

## **Biting theorems for Jacobians and their applications**

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**ABSTRACT.** — Weak continuity for sequences of Jacobians of vector-valued functions in  $W^{1,n}(\Omega; \mathbb{R}^n)$  in the sense of Chacon's Biting Lemma [13] is established via a maximal function method developed by Acerbi and Fusco [1]. A « div-curl » Lemma in the same spirit is also proved. The results are applied to the existence problem of nonlinear elastostatics and to a problem involving rank-one connections appearing in the theory of phase transitions.

*Key-words:* Weak convergence, Chacon's biting lemma,  $L^1$ , Jacobians, determinants, div-curl lemma, maximal function, nonlinear elasticity, rank-one connections.

**RÉSUMÉ.** — On démontre un théorème de continuité faible, au sens du lemme de morsure de Chacon, pour des suites de jacobiens de fonctions dans  $W^{1,n}(\Omega; \mathbb{R}^n)$ , ainsi qu'un lemme « div-rot ». Pour ce faire, on utilise la méthode de fonction maximale introduite par Acerbi et Fusco.

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## 0. INTRODUCTION

This paper establishes some « biting » theorems for Jacobians and a « biting div-curl lemma » in compensated compactness based on Chacon's biting lemma (see [13], [4], [10]). Chacon's biting lemma is the following (in the form of Ball and Murat [10]):

LEMMA. — (Brooks and Chacon [13], Ball and Murat [10], Balder [4]) : Let  $(\Omega, \mathfrak{F}, \mu)$  be a finite positive measure space,  $X$  a reflexive Banach space and let  $\{f_i\}$  be a bounded sequence in  $L^1(\Omega; X)$  i. e.,

$$\sup_j \int_{\Omega} \|f_j\|_X d\mu = C_0 < \infty.$$

Then there exist a function  $f \in L^1(\Omega; X)$ , a subsequence  $\{f_{\nu}\}$  of  $\{f_j\}$  and a nonincreasing sequence of sets  $E_k \in \mathfrak{F}$ , with  $\lim_{\nu \rightarrow \infty} \mu(E_k) = 0$ , such that

$$f_{\nu} \rightharpoonup f \quad \text{weakly in } L^1(\Omega \setminus E_k; X)$$

as  $\nu \rightarrow \infty$  for every fixed  $k$ .

In the above,  $L^1(\Omega; X)$  denotes the Banach space of (equivalence classes of) strongly measurable mappings  $g: \Omega \rightarrow X$  with finite norm

$$\|g\|_1 = \int_{\Omega} \|g\|_X d\mu.$$

The result is a useful tool in some variational problems where there is only an  $L^1$  bound on minimizing sequences. One such use has recently been made by Lin [20] in a study of the pure traction problem of nonlinear thermoelasticity. One can also establish some connections between Chacon's biting lemma and Young measures (see [11]). However this lemma says nothing about weak continuity of sequences of functions with partial derivative constraints. One may compare it with the results in the theory of compensated compactness, see e. g. [22], [29].

Suppose  $\Omega \subset \mathbb{R}^N$  is open, bounded and smooth, and that  $u_j: \Omega \rightarrow \mathbb{R}^N$  is a bounded sequence in  $W^{1,m}(\Omega; \mathbb{R}^N)$  with  $2 \leq m \leq \min\{n, N\}$  a positive integer. Denote by  $J_m(Du)$  the  $m \times m$  minors of  $Du$ , where

$$Du(x) = (\partial u_i / \partial x_{\alpha}), \quad i = 1, \dots, N; \quad \alpha = 1, \dots, n.$$

We show in this paper that, roughly speaking, if  $u_j \rightharpoonup u$  weakly in  $W^{1,m}(\Omega; \mathbb{R}^n)$ , there exists a subsequence  $(u_{\nu})$  of  $(u_j)$ , such that

$$J_m(Du_{\nu}) \rightharpoonup J_m(Du) \quad \text{weakly in } L^1(\Omega \setminus E_k) \quad \text{for all } k$$

where  $\{E_k\}$  is the sequence of measurable sets in Chacon's biting lemma. In other words, we have more information in the biting lemma:

$f = J_m(Du)$  provided some restrictions on partial derivatives of the sequences  $(u_j)$  are assumed.

It should be noted that, in general, it is false that  $u_j \rightharpoonup u$  in  $W^{1,m}(\Omega; \mathbb{R}^N)$  implies that there exists a subsequence  $(u_{j'})$  of  $(u_j)$  such that

$$J_m(Du_{j'}) \rightharpoonup J_m(Du) \quad \text{in } L^1(\Omega) \text{ weakly}$$

(see Ball and Murat [9], Counterexample 7.1, 7.3). Hence our result is almost optimal if no further restriction on  $J_m(Du_j)$  is assumed. Our result may be compared to that of Reshetnyak [26], [27], who shows that

$$J_m(Du_j) \overset{*}{\rightharpoonup} J_m(Du) \quad \text{in the sense of measure.}$$

The difference between our biting weak continuity theorems and the result of Reshetnyak is significant: the sets of test functions for weak convergence in the two cases may be disjoint, since the sets  $\{E_k\}$  to be « bitten » in Chacon's lemma may be very « bad » (see e. g. [10]). The weak continuity of Jacobians for Sobolev functions plays an important role in many fields of mathematics and mechanics, especially in nonlinear elastostatics and in the theory of phase transitions (see Ball [5], [6], Ball and Murat [9], Ciarlet [15], Nečas [24], Giaquinta, Modica and Souček [18], Ball and James [8], Kinderlehrer [19], Chipot and Kinderlehrer [14]). In this paper, we show through some examples the applications of biting theorems to nonlinear elasticity; specifically, we extend a result of Ball and Murat [9] on the existence of minimizing problems in nonlinear elasticity to the case where the material is inhomogeneous, and partially answer a question of Ball and James [8, p. 25] related to the theory of fine phase mixtures. In the same spirit, we study « biting » weak continuity problems in compensated compactness theory and prove a « biting » div-curl lemma (for the original div-curl lemma see Murat [22], [23], Tartar [29]).

The method we use involves the Hardy-Littlewood maximal function and the approximation of  $W^{1,p}$  functions by  $W^{1,\infty}$  functions as developed by Acerbi and Fusco [1], [2] in the study of semicontinuity problems in the calculus of variations.

In Section 1, notation and preliminaries are given which will be used in the proof of main results. We state and prove our « biting » theorems in Section 2 and finally in Section 3 we show how the biting theorems can be used to study minimizing problems in nonlinear elasticity and to study weak convergence problems in the theory of fine phase mixtures.

## 1. NOTATION AND PRELIMINARIES

If  $u \in \mathbb{R}^n$ , then  $|u|$  is its euclidean norm. When  $P$  is a  $N \times n$  matrix,  $|P|$  is the norm of  $P$  when regarded as a vector in  $\mathbb{R}^{nN}$ . We denote by

$M^{N \times n}$  the set of all  $N \times n$  real matrices endowed with this norm, while we denote by  $M^n$ ,  $M_+^n$  the set of all  $n \times n$  matrices and  $n \times n$  matrices with  $\det P > 0$  respectively, where  $P \in M^n$ .  $\det P$  denotes the determinant of  $P$ ,  $\text{adj } P$  denotes the adjugate matrix of  $P$ . The Lebesgue measure of a measurable subset  $S$  of  $\mathbb{R}^n$  will be denoted by  $\text{meas}(S)$ .

Let  $\Omega \subset \mathbb{R}^n$  be a measurable subset,  $1 \leq p \leq +\infty$ ,  $N \geq 1$ . We define  $L^p(\Omega; \mathbb{R}^N)$  to be the collection of  $N \rightarrow$  tuples  $(f^{(1)}, \dots, f^{(N)})$  of real functions in  $L^p(\Omega)$ . Analogously, when  $\Omega \subset \mathbb{R}^n$  is open, we say that  $u \in W^{1,p}(\Omega; \mathbb{R}^N)$  if  $u$  belongs to  $L^p(\Omega; \mathbb{R}^N)$  together with its distributional derivatives  $\partial u^{(i)}/\partial x_\alpha$ ,  $1 \leq i \leq N$ ,  $1 \leq \alpha \leq n$ . We also denote these derivatives by  $u_\alpha^{(i)}$  for simplicity. The  $N \times n$  matrix of these derivatives will be denoted by the symbol  $Du$ ;  $W^{1,p}(\Omega; \mathbb{R}^N)$  becomes a Banach space if it is endowed with the norm

$$\begin{aligned} \|u\|_{W^{1,p}(\Omega; \mathbb{R}^N)} &= \| |u| \|_{L^p(\Omega)} + \| |Du| \|_{L^p(\Omega)}, \quad 1 \leq p < +\infty \\ \|u\|_{W^{1,\infty}(\Omega; \mathbb{R}^N)} &= \text{ess sup } |u| + \text{ess sup } |Du| \end{aligned}$$

where

$$|u|(x) = |u(x)|, \quad |Du|(x) = |Du(x)|$$

and sometimes we write

$$W^{1,2}(\Omega; \mathbb{R}^N) = H^1(\Omega; \mathbb{R}^N), \quad H^{-1}(\Omega; \mathbb{R}^N) = \text{dual space of } H^1(\Omega; \mathbb{R}^N).$$

For an open set  $\Omega \subset \mathbb{R}^n$ , we say  $u \in C_0^k(\Omega; \mathbb{R}^N)$  if each  $u^{(i)}$  is a  $C^k$  function on  $\Omega$  with compact support, where  $k$  is a positive integer or  $+\infty$ .

We will use the minors of the matrix  $Du$  and we introduce the following notations. Let  $A \in M^{N \times n}$ ,  $I, J, \dots$  be multi-indices, i. e.,  $I = \{i_1, \dots, i_r\}$ ,  $J = \{j_1, \dots, j_r\}$ ,  $i_k, j_s$  integers,  $1 \leq i_1 < i_2 < \dots < i_r \leq N$ ,  $1 \leq j_1 < j_2 < \dots < j_r \leq n$ ;  $1 \leq r \leq \min\{N, n\}$ . We write  $|I| = |J| = r$ . Let  $A_{IJ}$  be the submatrix of  $A$  consisting of rows  $i_1, \dots, i_r$  and columns  $j_1, \dots, j_r$ , i. e.  $A_{IJ} = (a_{ij})$ ,  $i \in I, j \in J$  and denote by  $M'_{IJ}(A) = \det A_{IJ}$ . Throughout this paper,  $\rightarrow$  will denote weak convergence and  $\overset{*}{\rightarrow}$  denote weak\* convergence, while  $\overset{b}{\rightarrow}$  denotes « biting weak convergence » in  $L^1(\Omega)$ , i. e.  $u_j \overset{b}{\rightarrow} u$  as  $j \rightarrow \infty$  if and only if there exists a nonincreasing sequence of measurable sets  $(E_k)$ ,  $E_k \subset \Omega$  with  $\text{meas}(E_k) \rightarrow 0$ , such that

$$u_j \rightarrow u \quad \text{in } L^1(\Omega \setminus E_k) \quad \text{for } k = 1, 2, \dots$$

Finally we apply throughout this paper the summation convention that Latin indices are summed from 1 to  $N$  and Greek indices are summed from 1 to  $n$ . Sometimes, both Greek and Latin indices are summed from 1 to  $m$  ( $2 \leq m \leq \min\{n, N\}$ ).

From now on,  $\Omega \subset \mathbb{R}^n$  denotes a nonempty open bounded set.

DEFINITION 1.1. —  $f: \Omega \times \mathbb{R}^N \times \mathbb{M}^{N \times n} \rightarrow \mathbb{R}$  is a Caratheodory function if the following conditions are satisfied:

- i) for every  $(u, P) \in \mathbb{R}^N \times \mathbb{M}^{N \times n}$ ,  $x \rightarrow f(x, u, P)$  is measurable;
- ii) for almost all  $x \in \Omega$ ,  $(u, P) \rightarrow f(x, u, P)$  is continuous.

The following result can be found in [16]; it is called the Arzela-Young theorem by von Neumann (1950) [25, p. 32].

LEMMA 1.2. — Let  $G \subset \mathbb{R}^n$  be measurable with  $\text{meas}(G) < \infty$ . Assume that  $(M_k)$  is a sequence of measurable subsets of  $G$  such that for some  $\epsilon > 0$ , the following estimate holds:

$$\text{meas}(M_k) \geq \epsilon \quad \text{for all } k \in \mathfrak{N}, \quad (\text{the set of all positive integers}).$$

Then a subsequence  $M_{k_h}$  can be selected such that  $\bigcap_{h \in \mathfrak{N}} M_{k_h} \neq \emptyset$ .

The next result of Acerbi and Fusco [1] characterizes the concentration of singular points for a bounded sequence in  $L^1(\mathbb{R}^n)$ .

LEMMA 1.3. — Let  $(\phi_k)$  be a bounded sequence in  $L^1(\mathbb{R}^n)$ . Then to each  $\epsilon > 0$ , there exists a triple  $(A_\epsilon, \delta, S)$ , where  $A_\epsilon$  is measurable and  $\text{meas}(A_\epsilon) < \epsilon$ ,  $\delta > 0$ , and  $S$  is an infinite subset of  $\mathfrak{N}$ , such that for all  $k \in S$

$$\int_B |\phi_k(x)| dx < \epsilon$$

whenever  $B$  and  $A_\epsilon$  are disjoint and  $\text{meas}(B) < \delta$ .

Let  $r > 0$  and  $x \in \mathbb{R}^n$ , set  $B(x, r) = \{y \in \mathbb{R}^n : |y - x| < r\}$  and  $\text{meas}(B(x, r)) = \omega_{n-1} \times r^n$ .

DEFINITION 1.4. — (The Maximal Function). Let  $u \in C_0^\infty(\mathbb{R}^n)$ , we define

$$(M^*u)(x) = (Mu)(x) + \sum_{\alpha=1}^n (Mu_{,\alpha})(x)$$

where we set

$$(Mf)(x) = \sup_{r>0} \frac{1}{\omega_{n-1} r^n} \int_{B(x,r)} |f(y)| dy$$

for every locally summable  $f$ , where  $\omega_{n-1}$  is the volume of the  $n - 1$  dimensional unit sphere.

LEMMA 1.5. — If  $u \in C_0^\infty(\mathbb{R}^n)$ , then  $M^*u \in C^0(\mathbb{R}^n)$  and

$$|u(x)| + \sum_{\alpha=1}^n |u_{,\alpha}| \leq (M^*u)(x)$$

for all  $x \in \mathbb{R}^n$ . Moreover (see [28]) if  $p > 1$ , then

$$\| M^*u \|_{L^p(\mathbb{R}^n)} \leq c(n, p) \| u \|_{W^{1,p}(\mathbb{R}^n)}$$

and if  $p \geq 1$ , then

$$\text{meas} (\{ x \in \mathbb{R}^n : (M^*u)(x) \geq \lambda \}) \leq \frac{c(n, p)}{\lambda^p} \| u \|_{W^{1,p}(\mathbb{R}^n)}^p$$

for all  $\lambda > 0$ .

LEMMA 1.6. — (see [1]). Let  $u \in C_0^\infty(\mathbb{R}^n)$  and  $\lambda > 0$ , and set

$$H^\lambda = \{ x \in \mathbb{R}^n : (M^*u)(x) < \lambda \}.$$

Then for every  $x, y \in H^\lambda$  we have

$$\frac{|u(x) - u(y)|}{|x - y|} \leq C(n)\lambda.$$

LEMMA 1.7. — Let  $X$  be a metric space,  $E$  a subspace of  $X$ , and  $k$  a positive real number. Then any  $k$ -Lipschitz mapping from  $E$  into  $\mathbb{R}$  can be extended to a  $k$ -Lipschitz mapping from  $X$  into  $\mathbb{R}$ .

For the proof see [17, p. 298].

## 2. BITING THEOREMS

In this section, we state and prove the main theorems of this paper. We assume throughout this section that  $n \geq 2$ ,  $2 \leq m \leq \min \{ N, n \}$  be an integer,  $N \geq 2$ ,  $p \geq \max \{ m - 1, 2 \}$ .

THEOREM 2.1. — Let  $\Omega \subset \mathbb{R}^n$  be a bounded set with smooth boundary and assume that  $(u_j) \subset W^{1,p}(\Omega; \mathbb{R}^N)$  is a bounded sequence such that  $u_j \rightarrow u$  in  $W^{1,p}(\Omega; \mathbb{R}^N)$  and all  $M_{I',J'}(Du_j)$  are bounded in  $L^q(\Omega)$  with  $|I'| = |J'| = m - 1, \frac{1}{p} + \frac{1}{q} \leq 1$ . Then for all multi-indices  $I = (i_1, \dots, i_m)$ ,  $J = (j_1, \dots, j_m)$ ,  $1 \leq i_1 < i_2 < \dots < i_m \leq n$ ,  $1 \leq j_1 < \dots < j_m \leq N$ , there exists a subsequence  $(u_\nu)$  of  $(u_j)$  such that

$$(2.1) \quad M_{I,J}^m(Du_\nu) \xrightarrow{b} M_{I,J}^m(Du) \text{ on } \Omega \text{ as } \nu \rightarrow \infty$$

COROLLARY 2.2. — When  $p = m$ , we have  $M_{I',J'}(Du_j)$  bounded in  $L^{\frac{m}{m-1}}(\Omega)$ ,  $|I'| = |J'| = m - 1$ . This ensures that  $u_j \rightarrow u$  in  $W^{1,m}(\Omega; \mathbb{R}^N)$  implies that there exists a subsequence  $(u_\nu)$  such that  $M_{I,J}(Du_\nu) \xrightarrow{b} M_{I,J}(Du)$ . Moreover, when  $m = n = N$  and  $\{ u_j \}$  is bounded in  $W^{1,n}(\Omega; \mathbb{R}^n)$  in Theorem 2.1, we have

$$(2.2) \quad \det Du_\nu \xrightarrow{b} \det Du \text{ on } \Omega.$$

REMARK 2.3. — *If the assumption  $q \geq p/(p - 1)$  in Theorem 2.1 is dropped, leaving only  $q \geq 1$ , one cannot deduce that*

$$M_{IJ}^m(Du_\nu) \stackrel{b}{\rightarrow} M_{IJ}^m(Du)$$

*for some subsequence  $(u_\nu)$  of  $(u_j)$  even if  $M_{IJ}^m$  has a meaning in  $L^1(\Omega)$ , see Ball and Murat [9, counterexample 7.4] and even in the case  $p = m$ , it is not true in general that*

$$M_{IJ}^m(Du_\nu) \rightarrow M_{IJ}^m(Du) \quad \text{in } L^1(\Omega)$$

([9, Counterexample 7.1, 7.3]).

A new proof of Corollary 2.2 can be obtained by using  $W^{1,p}$  weak lower semicontinuity result due to Acerbi and Fusco [1] instead of the Maximal Function Method (see [11] where we study lower semicontinuity problems in the calculus of variations using Chacon's biting lemma and the theory of young measures).

In order to prove the theorem, we need some weak convergence theorems for Jacobians from [4], [5], [18], [21], [7] and we use  $\text{adj}_{IJ}(A)$  to denote the adjugate matrix of  $A_{IJ}$  thought of as an  $r \times r$  matrix.

THEOREM A. — (see [4], [18], [21]) *If  $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ , then  $\text{adj}_{IJ}Du \in L^{\frac{p}{p-1}}(\Omega; M^m)$ . Furthermore*

$$(2.3) \quad \left. \begin{array}{l} u_j \rightarrow u \text{ in } W^{1,p}(\Omega; \mathbb{R}^N) \\ \text{adj}_{IJ}Du_j \rightarrow H \text{ in } L^q(\Omega; M^m), \quad q \geq 1 \end{array} \right\} \text{ imply } H = \text{adj}_{IJ}Du$$

THEOREM B. — *Let  $u \in W^{1,p}(\Omega; \mathbb{R}^N)$  such that  $\text{adj}_{IJ}Du \in L^q$ , where*

$$\frac{1}{p} + \frac{1}{q} \leq 1.$$

*Then  $M_{IJ}(Du)$  given by*

$$(2.4) \quad M_{IJ}(Du) = u_{,j_s}^i (\text{ad}_{IJ}(Du))_{j_s}^i$$

*is in  $L^1(\Omega)$ . Furthermore*

$$\left. \begin{array}{l} u_j \rightarrow u \text{ in } W^{1,p}(\Omega; \mathbb{R}^N) \\ \text{adj}_{IJ}(Du_j) \rightarrow H \text{ in } L^q(\Omega; M^m) \\ M_{IJ}(Du_j) \rightarrow \delta \text{ in } L^1(\Omega) \end{array} \right\} \text{ imply } \left\{ \begin{array}{l} H = \text{adj}_{IJ}(Du) \\ \delta = M_{IJ}(Du), \text{ a. e. on } \Omega \end{array} \right.$$

We can now state an abstract version of Theorem 2.1, i. e., a « biting » div-curl lemma in compensated compactness theory. The original div-curl lemma is the following (see Murat [22], [23], Tartar [29], Bensoussan, Lions and Papanicolaou [12]).

THEOREM C. — Let  $\Omega$  be an open set of  $\mathbb{R}^n$  and  $(u_j), (w_j)$  be two sequences in  $L^2(\Omega; \mathbb{R}^n)$  such that

$$(2.5) \quad \begin{aligned} u_j &\rightharpoonup u \text{ in } L^2(\Omega; \mathbb{R}^n) \\ w_j &\rightharpoonup w \text{ in } L^2(\Omega; \mathbb{R}^n) \end{aligned}$$

$$(2.6) \quad \begin{aligned} \text{curl } u_j &\text{ is bounded in } L^2(\Omega; \mathbb{R}^n) \text{ (or compact in } H^{-1}(\Omega; \mathbb{R}^{n^2})) \\ \text{div } w_j &\text{ is bounded in } L^2(\Omega; \mathbb{R}^n) \text{ (or compact in } H^{-1}(\Omega)) \end{aligned}$$

Let  $u \cdot w$  denote the inner product of  $u$  and  $w$  in  $\mathbb{R}^n$ , i. e.,

$$u \cdot w = \sum_{i=1}^n u^{(i)} w^{(i)}.$$

Then

$$u_j \cdot w_j \xrightarrow{*} u \cdot w \text{ in the sense of measure}$$

where  $(\text{curl } u)_{ij} = \frac{\partial u^{(i)}}{\partial x_j} - \frac{\partial u^{(j)}}{\partial x_i}$ ,  $\text{div } w = \sum_{i=1}^n \frac{\partial w^{(i)}}{\partial x_i}$ .

REMARK 2.4. — Reshetnyak [26], [27] shows that  $\det Du_j \xrightarrow{*} \det Du$  in the sense of measure, as  $u_j \rightharpoonup u$  in  $W^{1,n}(\Omega; \mathbb{R}^n)$ ; see also Ball, Currie and Olver [7, Theorem 3.4]. However, this result depends heavily on the divergence theorem, hence cannot be used for problems where we only have  $\det Du_j \xrightarrow{b} \chi$  on  $\Omega$  (see Proposition 3.3 below).

REMARK 2.5. — The conclusion of Theorem C is still true in the case where  $L^2$  is replaced in (2.5) and (2.6) by  $L^p$  spaces:  $L^p$  for  $u_j$  and  $\text{curl } u_j$ ,  $L^{p'}$  for  $w_j$  and  $\text{div } w_j$  with  $\frac{1}{q} + \frac{1}{p'} = 1$ , see Murat [22, section 2].

THEOREM 2.6. — Let  $\Omega$  be an open bounded subset of  $\mathbb{R}^n$  and  $(u_j), (w_j)$  be two sequences of  $L^2(\Omega; \mathbb{R}^n)$  satisfying (2.5) and (2.6). Then there exist subsequences  $(u_\nu), (w_\nu)$  of  $(u_j), (w_j)$  respectively such that

$$u_\nu \cdot w_\nu \xrightarrow{b} u \cdot w \text{ on } \Omega.$$

Proof of Theorem 2.1. — We prove the theorem in the case where  $M_{I\bar{J}}^m$  is the  $m$ -order minor consists of first  $m$  rows and first  $m$  columns and denote it by  $J(A)$ . Denote by  $A_{I\bar{I}}$  the submatrix of  $A$  consists of first  $m$  columns and first  $m$  rows. The transposed matrix of cofactors of  $A_{I\bar{I}}$  is denoted by  $\text{adj}_I A$  for simplicity, where  $I = \{1, \dots, m\}$ . Since we consider  $J(Du_j)$  as

$$J(Du_j) = u_{j,\alpha}^{(1)} (\text{adj}_I(Du_j))_\alpha^1$$

we have

$$\|J(Du_j)\|_{L^1(\Omega)} \leq \|Du_j^{(1)}\|_{L^p(\Omega)} \|\text{adj}_I Du_j\|_{L^{\frac{p}{p-1}}(\Omega)}$$

hence  $J(Du_j)$  is a bounded sequence in  $L^1(\Omega)$ . By Chacon's biting lemma, there exists a subsequence  $(u_\nu)$  of  $(u_j)$  such that

$$J(Du_\nu) \xrightarrow{b} \chi \text{ on } \Omega \text{ as } \nu \rightarrow \infty.$$

On the other hand, if we set

$$z_\nu = u_\nu^{(1)} - u^{(1)}$$

we have

$$z_\nu \rightarrow 0 \text{ in } W^{1,p}(\Omega)$$

and

$$(2.7) \quad J(Du_\nu) = z_{\nu,\alpha} (\text{adj}_I Du_\nu)_\alpha^1 + u_{,\alpha}^{(1)} (\text{adj}_I Du_\nu)_\alpha^1,$$

and by the assumption and Theorem A,

$$(\text{adj}_I Du_\nu)_\alpha^1 \rightarrow (\text{adj}_I Du)_\alpha^1 \text{ in } L^q(\Omega)$$

so that

$$(2.8) \quad z_{\nu,\alpha} (\text{adj}_I Du_\nu)_\alpha^1 \xrightarrow{b} \chi - J(Du) := \chi'$$

and we will prove that  $\chi' = 0$  a. e. on  $\Omega$ . By an extension theorem (Adams [3, Theorem 4.26]) we may assume  $(z_\nu)$  to be defined on  $\mathbb{R}^n$  with  $\|z_\nu\|_{W^{1,p}(\mathbb{R}^n)}$  bounded uniformly with respect to  $\nu$ . Since  $C_0^\infty(\mathbb{R}^n)$  is dense in  $W^{1,p}(\mathbb{R}^n)$ , there exists a sequence  $(w_\nu) \subset C_0^\infty(\mathbb{R}^n)$  such that

$$\|z_\nu - w_\nu\|_{W^{1,p}(\mathbb{R}^n)} < \frac{1}{\nu}$$

hence we may assume the sequence  $(z_\nu)$  to be in  $C_0^\infty(\mathbb{R}^n)$  and to be bounded in  $W^{1,p}(\mathbb{R}^n)$ . For every fixed  $\epsilon > 0$ , apply Lemma 1.3 to the sequence  $((M^*)z_\nu(x))^p$ . This gives a subsequence (still denoted by  $(z_\nu)$ ), a set  $A_\epsilon \subset \Omega$  with  $\text{meas}(A_\epsilon) < \epsilon$  and a real number  $\delta > 0$  such that

$$\int_B (M^*z_\nu(x))^p dx < \epsilon$$

for all  $\nu$  and for every  $B \subset \Omega \setminus A_\epsilon$ , with  $\text{meas}(B) < \delta$ . By Lemma 1.5, we may take  $\lambda > 0$  so large that for all  $\nu$

$$(2.9) \quad \text{meas} \{x \in \mathbb{R}^n : (M^*z_\nu)(x) \geq \lambda\} < \min\{\epsilon, \delta\},$$

For all  $\nu$ , set

$$H_\nu^\lambda = \{x \in \mathbb{R}^n : (M^*z_\nu)(x) < \lambda\}.$$

Lemma 1.6 ensures that for all  $x, y \in H_\nu^\lambda$

$$\frac{|z_\nu(y) - z_\nu(x)|}{|y - x|} \leq C(n)\lambda.$$

Let  $g_\nu$  be a Lipschitz function extending  $z_\nu$  outside  $H_\nu^\lambda$  with Lipschitz constant not greater than  $C(n)\lambda$  (Lemma 1.7). Since  $H_\nu^\lambda$  is an open set, we have

$$g_\nu(x) = z_\nu(x), \quad Dg_\nu(x) = Dz_\nu(x)$$

for all  $x \in H_\nu^\lambda$  and

$$\|Dg_\nu\|_{L^\infty(\mathbb{R}^n)} \leq C(n)\lambda.$$

We may also assume

$$\|g_\nu\|_{L^\infty(\mathbb{R}^n)} \leq \|z_\nu\|_{L^\infty(H^1(\Omega; \mathbb{R}^n))} \leq \lambda$$

and may suppose that at least for a subsequence

$$g_\nu \rightharpoonup^* v \text{ in } W^{1,\infty}(\Omega).$$

Let  $(E_k)$  be the sets given by Chacon's lemma: we have for any fixed  $\phi \in L^\infty(\Omega)$  with  $0 \leq \phi \leq 1$  and fixed  $k \geq 1$

$$(2.10) \quad \int_{\Omega \setminus (E_k \cup A_\epsilon)} \phi(z_{\nu,\alpha}(\text{adj}_I Du_\nu)_\alpha^1) dx = \int_{\Omega \setminus (E_k \cup A_\epsilon)} \phi(g_{\nu,\alpha}(\text{adj}_I Du_\nu)_\alpha^1) dx + \\ + \int_{\Omega \setminus (E_k \cup A_\epsilon \cup H_\nu^\lambda)} \phi(z_{\nu,\alpha}(\text{adj}_I Du_\nu)_\alpha^1) dx - \int_{\Omega \setminus (E_k \cup A_\epsilon \cup H_\nu^\lambda)} \phi(z_{\nu,\alpha}(\text{adj}_I Du_\nu)_\alpha^1) dx.$$

Now

$$(2.11) \quad \left| \int_{\Omega \setminus (E_k \cup A_\epsilon \cup H_\nu^\lambda)} \phi(z_{\nu,\alpha}(\text{adj}_I Du_\nu)_\alpha^1) dx \right| \\ \leq \|Dz_\nu\|_{L^p(\Omega \setminus (E_k \cup A_\epsilon \cup H_\nu^\lambda))} \|\text{adj}_I Du_\nu\|_{L^p(\Omega \setminus (E_k \cup A_\epsilon \cup H_\nu^\lambda))} \\ \leq C \|(M^*z_\nu)(x)\|_{L^p(\Omega \setminus (E_k \cup A_\epsilon \cup H_\nu^\lambda))} \leq C\epsilon^{\frac{1}{p}}$$

since  $\Omega \setminus (E_k \cup A_\epsilon \cup H_\nu^\lambda) \cap A_\epsilon = \emptyset$  and  $\text{meas}(\Omega \setminus (E_k \cup A_\epsilon \cup H_\nu^\lambda)) \leq \min\{\epsilon, \delta\}$ , (consult (2.9)) where  $p' = \frac{p}{p-1}$  and  $C$  depends only on  $n$  and the  $L^{p'}(\Omega)$  norm of  $\text{adj}_I Du_\nu$ . Also

$$(2.12) \quad \left| \int_{\Omega \setminus (E_k \cup A_\epsilon \cup H_\nu^\lambda)} \phi(g_{\nu,\alpha}(\text{adj}_I Du_\nu)_\alpha^1) dx \right| \leq C \|Dg_\nu\|_{L^p(\Omega \setminus (E_k \cup A_\epsilon \cup H_\nu^\lambda))} \leq C\epsilon^{\frac{1}{p}}$$

since on  $\Omega \setminus (E_k \cup A_\epsilon \cup H_\nu^\lambda)$  we have  $|Dg_\nu(x)| \leq C(n)\lambda \leq C(n)(M^*z_\nu)(x)$ . Now choose an open set  $\Omega' \subset \Omega$  containing  $\Omega \setminus (E_k \cup A_\epsilon)$  such that

$$(2.13) \quad \int_{\Omega' \setminus (\Omega \setminus (E_k \cup A_\epsilon))} |Dg_\nu|^p dx \leq \epsilon$$

(this is possible since the functions  $g_\nu$  are uniformly bounded in  $W^{1,\infty}(\mathbb{R}^n)$ ). By (2.8), (2.11), (2.12) and (2.13) we have now

$$(2.14) \quad -C'\epsilon^{\frac{1}{p}} \leq \int_{\Omega \setminus (E_k \cup A_k)} \phi(z_{\nu,\alpha}(\text{adj}_I Du_\nu)_\alpha^1) dx - \int_{\Omega'} \phi(g_{\nu,\alpha}(\text{adj}_I Du_\nu)_\alpha^1) dx \leq C'\epsilon^{\frac{1}{p}}$$

Denote by  $\tilde{u} = (g_\nu, u_\nu)$  the function whose first component is  $g_\nu$ , while the others are the same as those of  $u_\nu$ : then  $J(D\tilde{u}_\nu) = g_{\nu,\alpha}(\text{adj}_I Du_\nu)_\alpha^1$  is bounded in  $L^{p'}(\Omega')$ , and as  $\tilde{u}_\nu$  converges weakly in  $L^p$  to  $(\nu, u)$ , we have by Theorem B

$$J(D\tilde{u}_\nu) \rightharpoonup \nu_{,\alpha}(\text{adj}_I Du_\nu)_\alpha^1 \text{ in } L^{p'}(\Omega').$$

Passing to the limit in (2.14), we have

$$(2.15) \quad -C'\epsilon^{\frac{1}{p}} \leq \int_{\Omega \setminus (E_k \cup A_k)} \phi \chi' dx - \int_{\Omega' \cap G} \phi \nu_{,\alpha}(\text{adj}_I Du)_\alpha^1 dx \leq C'\epsilon^{\frac{1}{p}}$$

where  $G = \{x \in \Omega, \nu(x) \neq 0\}$ . Following the argument of Acerbi and Fusco [1, p. 139, 140], we have

$$\text{meas}(G) \leq \delta.$$

Thus we have

$$(2.16) \quad \int_{\Omega' \cup G} |Dv|^p dx = \int_{(\Omega' \cup G) \setminus (\Omega \setminus (E_k \cup A_k))} |Dv|^p dx + \int_{(\Omega' \cup G) \cup (\Omega \setminus (E_k \cup A_k))} |Dv|^p dx := I_1 + I_2.$$

By (2.13) and weak lower semicontinuity of  $L^p$  norms, ( $p > 1$ )

$$(2.17) \quad I_1 \leq \int_{\Omega' \setminus (\Omega \setminus (E_k \cup A_k))} |Dv|^p dx \leq \liminf_{\nu \rightarrow \infty} \int_{\Omega' \setminus (\Omega \setminus (E_k \cup A_k))} |Dg_\nu|^p dx \leq \epsilon$$

while

$$(2.18) \quad \int_{(\Omega' \cup G) \cup (\Omega \setminus (E_k \cup A_k))} |Dg_{\nu,\alpha}|^p dx \leq \int_{\Omega \setminus (E_k \cup A_k) \cup G \setminus H_k^\delta} C(n)^p \lambda^p dx + \int_{\Omega \setminus (E_k \cup A_k) \cap G \cap H_k^\delta} |Dz_\nu|^p dx \leq C(n)^p \int_{\Omega \setminus (E_k \cup A_k) \cap G \setminus H_k^\delta} (M^* z_\nu(x))^p dx + \epsilon \leq C_1 \epsilon$$

so that we have

$$(2.19) \quad I_2 \leq \int_{\Omega \setminus (E_k \cup A_k) \cup G} |Dv|^p dx \leq C_1 \epsilon.$$

Thus we have from (2.15)-(2.19)

$$(2.20) \quad -\tilde{C}\epsilon^{\frac{1}{p}} - \int_{A_\epsilon} |\chi'| dx \leq \int_{\Omega \setminus E_k} \phi \chi' dx \leq \int_{A_\epsilon} |\chi'| dx + \tilde{C}\epsilon^{\frac{1}{p}}.$$

Let  $\epsilon \rightarrow 0$ . We obtain

$$\int_{\Omega \setminus E_k} \phi \chi' dx = 0 \quad \text{for all } \phi \in L^\infty(\Omega) \text{ with } 0 \leq \phi \leq 1 \text{ and all } E_k.$$

Hence  $\chi' = 0$  a. e. on  $\Omega$ .

Q. E. D.

*Proof of Theorem 2.4.* — We use the idea of Murat [22, Section 2] in proving the div-curl lemma. Firstly, by Chacon's biting lemma we have a subsequence  $u_\nu \cdot w_\nu \xrightarrow{b} \chi$  on  $\Omega$ , and what we should prove is that  $\chi = u \cdot w$ . Now localize  $u_\nu, w_\nu$  by multiplying by a  $C_0^\infty(\mathbb{R}^n)$  function  $\theta$ . Then  $\theta u_\nu, \theta w_\nu$  satisfy

$$(2.21) \quad \theta u_\nu \rightharpoonup \theta u, \quad \theta w_\nu \rightharpoonup \theta w \text{ in } L^2(\Omega)$$

$$(2.22) \quad (\theta u_\nu) \cdot (\theta w_\nu) \xrightarrow{b} \theta^2 \chi \text{ on } \Omega$$

$$(2.23) \quad [\text{curl}(\theta u_\nu)]_{ij} = \theta [\text{curl} u_\nu]_{ij} + \theta_{,j} u_\nu^{(i)} - \theta_{,i} u_\nu^{(j)}$$

bounded in  $L^2(\Omega; \mathbb{R}^{n^2})$  (or compact in  $H^{-1}(\Omega; \mathbb{R}^{n^2})$ )

$$(2.24) \quad \text{div}(\theta w_\nu) = (D\theta) \cdot w_\nu + \theta \text{div} w_\nu$$

bounded in  $L^2(\Omega)$  (or compact in  $H^{-1}(\Omega)$ )

so that we can extend  $\theta u_\nu, \theta w_\nu$  to all of  $\mathbb{R}^n$  by zero outside  $\Omega$ . By a result of Murat [22, Lemma 2], we may decompose  $\theta u_\nu$  as

$$(2.25) \quad \theta u_\nu = h_\nu + Dy_\nu$$

where  $h_\nu$  is defined by  $\text{curl}(h_\nu) = \text{curl}(u_\nu), \text{div}(h_\nu) = 0$  in  $\mathbb{R}^n$  and the term  $h_\nu$  turns out to be bounded in  $H^1(\mathbb{R}^n; \mathbb{R}^n)$  (see [22, Lemma 2]), thus compact in  $L^2_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^n)$ . We may also assume that  $y_\nu \rightharpoonup y$  in  $H^1(\Omega), h_\nu \rightarrow h$  in  $L^2(\Omega; \mathbb{R}^n)$  strongly and passing to the limit in (2.25) gives

$$(2.26) \quad \theta u = h + Dy.$$

Let  $z_\nu = y_\nu - y$ , we have  $z_\nu \rightarrow 0$  in  $H^1(\Omega)$ , and we may assume as in the proof of Theorem 2.1 that  $z_\nu \in C_0^\infty(\mathbb{R}^n)$  and bounded in  $H^1(\mathbb{R}^n)$ . Therefore, for  $\epsilon > 0$ , we have  $A_\epsilon \subset \Omega, \text{meas}(A_\epsilon) \leq \epsilon, \delta > 0$  and a subsequence, still denoted by  $z_\nu$  such that

$$\int_B (M^* z_\nu)^2(x) dx \leq \epsilon$$

whenever  $\text{meas}(B) \leq \delta, B \cap A_\epsilon = \emptyset$ .

We have now

$$(2.27) \quad Dz_\nu \cdot \theta w_\nu = \theta^2 u_\nu \cdot w_\nu - Dy \cdot \theta w_\nu - h_\nu \cdot \theta w_\nu \stackrel{b.}{=} \theta^2 \chi - Dy \cdot \theta w - h \cdot \theta w \\ = \theta^2 (\chi - u \cdot w) := \chi'$$

since  $\theta w_\nu \rightharpoonup$  in  $L^2(\Omega)$  and  $h_\nu \rightarrow h$  strongly in  $L^2(\Omega)$ , what we should prove is that  $\chi' = 0$  a. e. on  $\Omega$ . As in the proof of Theorem 2.1,  $\theta w_\nu$  plays the role of  $\text{adj}_T Du_\nu$  with  $p = p' = 2$ . Hence we have (we adopt the notation in the proof of theorem 2.1)

$$(2.28) \quad -C\epsilon^{\frac{1}{p}} \geq \int_{\Omega \setminus (E_\epsilon \cup A_\epsilon)} \phi Dz_\nu \cdot \theta w_\nu dx - \int_{\Omega'} \phi Dg_\nu \cdot \theta w_\nu dx \leq C\epsilon^{\frac{1}{p}}.$$

Since  $Dg_\nu \cdot \theta w_\nu \in L^2(\Omega)$  and bounded,  $Dg_\nu, \theta w_\nu$  satisfy all the assumptions of Theorem C, hence

$$(2.29) \quad Dg_\nu \cdot \theta w_\nu \xrightarrow{*} Dv \cdot \theta w \quad \text{in the sense of measure.}$$

On the other hand, there exists a subsequence, which we still denote by the subscript  $\nu$ , such that

$$(2.30) \quad Dg_\nu \cdot \theta w_\nu \rightharpoonup \Theta \in L^2(\Omega').$$

Then, (2.29) and (2.30) imply that  $\Theta = Dv \cdot \theta w$  a. e. on  $\Omega$  and we can pass to the limit in (2.28) to obtain

$$(2.31) \quad -C\epsilon^{\frac{1}{p}} \leq \int_{\Omega \setminus (E_\epsilon \cup A_\epsilon)} \phi \chi' dx - \int_{\Omega'} Dv \cdot \theta w dx \leq C\epsilon^{\frac{1}{p}}$$

and we have  $\int_{\Omega'} |Dv|^2 dx \leq C'\epsilon$  so that from (2.31) and the above estimate we have now

$$(2.32) \quad -C_1\epsilon^{\frac{1}{p}} - \int_{A_\epsilon} \chi' \leq \int_{\Omega \setminus E_\epsilon} \phi \chi' dx \leq C_1\epsilon^{\frac{1}{p}} + \int_{A_\epsilon} \chi' dx.$$

Letting  $\epsilon \rightarrow 0$  as before and comparing (2.32) with the last few lines of the proof of Theorem 2.1, we conclude that  $\chi' = 0$  a. e. for each fixed  $\phi$ , which ensures that  $\chi = u \cdot w$  a. e. on  $\Omega$ .

Q. E. D.

### 3. APPLICATIONS

In this section we apply Theorem 2.1 and Corollary 2.2 to particular problems in nonlinear elasticity and the theory of fine phase mixtures. Firstly, we prove an existence result in nonlinear elasticity for inhomogeneous materials which generalizes a result proved by Ball and Murat [9, Theorem 6.1] for homogeneous materials using quite different principles.

For simplicity, we only consider the case where the applied body is zero and we only consider pure displacement problems.

**THEOREM 3.1.** — *Let  $W: \Omega \times M_+^3 \rightarrow [0, +\infty[$  be a stored energy function such that*

*1) (Polyconvexity) there exists a Caratheodory function  $G: \Omega \times M^3 \times M^3 \times ]0, +\infty[ \rightarrow [0, +\infty[$  such that*

$$(3.1) \quad W(x, F) = G(x, F, \text{adj } F, \det F) \quad \text{for all } (x, F) \in \Omega \times M_+^3;$$

*and for almost every  $x \in \Omega, (F, H, \delta) \rightarrow G(x, F, H, \delta), (F, H, \delta) \in M^3 \times M^3 \times ]0, +\infty[$  is continuous and convex.*

*2) (Continuity into  $\mathbb{R} \cup +\infty$ ) if  $F_j \rightarrow F$  in  $M_+^3, H_j \rightarrow H$  in  $M_+^3$  and  $\delta_j \rightarrow 0_+,$  then*

$$(3.2) \quad \lim_{j \rightarrow \infty} G(x, F_j, H_j, \delta_j) = +\infty \quad \text{for a. e. } x \in \Omega$$

*3) (Coerciveness) there exists  $b > 0, p \geq 2, q \geq \frac{p}{p-1},$  such that for almost all  $x \in \Omega$  and all  $(F, H, \delta) \in M^3 \times M^3 \times ]0, +\infty[$ ,*

$$(3.3) \quad G(x, F, H, \delta) \geq b(|F|^p + |H|^q).$$

Suppose  $\Omega \subset \mathbb{R}^n$  is open bounded and smooth with boundary  $\partial\Omega.$  Let

$$(3.4) \quad I(u) = \int_{\Omega} W(x, Du(x)) dx$$

and  $\mathcal{U} \subset W^{1,p}(\Omega; \mathbb{R}^3)$  be defined by

$$(3.5) \quad \mathcal{U} = \{u \in W^{1,p}(\Omega; \mathbb{R}^3) : \text{adj } Du \in L^q(\Omega; M^3), \det Du > 0 \text{ a. e.}, u = u_0 \text{ on } \partial\Omega\}$$

with  $u_0 \in W^{1,p}(\Omega; \mathbb{R}^n).$  Assume that  $\mathcal{U} \neq \emptyset$  and that

$$\inf_{u \in \mathcal{U}} I(u) < +\infty.$$

Then the problem: Find  $u \in \mathcal{U}$  such that

$$(3.6) \quad I(u) = \inf_{v \in \mathcal{U}} I(v)$$

has at least one solution.

**REMARK 3.2.** — *As S. Müller has remarked to me, a simpler proof of theorem 3.1 in the case  $2 \leq p < 3$  can be given by using a remark made independently by M. Esteban and V. Sverak (personal communication) that  $|\det P| \leq \text{const. } |\text{adj } P|^{\frac{n}{n-1}},$  for all  $P \in M^n, (n \geq 2).$  In fact we then have that  $\det Du_j$  is bounded in  $L^r(\Omega)$  for some  $r > 1,$  so that standard lower semi-continuity results based on weak  $L^1$  convergence*

(Theorem B, also see Ball [5], [6]) gives the result. However, this idea does not work for the case  $p=3$ ,  $n=3$ .

*Proof of Theorem 3.1.* — Define

$$(3.7) \quad \bar{G}(x, F, H, \delta) = \begin{cases} G(x, F, H, \delta) & \text{if } \delta > 0 \\ +\infty & \text{if } \delta \leq 0. \end{cases}$$

Then  $\bar{G}$  is easily seen to be convex in  $(F, H, \delta) \in M^3 \times M^3 \times ]-\infty, +\infty[$  and continuous into  $[0, +\infty[$  for a. e.  $x \in \Omega$ . Thus the functional

$$(3.8) \quad \bar{I}(u; \Omega) = \int_{\Omega} \bar{G}(x, Du, \text{adj } Du, \det Du) dx$$

is well defined and we write

$$(3.9) \quad \bar{I}(u; S) = \int_S \bar{G}(x, Du, \text{adj } Du, \det Du) dx$$

for every measurable set  $S \subset \Omega$  if the right hand side of (3.9) makes sense.

Let  $u \in W^{1,p}(\Omega; \mathbb{R}^3)$  with  $\text{adj } Du \in L^q(\Omega; M^3)$ , then  $\det Du \in L^1(\Omega)$  (here  $\det Du$  is defined pointwise, i. e. by  $\det Du = u_{,\alpha}^{(1)} (\text{adj } Du)_{\alpha}^{(1)}$ ). If  $\bar{I}(u; \Omega) < +\infty$ , then it follows that  $\det Du(x) > 0$  a. e. If  $u \in \mathcal{U}$ , then  $I(u) = \bar{I}(u; \Omega)$ . Thus the original problem is equivalent to minimizing  $\bar{I}(u; \Omega)$  over  $\mathcal{U}$ .

For  $u \in \mathcal{U}$ , by 3) we have

$$(3.10) \quad \bar{I}(u; \Omega) \geq b \left( \int_{\Omega} |Du|^p dx + \int_{\Omega} |\text{adj } Du|^q dx \right).$$

Let  $u_j \in \mathcal{U}$  be a minimizing sequence of  $\bar{I}(\cdot, \Omega)$ . Noting that  $u_j = u_0$  on  $\partial\Omega$ , from (3.10) it follows that  $u_j$ ,  $\text{adj } Du_j$ ,  $\det Du_j$  are bounded in  $W^{1,p}(\Omega; \mathbb{R}^3)$ ,  $L^q(\Omega; M^3)$  and  $L^1(\Omega)$  respectively. By Theorem 2.1, a subsequence  $(u_{\nu})$  can be found such that

$$\begin{aligned} u_{\nu} &\rightharpoonup u \text{ in } W^{1,p}(\Omega; \mathbb{R}^3) \\ \text{adj } Du_{\nu} &\rightharpoonup \text{adj } Du \text{ in } L^q(\Omega; M^3) \\ \det Du_{\nu} &\xrightarrow{b} \det Du \text{ on } \Omega. \end{aligned}$$

This means that there exists a sequence of nonincreasing measurable sets  $(E_k)$  with  $\text{meas}(E_k) \rightarrow 0$  as  $k \rightarrow \infty$ , such that

$$\det Du_{\nu} \rightharpoonup \det Du \text{ in } L^1(\Omega \setminus E_k) \text{ for every fixed, as } \nu \rightarrow \infty.$$

Since  $\bar{G}(x, \cdot, \cdot, \cdot)$  is convex and continuous into  $[0, +\infty[$ , it follows by a weak lower semi-continuity theorem of Ekeland and Temam [17, Theorem 2.1] that

$$\bar{I}(u; \Omega \setminus E_k) \leq \liminf_{\nu \rightarrow \infty} \bar{I}(u_{\nu}; \Omega \setminus E_k) \leq \liminf_{\nu \rightarrow \infty} \bar{I}(u_{\nu}; \Omega) := \mu$$

for all  $E_k$ , and we also have  $\text{meas}(\bigcap_{k=1}^{\infty} E_k) = 0$  and

$$\bar{I}(u; \Omega \setminus E_k) = \bar{I}(u; \Omega \setminus E_1) + \sum_{s=1}^k \bar{I}(u; E_s \setminus E_{s+1}) \leq \mu$$

hence

$$\bar{I}(u; \Omega \setminus \sum_{k=1}^{\infty} E_k) \leq \mu$$

and we conclude that

$$\bar{I}(u; \Omega) \leq \mu$$

so that  $u \in \mathcal{U}$  and it follows that  $\bar{I}(\cdot; \Omega)$  hence  $I$  attains a minimum.

Q. E. D.

In [8], Ball and James establish the following result

**PROPOSITION.** — (see [8, p. 23-24]). *Let  $\Omega \subset \mathbb{R}^n$  be bounded and connected. Let  $p > 2$ , and let  $A, B \in M^{N \times n}$  be distinct. Let  $y_j \rightarrow y$  in  $W^{1,p}(\Omega; \mathbb{R}^N)$  and suppose that for every  $\epsilon > 0$ ,*

$$(3.11) \quad \lim_j \text{meas}(\{x \in \Omega : |Dy_j(x) - A| > \epsilon \text{ and } |Dy_j(x) - B| > \epsilon\}) = 0.$$

Then

$$(3.12) \quad Dy(x) = \lambda(x)A + (1 - \lambda(x))B \text{ a. e. } x \in \Omega$$

for some measurable function  $\lambda$  satisfying  $0 \leq \lambda(x) \leq 1$  a. e., and one of the following possibilities holds:

- i)  $\lambda(x) = 1$  a. e., and  $Dy_j(x) \rightarrow A$  in measure,
- ii)  $\lambda(x) = 0$  a. e., and  $Dy_j(x) \rightarrow B$  in measure,
- iii)  $\lambda$  equals neither 0 a. e. nor 1 a. e. and

$$(3.13) \quad A - B = c \otimes n$$

for some  $c \in \mathbb{R}^N$  and  $n \in \mathbb{R}^n, |n| = 1$ .

We can now establish this proposition in the case  $p = 2$ . In [8], ball and James ask the question whether the proposition is true when  $1 \leq p \leq 2$  (see remarks on [8, p. 25]).

**PROPOSITION 3.3.** — *The above proposition of Ball and James is still true in the case  $p=2$ .*

*Proof.* — In a similar way to the argument on [8], let  $\Omega_A^{j,\epsilon} = \{x \in \Omega : |Dy_j(x) - A| \leq \epsilon\}$ ,  $\Omega_B^{j,\epsilon} = \{x \in \Omega : |Dy_j(x) - B| \leq \epsilon\}$  and let  $\chi_A^{j,\epsilon}, \chi_B^{j,\epsilon}$  denote the characteristic functions of  $\Omega_A^{j,\epsilon}, \Omega_B^{j,\epsilon}$  respectively. Then for  $\epsilon$  sufficiently small that  $\Omega_A^{j,\epsilon}, \Omega_B^{j,\epsilon}$  are disjoint,

$$(3.14) \quad Dy_j(x) = \chi_A^{j,\epsilon}(x)(A + \theta^{j,\epsilon}(x)) + \chi_B^{j,\epsilon}(x)(B + \psi^{j,\epsilon}(x)) + (1 - \chi_A^{j,\epsilon}(x) - \chi_B^{j,\epsilon}(x))Dy_j(x) \text{ a. e. } x \in \Omega$$

where  $\theta^{j,\epsilon}(x) := Dy_j(x) - A$ ;  $\psi^{j,\epsilon}(x) := Dy_j(x) - B$ . Since  $\chi_A^{j,\epsilon}$ ,  $\chi_B^{j,\epsilon}$  are uniformly bounded and

$$|\chi_A^{j,\epsilon}(x)\theta^{j,\epsilon}(x) + \chi_B^{j,\epsilon}(x)\psi^{j,\epsilon}(x)| \leq \epsilon,$$

there exists a subsequence, again denoted by the same subscript  $j$ , such that

$$\chi_A^{j,\epsilon}(x) \xrightarrow{\text{a.e.}} \lambda_A^\epsilon, \quad \chi_B^{j,\epsilon}(x) \xrightarrow{\text{a.e.}} \lambda_B^\epsilon \quad \text{in } L^\infty(\Omega).$$

and

$$\chi_A^{j,\epsilon}\theta^{j,\epsilon} + \chi_B^{j,\epsilon}\psi^{j,\epsilon} \xrightarrow{\text{a.e.}} H^\epsilon \quad \text{in } L^\infty(\Omega; M^{N \times n})$$

where  $|H^\epsilon(x)| \leq \epsilon$  a. e. in  $\Omega$ .

Since by (3.11),  $\lim_{j \rightarrow \infty} \int_\Omega (1 - \chi_A^{j,\epsilon}(x) - \chi_B^{j,\epsilon}(x)) dx = 0$ , it follows that

$$(3.15) \quad 0 \leq \lambda_A^\epsilon(x) = 1 - \lambda_B^\epsilon(x) \leq 1.$$

Also

$$\begin{aligned} \int_\Omega (1 - \chi_A^{j,\epsilon} - \chi_B^{j,\epsilon}) |Dy_j(x)| dx &\leq \\ &\leq \left( \int_\Omega (1 - \chi_A^{j,\epsilon} - \chi_B^{j,\epsilon}) dx \right)^{\frac{1}{2}} \left( \int_\Omega |Dy_j(x)|^2 dx \right)^{\frac{1}{2}} \end{aligned}$$

so that the last term in (3.14) tends to zero as  $j \rightarrow \infty$  strongly in  $L^1(\Omega)$ . Passing to the limit  $j \rightarrow \infty$  in (3.14) we obtain

$$Dy(x) = \lambda_A^\epsilon(x)A + (1 - \lambda_B^\epsilon(x))B + H^\epsilon(x) \text{ a. e. } x \in \Omega$$

From (3.15) there exists a subsequence  $\epsilon_h \rightarrow 0$  such that  $\lambda_A^{\epsilon_h} \xrightarrow{\text{a.e.}} \lambda(\cdot)$  in  $L^\infty(\Omega)$ , where  $0 \leq \lambda(x) \leq 1$  a. e. Passing to the limit  $h \rightarrow \infty$  in (3.16) we obtain (3.12). We remark in particular that

$$(3.16) \quad Dy - A = (1 - \lambda)(B - A).$$

If  $\lambda(x) = 1$  a. e. or  $0$  a. e., *i*), *ii*) follow from the argument of Ball and James [8, p. 24]. If  $\lambda(x)$  takes only the values  $0$  and  $1$  a. e., *iii*) also follows from [8, p. 24]. Hence we need only consider the case when  $0 < \lambda(x) < 1$  on a set of positive measure. Since (3.13) says nothing if  $N = 1$  or  $n = 1$ , we suppose also that  $N \geq 2$ ,  $n \geq 2$ . If  $M \in M^{N \times n}$ , here we denote by  $J(M)$  some  $2 \times 2$  minor of  $M$ . If we set

$$G = \{ x \in \Omega : 0 < \lambda(x) < 1 \}$$

we have  $\text{meas}(G) > 0$ . Then there exists a ball  $B \subset \Omega$ , such that  $\text{meas}(G \cap B) > 0$  and all arguments above work on  $B$ ; hence we may assume that  $\partial\Omega$  is sufficiently smooth that assumption on  $\partial\Omega$  of Theorem 2 is satisfied. By Corollary 2.2, there exists a subsequence  $(y_\nu)$  of  $(y_j)$  such that

$$J(Dy_\nu - A) \xrightarrow{b} J(Dy - A) \text{ on } \Omega, \text{ as } \nu \rightarrow \infty.$$

therefore, there exists a nonincreasing sequence  $(E_k)$  of measurable subsets of  $\Omega$ ,  $\text{meas}(E_k) \rightarrow 0$ , such that

$$(3.17) \quad J(Dy_\nu - A) \rightarrow J(Dy - A) \text{ in } L^1(\Omega \setminus E_k) \text{ for every fixed } k.$$

Now, for  $\epsilon > 0$  sufficiently small that  $\Omega_A^{\nu,\epsilon} \cap \Omega_B^{\nu,\epsilon} = \emptyset$ , we have by Lemma 1.3 that there exist  $A_\epsilon \subset \Omega$  with  $\text{meas}(A_\epsilon) < \epsilon$ , a subsequence, still denoted by  $(y_\nu)$ , and  $\delta > 0$ , such that

$$(3.18) \quad \int_S |Dy_\nu - A|^2 dx < \epsilon \text{ for all } \nu > 0$$

where  $S \subset \Omega$ ,  $S \cap A_\epsilon = \emptyset$  and  $\text{meas}(S) < \delta$ . Hence we have, for every fixed  $\phi \in L^\infty(\Omega)$  with  $0 \leq \phi(x) \leq 1$ ,

$$(3.19) \quad \begin{aligned} \int_{\Omega \setminus (E_k \cup A_\epsilon)} \phi J(Dy_\nu - A) dx &= \int_{\Omega \setminus (E_k \cup A_\epsilon)} \phi \chi_A^{\nu,\epsilon}(x) J(\theta^{\nu,\epsilon}(x)) dx + \\ &+ \int_{\Omega \setminus (E_k \cup A_\epsilon)} \phi \chi_B^{\nu,\epsilon}(x) J(B - A + \phi^{\nu,\epsilon}(x)) dx + \\ &+ \int_{\Omega \setminus (E_k \cup A_\epsilon)} \phi (1 - \chi_A^{\nu,\epsilon}(x) - \chi_B^{\nu,\epsilon}(x)) J(Dy_\nu(x) - A) dx := I_1 + I_2 + i_3. \end{aligned}$$

We have

$$\begin{aligned} |I_1| &= \left| \int_{\Omega \setminus (E_k \cup A_\epsilon)} \phi \chi_A^{\nu,\epsilon}(x) J(\theta^{\nu,\epsilon}(x)) dx \right| \leq \epsilon^2 \text{meas}(\Omega) \\ |I_3| &\leq \int_{(\Omega \setminus (E_k \cup A_\epsilon)) \setminus (\Omega_A^{\nu,\epsilon} \cup \Omega_B^{\nu,\epsilon})} |Dy_\nu(x) - A|^2 dx \leq \epsilon \end{aligned}$$

whenever  $\nu$  is sufficiently large, since (3.11) implies

$$\lim_{\nu \rightarrow \infty} \text{meas} [(\Omega \setminus (E_k \cup A_\epsilon)) \setminus (\Omega_A^{\nu,\epsilon} \cup \Omega_B^{\nu,\epsilon})] = 0.$$

$$\begin{aligned} I_2 &= \int_{\Omega \setminus (E_k \cup A_\epsilon)} \phi \chi_B^{\nu,\epsilon} J(B - A) dx + \int_{\Omega \setminus (E_k \cup A_\epsilon)} \phi_B^{\nu,\epsilon} J(\psi^{\nu,\epsilon}(x)) dx + \\ &+ \int_{\Omega \setminus (E_k \cup A_\epsilon)} \phi \chi_B^{\nu,\epsilon} C(B - A, \phi^{\nu,\epsilon}(x)) dx \end{aligned}$$

where

$$\begin{aligned} &\left| \int_{\Omega \setminus (E_k \cup A_\epsilon)} \phi \chi_B^{\nu,\epsilon} C(B - A, \psi^{\nu,\epsilon}) dx \right| \\ &\leq |B - A| (\text{meas}(\Omega))^{\frac{1}{2}} \left( \int_{\Omega \setminus (E_k \cup A_\epsilon)} \chi_B^{\nu,\epsilon} |\psi^{\nu,\epsilon}(x)|^2 dx \right)^{\frac{1}{2}} \\ &\leq |B - A| \epsilon \text{meas}(\Omega) \end{aligned}$$

while

$$\left| \int_{\Omega \setminus (E_k \cup A_\epsilon)} \phi \chi_B^{\nu, \epsilon} J(\psi^{\nu, \epsilon}(x)) dx \right| \leq \text{meas}(\Omega) \epsilon^2$$

(we may assume that  $\epsilon < 1$ ) so that we have, from (3.19)

$$(3.20) \quad -C\epsilon \leq \int_{\Omega \setminus (E_k \cup A_\epsilon)} \phi J(Dy_\nu - A) dx - \int_{\Omega \setminus (E_k \cup A_\epsilon)} \phi \chi_B^{\nu, \epsilon} J(B - A) dx \leq C\epsilon$$

whenever  $\nu$  sufficiently large. Passing to the limit  $\nu \rightarrow \infty$  in (3.20) and applying Corollary 2.2 the term  $J(Dy_\nu - A)$  on  $\Omega \setminus (E_k \cup A_\epsilon)$  lead to

$$(3.21) \quad -C\epsilon \leq \int_{\Omega \setminus (E_k \cup A_\epsilon)} \phi J(Dy - A) dx - \int_{\Omega \setminus (E_k \cup A_\epsilon)} \phi(1 - \lambda_A^\epsilon) J(B - A) dx \leq C\epsilon;$$

and we also have

$$(3.22) \quad \left| \int_{\Omega \setminus (E_k \cap A_\epsilon)} \phi(1 - \lambda_A^\epsilon) J(B - A) dx \right| \leq |B - A|^2 \text{meas}(A_\epsilon) \leq |B - A|^2 \epsilon$$

and

$$(3.23) \quad \left| \int_{\Omega \setminus (E_k \cup A_\epsilon)} \phi J(Dy - A) dx \right| \leq \int_{A_\epsilon} |Dy - A|^2 dx.$$

Thus we have, from (3.21), (3.22) and (3.23)

$$(3.24) \quad -C'\epsilon - \int_{A_\epsilon} |Dy - A|^2 dx \leq \int_{\Omega \setminus E_k} \phi J(Dy - A) dx - \int_{\Omega \setminus E_k} \phi(1 - \lambda_A^\epsilon) J(B - A) dx \leq C'\epsilon + \int_{A_\epsilon} |Dy - A|^2 dx.$$

Let  $\epsilon_h \rightarrow 0$  be the subsequence such that  $\lambda_A^{\epsilon_h} \rightarrow \lambda(\cdot)$  in  $L^\infty(\Omega)$ . Passing to the limit  $h \rightarrow \infty$  in (3.24), we have

$$\int_{\Omega \setminus E_k} \phi J(Dy - A) dx = \int_{\Omega \setminus E_k} \phi(1 - \lambda(x)) J(B - A) dx$$

for all  $\phi \in L^\infty(\Omega)$ ,  $0 \leq \phi(x) \leq 1$ , for all  $E_k$ ,  $k > 0$ , hence

$$(3.25) \quad J(Dy - A) = (1 - \lambda(x)) J(B - A) \text{ a. e. on } \Omega.$$

However, from (3.16) we have

$$(3.26) \quad J(Dy - A) = (1 - \lambda(x))^2 J(B - A) \text{ a. e. on } \Omega.$$

Compare (3.25) with (3.26), we obtain

$$\lambda(x)(1 - \lambda(x)) J(B - A) = 0 \text{ a. e. on } \Omega.$$

Since  $G$  is of positive measure, it follows that  $J(B - A) = 0$ . Since a matrix all of whose  $2 \times 2$  minors vanish is rank one, this completes the proof. Q. E. D.

REMARK 3.4. — *The assumptions of Proposition 3.2 are equivalent to assuming that the Young measure  $(\nu(x))$ ,  $x \in \Omega$ , relating to the sequence  $(Dy_j)$  satisfies  $\text{supp } (\nu(x)) \subset \{A, B\}$  (see Ball and James [8], Chipot and Kinderlehrer [14] and Kinderlehrer [19]).*

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