fact that the system can move, in a neighbourhood of 0, at a spead which is almost infinite with a neglectible cost.

Strongly motivated by the example *b*), we conclude the first part by a uniqueness result for the Bellman Equation under assumptions on the dynamics, the running cost and the discount factor which generalize the example *b*) above. The proof of this result is based on the approach by rescaled Hamiltonians and on the change of variables, $v = -e^{-u}$, introduced by S. N. Kruzkov [*20*] and used by several authors in the context of viscosity solutions (cf. [*12*], [*15*], [*21*], [*22*]).

The second section is devoted to the study of problem like (P) for general first-order Hamilton-Jacobi equation—i. e. for non-necessarely convex Hamiltonians—. We prove existence and uniqueness results in three typical cases: the first one is when H is, roughly speaking, the restriction on $D = \{(x, t, p) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N / p \in C(x)\}$ of an Hamiltonian which satisfies the assumptions giving the uniqueness of viscosity solutions for first-order H. J. equations in $\mathbb{R}^N$, the second case is when $H \to +\infty$ when $p \to \partial C(x)$ and finally, the third case is when H does not depend on p. Concerning $C(\cdot)$, we impose some continuity and starshaped assumptions. In the two first cases, we prove existence and comparison results for *bounded continuous solutions;* in the third one, we obtain them for continuous solutions only *bounded from above*. Let us precise that all our method can be easily extended to problems given in a domain (different of $\mathbb{R}^N$) with Dirichlet boundary conditions, to state-constraints problems and to exit time problems. We refer the reader to the bibliography for references on such problems. Some results concerning related problems are obtained by E. N. Barron [*7*] in the case of the monotone control problem and by S. Delaguiche [*13*], in an economical context, where H is convex and C does not depend on x. Our third result is inspired by the works of S. N. Kruzkov [*20*], M. G. Crandall and P. L. Lions [*12*], P. L. Lions [*22*], H. Ishii [*15*] and J. M. Lasry and P. L. Lions [*21*], on the uniqueness properties of the equation

$$\text{(E)} \qquad \begin{cases} H(x, Du) = 0 & \text{in } \Omega, \\ u = \psi & \text{on } \partial\Omega. \end{cases}$$

The third part is devoted to show some applications of our results and our methods which, *a priori*, may be different from the motivation coming from example *b*). We consider three main applications; the first one concerns the equation (E): under geometrical and continuity assumptions on H, we prove a comparison result for viscosity sub and supersolution of (E) bounded from below. This result slightly extends those of [*20*], [*12*], [*22*], [*15*] and [*21*]. The second application concerns cheap control problems closely connected to [*7*]. Finally, we mention the connections of ($\mathcal{P}$) with some non-convex problems in the Calculus of Variations (cf. E. Mascolo [*24*], [*25*], [*26*] and E. Mascolo and R. Schianchi [*27*], [*28*], [*29*]). Our work provides some uniqueness results to them.

I. ON UNBOUNDED CONTROL PROBLEMS

We are interested in this part in unbounded control problems in $\mathbb{R}^N$. We recall that « unbounded » means both that the space of control is not compact and that the dynamic, the running cost and the discount factor may have unbounded $W^{1,\infty}$ norm. The new point is that it is generally impossible to derive a Bellman Equation directly from the Dynamic Programming Principle and that the Hamiltonian (that we can write formally by analogy with the classical case) may be discontinuous and even be $+\infty$ at almost all points! In a first section, we describe a general approach of such problems; we show that the value function is a viscosity solution of the Bellman Equation for the formal Hamiltonian mentionned above in the sense of H. Ishii [*16*]. And even it is the maximum viscosity subsolution of this equation. An other approach consists in dealing with rescaled Hamiltonians; in particular, this permits to deal with locally bounded Hamiltonians. A second section is devoted to prove a comparison result for the Bellman Equation under assumptions on the dynamics, the running cost and the discount factor which generalize example *b*) of the introduction. Let us recall that the idea of the proof is based both on the approach by rescaled Hamiltonians and on the change of variables of S. N. Kruzkov [*20*] (cf. also [*12*], [*15*] [*21*], [*22*]). This method can be extended to more general context, in particular for differential games.

a. The general approach.

In order to be more specific, let us describe the control problem. We consider a system which state is given by the solution y_x of the following O. D. E.

$$(2) \qquad dy_x(t) + b(y_x(t), v(t))dt = 0, \qquad y_x(0) = x \in \mathbb{R}^n.$$

For each trajectory $y_x(\cdot)$ and each control $v(\cdot)$, the cost function is defined by

$$J(x, v(\cdot)) = \int_0^{+\infty} f(y_x(t), v(t)) \exp\left(-\int_0^t c(y_x(s), v(s))ds\right)dt$$

and we are interested in the value function

$$u(x) = \operatorname*{Inf}_{v(\cdot) \in L^\infty(0,\infty,V)} \{ J(x, v(\cdot)) \} \tag{3}$$

V is a metric space, f, c, b are given functions from $\mathbb{R}^N \times V$ into respectively $\mathbb{R}$, $\mathbb{R}$ and $\mathbb{R}^N$. Our aim consists both in giving a Bellman Equation for u and in proving that this equation characterizes u in the sense either that u is the unique solution of this equation or at least (as it will be the case) that u is the maximum subsolution of this equation. By analogy with the classical case, we will consider the Hamiltonian

$$H(x, t, p) = \sup_{v \in V} \{ b(x, v)\cdot p + c(x, v)\cdot t - f(x, v) \} \tag{4}$$

and the rescaled Hamiltonians H_Φ, defined for any real valued function Φ on $\mathbb{R}^N \times V$ by

$$H_\Phi(x, t, p) = \sup_{v \in V} \left\{ \frac{b(x, v)\cdot p + c(x, v)\cdot t - f(x, v)}{\Phi(x, v)} \right\} \tag{5}$$

H and H_Φ are, in general, discontinuous, H may be $+\infty$ at some points; H_Φ are introduced to deal with locally bounded Hamiltonians for a suitable choice of Φ. We are going to recall the definition of viscosity solutions for discontinuous solutions and discontinuous Hamiltonians. We need the following definition.

DEFINITION 1. — Let v be a locally bounded function in $\mathbb{R}^N$. The lower semi-continuous envelope of v (l. s. c. in short) is the function $v_\star$ defined by

$$v_\star(x) = \liminf_{y \to x} v(y)$$

The upper semicontinuous envelope of v (u. s. c. in short) is the function $v^\star$ defined by

$$v^\star(x) = \limsup_{y \to x} v(y)$$

Now, we recall the Ishii [*16*]'s definition of viscosity sub and supersolutions of

$$H(x, u, Du) = 0 \quad \text{in} \quad \mathbb{R}^N \tag{HJ}$$

where H is a function which takes its values in $\mathbb{R} \cup \{ -\infty, +\infty \}$.

DEFINITION 2. — Let v be a locally bounded function in $\mathbb{R}^N$, we say that v is viscosity subsolution (resp. supersolution) of (HJ) iff

$$i)\quad \begin{cases} \forall x_0 \in \mathbb{R}^N,\ \forall \phi \in C^1(\mathbb{R}^N) \quad \text{if} \quad x_0 \text{ is a local maximum point of } v^\star - \phi, \\ \text{then} \\ \qquad H_\star(x_0, v^\star(x_0),\ D\phi(x_0)) \leqslant 0 \end{cases}$$

(resp.

$$ii)\quad \begin{cases} \forall x_0 \in \mathbb{R}^N, \forall \phi \in C^1(\mathbb{R}^N) \quad \text{if} \quad x_0 \text{ is a local minimum point of } v_\star - \phi, \\ \text{then} \\ \qquad H^\star(x_0, v_\star(x_0),\ D\phi(x_0)) \geqslant 0. \end{cases}$$

v is a viscosity solution of (HJ) iff it satisfies both *i*) and *ii*).

REMARK I.1. — In this definition, $H_\star$ and $H^\star$ may be $+\infty$ or $-\infty$ with the natural properties $-\infty \leqslant 0 \leqslant +\infty$.

To state our results, we need the following assumptions

$$(6)\quad \begin{cases} \text{For } \varphi = b_i\ (1 \leqslant i \leqslant n),\ f \text{ and } c, \text{ we have} \\ \varphi \in C(\mathbb{R}^N \times V), \forall v \in V, \varphi(\cdot, v) \in W^{1,\infty}(\mathbb{R}^N) \\ \| \varphi(\cdot, v) \|_{1,\infty} \leqslant C(K) \quad \text{if} \quad v \in K, \quad \forall K \text{ compact subset of } V. \end{cases}$$

$$(7)\quad c(x, v) \geqslant \lambda(K) > 0 \quad \text{in} \quad \mathbb{R}^N \times K, \qquad \forall K \text{ compact subset of } V.$$

THEOREM I.1. — Assume (6), (7) and that u defined by (3) is bounded. Then, u is a viscosity solution of

$$(8)\qquad H(x, u, Du) = 0 \quad \text{in} \quad \mathbb{R}^N,$$

and of

$$(9)\qquad H_\Phi(x, u, Du) = 0 \quad \text{in} \quad \mathbb{R}^N,$$

for any real-valued function Φ on $\mathbb{R}^N \times V$ such that $\Phi > 0$ in $\mathbb{R}^N \times V$. Moreover, u is the maximum viscosity subsolution of (8) and (9).

REMARK II.2. — Let w be a bounded function in $\mathbb{R}^N$; then, it is easy to see that w is a viscosity subsolution of (8) iff w is a viscosity subsolution of (9). If w is a viscosity supersolution of (8), then it is also of (9) but the converse is false, in general; so that (8) contains more informations than (9).

The first comment that we can do is on the definition of viscosity solution for discontinuous Hamiltonians of H. Ishii [*16*]. The study of unbounded control problems provides some justification of this notion. Indeed, consider the following example: $b(x, v) = v \in \mathbb{R}^N$, $f(x, v) = f(x) \in W^{1,\infty}(\mathbb{R}^N)$ and $c(x, v) \equiv 1$ in $\mathbb{R}^N$, then

$$H(x, t, p) = \begin{cases} +\infty & \text{if} \quad p \neq 0 \\ t - f(x) & \text{if} \quad p = 0 \end{cases}$$

and $u \equiv \operatorname*{Inf}_{\mathbb{R}^N} f$. The important point is that $H^\star \equiv +\infty$! So, we have no condition of supersolution. The second remark is that u is not supersolution with the definition

$$H_\star(x, u, Du) \geqslant 0 \quad \text{in} \quad \mathbb{R}^n$$

(except if $f \equiv \inf_{\mathbb{R}^N} f$!). No stronger notion of solutions than the Ishii's one seems possible in this case. The final remark is that if the infimum of f is not achieved, there is no optimal trajectory even in the weak sense of a jump; this is certainly why there is no condition of supersolution.

Let us remark that if c satisfies

$$(10) \qquad c(x, v) \geqslant \lambda(1 + \| b(\cdot, v) \|_{1,\infty} + \| c(\cdot, v) \|_{1,\infty} + \| f(\cdot, v) \|_{1,\infty})$$

for some $\lambda > 0$ then the Hamiltonian H_Φ with

$$\phi(x, v) \equiv \phi(v) = 1 + \| b(\cdot, v) \|_{1,\infty} + \| c(\cdot, v) \|_{1,\infty} + \| f(\cdot, v) \|_{1,\infty}$$

satisfies the assumptions of the uniqueness result for bounded viscosity solutions (cf. M. G. Crandall, H. Ishii and P. L. Lions [*11*]). Therefore, u is the unique bounded viscosity solution of (9). Curiously, this type of assumption can be found in a certain class of exit time or stopping time problems where $c \equiv 0$ after the change of variable of S. N. Kruzkov [*20*], $v = \psi(u) = -e^{-u}$. Let us give the following typical example in a geodesic problem

$$u(x) = \operatorname*{Inf}_{(\xi,t)} \left\{ \int_0^t \{ V(\xi) \, | \dot{\xi} |^2 + f(\xi) \} \, ds / \xi(0) = x, \, \xi(t) = 0 \right\},$$

$$v(x) = \psi\left(\operatorname*{Inf}_{(\xi,t)} \left\{ \int_0^t \{ V(\xi) \, | \dot{\xi} |^2 + f(\xi) \} \, ds / \xi(0) = x, \, \xi(t) = 0 \right\} \right).$$

Since ψ is non-decreasing, we have

$$v(x) = \operatorname*{Inf}_{(\xi,t)} \left\{ -\exp\left(-\int_0^t (V(\xi) \, | \dot{\xi} |^2 + f(\xi)) ds \right) / \xi(0) = x, \, \xi(t) = 0 \right\},$$

v is the value function of an exit time problem, where the running cost is zero, the exit cost -1, $b(x, v) = v \in \mathbb{R}$ and

$$c(x, v) = V(x) \, | v |^2 + f(x).$$

If $V, f \in W^{1,\infty}(\mathbb{R}^N)$ and $V(x) \geqslant \epsilon > 0$ in $\mathbb{R}^N$, $f(x) \geqslant \epsilon > 0$ in $\mathbb{R}^N$ then (10) is satisfied. (Cf. for this type of result, M. G. Crandall and P. L. Lions [*10*], H. Ishii [*15*], J.M. Lasry and P. L. Lions [*21*] and P. L. Lions [*22*] or the third part.)

Proof of theorem I.1. — The proof consists in approximating the problem by classical deterministic control problems and to pass to the limit by the stability result of H. Ishii [*16*] or G. Barles and B. Perthame [*5*].

Let $(V_R)_{R \geqslant 0}$ be a sequence of compact subsets of V such that $V_R \subset V_{R'}$ if $R' \geqslant R$ and $\bigcup_{R \geqslant 0} V_R = V$ and let u_R, H_R be defined as u and H by (3) and (4) but changing V in V_R. Finally, let w be a bounded viscosity subsolution of (8).

We recall that, under the assumptions of theorem I.1, u_R is the unique bounded viscosity solution and the maximum bounded viscosity subsolution of

$$(11) \qquad H_R(x, u, Du) = 0 \quad \text{in} \quad \mathbb{R}^N$$

(cf. for the proof of this claim, P. L. Lions [*22*]).

Since $H_R \leqslant H_{R'} \leqslant H$ if $R \leqslant R'$, then w is still viscosity subsolution for H_R and $H_{R'}$, and $u_{R'}$ is still a viscosity subsolution for H_R; hence

$$(12) \qquad w \leqslant u_{R'} \leqslant u_R$$

By the stability results of [*16*], [*5*], since u_R is uniformly bounded and non-increasing, u, defined by

$$u = \operatorname*{Inf}_R u_R$$

is viscosity solution of (8) because $H = \sup_R H_R$ and by (12)

$$w \leqslant u.$$

Hence, u is the maximum viscosity subsolution of (8).

The proof for H_Φ is exactly the same; so, we will skip it. Let us just mention that—with obvious notations—the Hamiltonian H_Φ^R is the Hamiltonian of the control problem obtained by making the time change and considering the new time τ given by

$$\tau = \int_0^t \Phi(y(s), v(s))ds$$

In the case when ϕ depends only on $b(x, v)$ (for example, $\phi(x, v) = 1 + |b(x, v)|$), the interpretation is very simple: we want to see the dynamics at a time-scale connected to its speed.

REMARK I.3. — Let us conclude this part by mentioning the generalization of the examples *a*), *b*), *c*) of the Introduction in the case when $c(x, v) \equiv \lambda > 0$.

Example *a*) corresponds to the case

$$(A1) \qquad f(x, v)(1 + |b(x, v)|)^{-1} \to +\infty \quad \text{when} \quad |b(x, v)| \to +\infty \quad \text{uniformly for} \quad x \in \mathbb{R}^N.$$

Example $b)$ corresponds to the case

$$\text{(A2)} \quad \begin{cases} \text{There exists constants } C_1, C_2 > 0, D_1, D_2 \in \mathbb{R} \text{ such that} \\ \quad C_1 |b(x, v)| + D_1 \leqslant f(x, v) \leqslant C_2 |b(x, v)| + D_2 \end{cases}$$

Example $c)$ corresponds to the case

$$\text{(A3)} \quad \begin{cases} \text{There exists } x_0 \in \mathbb{R}^N \text{ such that} \\ \quad f(x_0, v)(1 + |b(x_0, v)|)^{-1} \to 0 \quad \text{when} \quad |b(x_0, v)| \to +\infty. \end{cases}$$

Let us recall that $c)$ leads to non uniqueness features and so, the next section is more particularly motivated by examples $a)$ or $b)$; and especially $b)$.

b. A uniqueness result for the Bellman Equation.

The aim of this section is to give a simple uniqueness, and even comparison result for the Bellman Equation (8) in the particular case when $c(x, v) \equiv \lambda \geqslant 0$, when (A2) holds and under restrictive assumptions on the lipschitz constants in x of b and f. Our method is based on the use of rescaled Hamiltonians and of the change of variable of S. N. Kruskov [20] (cf. also [12], [15], [21], [22]), $v = -e^{-\alpha u}$, with a suitable choice of $\alpha > 0$. We need the following assumption

$$(6') \quad \begin{cases} \text{For } \varphi = b_i \ (1 \leqslant i \leqslant n) \text{ and } f, \quad \text{we have} \\ \qquad \varphi \in C(\mathbb{R}^N \times V), \qquad \forall v \in V, \qquad \varphi(\cdot, v) \in W^{1,\infty}(\mathbb{R}^N) \\ \qquad \qquad \| \varphi(\cdot, v) \|_{L^\infty(\mathbb{R}^N)} \leqslant C(K). \\ \text{for all compact subset K of V and } v \in K. \text{ There exists } C_1 > 0 \\ \text{and } C_2 \in \mathbb{R} \text{ such that} \\ \qquad |\nabla_x \varphi(x, v)| \leqslant C_1 |\varphi(x, v)| + C_2 \quad \text{in} \quad \mathbb{R}^N \times V. \end{cases}$$

Our result is the following.

THEOREM I.2. — Assume that (6′) and (A2) holds, that $c(x, v) \equiv \lambda \geqslant 0$ and that either $\lambda > 0$ or $D_1 > 0$. If u_1 is a bounded u.s.c. subsolution of (8) and if u_2 is a bounded l.s.c. supersolution of (8) then

$$u_1 \leqslant u_2 \quad \text{in} \quad \mathbb{R}^N.$$

In particular, u given by (3) is the unique viscosity solution of (8) and is in $BUC(\mathbb{R}^N)$.

Proof of theorem I.2. — By remark II.2, u_1 and u_2 are respectively viscosity sub and supersolutions of (9) for ϕ given by

$$\phi(x, v) = 1 + |b(x, v)|.$$

Now, let us define ω_1 and $\omega_2 = -e^{-\alpha u_1}$, $\omega_2 = -e^{-\alpha u_2}$ for $\alpha > 0$. Then, ω_1 and ω_2 are respectively sub and supersolutions of

$$\tilde{H}(x, \omega, D\omega) = 0 \quad \text{in} \quad \mathbb{R}^N \tag{9'}$$

where

$$\tilde{H}(x, t, p) = \sup_{v \in V} \left\{ \frac{-b(x, v)p + \lambda \log(-t)t + \alpha f(x, v)t}{1 + |b(x, v)|} \right\}$$

It is easy to check that $\tilde{H}$ satisfies the classical continuity assumptions in x and p which ensure the classical comparison result for (9') (cf. [*9*], [*10*], [*11*], [*22*]). Only the monotonicity assumption in t is not completely clear. To check it, let us differentiate $\lambda \log(-t)t + \alpha f(x, v)t$ with respect to t; this yields

$$\lambda [\log(-t) + 1] + \alpha f(x, v).$$

Since we just need the monotonicity of $\tilde{H}$ in t in the interval $[-e^{-\alpha m}; -e^{-\alpha M}]$ where $m = \text{Min}\left(\inf_{\mathbb{R}^N} u_1; \inf_{\mathbb{R}^N} u_2\right)$ and $M = \text{Max}\left(\sup_{\mathbb{R}^N} u_1, \sup_{\mathbb{R}^N} u_2\right)$ then in this interval

$$-\alpha M \leqslant \log(-t) \leqslant -\alpha m$$

So, if $D_1 > 0$, it is enough to take α such that $-\alpha M + 1 \geqslant 0$, because by (A2)

$$\lambda [\log(-t) + 1] + \alpha f(x, v) \geqslant \alpha f(x, v) \geqslant \gamma[1 + |b(x, v)|]$$

for some $\gamma > 0$. In the other case, if $\lambda > 0$, we first change u_1, u_2 in $u_1 + K$, $u_2 + K$ for $K > 0$ large enough; f is changed in $f + \lambda K$ and D_1 in $D_1 + \lambda K$. By the comparison results of [*9*], [*10*], [*11*] or [*12*], we have

$$\omega_1 \leqslant \omega_2$$

Hence

$$u_1 \leqslant u_2.$$

Moreover, the unique viscosity solution of (9') (which is u given by (3)) is in $BUC(\mathbb{R}^N)$. So, we have a regularity result for u.

In the next section, we will study uniqueness properties for problems like ($\mathcal{P}$) in order to get more general results for the Bellman Equation and to extend them to general first-order Hamilton-Jacobi Equations.

II. HAMILTON-JACOBI EQUATIONS WITH GRADIENT CONSTRAINTS

We consider in this part existence and uniqueness results for bounded viscosity solution of the problem ($\mathcal{P}$) in $\mathbb{R}^N$. Our main motivation comes from the example b) of the Introduction; an example of such a situation in an economical context is described in S. Delaguiche [13], where, also, uniqueness and existence results are given in the case of convex Hamiltonians. The reason why we only deal with bounded solutions is that, if C does not depend on x and if $p \notin \text{Int } C$, then all affine maps $x \to a + p \cdot x$, for all $a \in \mathbb{R}$, are viscosity supersolution of $(\mathcal{P})$ and so, no comparison result is possible if we accept such maps. The boundedness is a way to avoid this difficulty but other ways can be imagined (solutions bounded from below or satisfying $\frac{u(x)}{|x|} \to 0$ when $|x| \to +\infty$ as in [13],... etc.). We will give, at the end of this part, a particular result in the case when H does not depend on p for solutions bounded from above.

In all the following, we are interested in the problem in $\mathbb{R}^N$ and we will also assume that $H(x, t, p)$ is of the form $H(x, p) + t$, for the sake of simplicity. Our methods, which are in this section more particularly inspired by M. G. Crandall, L. C. Evans and P. L. Lions [9], H. Ishii [15], S. N. Kruzkov [20], G. Barles and B. Perthame [5] and G. Barles [1], [2], can be easily adapted to treat problems in $\mathbb{R}^N$ with more general assumptions (cf. H. Ishii [15], [18], [19], G. Barles and P. L. Lions [4]) and others problems in bounded domains (cf. references above and the bibliography) and in particular state-constraints problems (cf. M. H. Soner [30], I. Capuzzo-Dolcetta and P. L. Lions [8]) and exit time problems (cf. H. Ishii [15], G. Barles and B. Perthame [5], [6]). We will investigate below three cases, which are interesting for the applications; the first one is when H is, roughly speaking, the restriction to $C(x)$ of a continuous Hamiltonian defined on all $\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N$, the second one is when $H \to +\infty$ when $p \to \partial C(x)$ and the third one is the particular case when H does not depend on p. In this last case, we will give some comparison results concerning unbounded sub and supersolutions. In order to be more specific, we need the following assumptions.

$$(H_1^\epsilon) \quad \begin{cases} \text{There exists a modulus } m_1^\epsilon \text{ such that} \\ |H(x,p)-H(x,q)| \leqslant m_1^\epsilon(|p-q|) \quad \text{for} \quad x \in \mathbb{R}^N, \quad |p| \leqslant 1/\epsilon, \\ |q| \leqslant 1/\epsilon, \quad d(p, \partial C(x)) \geqslant \epsilon, \quad d(q, \partial C(x)) \geqslant \epsilon, \quad p, q \in C(x). \end{cases}$$

$$(H_2^\epsilon) \quad \begin{cases} \text{There exists a modulus } m_2^\epsilon \text{ such that} \\ |H(x,p)-H(y,q)| \leqslant m_2^\epsilon(|x-y|(1+|p|)) \quad \text{for} \quad x, y \in \mathbb{R}^N, \\ d(p, \partial C(x)) \geqslant \epsilon, \quad d(q, \partial C(x)) \geqslant \epsilon, \quad p, q \in C(x). \end{cases}$$

$$(H_3) \qquad H(x, p) \to +\infty \quad \text{when} \quad p \to \partial C(x) \quad \text{uniformly for } x \in \mathbb{R}^N.$$

$$(\mathrm{H}_4)\quad \begin{cases} \text{There exists a continuous function } \varphi(x, p) \text{ in } \mathbb{R}^{\mathrm{N}} \times \mathbb{R}^{\mathrm{N}} \text{ such that} \\ \varphi(x, p) \leqslant \alpha(\eta) < 0 \ \text{ if } d(p, [\mathrm{C}(x)]^c) \geqslant \eta > 0 \ \text{ and } \ \varphi(x, p) \geqslant \beta(\eta) > 0 \\ \text{if } \ d(p, \mathrm{C}(x)) \geqslant \eta > 0, \quad \text{for all } \ x \in \mathbb{R}^{\mathrm{N}}, \quad p \in \mathbb{R}^{\mathrm{N}}. \end{cases}$$

$$(\mathrm{H}_5)\quad \begin{cases} \text{Let } \varphi \text{ be the function defined in } (\mathrm{H}_4). \text{ There exists} \\ \phi \in \mathrm{C}^1(\mathbb{R}^{\mathrm{N}}) \cap \mathrm{W}^{1,\infty}(\mathbb{R}^{\mathrm{N}}) \quad \text{such that} \quad \mathrm{D}\phi \in \mathrm{BUC}(\mathbb{R}^{\mathrm{N}}) \quad \text{and} \\ \qquad \varphi(x, \mu p + (1 - \mu)\mathrm{D}\phi(x)) \leqslant \mu - 1, \\ \text{for } \ x \in \mathbb{R}^{\mathrm{N}}, \quad p \in \mathrm{C}(x) \ \text{ and } \ \mu \leqslant 1. \end{cases}$$

$$(\mathrm{H}_5')\quad \begin{cases} \text{Let } \varphi \text{ be the function defined in } (\mathrm{H}_4). \text{ There exists} \\ \phi \in \mathrm{C}^1(\mathbb{R}^{\mathrm{N}}) \cap \mathrm{W}^{1,\infty}(\mathbb{R}^{\mathrm{N}}) \quad \text{such that} \quad \mathrm{D}\phi \in \mathrm{BUC}(\mathbb{R}^{\mathrm{N}}) \quad \text{and} \\ \qquad \varphi(x, \mu p + (1 - \mu)\mathrm{D}\phi(x)) \leqslant \mathrm{K}(\mu) < 0 \\ \text{for } \ x \in \mathbb{R}^{\mathrm{N}}, \quad p \in \mathrm{C}(x) \ \text{ and } \ \mu \geqslant 1. \end{cases}$$

$$(\mathrm{H}_6)\quad \begin{cases} \text{There exists modulus } m_3 \text{ and } m_4^{\mathrm{R}}, \text{ for all } 0 < \mathrm{R} < \infty \text{ such that} \\ |\varphi(x, p) - \varphi(y, q)| \leqslant m_3(|x - y|(1 + \gamma|p|)) + m_4^{\mathrm{R}}(|p - q|), \\ \text{for } x, y \in \mathbb{R}^{\mathrm{N}}, |p|, |q| \leqslant \mathrm{R} \text{ with } \gamma = 1 \text{ if } \mathrm{C}(x) \text{ is uniformly} \\ \text{bounded for } x \in \mathbb{R}^{\mathrm{N}} \text{ or if C does not depend on } x \text{ or finally} \\ \text{if either H does not depend on } p \text{ or is convex in } p; \text{ with } \gamma = 0 \\ \text{and } m_3(r) = \mathrm{L}\cdot r \ (\mathrm{L} \in \mathbb{R}) \text{ in the other cases.} \end{cases}$$

Our results are the followings

THEOREM II.1. — Assume that H and C(·) satisfy either

a) (H_1^0), (H_2^0), (H_4), (H_5), (H_6),

or

b) (H_1^ϵ), (H_2^ϵ), (H_3), (H_4), (H_6), $\forall \epsilon > 0$.

Comparison. — Let v and w be respectively a bounded u.s.c. viscosity subsolution of $(\mathcal{P})$ and a bounded l.s.c. viscosity supersolution of $(\mathcal{P})$, then

$$v \leqslant w.$$

Existence. — Assume, in addition, that $\mathrm{H}(x, 0)$ is bounded, then there exists a unique viscosity solution u of $(\mathcal{P})$ in $\mathrm{C}_b(\mathbb{R}^{\mathrm{N}})$.

THEOREM II.2. — Assume that H does not depend on p, that $\mathrm{H}(x)$ is uniformly continuous and bounded from above and that C(·) satisfies

(H_4), (H'_5) and (H_6) with m_4^R which does not depend in R if $D\phi \equiv 0$. Then, all the conclusion of theorem II.1 hold replacing bounded by bounded from above.

REMARK II.1. — We can always reduce the problem $(\mathcal{P})$ to the first-order H. J. Equation

$$\tilde{H}(x, u, Du) = 0 \quad \text{in} \quad \mathbb{R}^N \tag{13}$$

with

$$\tilde{H}(x, t, p) = \operatorname{Max}\,(H(x, p) + t;\ \varphi(x, p)),$$

using in the case when $H(x, p) \to +\infty$ when $p \to \partial C(x)$ the ideas of [*2*]. Our results use only the particular form of $\tilde{H}$ and some uniqueness properties of the equation

$$\varphi(x, Du) = 0 \quad \text{in} \quad \mathbb{R}^N.$$

This point will be precised in the proof of theorem II.2 and in part III. We have prefered to keep the geometrical form $(\mathcal{P})$ to point out that we have *a priori* the choice of φ. The above (complicated) assumptions give precise formulations on our continuity and starshapedness requirements on $C(\cdot)$. Concerning H, they are the translation of the classical uniqueness assumptions but become more complicated since H is only defined on $C(\cdot)$.

REMARK II.2. — As precised above, one can obtained more general results by weakening the assumptions on H and φ as in H. Ishii [*18*], [*19*] and in particular by assuming that m_1^ε, m_2^ε, m_3, m_4^R depend strongly in H or φ as in G. Barles and P. L. Lions [*4*].

REMARK II.3. — One can treat in the same way problems of the form

$$(\tilde{\mathcal{P}}) \quad \begin{cases} H(x, u, Du) \leqslant 0 & \text{if} \quad Du \in \operatorname{Int} C(x) \quad \text{in} \quad \mathbb{R}^N, \\ Du \in C(x) \quad \text{and} & H(x, u, Du) \geqslant 0 \quad \text{in} \quad \mathbb{R}^N. \end{cases}$$

These problems are connected to first-order H. J. Equation of the form

$$\operatorname{Min}\,(H(x, u, Du);\quad \varphi(x, Du)) = 0 \quad \text{in} \quad \mathbb{R}^N,$$

and it is easy to check that u is viscosity solution of $(\tilde{\mathcal{P}})$ iff $-u$ is viscosity solution of $(\mathcal{P})$ with $-C(x)$ and $-H(x, -t, -p)$.

REMARK II.4. — The assumptions (H'_5) of theorem II.2 are optimal as one can see by looking at the following example

1. $\operatorname{Max}\,(u, |\nabla u| - 1) = 0$ in $\mathbb{R}^N - \{0\}$, $u(0) = 0$.
1. $\operatorname{Max}\,(u, 1 - |\nabla u|) = 0$ in $\mathbb{R}^N - \{0\}$, $u(0) = 0$.

For 2., (H'_5) is satisfied and there is a unique viscosity solution which is $-|x|$. For 1., (H'_5) is not satisfied and the comparison result is false : take

$$v(x) = -|x|$$
$$w(x) = \text{Inf}(-|x|; p\cdot x) \quad \text{with} \quad |p| > 1.$$

Proof of theorem I.1.

Proof of the comparison result. — We prove it under the assumptions *a*), the proof with *b*) being easier. Before going into technical details, let us explain the heuristic idea of the proof. We are interested in $\underset{\mathbb{R}^N}{\text{Sup}}\,(v-w)$. Assume that this supremum is achieved at x_0 then

$$\nabla v(x_0) = \nabla w(x_0)$$

We know that $\nabla v(x_0) \in C(x_0)$; if $\nabla v(x_0) \in \text{Int}\, C(x_0)$, then $\nabla w(x_0) \in \text{Int}\, C(x_0)$, we have an inequality for w and we can conclude. (In particular, this is always the case with (H_3)). On the contrary, we approach this supremum by

$$\sup_{\mathbb{R}^N} (\mu v + (1-\mu)\phi - w)$$

Since $\mu\nabla v + (1-\mu)\nabla\phi \in \text{Int}\, C(x)$, $\forall x \in \mathbb{R}^N$, we are done.

Now, we give the details. The proof consists in adapting the proof of the classical comparison result of (here) M. G. Crandall, L. C. Evans and P. L. Lions [9] and in playing with the parameters as in [1]. We introduce the function

$$\psi(x, y) = \mu v(x) + (1-\mu)\phi(x) - w(y) - \frac{|x-y|^2}{\epsilon^2} - \alpha|x|^2 - \alpha|y|^2$$

where $\mu < 1$ and $\alpha > 0$, $\epsilon > 0$. μ is devoted to tend to 1, α and ϵ to 0. They will be choosen later. For the sake of clarity, we omit the dependence of ψ in μ, α, ϵ as also for $(\bar{x}, \bar{y})$ a point where the maximum of ψ is achieved. Since v is is viscosity subsolution of $(\mathcal{P})$, we have

$$\lambda_1(\bar{x}, \bar{y}) = -\frac{(1-\mu)}{\mu} D\phi(\bar{x}) + \frac{2}{\mu}\left(\frac{\bar{x}-\bar{y}}{\epsilon^2}\right) + \frac{2\alpha\bar{x}}{\mu} \in C(\bar{x})$$

and

$$H(\bar{x}, \lambda_1(\bar{x}, \bar{y})) + v(\bar{x}) \leqslant 0.$$

Now, for w

$$\lambda_2(\bar{x}, \bar{y}) = \frac{2(\bar{x}-\bar{y})}{\epsilon^2} - 2\alpha\bar{y} = \mu_n\lambda_1(\bar{x}, \bar{y}) + (1-\mu_n)D\phi(\bar{x}) - 2\alpha(\bar{x}+\bar{y})$$

We estimate $\varphi(\bar{y}, \lambda_2(\bar{x}, \bar{y}))$; we obtain

$$\varphi(\bar{y}, \lambda_2(\bar{x}, \bar{y})) \leqslant (\mu - 1) + m_3(|\bar{x} - \bar{y}|(1 + \gamma|\lambda_2(\bar{x} - \bar{y})|)) + m_4^{R^\epsilon}(2\alpha|\bar{x} + \bar{y}|)$$

where R^ϵ depends only on ϵ. Recall that there exists C depending only on $\| v \|_\infty$ and $\| w \|_\infty$ and $\delta(\epsilon, \alpha) \to 0$ when $(\epsilon, \alpha) \to (0, 0)$ such that

$$|\bar{x} - \bar{y}| \leqslant \delta(\epsilon, \alpha)\epsilon$$

$$|\bar{x}| + |\bar{y}| \leqslant C\alpha^{-1/2}$$

and $\delta(\epsilon, \alpha) \leqslant C$. Using these inequalities, we have, say, for $\alpha \leqslant 1$

$$\varphi(y, \lambda_2)(\bar{x}, \bar{y})) \leqslant (\mu - 1) + m_3(C \cdot \epsilon + \gamma(C\epsilon + C\delta(\epsilon, \alpha)) + m_4^{R^\epsilon}(2C\alpha^{1/2})$$

We fix $\mu < 1$. Then exists $\epsilon_0 > 0$ (which depends on μ) such that

$$(\bigstar) \qquad (\mu - 1) + m_3(C\epsilon + \gamma(C\epsilon + \delta(\epsilon, 0)) < 0$$

for $\epsilon \leqslant \epsilon_0$. Take $\epsilon = \epsilon_0$, then for α small enough, we have

$$\varphi(y, \lambda_2(\bar{x}, \bar{y})) < 0$$

Hence, $\lambda_2(\bar{x}, \bar{y}) \in \text{Int } C(\bar{y})$ and since w is a viscosity supersolution of $(\mathcal{P})$, this yields

$$H(\bar{y}, \lambda_2(\bar{x}, \bar{y})) + w(\bar{y}) \geqslant 0$$

therefore

$$v(\bar{x}) - w(\bar{y}) \leqslant H(\bar{y}, \lambda_2(\bar{x}, \bar{y})) - H(\bar{x}, \lambda_1(\bar{x}, \bar{y}))$$

By classical computations, if we denote by $M(\mu, \epsilon, \alpha)$ the supremum of ψ, we get

$$M(\mu, \epsilon_0, \alpha) \leqslant |\mu - 1|(\| v \|_\infty + \| \phi \|_\infty) + m_2(|\bar{x} - \bar{y}|(1 + |\lambda_2(\bar{x}, \bar{y})|)) + m_1(|\lambda_2(\bar{x}, \bar{y}) - \lambda_1(\bar{x}, \bar{y})|)$$

Letting α go to 0 and using the estimates above, we obtain

$$(14) \quad M(\mu, \epsilon_0, \alpha_0) \leqslant |\mu - 1|(\| v \|_\infty + \| \phi \|_\infty) + m_2(C\epsilon_0 + 2\delta^2(\epsilon_0, 0)) + m_1\left(\frac{|\mu - 1|}{\mu}\left(\| D\phi \|_\infty + \frac{2\delta(\epsilon_0, 0)}{\epsilon_0}\right)\right)$$

The additional difficulty to [*1*] is that, because of $(\bigstar)$, we can not fix ϵ_0 and let μ go to 1. In the general case, to pass to the limit in the m_1-term, we have to assume that $(\bigstar)$ holds for a sequence $(\mu^n, \epsilon^n) \to (1, 0)$ such that $\dfrac{|\mu^n - 1|}{\epsilon^n}$ is bounded. Using this sequence in (14), we conclude since

$$\sup_{\mathbb{R}^N} (v - w) = \lim_n M(\mu^n, \epsilon^n, 0).$$

This situation holds in particular if $\gamma = 0$ and $m_3(r) = L \cdot r$ $(L > 0)$ or if C does not depend on x where $m_3 \equiv 0$. In the case where $C(x)$ is uniformly bounded then $\delta(\epsilon_0, 0) \leqslant K\cdot\epsilon_0$ ($K > 0$) and we can first let $\epsilon_0 \to 0$ and then $\mu \to 1$. If H does not depend on p, $m_1 \equiv 0$ and the difficulty disappears. Finally, if H is convex, then the function

$$v_\mu = \mu v + (1-\mu)\phi + -(1-\mu)\,\|\,H(x, D\phi) + \phi\,\|_\infty, \qquad \text{for} \quad \mu < 1$$

is still a viscosity subsolution of ($\mathcal{P}$) and

$$\varphi(x, Dv_\mu) \leqslant \mu - 1 \qquad \text{in} \quad \mathbb{R}^N.$$

We fix μ and we compare directly v_μ and w (by the method above with « $\mu = 1$ »!); $v_\mu \leqslant w$ and letting $\mu \to 1$, we conclude. It is worth noting that in this last case, we can replace (H_1^0) by (H_1^ϵ), ($\forall\epsilon > 0$). This result in the convex case complements (together with *b*) of theorem II.1) the result of theorem I.2. This ends the proof of the comparison result. We skip the proof of the existence result which can be obtained by the Perron's method for H. J. Equation described in H. Ishii [*17*]. Let us just remark that $-\,\|\,H(\cdot, 0)_\infty$ and $+\,\|\,H(\cdot, 0)\,\|_\infty$ are respectively viscosity sub and supersolutions of ($\mathcal{P}$) and that the continuity of the solution comes directly from the above comparison result.

Proof of theorem II.2. — In the case when $\phi \equiv 0$, the idea of the proof is to reduce the problem to the situation of

$$\varphi(x, p) = 1 - \lambda(x, p)$$

with λ positively homogeneous of degree 1 in p and then to do the change of variable of S. N. Kruzkov [*20*]'s type, $v = e^u$. This idea slightly extends or complements some results for this type of problems of M. G. Crandall and P. L. Lions [*10*], H. Ishii [*15*], J. M. Lasry and P. L. Lions [*21*] and P. L. Lions [*22*]. It is based upon the following lemma.

LEMMA II.1. — Under the assumptions of theorem II.2, let λ be defined by

$$\lambda(x, p) = \text{Inf}\left\{ \nu \geqslant 0 / \varphi\left(x, \frac{(\nu - 1)}{\nu}\, D\phi(x) + \frac{p}{\nu}\right) \geqslant 0 \right\}$$

Then:

i) $\lambda(x, p)$ is well-defined (i. e. $\lambda(x, p) < +\infty$, $\forall x, p \in \mathbb{R}^N$)

ii) $\lambda(x, \mu p + (1-\mu)D\phi(x)) = \mu\lambda(x, p)$, for all $\mu \geqslant 0$

iii) u is a viscosity subsolution (resp. supersolution) of

$$\text{Max}\,(u + H(x);\ \varphi(x, Du)) = 0 \qquad \text{in} \quad \mathbb{R}^N$$

iff u is a viscosity subsolution (resp. supersolution) of

$$\text{Max}\,(u + H(x);\ 1 - \lambda(x, Du)) = 0 \qquad \text{in} \quad \mathbb{R}^N.$$

iv) There exists two modulus ω_1 and ω_2^R such that

$$\lambda(x, p) - \lambda(y, q) \leqslant [\omega_1(|x-y|(1+|p|)) + \omega_2^R(|p-q|)](1+\lambda(x,p)), \quad \forall x, y, p, q \in \mathbb{R}^N.$$

We leave the proof of this lemma to the reader. Let us just precise that *i*) comes from the fact that $D\phi(x) \in \text{Int}\,(C(x))^c$, *ii*) and *iii*) a simple consequence of the definition of $\lambda(x, p)$ and of (H_5'), *iv*) use both (H_5') and (H_6). Now, we prove the theorem by using this lemma. We make the change of variable

$$v = \phi + e^{(u-\phi)}$$

Since ϕ is bounded and C^1, and since u is bounded from above, v is bounded and viscosity solution of

$$\text{(15)} \quad \text{Max}\,(v - \phi - e^{(H-\phi)};\ -\lambda(x, Dv) + v - \phi) = 0 \quad \text{in } \mathbb{R}^N.$$

Using a straightforward extension of the results and the methods of G. Barles and P. L. Lions [*4*], one concludes easily that there exists a unique bounded continuous solution of (15) and that the comparison result of theorem II.2 is true since one can compare viscosity sub and supersolutions of (15). And the proof is complete.

REMARK II.6. — Lemma II.1 justifies *a posteriori* the assumption (H_5) in which « $\mu - 1$ » is a general as « $K(\mu) > 0$ ».

III. SOME EXAMPLES OF APPLICATIONS OF FIRST-ORDER HAMILTON-JACOBI EQUATION WITH GRADIENT-CONSTRAINTS

This section is devoted to present some applications of the results and the methods of the preceeding part. Of course, we mean applications which, *a priori,* do not come from unbounded control problems.

a. Applications to first-order H. J. Equations.

We are interested in problems like

$$\text{(16)} \qquad \begin{cases} H(x, Du) = 0 & \text{in } \Omega, \\ u = \psi & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded or unbounded domain in $\mathbb{R}^N$, ψ, H are given continuous functions. We need the following assumption on H.

$$
(17) \quad \begin{cases} \text{There exists } \phi \in C^1(\Omega) \cap W^{1,\infty}(\bar{\Omega}) \text{ such that } D\phi \in BUC(\Omega), \\ H(x, D\phi(x)) \leqslant \beta < 0 \text{ in } \Omega, \text{ and such that : } \forall x \in \mathbb{R}^N, \forall p \in \mathbb{R}^N \text{ if} \\ \text{there exists } \mu_0 > 0 \text{ s. t.} \\ \qquad H(x, D\phi(x) + \mu_0(p - D\phi(x))) = 0 \\ \text{then} \\ \qquad H(x, D\phi(x) + \nu\mu_0(p - D\phi(x))) \begin{cases} > 0 & \text{if } \nu > 1 \\ \leqslant K(\nu) < 0 & \text{if } \nu < 1 \end{cases} \\ \text{where K is independent of } x \text{ and } p. \end{cases}
$$

Our result is the following

THEOREM III.1. — Assume (H_1^0), (H_2^0) with $C(x) \equiv \mathbb{R}^N$ and (17). Let v be a viscosity subsolution of (16) bounded from below and w be a viscosity supersolution of (16) bounded from below, then

$$v \leqslant w.$$

A lot of variants of this result can be considered: in particular if we can take $\phi \equiv 0$, we may relax the assumption (H_2^0) by assuming that $H(x, p)$ is only uniformly continuous in $\mathbb{R}^N \times B_R$ ($\forall R < \infty$). This result extends or complements some result of M. G. Crandall and P. L. Lions [*10*], H. Ishii [*15*], J. M. Lasry and P. L. Lions [*21*] or P. L. Lions [*22*].

REMARK III.1. — The geometrical assumption (17) is necessary to have a uniqueness result for (16) as it was remarked in the works mentioned above. For the sake of completeness, let us give the following example

$$
(18) \qquad \begin{cases} (|u'| - 1)(|u'| - n(x))^2 = 0 \quad \text{in} \quad (0, 1), \\ u(0) = 0, \qquad u(1) = 1 \end{cases}
$$

where n is a lipschitz function, $n(x) \geqslant \alpha > 0$ in $(0, 1)$ and

$$\int_0^1 n(t)dt = 1.$$

Then $u(x) = x$ and $v(x) = \int_0^x n(t)dt$ are viscosity solutions of (18).

Proof of theorem III.1. —We just sketch the proof which is almost exactly the same as the one of theorem II-2. We consider $\lambda(x, p)$ defined by

$$\lambda(x, p) = \text{Inf}\left\{ \nu \geqslant 0/H\left(x, \frac{(\nu - 1)}{\nu} D\phi(x) + \frac{p}{\nu}\right) \leqslant 0 \right\}$$

Then, a analogous lemma to lemma II.1 holds. In particular, v and w are respectively viscosity sub and supersolutions of

$$\begin{cases} \lambda(x, Du) = 1 & \text{in} \quad \Omega, \\ u = \psi & \text{on} \quad \partial\Omega. \end{cases}$$

Finally the change of variable

$$z = \phi - e^{-(u-\phi)}$$

concludes.

b. On some cheap control problems.

We are interested here in some cheap impulse control problems in $\mathbb{R}^N$. More precisely, we consider the Q. V. I.

$$\text{(19)} \qquad \operatorname{Max}\,(H(x, Du) + u\,;\ u - Mu) = 0 \quad \text{in} \quad \mathbb{R}^N,$$

where H is a continuous Hamiltonian and M is defined by

$$Mu(x) = \operatorname*{Inf}_{\xi \geqslant 0}\,(u(x+\xi) + c(\xi))$$

where c is continuous in $(\mathbb{R}^+)^N$. We are more particularly interested in the case when *c is convex and* $c(0) = 0$. Our first result shows clearly the connections between (19) and gradient-constraints problems.

LEMMA III.1. — Let u be a u. s. c. bounded function in $\mathbb{R}^N$ and let c be convex, $c(0) = 0$, then the two following propositions are equivalent

i) $\qquad \forall \xi \neq 0, \qquad u(x) \leqslant u(x+\xi) + c(\xi) \quad \text{in} \quad \mathbb{R}^N.$

ii) $\qquad \nabla u(x) \in -\partial c(0) \quad \text{in} \quad \mathbb{R}^N.$

Of course, *ii)* has to be understood in the viscosity subsolution sense. This lemma means that the Q. V. I. and the problem $(\mathcal{P})$ associated to H and $-\partial c(0)$ have the same viscosity subsolutions. Since, for all u

$$Mu \leqslant u \qquad \text{in} \quad \mathbb{R}^N$$

all the functions are viscosity supersolutions of the Q. V. I. and so we can only have for the Q. V. I. a maximum subsolution. On the contrary, if $-\partial c(0)$ satisfies (H_4) and (H_5), $(\mathcal{P})$ is well-posed, and the unique viscosity solution of $(\mathcal{P})$ is the maximum subsolution of the Q. V. I.

It is enough to say when $-\partial c(0)$ and more particularly

$$\varphi(p) = d(p, [-\partial c(0)]^c) \quad \text{if} \quad p \in -\partial c(0), \qquad -d(p, \partial c(0)) \quad \text{if} \quad p \notin \partial c(0)$$

satisfies (H_4) and (H_5) with $\phi = 0$.

PROPOSITION III.3. — A necessary and sufficient condition for $-\partial c(0)$ to satisfy (H_4) and (H_5) is

$$p \in \operatorname{Int}(\mathbb{R}^+)^N \cap \partial c(0). \tag{20}$$

The proof of this result is obvious after the following remark

$$\partial c(0) = [\partial c(0) \cap (\mathbb{R}^+)^N] + (\mathbb{R}^-)^N$$

This result holds typically for functions like $c(\xi) = a \mid \xi \mid$ $(a > 0)$. Then we can state our existence and uniqueness result.

PROPOSITION III.4. — Assume that H satisfies the assumption of theorem II.1, that c is convex, $c(0) = 0$ and that (20) holds, then (19) has a maximum solution (and subsolution) u in $C_b(\mathbb{R}^N)$ which is the unique viscosity solution of $(\mathcal{P})$ with H and $-\partial c(0)$.

This result is a simple application of theorem II.1 and of the remarks above. So, it is enough to prove lemma III.1.

Proof of lemma III.1. — *i*) $\Rightarrow$ *ii*) is very easy; hence, we just prove the reverse implication.

We introduce the function u^ϵ defined in $\mathbb{R}^N$ by

$$u^\epsilon(x) = \operatorname*{Sup}_{y \in \mathbb{R}^N} \left(u(y) - \frac{|y - x|^2}{\epsilon^2} \right)$$

for $\epsilon > 0$. This procedure is called sup-convolution and was introduced in the frame of Hamilton-Jacobi Equation by J. M. Lasry and P. L. Lions [*21*]. Following [*21*], one checks easily that $u^\epsilon \in W^{1,\infty}(\mathbb{R}^N)$, $u^{\epsilon'} \leqslant u^\epsilon$ if $\epsilon' \geqslant \epsilon$ and $u = \operatorname*{Inf}_{\epsilon > 0} u^\epsilon$. Moreover

$$\nabla u^\epsilon \in -\partial c(0) \quad \text{in} \quad \mathbb{R}^N$$

in the viscosity subsolution sense and, in particular, almost everywhere. By a regularisation of u^ϵ, using the convolution by a non-negative approximation of the unity, we may assume that $u^\epsilon \in C^\infty(\mathbb{R}^N)$. Now, we compute $u^\epsilon(x + \xi) - u^\epsilon(x)$

$$u^\epsilon(x + \xi) - u^\epsilon(x) = \int_0^1 \nabla u^\epsilon(x + t\xi)\xi dt$$

But, $\nabla u^\epsilon(x + t\xi) \in -\partial c(0)$, then

$$\nabla u^\epsilon(x + t\xi)\xi \geqslant -c(\xi)$$

for all $t \in (0, 1)$. Finally

$$u^\epsilon(x + \xi) - u^\epsilon(x) \geqslant -c(\xi)$$

Letting $\epsilon \to 0$, we get the result.

REMARK III.1. — The condition (20) of the proposition III.3 seems necessary as the following example shows

$$\text{Max}\,(u, u_x) = 0 \quad \text{in} \quad \mathbb{R}^N.$$

All the non positive constants are solutions. Nevertheless, (20) is strictly connected to the unboundedness of the domain: in bounded domains (and even in more general case, cf. [7]), we can build suitable function ϕ like $\phi(x) = p \cdot x$, $p \in \text{Int}\,(-c(0))$ in order to have (H_4) and (H_5).

c. Non convex problems in the calculus of variations.

In several works, E. Mascolo [24], [25], [26] and E. Mascolo and R. Schianchi [27], [28], [29] studied problems of the type

$$\text{Inf}\left(\int_\Omega f(x, Du)dx;\ Du \in W^{1,\infty}(\Omega), \qquad u = u_0 \ \text{on} \ \partial\Omega\right)$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$, u_0 is the boundary condition and $f(x, p)$ is a continuous function *which is not, a priori, convex in p.* They showed that, under suitable assumptions this problem leads to the equation

$$(21) \qquad \begin{cases} Du \in \partial C(x) & \text{a. e. in } \Omega, \\ u = u_0 \ \text{on} \ \partial\Omega \end{cases}$$

where $C(x)$ is, for all $x \in \Omega$, a bounded convex subset of $\mathbb{R}^N$. They obtained for (21) existence results based on the study of P. L. Lions [22] on compatibility conditions with boundary data which gives « explicit » formula for the maximum viscosity subsolution of (21). Our aim is to show the connections between $(\mathcal{P})$ and (21), and to use them to get uniqueness results for (21). Our first result is the following

PROPOSITION III.4. — If u is a viscosity solution of

$$(22) \qquad \begin{cases} Du \in C(x) \quad \text{and} \quad u \leqslant M, & \text{in } \Omega \\ u \geqslant M \quad \text{if} \quad Du \in \text{Int}\, C(x), & \text{in } \Omega \\ u = u_0 \ \text{on} \ \partial\Omega \end{cases}$$

where M is a constant large enough—say $M = \| u \|_\infty + 1$—, u is a solution of (21). Conversely, if u is a solution of (21), u is a viscosity subsolution of (22).

The proof consists only in remarking that, if u is viscosity solution of (22), $Du \in C(x)$ a. e. in Ω and $Du \notin \text{Int}\, C(x)$ a. e. in Ω in $M > \| u \|_\infty$. Hence, $Du \in \partial C(x)$ a. e. For the converse implication, (21) implies that $Du \in C(x)$ a. e. and since $C(x)$ is convex, u is a viscosity subsolution of (22).

Then, the uniqueness result directly comes from theorem II.2. In fact, since Ω and C are bounded, we can even relax the assumptions of this theorem to obtain

PROPOSITION III.5. — Assume that $C(x)$ is convex and uniformly bounded for $x \in \Omega$, that $(x, p) \to d(p, \partial C(x))$ is continuous, and that there exists $\phi \in C^1(\Omega)$ such that $D\phi(x) \in \text{Int } C(x)$ for all $x \in \bar{\Omega}$; then if there exists a viscosity solution of (21), it is unique in $C(\bar{\Omega})$ and it is the maximum solution of (21).

We leave the proof of this proposition to the reader since it is essentially routine adaptations of the ideas of the second section.

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