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A new approach to initial traces in nonlinear filtration

by

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ABSTRACT. — For weak solutions of equations of the type of nonlinear filtration in $\mathbf{R}^N \times (0, T)$, $0 < T < \infty$, we prove precise sup-estimates and local and global Harnack type inequalities. These estimations permit to identify the initial traces and describe the behavior of such solutions as $|x| \to \infty$.

The main point is to introduce a new approach, free of the specific features of the porous medium equation such as homogeneity, scaling, quasi-convexity, etc. This approach on one hand allows generalizations to a large variety of equations and on other yields new results on gradient averages.

Key-words: Radon measures, Harnack inequality degenerate parabolic equations.

RÉSUMÉ. — On démontre des inégalités de Harnack locales et globales, ainsi des majorations pour des solutions faibles des équations de filtrage non linéaire dans $\mathbf{R}^N \times (0, T)$, pour $T < \infty$. Ces estimations permettent d'identifier les traces initiales et de décrire le comportement des solutions quand $|x| \to \infty$.

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1. INTRODUCTION AND RESULTS

Motivated by a recent paper of Dahlberg-Kenig [8], we will make a few remarks on Harnack inequality and initial traces for non-negative solutions of the filtration equation

(1.1)
$$u_t - \Delta\varphi(u) = 0, \qquad \mathcal{D}'(S_T),$$
$$S_T \equiv \mathbf{R}^N \times (0, T), \qquad 0 < T \leq +\infty,$$

where

(1.2) $\varphi: \mathbf{R}^+ \to \mathbf{R}^+$ is absolutely continuous, non-decreasing, $\varphi(0)=0$,
$$\varphi(1) = 1 \quad \text{and} \quad 1 + \Lambda^{-1} \leq \frac{\varphi'(s)s}{\varphi(s)} < \Lambda, \quad \text{a. e. } s > 0,$$
for some given $\Lambda > 2$.

We will consider

a) Weak solutions of the Cauchy problem (1.1) with initial datum $u_0 \in L^p_{loc}(\mathbf{R}^N)$, $u \in L^\infty(0, T; L^p_{loc}(\mathbf{R}^N))$, $\varphi(u) \in L^q(0, T; W^{1,q}_{loc}(\mathbf{R}^N))$, for some $p, q \geq 1$.

b) Continuous distributional solutions in S_T with no reference to initial data. We let B_T be the set of all such solutions.

Both notions are standard [3], [4], [6], [8], [10].

For these solutions we will prove sup-estimates and quantitative Harnack inequalities. The results of this paper are basically known, even though in a weaker form.

Our aim is to adopt an entirely different approach that does not make use of homogeneity [1], [3], scaling properties [8], quasi-convexity [1], [3], symetrization [8], etc., in order to single out the features of the degeneracy exhibited by (1.1). For example, our approach shows that (1.1)-(1.2), for the issues of solvability of the Cauchy problem and existence of initial traces, is not very different form $u_t = \Delta u^m$, $m > 1$, both in terms of results and proofs.

Our methods supply precise L^∞-bounds, give a quantitative Harnack inequality and can be extended to general anisotropic operators (see Section 4). They also give new results on gradient averages and local Harnack estimates, and we feel they are simpler being only based on geometrical ideas.

1. *i*) The Cauchy problem.

If $f \in L^1_{loc}(\mathbf{R}^N)$, $r > 0$, let
$$|||f|||_r \equiv \sup_{\rho \geq r} [\Phi^{-1}(\rho^2)]^{-1} \fint_{B_\rho} |f| \, dx,$$

where

$$B_\rho \equiv \{ |x| < \rho \}, \qquad \fint_{B_\rho} |f|\,dx = \rho^{-N} \int_{B_\rho} |f|\,dx,$$

and

$$\Phi(s) = \frac{\varphi(s)}{s}, \qquad s > 0. \tag{1.3}$$

With $\gamma = \gamma(N, \Lambda)$ we denote quantitative constants that can be determined *a priori* only in terms of N, Λ.

Suppose u solves the Cauchy problem (1.1) with initial datum.

$$u_0 \in C_0^\infty(\mathbf{R}^N), \qquad u_0 \geq 0. \tag{1.4}$$

PROPOSITION 1. — *$\exists \gamma_i = \gamma_i(N, \Lambda)$, $i = 0, 1, 2$ such that $\forall r > 0$, $\forall \rho > r$ and $\forall 0 < t < T_r$,*

$$T_r \equiv \gamma_0 \inf_{\rho \geq r} \rho^2 / \Phi\left(\fint_{B_\rho} u_0\,dx \right), \qquad \gamma_0 \in (0, 1), \tag{1.5}$$

there holds

$$||| u(\cdot, t) |||_r \leq \gamma_1 ||| u_0 |||_r, \tag{1.6}$$

$$\left[\frac{t}{\rho^2} \Phi(\| u(\cdot, t) \|_{\infty, B_\rho}) \right]^{N/2} \frac{\| u(\cdot, t) \|_{\infty, B_\rho}}{\Phi^{-1}(\rho^2)} \leq \gamma_2 ||| u_0 |||_r. \tag{1.7}$$

REMARK 1.1. — If $\varphi(s) = s^m$, $m > 1$, (1.7) implies

$$\| u(\cdot, t) \|_{\infty, B_\rho} < \gamma' t^{-\frac{N}{\kappa}} \rho^{\frac{2}{m-1}} ||| u_0 |||_r^{\frac{2}{\kappa}}, \tag{1.7)'}$$

$\kappa = N(m - 1) + 2$.

A consequence is existence of distributional solutions of (1.1) with initial datum a σ-finite measure μ satisfying

$$||| \mu |||_r \equiv \sup_{\rho \geq r} [\Phi^{-1}(\rho^2)]^{-1} \fint_{B_\rho} d\mu < +\infty, \qquad \text{for some } r > 0.$$

The solution $u \in C^\alpha_{loc}(S_{T(\mu)})$, $\alpha = \alpha(N, \Lambda) \in (0, 1)$,

$$T(\mu) = \lim_{r \to +\infty} T_r(\mu); \qquad T_r(\mu) \equiv \gamma_0 \inf_{\rho \geq r} \left[\rho^2 / \Phi\left(\fint_{B_\rho} d\mu \right) \right].$$

Approximating μ, as in [3], by $u_{0,n} \in C_0^\infty(\mathbf{R}^n)$, Proposition 1 supplies uniform $L^\infty_{loc}(S_{T_r(\mu)})$-bounds $\forall r > 0$, and Theorem 2 of [6] gives uniform $C^\alpha_{loc}(S_{T_r(\mu)})$-estimates.

PROPOSITION 2. — *For every weak solution* $u \in B_T$ *of (1.1) in* S_T *satisfying*

$$\sup_{0<t\leq T} ||| u(\cdot, t) |||_r < +\infty,$$

there holds

$$\sup_{\rho\geq r} \frac{\| u(\cdot, t) \|_{\infty, B_\rho}}{\Phi^{-1}(\rho^2)} < +\infty, \qquad \forall 0 < t < T.$$

The proof is a minor variant of the proof of (1.7).

1. *ii*) Distributional solutions in S_T.

Harnack inequality. — $\exists \gamma = \gamma(N, \Lambda)$ such that $\forall u \in B_{T+\epsilon}$ for some $\epsilon \in (0, 1)$, $\forall \tau \in \left(0, \frac{T}{4}\right)$, $\forall \rho > 0$

$$(1.8) \quad \fint_{B_\rho} u(x, \tau)dx \leq \gamma \left\{ \Phi^{-1}\left(\frac{\rho^2}{T}\right) + \left[\frac{T}{\rho^2} \Phi(u(0, T))\right]^{\frac{N}{2}} u(0, T) \right\}.$$

It follows that each $u \in B_T$ has as initial trace a σ-finite Borel measure $\mu(u)$, $||| \mu(u) |||_r < +\infty$, $\forall \rho > 0$ (see [*1*]). Such a measure is unique (as shown in Section 4 in a more general setting). Moreover, combining the beautiful result of M. Pierre [*10*] on uniqueness of solutions of (1.1) with finite measures as initial datum, with the approximation procedure of [*8*], such a μ uniquely determines u.

REMARK 1.2. — Inequality (1.7) is both a sup-estimate and a statement about the behavior of $u(x, t)$ as $|x| \to \infty$. In particular, (1.8) and Proposition 2 imply that every $u \in B_T$ grows as $|x| \to \infty$, $\forall 0 < t < T$ at most as prescribed by (1.7).

REMARK 1.3. — Inequality (1.8) holds for non-negative locally bounded weak solutions in S_T, with locally finite energy, i. e., $D\varphi(u) \in L^2_{loc}(S_T)$. Indeed such solutions are locally Hölder continuous in S_T [*6*].

REMARK 1.4. — The proof shows that $u(0, T)$ in (1.8) can be replaced by $\inf_{|x|<\rho} u(x, T)$.

Combining the techniques of [*5*] and Section 3 here, we obtain the

Local Harnack estimate. — Let G be an open of $\mathbf{R}^{N+1}$ and $u \geq 0$ a *local* weak solution of (1.1) in G, $D\varphi(u) \in L^2_{loc}(G)$, $u \in L^\infty_{loc}(G)$. Let $P_0 \equiv (x_0, t_0) \in G$, $u(P_0) > 0$, $\rho > 0$.

There exist constants $C_i = C_i(N, \Lambda)$, $i = 0, 1$ such that

$$(1.9) \qquad \begin{cases} u(P_0) \le C_0 \inf\limits_{|x-x_0|<\rho} u(x, t_0 + \theta), \\ \theta = C_1 \rho^2/\Phi(u(P_0)), \end{cases}$$

provided the box $Q \equiv \{|x-x_0| < \rho\} \times \{t_0-\theta, t_0+\theta\}$ is all contained in G.

If $\Phi(s) = 1$, $\forall s \ge 0$, (1.9) coincides with the classical Harnack inequality for non-negative solutions of the heat equation, in the form of Krylov-Safonov [*9*].

1. *iii*) **Comments on the estimates.**

Estimate (1.7)′ is due to Bénilan-Crandall-Pierre [*3*]. For φ as in (1.2) an L^∞-bound in terms of $|||u_0|||_r$ is derived in [*8*]. Its form is not tractable, and it does not seem to imply (1.7)′ when $\varphi(s) = s^m$, $m > 1$.

We feel the form of (1.7) is the natural one in view of (1.8).

The Harnack inequality (1.8) is due to Aronson-Caffarelli [*1*] if $\varphi(u) = u^m$, $m > 1$.

Dahlberg-Kenig [*8*] proved (1.8) for φ as in (1.2), with a qualitative constant γ. Their proof is based on a blow-up technique which typically does not give quantitative estimates.

It is clear that (1.8) implies (1.7). It is less clear that the converse is true. Our approach shows that (1.8) is implied by (1.7).

Since our methods are free of specific properties of (1.1) (regularizing effects, scaling, etc.), extensions to general operators are possible. We have chosen to present the main ideas in the setting of (1.1) and will collect extensions and new results on gradient averages in Section 4.

1. *iv*) **A generalization of** (1.2).

A slightly more general non-linearity φ is allowed in [*8*]. Namely, (1.2) is assumed to hold $\forall s \ge 1$ and

$$(1.2)' \qquad \Lambda^{-1} \le \frac{\varphi'(s)s}{\varphi(s)} \le \Lambda, \qquad \text{a. e.} \quad s \in (0, 1).$$

Our methods cover such a case. To keep the presentation as clear as possible, we will work with (1.2) and indicate in Section 4 the few modifications needed to allow the behavior (1.2)′ near zero.

1. ν) Some elementary inequalities.

We list a few simple inequalities, that are a direct consequence of (1.2) and that are needed as we proceed:

$$(1.10)\qquad \begin{cases} h^{\frac{1}{\Lambda}}\Phi(s) \le \Phi(hs) \le h^{\Lambda-1}\Phi(s), & \forall s>0, \quad \forall h \ge 1; \\ h^{\Lambda-1}\Phi(s) \le \Phi(hs) \le h^{\frac{1}{\Lambda}}\Phi(s), & \forall s>0, \quad \forall h \in (0,1). \end{cases}$$

We sketch the proof of the estimates above; from (1.2) it follows

$$\frac{1}{\Lambda s} \le \frac{\Phi'(s)}{\Phi(s)} \le \frac{\Lambda-1}{s}, \qquad \text{a. e.} \quad s>0.$$

Integrating over (s, hs) (or (hs, s)) we get (1.10).

2. PROOF OF PROPOSITION 1

Let u be a weak solution of

$$(2.1)\qquad u_t - \Delta\varphi(u) = 0 \quad \text{in} \quad S_\infty; \qquad u(x,0) = u_0(x) \in C_0^\infty(\mathbf{R}^N).$$

LEMMA 2.1. — $\exists \gamma = \gamma(N, \Lambda) : \forall \sigma \in (0,1), \forall \rho > 0, \forall t > 0$ *satisfying*

$$(2.2)\qquad \sup_{0\le\tau\le t} \frac{\tau}{\rho^2}\,\Phi(\| u(\cdot,\tau)\|_{\infty, B_{(1+\sigma)\rho}}) \le 1,$$

there holds

$$(2.3)\qquad \left[\frac{t}{\rho^2}\,\Phi(\|u\|_{\infty, B_\rho})\right]^{\frac{N}{2}} \| u(\cdot,t)\|_{\infty, B_\rho} \le \frac{\gamma}{\sigma^{2(N+2)}} \int_0^t \!\!\!\!\!\!- \int_{B_{(1+\sigma)\rho}} \!\!\!\!\!\!\!\!\!\!\!\!\!\!- \quad u(x,\tau)\,dx\,d\tau.$$

Proof. — Let $t>0$, $\rho>0$, $\sigma \in \left(0, \frac{1}{2}\right)$ be fixed; consider the sequences, $n = 0, 1, 2, \dots$

$$t_n = \frac{t}{2} - \frac{\sigma t}{2^{n+1}}; \qquad \rho_n = \rho + \frac{\sigma\rho}{2^n}; \qquad k_n = k - \frac{k}{2^{n+1}}, \qquad k>0,$$

and set

$$B_n \equiv B_{\rho_n}, \qquad Q_n \equiv B_n \times (t_n, t), \qquad z_n(x,t) \equiv (u(x,t) - k_n)_+ .$$

Let ζ_n be a piecewise smooth cutoff function in Q_n such that

$$\zeta_n \equiv 1 \quad \text{on} \quad Q_{n+1}; \qquad |D\zeta_n| \le 2^{n+1}/\sigma\rho; \qquad 0 \le \zeta_{n,t} \le \frac{2^{n+2}}{\sigma t},$$

and in the weak formulation of (2.1) take testing functions $z_{n+1}\zeta_n^2$, $n = 0, 1, 2, \dots$, modulo a local regularization. By standard calculations

$$(2.4)\quad \sup_{t_n \le \tau \le t} \int_{B_n(\tau)} z_{n+1}^2 \zeta_n^2 dx + \iint_{Q_n} \Phi(u) \,|\, Dz_{n+1}\zeta_n \,|^2 dx d\tau$$
$$\le \frac{\gamma 2^{2n}}{\sigma^2} \left\{ \rho^{-2} \iint_{Q_n} \Phi(u) z_n^2 dx d\tau + \frac{1}{t} \iint_{Q_n} z_n^2 dx d\tau \right\},$$

where $\gamma = \gamma(N, \Lambda)$. Observe that

$$\iint_{Q_n} \Phi(u) \,|\, Dz_{n+1}\zeta_n \,|^2 dx d\tau \ge \gamma \Phi(k) \iint_{Q_n} |\, Dz_{n+1}\zeta_n \,|^2 dx d\tau$$

and that the term in brackets in (2.4) is majorized by

$$\frac{4}{t} \left\{ 1 + \sup_{0 \le \tau \le t} \frac{\tau}{\rho^2} \Phi(\| u(\cdot, \tau) \|_{\infty, B_{(1+\sigma)\rho}}) \right\} \iint_{Q_n} z_n^2 dx d\tau \le 8t^{-1} \iint_{Q_n} z_n^2 dx d\tau.$$

We conclude

$$(2.5)\quad \sup_{t_n \le \tau \le t} \int_{B_n(\tau)} z_{n+1}^2 \zeta_n^2 dx + \Phi(k) \iint_{Q_n} |\, Dz_{n+1}\zeta_n \,|^2 dx d\tau$$
$$\le \frac{\gamma 2^{2n}}{\sigma^2 t} \iint_{Q_n} z_n^2 dx d\tau; \qquad \gamma = \gamma(N, \Lambda), \qquad n = 0, 1, 2, \dots$$

By the embedding of [*11*], page 74-75

$$\iint_{Q_{n+1}} z_{n+1}^2 dx d\tau \le \iint_{Q_n} (z_{n+1}\, \zeta_n)^2 dx d\tau$$
$$\le |\, A_{n+1} \,|^{\frac{2}{N+2}} \left(\iint_{Q_n} (z_{n+1}\zeta_n)^{2\frac{N+2}{N}} dx\, d\tau \right)^{\frac{N}{N+2}}$$
$$\le \gamma \,|\, A_{n+1} \,|^{\frac{2}{N+2}} \left\{ \sup_{t_n \le \tau \le t} \int_{B_n(\tau)} (z_{n+1}\zeta_n)^2 dx \right\}^{\frac{2}{N+2}}$$
$$\left\{ \iint_{Q_n} |\, Dz_{n+1}\zeta_n \,|^2 dx d\tau \right\}^{\frac{N}{N+2}},$$

where

$$A_{n+1} \equiv \{ (x, t) \in Q_n \,|\, u(x, t) > k_{n+1} \}; \qquad |\, A_{n+1} \,| = \text{meas}\, A_{n+1}.$$

Combining this with (2.5) and observing that

$$\iint_{Q_n} z_n^2 dx d\tau \geq (k_{n+1} - k_n)^2 \,|\, A_{n+1} \,|\, > 2^{-(2n+2)} k^2 \,|\, A_{n+1} \,|\,,$$

we deduce

$$(2.6) \quad \iint_{Q_{n+1}} z_{n+1}^2 dx d\tau \leq \frac{\gamma 2^{4n}}{\sigma^2 t} \left[k^2 \Phi(k)^{\frac{N}{2}} \right]^{-\frac{2}{N+2}} \left(\iint_{Q_n} z_n^2 dx d\tau \right)^{1+\frac{2}{N+2}}, \quad n = 0, 1, 2, \ldots.$$

If k is chosen so large as to satisfy

$$(2.7) \quad \iint_{Q_0} u^2 dx d\tau < \tilde{\gamma} \left[k^2 \Phi(k)^{\frac{N}{2}} \right] (t\sigma^2)^{\frac{N+2}{2}},$$

then $\iint_{Q_n} z_n^2 dx d\tau \to 0$ as $n \to +\infty$ (see Lemma 5.6 of [*11*], p. 95); here $\tilde{\gamma}$ is a (small) constant depending only upon N and γ in (2.6). We have proved $\| u \|_{\infty, Q(\rho, t_\infty)} \leq k$, where

$$t_\infty = \frac{t}{2}, \qquad Q(s, \tau) = B_s \times (\tau, t), \qquad \forall t_0 \leq \tau \leq t_\infty, \qquad \forall \rho \leq s \leq \rho_0 .$$

Choosing k from (2.7) taken with equality,

$$(2.8) \quad \| u \|^2_{\infty, Q(\rho, t_\infty)} \left[t \Phi(\| u \|_{\infty, Q(\rho, t_\infty)}) \right]^{\frac{N}{2}} \leq \frac{\gamma \sigma^{-(N+2)}}{t} \int_{t_0}^{t} \int_{B_{(1+\sigma)\rho}} u^2 dx d\tau .$$

Multiply the left-hand side of (2.8) by

$$\left[t \Phi(\| u \|_{\infty, Q(\rho, t_\infty)}) \right]^{\frac{N}{2}}$$

and the right-hand side by

$$\left[t \Phi(\| u \|_{\infty, Q(\rho_0, t_0)}) \right]^{\frac{N}{2}},$$

and $\forall 0 < \rho < s \leq \rho_0$, $\forall t_0 \leq \tau \leq t_\infty$, set

$$(2.9) \quad f(s, \tau) = \left\{ \left[t \Phi(\| u \|_{\infty, Q(s, \tau)}) \right]^{\frac{N}{2}} \| u \|_{\infty, Q(s, \tau)} \right\}^2 .$$

The from (2.8), modified as above,

$$(2.10) \quad f(\rho, t_\infty) \leq \gamma \left(\frac{\rho_0}{\rho_0 - \rho} + \frac{t_\infty}{t_\infty - t_0} \right)^{N+2} \sqrt{f(\rho_0, t_0)} \,\, \fint_0^t \fint_{B_{\rho_0}} u(x, \tau) dx d\tau,$$

and one realizes immediately that (2.10) holds true, with the same constant $\gamma = \gamma(N, \Lambda)$, with ρ, ρ_0 replaced by s, s', $\forall 0 < \rho \leq s < s' \leq \rho_0$, and t_0, t_∞ replaced by τ, τ', $\forall t_0 \leq \tau < \tau' \leq t_\infty$.

Therefore by Cauchy inequality, $\forall \delta \in (0, 1)$

$$(2.11) \qquad f(s, \tau') \leq \delta f(s', \tau) + M_0\left(\frac{s'}{s'-s} + \frac{\tau'}{\tau'-\tau}\right)^{2(N+2)},$$

where

$$M_0 = \frac{\gamma}{\delta}\left(\int_0^t \int_{B_{\rho_0}} u(x, \tau)\,dx\,d\tau\right)^2.$$

Set

$$s_0 = \rho \quad \text{and} \quad s_{i+1} - s_i = (1-\sigma)\sigma^i(\rho_0 - \rho), \quad i = 0, 1, 2, \ldots$$
$$\tau_0 = t_\infty \quad \text{and} \quad \tau_i - \tau_{i+1} = (1-\sigma)\sigma^i(t_\infty - t_0), \quad i = 0, 1, 2, \ldots.$$

Then from (2.11)

$$f(s_i, \tau_i) \leq \delta f(s_{i+1}, \tau_{i+1}) + M_1\sigma^{-2(N+2)i}; \qquad i = 0, 1, 2, \ldots,$$

with $M_1 = \gamma M_0[\sigma(1-\sigma)]^{-2(N+2)}$, and by iteration, $\forall m = 1, 2, \ldots$

$$f(\rho, t_\infty) \leq \delta^m f(s_m, \tau_m) + M_1 \sum_{i=0}^{m} [\delta\sigma^{-2(N+2)}]^i.$$

Choosing $\delta\sigma^{-2(N+2)} = \frac{1}{2}$ and letting $m \to +\infty$ proves the lemma for $\sigma \in \left(0, \frac{1}{2}\right)$, and hence $\forall \sigma \in (0, 1)$.

REMARK 2.1. — The proof shows that Lemma 2.1 holds true for every local weak solution of (1.1) in S_T.

2. *i*) Estimating the blow up time.

Let $r > 0$ be fixed and let $\tilde{t}_r$ be the supremum of all $t > 0$ such that

$$\sup_{0<\tau<t} \tau \sup_{\rho \geq r} \frac{\Phi(\| u(\cdot, \tau)\|_{\infty, B_\rho})}{\rho^2} \leq 1.$$

If $\tilde{t}_r$ is finite, $\exists \bar{\rho} \geq r$ and $0 < \bar{t} \leq \tilde{t}_r$ such that

$$\frac{\bar{t}}{\bar{\rho}^2}\,\Phi(\| u(\cdot, \bar{t})\|_{\infty, B_{\bar{\rho}}}) \geq \frac{1}{2}.$$

Since $u_0 \in C_0^\infty(\mathbf{R}^N)$, $u \in L^\infty(\mathbf{R}^N \times \mathbf{R}^+)$ and $\tilde{t}_r$ is bounded away from zero.

We will turn the qualitative knowledge of $\tilde{t}_r$ into a quantitative estimate.

Let

$$\| u(t) \|_\rho = \sup_{0 \le \tau \le t} \fint_{B_\rho} u(x, \tau) d\tau ,$$

and define

$$\lambda = \gamma \| u(t) \|_{2\rho} , \qquad \lambda = \lambda(\rho, t),$$

where $\gamma = \gamma(\mathrm{N}, \Lambda)$ is the constant of (2.3). Then from (2.3), if $t \le \tilde{t}_r$, $\rho \ge r$,

$$\| u(\cdot, t) \|_{\infty, \mathrm{B}_\rho} \le \mathrm{A}^{-1}[\lambda(\rho^2/t)^{\frac{\mathrm{N}}{2}}] , \tag{2.12}$$

$$\mathrm{A}(s) = [\Phi(s)]^{\frac{\mathrm{N}}{2}} s, \quad s > 0 . \tag{2.13}$$

From this

$$\left[\frac{t}{\rho^2} \Phi(\| u(\cdot, t) \|_{\infty, \mathrm{B}_\rho})\right]^{\frac{\mathrm{N}}{2}} \le \left(\frac{t}{\rho^2}\right)^{\frac{\mathrm{N}}{2}} \Phi^{\frac{\mathrm{N}}{2}}(\mathrm{A}^{-1}[\lambda(\rho^2/t)^{\frac{\mathrm{N}}{2}}]) \tag{2.14}$$

$$= \left(\frac{t}{\rho^2}\right)^{\frac{\mathrm{N}}{2}} \frac{\lambda(\rho^2/t)^{\frac{\mathrm{N}}{2}}}{\mathrm{A}^{-1}(\lambda(\rho^2/t)^{\frac{\mathrm{N}}{2}})} = \frac{\lambda}{\mathrm{A}^{-1}(\lambda(\rho^2/t)^{\frac{\mathrm{N}}{2}})}$$

For a constant $\mathrm{D} > 1$ to be chosen let $\hat{t} = \hat{t}(\rho)$ be the root of the equation

$$\rho^2 = \mathrm{D} t \Phi(\lambda(\rho, t)) ,$$

and set

$$t^* = \inf_{\rho \ge r} \hat{t}(\rho) .$$

Then $\forall 0 < t < t^*$ the righ-hand side of (2.14) is majorized by

$$\frac{\lambda}{\mathrm{A}^{-1}(\mathrm{D}^{\frac{\mathrm{N}}{2}} \mathrm{A}(\lambda))} \le \frac{\lambda}{\mathrm{A}^{-1}(\mathrm{A}(\bar{\mathrm{D}}\lambda))} = \frac{1}{\bar{\mathrm{D}}} < 2^{-\mathrm{N}} ,$$

where we have used (1.10), and $\bar{\mathrm{D}} = \mathrm{D}^{\frac{\mathrm{N}}{\mathrm{N}(\Lambda - 1) + 2}}$ can be chosen greater than 2^{N}. Obviously we have $t^* \le \tilde{t}_r$; indeed, if not from (2.14) with $t = \bar{t}$, $\rho = \bar{\rho}$ we derive a contradiction.

We summarize

LEMMA 2.2. — *Let $r > 0$ be fixed. $\forall \epsilon \in (0, 2^{-\mathrm{N}})$, $\exists \mathrm{C} = \mathrm{C}(\epsilon, \mathrm{N}, \Lambda)$ independent of τ such that $\forall 0 < t \le t^*$, where*

$$\frac{1}{t^*} = \mathrm{C} \sup_{\rho \ge r} \frac{\Phi(\| u(t^*) \|_\rho)}{\rho^2} , \tag{2.15}$$

we have

$$\text{i)} \qquad \frac{t}{\rho^2}\,\Phi(\| u(\cdot,t)\|_{\infty,B_\rho}) \le 1, \qquad \forall \rho \ge r > 0,$$

and (2.3) holds true. Moreover

$$\text{ii)} \qquad \frac{\| u(t)\|_\rho}{A^{-1}(t^{-\frac{N}{2}}\rho^N \| u(t)\|_\rho)} \le \epsilon, \qquad \forall \rho \ge r > 0.$$

Next define a cutoff function ζ in $B_{2\rho}$, such that

$$\zeta \equiv 1 \quad \text{in} \quad B_\rho, \qquad |\Delta\zeta| \le \gamma\rho^{-2}.$$

Then $\forall 0 < t < t^*$, $\forall \rho \ge r$, (2.1) implies

$$(2.16) \quad \fint_{B_{2\rho}} \zeta(x)u(x,t)dx = \fint_{B_{2\rho}} \zeta(x)u_0(x)dx + \int_0^t \fint_{B_{2\rho}} \Delta\zeta(x)\varphi(u)(x,\tau)dx\,d\tau$$
$$\le \fint_{B_{2\rho}} u_0(x)dx + \gamma \| u(t)\|_{2\rho} \int_0^t \frac{1}{\rho^2}\,\Phi(\| u(\cdot,\tau)\|_{\infty,B_{2\rho}})d\tau.$$

We estimate the last integral as follows. From Lemma 2.1 and (2.14)

$$\int_0^t \frac{1}{\rho^2}\,\Phi(\| u(\cdot,\tau)\|_{\infty,B_{2\rho}})d\tau \le \frac{\gamma}{\rho^2}\int_0^t \tau^{-1}\left[\frac{\xi}{A^{-1}(\tau^{-\frac{N}{2}}\xi)}\right]^{\frac{2}{N}} d\tau,$$

where $\xi = (2\rho)^N \| u\,(t)\|_{4\rho}$; indeed $\dfrac{\xi}{A^{-1}(\tau^{-\frac{N}{2}}\xi)}$ is increasing in τ, as it follows from

$$\text{a)} \qquad \frac{d}{d\tau}\left[A^{-1}(\tau^{-\frac{N}{2}}\xi)\right]^{-\frac{2}{N}} = \frac{(A^{-1})'(\tau^{-\frac{N}{2}}\xi)\tau^{-\frac{2}{N}-1}\xi}{\left[A^{-1}(\tau^{-\frac{N}{2}}\xi)\right]^{\frac{N+2}{N}}},$$

$$\text{b)} \qquad \frac{A(s)}{s} \le A'(s) \le \frac{\Lambda N + 2A(s)}{2}\,\frac{A(s)}{s},$$

$$\text{c)} \qquad \gamma^{-1}\frac{A^{-1}(s)}{s} \le (A^{-1})'(s) \le \gamma\frac{A^{-1}(s)}{s}; \qquad \gamma = \gamma(N,\Lambda).$$

Therefore

$$\frac{d}{d\tau}\left[A^{-1}(\tau^{-\frac{N}{2}}\xi)\right]^{-\frac{2}{N}} \ge \frac{\gamma^{-1}}{\tau}\left[A^{-1}(\tau^{-\frac{N}{2}}\xi)\right]^{-\frac{2}{N}},$$

and

$$\xi^{\frac{2}{N}} \int_0^t \tau^{-1}[A^{-1}(\tau^{-\frac{N}{2}}\xi)]^{-\frac{2}{N}} d\tau \leq \gamma \left[\frac{\xi}{A^{-1}(t^{-\frac{N}{2}}\xi)}\right]^{\frac{2}{N}},$$

since $A^{-1}(\tau^{-\frac{N}{2}}\xi) \to \infty$ as $\tau \to 0$. Substituting this in the integral estimate (2.16) and taking into account Lemma 2.2, we obtain

$$\fint_{B_\rho} u(x, t)dx \leq 2^N \fint_{B_{2\rho}} u_0(x)dx + \gamma\epsilon^{\frac{2}{N}} \| u(t) \|_{2\rho}. \tag{2.17}$$

Dividing by $\Phi^{-1}(\rho^2)$ and taking the supremum $\forall\rho \geq r$, we find, for ϵ sufficiently small ($\epsilon = \epsilon(N, \Lambda)$ quantitatively determined)

$$||| u(\cdot, t) |||_r \leq \gamma_1(N, \Lambda) ||| u_0 |||_r. \tag{2.18}$$

Also from (2.17)

$$\sup_{\rho \geq r} \frac{\Phi(\| u(t) \|_\rho)}{\rho^2} \geq \bar{\gamma}(N, \Lambda) \sup_{\rho \geq r} \frac{\Phi\left(\fint_{B_\rho} u_0 dx\right)}{\rho^2}. \tag{2.19}$$

The right-hand side of (2.19) is determined only in terms of the data.

Putting this in (2.15) we conclude that $\exists\gamma_0 = \gamma_0(N, \Lambda) \in (0, 1)$ such that Lemma 2.2 and Proposition 1 hold true $\forall 0 < t < T_r$

$$T_r^{-1} = \gamma_0^{-1} \sup_{\rho \geq r} \left[\Phi\left(\fint_{B_\rho} u_0 dx\right)/\rho^2\right]. \tag{2.20}$$

2. *ii*) The case supp $u_0 \subset B_\rho$.

First we remark that in (2.19) γ_0 can be taken smaller provided T_r is reduced accordingly.

Assume supp $u_0 \subseteq B_\rho$ and set

$$E_0 = \fint_{B_\rho} u_0(x)dx, \tag{2.21}$$

$$r = k\rho, \qquad k \geq 2, \tag{2.22}$$

and observe that

$$\begin{cases} ||| u_0 |||_r = k^{-N}E_0/\Phi^{-1}(k^2\rho^2), \\ \sup_{s \geq r} \dfrac{\Phi\left(\fint_{B_s} u_0 dx\right)}{s^2} = \dfrac{\Phi(k^{-N}E_0)}{k^2\rho^2}. \end{cases} \tag{2.23}$$

From (2.18) and (2.23)

$$(2.24)\qquad \fint_{B_{k\rho}} u(x, t)dx \le \gamma_1(N, \Lambda)k^{-N}E_0,$$

$\forall 0 < t < T_r$. Also reasoning as in (2.16), and employing Lemma 2.2 and (2.24)

$$\fint_{B_{k\rho}} u(x, t)dx \ge k^{-N}E_0 - \gamma\epsilon^{\frac{2}{N}}k^{-N}E_0;$$

therefore by taking ϵ possibly smaller and consequently a smaller γ_0 we have

LEMMA 2.3. — $\exists\gamma_0 = \gamma_0(N, \Lambda) : \forall 0 < t < T_r$,

$$(2.25)\qquad \frac{k^2\rho^2}{T_r} = \frac{1}{\gamma_0}\Phi(k^{-N}E_0),$$

there holds

$$(2.26)\qquad \frac{1}{2}k^{-N}E_0 \le \fint_{B_{k\rho}} u(x, t)dx \le 2^{N+1}k^{-N}E_0.$$

2. *iii)* Proof of Proposition 2.

Fix $\tilde{t}\in(0, T)$; we use (1.10) with $hs = \| u(\cdot, \tilde{t}) \|_{\infty, B_\rho}$, $s = \Phi^{-1}(\rho^2)$, $\forall\rho > r$. Then

$$(2.27)\qquad \frac{\| u(\cdot, \tilde{t}) \|_{\infty, B_\rho}}{\Phi^{-1}(\rho^2)} \le \max\left\{\left[\frac{\Phi(\| u(\cdot, \tilde{t}) \|_{\infty, B_\rho})}{\rho^2}\right]^{\Lambda}, 1\right\}.$$

Therefore it is obvious that we need only consider the case

$$(2.28)\qquad \frac{\Phi(\| u(\cdot, \tilde{t}) \|_{\infty, B_\rho})}{\rho^2} \ge 2.$$

Choose $\epsilon > 0$ so small that $[\tilde{t} - \epsilon, \tilde{t} + \epsilon] \subset (0, T)$. Working in the strip $\mathbf{R}^N \times (\tilde{t} - \epsilon, \tilde{t} + \epsilon)$, we may assume after a translation and time-dilation that (1.1) holds weakly in $S = \mathbf{R}^N \times (-2, 2)$, and that

$$(2.29)\qquad \sup_{-2<t<2} ||| u(\cdot, t) |||_r \equiv U_0 \le +\infty.$$

Define $\forall\rho \ge r$, $t \in (0, 1]$, $\sigma \in [0, 1)$

$$\begin{aligned} Q_\rho(\sigma, t) &= B_{\rho(1+\sigma)} \times (-(1+\sigma)t, (1+\sigma)t),\\ Q_\rho(t) &= Q_\rho(0, t),\\ Q_\rho[t_0, t_1] &= B_\rho \times (t_0, t_1) \quad \text{if} \quad t_0 < t_1,\\ Q_\rho[t_1, t_0] &= B_\rho \times (t_1, t_0) \quad \text{if} \quad t_1 < t_0. \end{aligned}$$

Since u is continuous, we can find a point $(x_0, t_0) \in \overline{Q_\rho(1)}$ such that

$$\| u \|_{\infty, Q_\rho(1)} = u(x_0, t_0), \tag{2.30}$$

and a $t_1 \in (-1, 1)$ such that

$$\frac{|t_0 - t_1|}{\rho^2} \Phi(\| u \|_{\infty, Q_\rho(\sigma, 1)}) = 1, \tag{2.31}$$

since (2.28) holds. Then from (2.31) and Lemma 2.1

$$\left[\frac{|t_0 - t_1|}{\rho^2} \Phi(\| u \|_{\infty, Q_\rho[t_0, t_1]})\right]^{\frac{N}{2}} \frac{\| u \|_{\infty, Q_\rho[t_0, t_1]}}{\Phi^{-1}(\rho^2)} \leq \gamma \sigma^{-2(N+2)} U_0. \tag{2.32}$$

Now notice that

$$\| u \|_{\infty, Q_\rho[t_0, t_1]} = u(x_0, t_0) = \| u \|_{\infty, Q_\rho(1)},$$

and substitute $[t_0 - t_1]$ in (2.32) from (2.31) to get

$$\left(\frac{\Phi(\| u \|_{\infty, Q_\rho(1)})}{\rho^2}\right)^{\frac{N}{2}} \frac{\| u \|_{\infty, Q_\rho(1)}}{\Phi^{-1}(\rho^2)} \leq \frac{\gamma U_0}{\sigma^{2(N+2)}} \left(\frac{\Phi(\| u \|_{\infty, Q_\rho(\sigma, 1)})}{\rho^2 (1 + \sigma)^2}\right)^{\frac{N}{2}}. \tag{2.33}$$

Set

$$Y(s) = \left[\frac{\Phi(\| u \|_{\infty, Q_\rho(s, 1)})}{s^2 \rho^2}\right]^{\frac{N}{2} + \frac{1}{\Lambda - 1}};$$

then, taking into account (1.10) again, (2.33) implies $\forall 1 \leq s \leq s' < 2$

$$Y(s) \leq \gamma U_0 (s' - s)^{-2(N+2)} Y(s')^{1-\epsilon_0},$$

with $\epsilon_0 = 1 - \dfrac{N(\Lambda - 1)}{N(\Lambda - 1) + 2}$. A simple interpolation argument proves the proposition.

3. PROOF OF THE HARNACK INEQUALITY

From [6], $u \in B_T \Rightarrow u \in C^\alpha_{loc}(S_T)$ for some $\alpha = \alpha(N, \Lambda) \in (0, 1)$. A consequence is the following. Let $P_0 \equiv (x_0, t_0) \in S_T$ and let

$$Q_\rho(P_0) \equiv \{ |x - x_0| < \rho \} \times \left(t_0 - \frac{\rho^2}{\Phi(M)}, t_0\right) \subset S_T,$$

where M is any positive number satisfying

$$\| u \|_{\infty, Q_\rho(P_0)} \leq M.$$

There exists $C = C(N, \Lambda) > 1$ and $\alpha = \alpha(N, \Lambda) \in (0, 1)$, such that $\forall 0 < s \leq \rho$

$$\underset{\{|x-x_0|<s\}}{\operatorname{osc}} u(x, t_0) \leq CM\left(\frac{s}{\rho}\right)^{\alpha}. \tag{3.1}$$

First we prove (1.8) for solutions of (1.1) in S_T with initial datum

$$u_0 \in L^1(B_\rho); \qquad u_0(x) = 0, \qquad |x| > \rho,$$

and with $u(x, \tau)$ replaced by $u_0(x)$. The general case will be recovered later.

Let $T, \rho > 0$ be fixed and let $k \geq 2$ be the smallest root of the equation

$$(k\rho)^2/T = \gamma_0^{-1}\Phi(k^{-N}E_0), \tag{3.2}$$

where E_0 is as in (2.21) and γ_0 as in Lemma 2.3. Then T is the time T_r defined in (2.20), (2.25). Since $\Phi(k^{-N}E_0) \to 0$ and $(k\rho)^2/T \to +\infty$ as $k \to +\infty$, (3.2) will have a solution $k > 2$ if

$$\frac{4\rho^2}{T} < \gamma_0^{-1}\Phi(2^{-N}E_0), \tag{3.3}$$

which we assume.

By Lemma 2.3

$$\fint_{B_{k\rho}} u(x, t)dx \geq \frac{1}{2} k^{-N}E_0, \qquad \forall 0 < t < T,$$

and $\exists P_0 \equiv (x_0, t_0), \quad |x_0| < k\rho, \quad \frac{T}{4} < t_0 \leq \frac{T}{2}$,

$$u(P_0) \geq \frac{1}{2} k^{-N}E_0. \tag{3.4}$$

Consider the box $Q(P_0) \equiv \{|x-x_0| < 2k\rho\} \times \left(t_0 - \frac{T}{8}, t_0\right)$, and let $M = \| u \|_{\infty, Q(P_0)}$. We claim that

$$\frac{T\Phi(M)}{(k\rho)^2} \geq L^{-1}; \qquad L = \frac{2^{N\Lambda+1}}{\gamma_0}. \tag{3.5}$$

If not, $\Phi(M) \leq L^{-1}(k\rho)^2/T$ and in view of (3.2)

$$2^{-N\Lambda}\Phi(k^{-N}E_0) \leq \Phi(2^{-N}k^{-N}E_0) \leq \Phi\left(\fint_{B_{k\rho}} u(x, t_0)dx\right)$$
$$\leq \Phi(M) \leq \gamma_0 2^{-(N\Lambda+1)}(k\rho)^2/T = \frac{2^{-N\Lambda}}{2}\Phi(k^{-N}E_0),$$

a contradiction.

We draw two consequences of (3.5). First

$$(3.6)\qquad \| u \|_{\infty, Q(P_0)} \leq D_0 k^{-N} E_0 ; \qquad D_0 = \gamma_2 [2^{10+\Lambda(4+N)}/\gamma_0]^{\frac{N}{2}},$$

and γ_2 is the constant in (1.7). Indeed from the sup estimate (1.7) for the ball $B_{k\rho}$, $\forall t_0 - \frac{T}{8} < \tau < t_0$, we have $\tau > \frac{T}{8}$ and

$$[\gamma_0 2^{-(N\Lambda+1)}]^{\frac{N}{2}} 8^{-\frac{N}{2}} M \leq \left[\frac{\tau}{(2k\rho)^2}\Phi(M)\right]^{\frac{N}{2}} \frac{M}{\Phi^{-1}((2k\rho)^2)} \leq \gamma_2 \frac{k^{-N}E_0}{\Phi^{-1}(k^2\rho^2)},$$

where we have used (2.23) also. The second consequence is that the box

$$Q_\rho^b(P_0) \equiv \{ | x - x_0 | < bk\rho \} \times \left(t_0 - \frac{(bk\rho)^2}{\Phi(M)}, t_0\right), \qquad b = \frac{1}{4}\sqrt{\gamma_0 2^{-N\Lambda-1}},$$

is contained in $Q(P_0)$. Thus $\| u \|_{\infty, Q_\rho^b(P_0)} \leq D_0 k^{-N} E_0$.

We may now apply (3.1) to conclude, $\forall 0 < s < bk\rho$; $\forall x : | x - x_0 | < s$

$$u(x, t_0) \geq u(P_0) - CD_0 k^{-N} E_0 \left(\frac{s}{bk\rho}\right)^\alpha \geq k^{-N} E_0 \left(\frac{1}{2} - CD_0 \left(\frac{s}{bk\rho}\right)^\alpha\right).$$

Choose $s = \epsilon(bk\rho)$ and $\epsilon \in (0, 1)$ so small that $\frac{1}{2} - CD_0\epsilon^\alpha = \frac{1}{4}$. Then if $\epsilon_1 = \epsilon_1(N, \Lambda) = \epsilon b$, we have

$$(3.1)\qquad u(x, t_0) \geq \frac{1}{4} k^{-N} E_0, \qquad \forall x : | x - x_0 | < \epsilon_1 k\rho .$$

3. *i*) Remarks on the case $\varphi(s) = s^m$, $m > 1$.

Inequality (3.7), which is a sole consequence of the sup-estimate (1.7) and the Hölder continuity (3.1), is essentially the Harnack inequality (1.8).

This is apparent in the case of power non-linearities, when one has the Barenblatt-Pattle [*2*] [*12*] solutions

$$B(x, t) = \frac{hR^N}{S(t)^{N/K}} \left\{ 1 - \left(\frac{| x - x_0 |}{S(t)^{1/K}}\right)^2 \right\}_+^{\frac{1}{m-1}},$$

$$S(t) = \left(\frac{2\kappa m}{m-1} h^{m-1} R^{N(m-1)}(t - t_0) + R^\kappa\right), \qquad t \geq t_0,$$

$$h = \frac{1}{4} k^{-N} E_0 ; \qquad R = \epsilon_1 k\rho ; \qquad \kappa = N(m - 1) + 2.$$

By the comparison principle $u \geq \mathrm{B}$, $t \geq t_0$. Take $\tilde{t} - t_0 - \tilde{\mathrm{C}}\mathrm{T}$ where $\tilde{\mathrm{C}}$ is so large that

$$\mathrm{S}^{1/\kappa}(\tilde{t}) = \left\{ \frac{2\kappa m}{(m-1)} \frac{\epsilon_1^{\mathrm{N}(m-1)}}{4^{m-1}} \tilde{\mathrm{C}}\gamma_0 \frac{\mathrm{T}\Phi(k^{-\mathrm{N}}\mathrm{E}_0)(k\rho)^{\mathrm{N}(m-1)}}{\gamma_0} + (\epsilon_1 k\rho)^\kappa \right\}^{\frac{1}{\kappa}}$$

$$= \text{by} \quad (3.2) = (k\rho) \left\{ \frac{\epsilon_1^{\mathrm{N}(m-1)}\gamma_0 2\kappa m}{(m-1)4^{m-1}} \tilde{\mathrm{C}} + \epsilon_1^\kappa \right\}^{\frac{1}{\kappa}} \geq 4(k\rho).$$

For such a $\tilde{t}$, the ball $|x - x_0| < \mathrm{S}^{1/\mathrm{K}}(\tilde{t})$ covers $x = 0$ and therefore $\exists c_0 = c_0(\mathrm{N}, m)$ such that

$$u(0, \tilde{t}) \geq \mathrm{B}(0, \tilde{t}) \geq c_0 k^{-\mathrm{N}}\mathrm{E}_0.$$

From (3.2), $k^2 = \gamma_0^{-1} \frac{\mathrm{T}}{\rho^2} \mathrm{E}_0^{m-1} k^{-\mathrm{N}(m-1)}$ and therefore since $\frac{\mathrm{T}}{4} < t_0 \leq \frac{\mathrm{T}}{2}$, $\exists \gamma = \gamma(\mathrm{N}, m)$ such that

$$\mathrm{E}_0^{2/\kappa} \leq \gamma(\mathrm{N}, m)\left(\frac{\mathrm{T}}{\rho^2}\right)^{\frac{\mathrm{N}}{\kappa}} u\left(0, \left(\tilde{\mathrm{C}} + \frac{1}{2}\right)\mathrm{T}\right). \tag{3.8}$$

If we take into account also the case in which (3.3) does not hold, we see that (1.8) follows from (3.8) after we redefine T.

The proof for general φ is not very different. It entails an expansion of the positivity set of u, via comparison functions.

The use of Hölder estimates to prove Harnack type inequalities is embedded in the work of Krylov-Safonov [*9*] (see also Safonov [*13*]).

3. *ii*) Subsolutions.

We will work with anisotropic operators in view of the generalizations of the next section.

From (1.2) or (1.2)′

$$\psi(s) = \varphi^{-1}(s); \qquad \Lambda^{-1} \leq \frac{\psi'(s)s}{\psi(s)} \leq \Lambda, \qquad \text{a.e. } s > 0. \tag{3.9}$$

Let $(a_{ij}) \equiv (a_{ij}(t))$, $i, j = 1, \ldots, \mathrm{N}$ be a time-dependent matrix satisfying

$$\Lambda^{-1}|\eta|^2 \leq a_{ij}(t)\eta_i\eta_j \leq \Lambda|\eta|^2, \qquad \forall \eta \in \mathbf{R}^\mathrm{N}, \qquad \forall t > 0. \tag{3.10}$$

We will construct weak subsolutions $v \in \mathrm{C}^{1,\alpha}_{\mathrm{loc}}(\mathrm{S_T})$ of

$$\mathcal{L}(v) \equiv \psi'(v)v_t - a_{ij}(t)v_{x_i x_j}.$$

Let

$$(3.11)\quad v(x,t)=\frac{hR^{2\xi}}{S(t)^{\xi}}F^{\theta}(x,t),\qquad h>0,\qquad R>0,$$

$$S(t)=\frac{(t-\bar{t})}{\psi'(h)}+R^2;\qquad t\geq\bar{t}>0;$$

$$F(x,t)\equiv(1-\|z\|^2)_+\|z\|=|x-\bar{x}|/S^{1/2}(t);\qquad \bar{x}\in\mathbf{R}^N,$$

$$\theta=1+\frac{1}{2(\Lambda-1)}.$$

LEMMA 3.1. — $\exists\xi=\xi(N,\Lambda)$ *independent of* h, $\mathbf{R}$ *such that* $\mathcal{L}(v)\leq 0$ *in* $\{\|z\|<1\}$, *for all* $t>\bar{t}$ *satisfying*

$$(3.12)\qquad 0<t-\bar{t}<\frac{1}{\xi}\psi'(h)R^2.$$

Proof. — First observe that (3.9) implies

$$(3.13)\qquad \epsilon^{\Lambda}\psi(h)\leq\psi(\epsilon h)<\epsilon^{\frac{1}{\Lambda}}\psi(h),\qquad h>0,\qquad \epsilon\in(0,1).$$

Calculating within $\{\|z\|<1\}$, $t>\bar{t}$, gives

$$v_{x_i}=-2\theta EF^{\theta-1}(x-\bar{x})_i,$$
$$v_{x_ix_j}=-2\theta EF^{\theta-1}\delta_{ij}+4\theta(\theta-1)F^{\theta-2}E\frac{(x-\bar{x})_i(x-\bar{x})_j}{S(t)},$$
$$v_t=-\frac{\xi EF^{\theta}}{\psi'(h)}+\frac{2\theta EF^{\theta-1}}{\psi'(h)}\|z\|^2,$$

where

$$E\equiv hR^{2\xi}/S^{\xi+1}(t).$$

Therefore, if $\mathcal{L}^*(v)=E^{-1}(v)\mathcal{L}(v)$, using (3.10),

$$(3.14)\quad \mathcal{L}^*(v)\leq\left(\frac{S(t)}{R^2}\right)^{\xi}\frac{v\psi'(v)}{h\psi'(h)}\left[-\xi+\frac{2\theta\|z\|^2}{F}\right]$$
$$-4\Lambda^{-1}\theta(\theta-1)F^{\theta-2}\|z\|^2+2\Lambda N\theta F^{\theta-1}.$$

For the times t as in (3.12), $S(t)^{1/2}\leq\left(\frac{1}{\xi}+1\right)^{1/2}R$ and

$$R^{2\xi}\leq S(t)^{\xi}\leq\left(1+\frac{1}{\xi}\right)^{\xi}R^{2\xi}\leq eR^{2\xi}.$$

Let $\delta\in(0,1)$ and examine $\mathcal{L}^*(v)$ on the sets $\{\|z\|^2<1-\delta\}$ and $\{\|z\|^2\geq 1-\delta\}$.

On the set $\{\| z \|^2 \geq 1-\delta\}$, $F \leq \delta$ and from (3.14), if $\mathcal{L}^{**}(v) \equiv F^{2-\theta}\mathcal{L}^*(v)$ it follows

$$\mathcal{L}^{**}(v) \leq 2\theta e F^{1-\theta} \frac{v\psi'(v)}{h\psi'(h)} - d_0 + \beta\delta, \tag{3.15}$$

where

$$d_0 = 4\Lambda^{-1}\theta(\theta - 1)(1 - \delta), \qquad \beta - 2\Lambda N\theta.$$

Now

$$\frac{v\psi'(v)}{h\psi'(h)} = \frac{v\psi'(v)}{\psi(v)} \frac{\psi(h)}{h\psi'(h)} \frac{\psi(v)}{\psi(h)} \leq \Lambda^2 \frac{\psi(v)}{\psi(h)}.$$

Next from the definition (3.11) we have

$$v \leq hF^\theta \qquad \text{and} \qquad F^\theta \in (0, 1).$$

Therefore by (3.13)

$$\frac{v\psi'(v)}{h\psi'(h)} \leq \Lambda^2 \frac{\psi(v)}{\psi(h)} \leq \Lambda^2 F^{\theta/\Lambda},$$

and from (3.15)

$$\begin{aligned}\mathcal{L}^{**}(v) &\leq 2\theta\epsilon\Lambda^2 F^{1-\theta+\frac{\theta}{\Lambda}} + \beta\delta - d_0 = 2\theta\epsilon\Lambda^2 F^{\frac{1}{2\Lambda}} + \beta\delta - d_0 \\ &\leq 2\theta\epsilon\Lambda^2 \delta^{\frac{1}{2\Lambda}} + \beta\delta - d_0 \leq 0\end{aligned}$$

if δ is chosen small enough depending only upon Λ, β and independent of ξ.

On $\{\| z \|^2 < 1 - \delta\}$, $F > \delta$, $v > ah$, $a = \dfrac{\delta^\theta}{e}$ and

$$\Lambda^2 \geq \frac{v\psi'(v)}{h\psi'(h)} = \frac{v\psi'(v)}{\psi(v)} \frac{\psi(h)}{h\psi'(h)} \frac{\psi(v)}{\psi(h)} \geq \frac{1}{\Lambda^2} a^\Lambda.$$

We will choose $\xi > 2\theta \| z \|^2/F$ so that

$$\begin{aligned}\mathcal{L}^*(v) &\leq \frac{v\psi'(v)}{h\psi'(h)} \left[- \xi + \frac{2\theta \| z \|^2}{F} \right] + \beta \\ &\leq \frac{1}{\Lambda^2} \left(\frac{\delta^\theta}{\epsilon}\right)^\Lambda (- \xi) + \frac{2\Lambda^2\theta(1 - \delta)}{\delta} + \beta \leq 0\end{aligned}$$

if $\xi = \xi(\Lambda, \beta, \delta) = \xi(N, \Lambda)$ is chosen sufficiently large. Then lemma is proved.

3. *iii*) Proof of (1.8) concluded.

We return to (3.7) and compare u with v constructed above, within the region

$$G \equiv \{ |x| \leq 4k\rho \} \times \left(\frac{T}{8}, T\right), \qquad \rho > 0, \qquad T > 0 \text{ fixed}.$$

Since the C^{α}_{loc}-estimates of [6] are stable under regularization of u and φ (the structure of (1.2) being kept) we may assume u is a classical solution of (1.1) in G with $\varphi \in C^2(\mathbf{R}^+)$. Setting

$$(3.16) \qquad \begin{cases} w = \varphi(u), \quad \psi = \varphi^{-1} \quad \text{we have} \\ \psi'(w)w_t - \Delta w = 0 \quad \text{in } G, \end{cases}$$

and (3.7) implies

$$(3.17) \qquad w(x, t_0) > h \equiv \varphi\left(\frac{1}{4}k^{-N}E_0\right), \qquad \forall x : |x - x_0| < \epsilon_1 k\rho.$$

We recall that k, E_0, T, ρ are linked by (3.2). Using v as comparison function, we now proceed as in the case of 3. *i*), the only technical difference being that by (3.12) we must advance in « small steps ».

Employing v with $(\bar{x}, \bar{t}) \equiv (x_0, t_0)$, $R = \epsilon_1 k\rho$, we find at level

$$t_1 = t_0 + \frac{1}{\xi}\psi'(h)R^2,$$

within the ball

$$|x - x_0| < \frac{1}{2}\left[\left(1 + \frac{1}{\xi}\right)^{1/2} + 1\right]R \equiv (1 + \eta)R, \qquad \eta = \eta(N, \Lambda) \in (0, 1),$$

that $w(x, t_1) \geq v(x, t_1) \geq \lambda h$, $\lambda = \lambda(N, \Lambda) \in (0, 1)$. We now repeat the process with $(\bar{x}, \bar{t})$ replaced by (x_0, t_1), R replaced by $R_1 = (1 + \eta)R$ and h replaced by $h_1 \equiv \lambda h$, and proceed in this fashion to find sequences

$$R_i = (1 + \eta)^i R; \qquad h_i = \lambda^i h; \qquad t_i = t_0 + \xi^{-1}\sum_{j=1}^{i-1}\psi'(h_j)R_j^2,$$

$i = 0, 1, 2, \ldots$, such that in the ball $|x - x_0| < R_i$, at the level t_i, $w(x, t_i) \geq \lambda^i h$.

Our aim is to cover $x = 0$. Since $|x_0| \leq k\rho$, we will choose i_0 from

$$2k\rho \leq R_{i_0} = (1 + \eta)^{i_0} R = (1 + \eta)^{i_0}\epsilon_1 k\rho,$$

i. e.,

$$(3.18) \qquad i_0 = \left[\frac{\ln 2\epsilon_1^{-1}}{\ln(1 + \eta)}\right] + 1.$$

For such a choice

$$t_{i_0} = t_0 + \frac{1}{\xi} \sum_{j=1}^{i_0-1} \psi'(h_j) R_j^2 \tag{3.19}$$

and

$$w(0, t_{i_0}) \geq \sigma_0 \varphi\left(\frac{1}{4} k^{-N} E_0\right), \qquad \sigma_0 = \lambda^{i_0}. \tag{3.20}$$

We examine the level t_{i_0}. By (3.9) and (3.13)

$$\frac{1}{\Lambda} \lambda^{(\Lambda-1)j} \frac{\psi(h)}{h} \leq \psi'(h_j) \leq \Lambda \lambda^{\frac{1-\Lambda}{\Lambda} j} \frac{\psi(h)}{h}.$$

Therefore since $R = \epsilon_1 k \rho$

$$t_0 + \bar{C}_1 \frac{\psi(h)}{h} (k\rho)^2 \leq t_{i_0} \leq t_0 + \bar{C}_2 \frac{\psi(h)}{h} (k\rho)^2, \tag{3.21}$$

where $\bar{C}_i = \bar{C}_i(N, \Lambda)$, $i = 1, 2$ are given by

$$\bar{C}_1 = (\xi\Lambda)^{-1} \epsilon_1^2 \sum_{j=1}^{i_0-1} \lambda^{(\Lambda-1)j} (1 + \eta)^{2j},$$

$$\bar{C}_2 = \frac{\Lambda}{\xi} \epsilon_1^2 \sum_{j=1}^{i_0-1} \lambda^{\frac{1-\Lambda}{\Lambda} j} (1 + \eta)^{2j}.$$

Next by (3.17)

$$\bar{C}_3 \frac{1}{\Phi(k^{-N} E_0)} \leq \frac{\psi(h)}{h} = \frac{\frac{1}{4} k^{-N} E_0}{\varphi\left(\frac{1}{4} k^{-N} E_0\right)} \leq \bar{C}_4 \frac{1}{\Phi(k^{-N} E_0)}, \tag{3.22}$$

for $\bar{C}_i = \bar{C}_i(\Lambda)$, $i = 3, 4$. Finally recalling (3.2) we find from (3.21) and (3.22) that there exist two constants $C_i = C_i(N, \Lambda)$, $i = 1, 2$, quantitatively determined, such that

$$t_0 + C_1 T \leq t_{i_0} \leq t_0 + C_2 T.$$

By a further application of the comparison function v, if necessary, we may assume

$$t_{i_0} = \left(\frac{1}{2} + C_2\right) T \equiv T^*.$$

We conclude from (3.20) that $\varphi(u(0, T^*)) \geq \sigma_0 \varphi\left(\frac{1}{4} k^{-N} E_0\right)$ and by (3.13)

$$u(0, T^*) \geq \frac{1}{4} \sigma_0^{\Lambda} k^{-N} E_0 \equiv \sigma_1 k^{-N} E_0, \tag{3.23}$$

which implies

$$(3.24)\qquad k^2 \geq [\mathrm{E}_0/u(0,\mathrm{T}^*)]^{\frac{2}{\mathrm{N}}}\sigma_1^{\frac{2}{\mathrm{N}}}.$$

From (3.23) and (3.2)

$$\Phi(u(0,\mathrm{T}^*)) \geq \sigma_2 \frac{k^2\rho^2}{\mathrm{T}}; \qquad \sigma_2 \equiv \sigma_1^{\Lambda}\gamma_0.$$

Using (3.24) in this last estimate

$$(3.25)\qquad \mathrm{E}_0 \leq \left(\sigma_1\sigma_2^{\frac{\mathrm{N}}{2}}\right)^{-1}\left[\frac{\mathrm{T}^*}{\rho^2}\Phi(u(0,\mathrm{T}^*))\right]^{\frac{\mathrm{N}}{2}} u(0,\mathrm{T}^*).$$

This holds under the assumption that (3.3) is verified. Combining the case when (3.3) fails with (3.25) and redefining T, the Harnack inequality follows, for compactly supported initial data.

The general case follows from observing that the solution u at hand, majorizes z, the unique solution of

$$(3.26)\qquad z_t - \Delta\varphi(z) = 0; \qquad z(x,0) = \begin{cases} u(x,\tau); & |x| < \rho, \\ 0; & |x| \geq \rho, \end{cases}$$

$\forall\rho > 0$, $\forall 0 < \tau < \frac{\mathrm{T}}{4}$. Indeed, by the construction procedure of Sabinina [14], z is the monotone limit of z_R as $\mathrm{R} \to +\infty$, where z_R solves (3.26) in $\{|x| < \mathrm{R}\} \times (0,\mathrm{T})$ with $z_\mathrm{R}(x,t) = 0$ on $|x| = \mathrm{R}$. By the comparison principle over bounded domains $z_\mathrm{R} \leq u$ over $\{|x| < \mathrm{R}\}\times(0,\mathrm{T})$, $\forall\mathrm{R} > \rho$.

4. EXTENSIONS AND NEW RESULTS

All the estimates of Proposition 1 hold true for non-negative subsolutions (see [6]).

$$u \in \mathrm{C}\left(0,\mathrm{T};\mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^\mathrm{N})\right); \qquad \varphi(u) \in \mathrm{L}^2_{\mathrm{loc}}\left(0,\mathrm{T};\mathrm{W}^{1,2}_{\mathrm{loc}}(\mathbf{R}^\mathrm{N})\right),$$

of

$$(4.1)\qquad \begin{cases} u_t - \operatorname{div}\vec{a}(x,t,u,\mathrm{D}\varphi(u)) \leq b(x,t,u,\mathrm{D}\varphi(u)) & \text{in } \mathrm{S}_\mathrm{T}, \\ u(x,0) = u_0(x) \in \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^\mathrm{N}); \quad |||u_0|||_r < +\infty & \text{for some } r > 0, \end{cases}$$

where

$$\Lambda^{-1}\Phi(u)\,|\mathrm{D}u|^2 \leq \vec{a}(x,t,u,\mathrm{D}\varphi(u))\cdot\mathrm{D}u \leq \Lambda\Phi(u)\,|\mathrm{D}u|^2,$$
$$|\vec{a}(x,t,u,\mathrm{D}\varphi(u))| \leq \Lambda(\Phi(u)\,|\mathrm{D}u| + 1),$$
$$|b(x,t,u,\mathrm{D}\varphi(u))| \leq \Lambda.$$

We assume u can be constructed as the local weak limit, in the specified class, of subsolutions growing no faster than $\Phi^{-1}(|x|^2)$ as $|x| \to \infty$. Working in S_T, $0 < T < +\infty$ fixed, (1.6) and (1.7) hold with u replaced by

$$w = \max(u, 1); \tag{4.2}$$

moreover, we assume $T \leq 1$, without loss of generality.

As for the proof, Lemma 2.1 carries over with minor changes. A difference occurs in estimating the blow up time. In the case of (1.1), the main tool was (2.17) which is a trivial consequence of (1.1), since the Laplacian permits a double integration by parts.

The analog of (2.17) here would be

$$\fint_{B_\rho} w(x, t)dx \leq 2^N \fint_{B_{2\rho}} w_0(x)dx + \frac{\gamma(N, \Lambda)}{\rho} \int_0^t \fint_{B_{2\rho}} \Phi(w)\,|Dw|\,dxd\tau, \tag{4.3}$$

which is obtained from (4.1) by integration against $x \to \zeta(x)$, the standard cutoff function in $B_{2\rho}$.

We will prove the following lemma in the appendix.

LEMMA 4.1. — $\forall \beta \in (0, 1)$ *such that* $\forall \rho > 0$ *and all* t *satisfying (2.2)*

$$\int_0^t \tau^{\beta-1}\sqrt{\Phi(\|w(\cdot, \tau)\|_{\infty, B_{2\rho}})}d\tau < +\infty,$$

we have

$$\int_0^t \fint_{B_\rho} \tau^\beta \frac{\Phi^{\frac{3}{2}}(w)}{w} |Dw|^2 dxd\tau \leq \gamma_1 \|w(t)\|_{2\rho} \int_0^t \tau^{\beta-1}\sqrt{\Phi(\|w(\cdot, \tau)\|_{\infty, B_{2\rho}})}d\tau, \tag{4.4}$$

$$\int_0^t \fint_{B_\rho} \Phi(w)\,|Dw|\,dxd\tau \leq \gamma_2 \|w(t)\|_{2\rho} \left[\frac{\rho^N \|w(t)\|_{2\rho}}{A^{-1}\left(\left(\frac{\rho^2}{t}\right)^{\frac{N}{2}} \|w(t)\|_{2\rho}\right)}\right]^{\frac{1}{N}}, \tag{4.5}$$

where $\|w(t)\|_\rho$ *and* $A(\cdot)$ *have been defined in* 2. *i*). *Here* $\gamma_1 = \gamma_1(N, \Lambda, \beta)$, $\gamma_2 = \gamma_2(N, \Lambda)$.

We notice that, owing to (2.3), $\beta = \dfrac{1}{2}$ is a suitable choice of β.

REMARK 4.1. — For weak solutions of equations like (4.1) with $b \equiv 0$ and

$$\Lambda^{-1}\Phi(u)\,|Du|^2 \leq \vec{a}(x, t, u, D\varphi(u))\cdot Du \leq \Lambda\Phi(u)\,|Du|^2,$$
$$|\vec{a}(x, t, u, D\varphi(u))| \leq \Lambda\Phi(u)\,|Du|,$$

estimates (4.4) and (4.5) hold for the function u itself. This class of equations includes (1.1) or, more generally,

$$u_t - (a_{ij}(x,t)\varphi(u)_{x_i,x_j} = 0,$$

with $\Lambda^{-1}|\eta|^2 \le a_{ij}(x,t)\eta_i\eta_j \le \Lambda|\eta|^2$, $\forall \eta \in \mathbf{R}^N$.

Remark 4.2. — Lemma 4.1 actually holds even if

$$|b(x,t,u,D\varphi(u))| \le \Lambda(1 + f(u)|Du|),$$

where

$$f^2(s)\frac{s}{\Phi(s)} \le \Lambda, \qquad \forall s \ge 1.$$

Substituting (4.5) in (4.3) we obtain

$$\|w(t)\|_\rho \le \gamma\|w_0\|_{2\rho} + \gamma\|w(t)\|_{2\rho}\left[\frac{\|w(t)\|_{2\rho}}{\Lambda^{-1}\left(\left(\frac{\rho^2}{t}\right)^{\frac{N}{2}}\|w(t)\|_{2\rho}\right)}\right]^{\frac{1}{N}},$$

Therefore by taking $t = \epsilon\rho^2/\Phi(\|w(t)\|_{2\rho}))$ the coefficient of the last term can be made small for small ϵ. The remainder of the proof of Proposition 1 stays unchanged.

Estimates (4.4)-(4.5) appear to be new, even in the case of (1.1) with $\varphi(s) = s^m$, $m > 1$.

4. *i*) Gradient estimates for $u_t = \Delta u^m$.

Using estimate (1.7)′ one can see that in Lemma 4.1 the choice $\beta = \dfrac{N(m-1)}{2k} + \epsilon$ is allowed $\forall \epsilon > 0$.

Combining (4.4)-(4.5) with (1.6)-(1.7)′ we obtain

LEMMA 4.2. — *Let β be as above and let $u \ge 0$ be a weak solution in* S_∞ *of*

$$u_t - \Delta u^m = 0, \qquad u(\cdot, 0) = u_0(\cdot) \in C_0^\infty(\mathbf{R}^N), \qquad m > 1.$$

Then $\exists \gamma_i = \gamma_i(N, m)$, $i = 0, 1, 2$: $\forall r > 0$, $\forall \rho \ge r$, $\forall\, 0 < t < T_r$,

$$\frac{1}{T_r} = \gamma_0^{-1}\sup_{\rho \ge r}|||u_0|||_r^{m-1} \qquad (\gamma_0 \in (0,1)),$$

there holds

$$(4.6)\qquad \int_0^t \fint_{B_\rho} \tau^\beta u^{-\frac{m+1}{2}} |Du^m|^2 dx d\tau \le \gamma_1 \frac{t^\epsilon}{\epsilon} \rho^{\frac{m+1}{m+1}} ||| u_0 |||_r^{1+\frac{m-1}{\kappa}}$$

$$(4.7)\qquad \int_0^t \fint_{B_\rho} |Du^m| dx d\tau \le \gamma_2 t^{\frac{1}{\kappa}} \rho^{\frac{m+1}{m+1}} ||| u_0 |||_r^{1+\frac{m-1}{\kappa}}.$$

REMARK 4.3. — It is apparent that (4.6)-(4.7) continue to hold for u_0 a σ-finite measure μ in $\mathbf{R}^N$ such that $||| \mu |||_r < +\infty$, for some $r > 0$.

The interest in (4.7) is that it gives a quantitative estimate of how fast the $L^1(B_\rho \times (0, t)$ norm of $|Du^m|$ « up to $t = 0$ », tends to zero as $t \searrow 0$.

The estimate is sharp in view of the Barenblatt-Pattle solutions [*2*], [*12*].

As for (4.6) write for $q \in (1, 2)$, formally

$$\iint_Q |Du^m|^q dx d\tau = \iint_Q \tau^\nu u^{-\alpha} |Du^m|^q \tau^{-\nu} u^\alpha dx d\tau$$

$$\le \left(\iint_Q \tau^{\nu\frac{2}{q}} |Du^m|^2 u^{-\alpha\frac{2}{q}} dx d\tau \right)^{\frac{q}{2}} \left(\iint_Q \tau^{-\nu\frac{2}{2-q}} u^{\alpha\frac{2}{2-q}} \right)^{\frac{2-q}{2}},$$

$Q \equiv B_\rho \times (0, t)$; $\alpha = \dfrac{(m+1)q}{4}$; $\nu = \dfrac{q}{2}\beta$. Then using (4.6) and adjusting q so that the last integral is finite (Proposition 1 is used here), we find

$$(4.8)\qquad |Du^m| \in L^q(B_\rho \times (0, t)), \qquad q < 1 + \frac{1}{Nm+1}.$$

Such « integrability up to zero » is optimal in view of the explicit solution of [*2*], [*12*].

A similar restriction holds for solutions of the heat equation, with initial datum a Dirac measure. It is worth mentioning the following consequence of (4.8).

Solving $u_t - \Delta u^m = 0$ with $u(\cdot, 0) = \mu \ge 0$, a σ-finite Borel measure in $\mathbf{R}^N$ such that $||| \mu |||_r < +\infty$, for some $r > 0$, one finds a unique solution u satisfying

$$|Du^m| \in L_{loc}^{1+\frac{1-\epsilon}{Nm+1}}(\mathbf{R}^N \times (0, T(\mu))),$$

$\forall \epsilon \in (0, 1)$, where $T(\mu)$ is defined in Section 1.

4. *ii*) About the Harnack inequality.

Estimate (1.8) continues to hold for non-negative weak solutions (see Remark 1.3) of

$$u_t - a_{ij}(t)\varphi(u)_{x_i x_j} = b(x, t, u, D\varphi(u)), \tag{4.9}$$

where the matrix a_{ij} satisfies (3.10) and

$$0 \leq b(x,t, u, D\varphi(u)) \leq \Lambda(1 + \Phi(u) \,|\, Du \,|). \tag{4.10}$$

Indeed a version of lemma 2.3 (which follows from Proposition 1) continues to hold—for the homogeneous equation—in view of the previous paragraph. Moreover the Hölder continuity of solutions of (4.9) is guaranteed by the results of [*6*]. Finally, the comparison functions are constructed in 3. *ii*) and the comparison principle can be applied since $b \geq 0$.

4. *iii*) Initial traces.

Every non-negative continuous distributional solution of (4.9) has, a initial trace, a unique σ-finite measure $\mu \geq 0$ in $\mathbf{R}^N$, satisfying $|||\, \mu \,|||_r < +\infty$, $\forall r > 0$.

Existence follows from the stated Harnack inequality and uniqueness follows from

LEMMA 4.3. — $\exists \gamma = \gamma(N, \Lambda)$ *such that* $\forall \epsilon \in (0, 1)$, $\forall \rho > 0$, $\forall t \in (\tau, T_0)$ *where*

$$T_0 \leq \min\left\{ 1 ; \gamma_0^{-1}\left[\rho^2/\Phi\left(\fint_{B_\rho} u(x, \tau)dx \right)\right]\right\}, \tag{4.11}$$

there holds

$$\fint_{B_{(1+\epsilon)\rho}} u(x, t)dx \geq \fint_{B_\rho} u(x, \tau) \left\{ (1+\epsilon)^{-N} - \frac{\gamma}{\epsilon} \left[\frac{\fint_{B_\rho} u(x, \tau)dx}{A^{-1}\left(\left(\frac{\rho^2}{t}\right)^{\frac{N}{2}} \fint_{B_\rho} u(x, \tau)dx\right)} \right]^{\frac{1}{N}} \right\}, \tag{4.12}$$

Here γ_0 *is the constant of Proposition 1.*

Uniqueness of initial traces.

On the right-hand side of (4.12) take $\tau = \sigma\rho^2/\Phi\left(\fint_{B_\rho} u(x, \tau)dx\right)$, $\sigma \in (0, \gamma_0)$. Then by the definition (2.13) of $A(\cdot)$ and (1.10) we obtain

$$\fint_{B_{(1+\epsilon)\rho}} u(x, t)dx \geq \fint_{B_\rho} u(x, \tau)dx \left\{ (1+\epsilon)^{-N} - \frac{\gamma}{\epsilon}\sigma^{\frac{1}{N\Lambda+2}} \right\},$$

$\forall \epsilon \in (0, 1)$, $\forall \sigma \in (0, \gamma_0)$, $\forall \tau \in (0, t)$.

The proof can be concluded as in [*1*].

Proof of lemma 4.3. — It suffices to prove the lemma for weak solutions of

$$z_t - (a_{ij}\varphi(z)_{x_i})_{x_j} = 0, \qquad t > 0, \qquad x \in \mathbf{R}^N,$$
$$z(x, 0) = \begin{cases} u(x, \tau), & |x| < \rho, \\ 0, & |x| \geq \rho. \end{cases}$$

If $\zeta(x)$ is the standard cutoff function on $B_{(1+\epsilon)\rho}$ which equals one on B_ρ,

$$\int_{B_{(1+\epsilon)\rho}} z(x, t)dx \geq \int_{B_\rho} u(x, \tau)dx - \frac{\gamma}{\epsilon\rho}\int_0^t \int_{B_{(1+\epsilon)\rho}} |D\varphi(z)|\, dx d\tau.$$

Taking into account Lemma 4.1, Remark 4.1, (1.5) and (1.6) we deduce that for all $0 < t < T_0 \equiv \min\{1; \gamma_0[\rho^2/\Phi(E_r)]\}$, $E_\tau = \fint_{B_\rho} u(x, \tau)dx$,

$$\fint_{B_{(1+\epsilon)\rho}} z(x, t)dx \geq E_\tau \left\{ (1+\epsilon)^{-N} - \frac{\gamma}{\epsilon\rho}\left[\frac{\rho^N E_\tau}{A^{-1}\left(\left(\frac{\rho^2}{t}\right)^{\frac{N}{2}} E_\tau\right)}\right]^{\frac{1}{N}} \right\}.$$

We remark that Lemma 4.3 is a sole consequence of the sup estimates of Proposition 1.

4. *iv*) About (1.2).

We finally comment on assumption (1.2)′ of Section 1. All our estimates are valid for it. As for Propositions 1 and 2, they still hold if u is replaced with $\max\{u; 1\}$. Indeed, in their proof in Section 2, (1.2)′ does not occur if one takes levels $k \geq 1$.

As for the comparison functions v in (3.11) needed to prove the Harnack inequality, we have constructed them in such a way that they are valid if (1.2)′ holds. The local Hölder continuity of solutions in the form (3.1) follows from a simple adaptation of the arguments of [6] and [15].

5. APPENDIX : PROOF OF LEMMA 4.1

In the weak formulation of (4.1) take the testing function

$$g = t^{\beta}\left(\int_1^w \frac{\Phi(s)}{\sqrt{\varphi(s)s}}\,ds\right)\zeta^3,$$

where ζ is the cutoff function in $B_{\frac{2}{3}\rho}$, $\zeta \equiv 1$ on B_ρ.

The following calculations can be made rigorous by means of a Steklov averaging process

$$\int_0^t\int_{B_{2\rho}} w_\rho \tau^{\beta}\left(\int_1^w \frac{\Phi(s)}{\sqrt{\varphi(s)s}}\,ds\right)\zeta^3 dx\,d\tau \geq -\beta\int_0^t \tau^{\beta-1}\int_{B_{2\rho}} w\left(\int_1^w \frac{\Phi(s)}{\sqrt{\varphi(s)s}}\,ds\right)\zeta^3 dx\,d\tau.$$

Using (1.2)

$$\begin{aligned}\int_1^w \frac{\Phi(s)}{\sqrt{\varphi(s)s}}\,ds &\leq \frac{2\Lambda}{\Lambda+1}\int_1^w s^{-\frac{1}{2}}\frac{d}{ds}\sqrt{\phi(s)}ds\\ &\leq \frac{2\Lambda}{\Lambda+1}\sqrt{\frac{\Phi(w)}{w}} + \frac{\Lambda}{\Lambda+1}\int_1^w \frac{\sqrt{\varphi(s)}}{s^{1+\frac{1}{2}}}\,ds\\ &= \frac{2\Lambda}{\Lambda+1}\sqrt{\Phi(w)} + \frac{\Lambda}{\Lambda+1}\int_1^w \frac{\Phi(s)}{\sqrt{\varphi(s)s}}\,ds,\end{aligned}$$

so that

$$\int_1^w \frac{\Phi(s)}{\sqrt{\varphi(s)s}}\,ds \leq 2\Lambda\sqrt{\Phi(w)}.$$

and that

$$\int_0^t \fint_{B_{2\rho}} w_\tau g\,dx\,d\tau \geq -\gamma\rho^{N}\,\|w(t)\|_{2\rho}\int_0^t \tau^{\beta-1}\sqrt{\Phi(\|w(\cdot,\tau)\|_{\infty,B_{2\rho}})}d\tau.$$

For the space part of the operator in (4.1), setting $Q \equiv B_{2\rho} \times (0, t)$,

$$(6.1)\quad \iint_Q \vec{a}(x, \tau, w, D\varphi(w))\cdot Dg\,dx\,d\tau \geq \frac{1}{2\Lambda}\iint_Q \tau^\beta \frac{\Phi^2(w)}{\sqrt{\varphi(w)w}}\,|Dw|^2\zeta^3 dx\,d\tau$$

$$- \gamma \iint_Q \frac{\tau^\beta}{\rho^2}\sqrt{\Phi(w)}\phi(w)\zeta dx\,d\tau - \gamma \iint_Q \tau^\beta \frac{\Phi(w)}{\sqrt{\varphi(w)w}}\,\zeta^3 dx\,d\tau$$

$$- \gamma \iint_Q \frac{\tau^\beta}{\rho}\,\sqrt{\Phi(w)}\zeta^2 dx\,d\tau.$$

Taking into account (2.3) we have

$$\iint_Q \frac{\tau^\beta}{\rho^2}\,\sqrt{\Phi(w)}\varphi(w)\zeta dx\,d\tau \leq \int_0^t \tau^{\beta-1}\,d\tau \int_{B_{\frac{3}{2}\rho}} \left[\frac{\tau}{\rho^2}\Phi(w)\right] w\sqrt{\Phi(w)}dx$$

$$\leq \gamma \,\|\, w(t)\,\|_{2\rho}\rho^N \int_0^t \tau^{\beta-1}\sqrt{\Phi(\|\, w(\cdot, \tau)\,\|_{\infty, B_{\frac{3}{2}\rho}})}d\tau.$$

Employing (2.2) it is easily seen that the same estimate holds for the last two integrals on the right-hand side of (6.1), as well as for the terms arising from the right-hand side of (4.1). Hence we have proved (4.4).

Next we take $\beta = \frac{1}{2}$ in the estimate above and use Hölder inequality to get

$$\iint_Q \Phi(w)\,|\,Dw\,|\,\zeta^2 dx\,d\tau = \iint_Q \tau^{\frac{1}{4}}\frac{\Phi(w)}{[\varphi(w)w]^{\frac{1}{4}}}\,|\,Dw\,|\,\tau^{-\frac{1}{4}}[\varphi(w)w]^{\frac{1}{4}}\zeta^2 dx\,d\tau$$

$$\leq \left(\iint_Q \tau^{\frac{1}{2}}\,\frac{\Phi^2(w)}{\sqrt{\varphi(w)w}}\,|\,Dw\,|^2\zeta^3 dx\,d\tau\right)^{\frac{1}{2}}\left(\iint_Q \tau^{-\frac{1}{2}}\sqrt{\Phi(w)w}\,\zeta dx\,d\tau\right)^{\frac{1}{2}}$$

$$\leq \gamma\rho^N \,\|\, w(t)\,\|_{2\rho} \int_0^t \tau^{-\frac{1}{2}}\sqrt{\Phi(\|\, w(\cdot, \tau)\,\|_{\infty, B_{\frac{3}{2}\rho}})}d\tau,$$

and it remains to estimate the last integral.

By Lemma 2.1 and the definition of $A(\cdot)$, by suitably modifying γ,

$$\int_0^t \tau^{-\frac{1}{2}}\sqrt{\Phi(\|\, w(\cdot, \tau)\,\|_{\infty, B_{\frac{3}{2}\rho}})}d\tau \leq \gamma \int_0^t \left[\frac{\xi}{A^{-1}(\tau^{-\frac{N}{2}}\xi)}\right]^{\frac{1}{N}} d\tau,$$

$\xi = \rho^N \,\|\, w(t)\,\|_{2\rho}$.

The last integral is estimated as in Section 2. *i*), to yield

$$\int_0^t \tau^{-1} \left[\frac{\xi}{A^{-1}(\tau^{-\frac{N}{2}}\xi)} \right]^{\frac{1}{N}} d\tau \le \gamma \int_0^t \left[\frac{\xi}{A^{-1}(t^{-\frac{N}{2}}\xi)} \right]^{\frac{1}{N}} d\tau,$$

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