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# A generalization of a theorem of H. Brezis & F. E. Browder and applications to some unilateral problems

by

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ABSTRACT. — The first Section of this paper is devoted to prove the following Theorem, which extends previous results of H. Brezis and F. E. Browder: Let $w \in W_0^{m,p}(\Omega)$, $w \geq 0$ a. e. in $\Omega$ and $T \in W^{-m,p'}(\Omega)$, $T = \mu + h$ where $\mu$ is a positive Radon measure and $h \in L^1_{loc}(\Omega)$ is such that $hw \geq -|\Phi|$ a. e. in $\Omega$ for some $\Phi \in L^1(\Omega)$; then $w$ belongs to $L^1(\Omega; d\mu)$, $hw$ belongs to $L^1(\Omega)$ and $\langle T, w \rangle = \int_\Omega w d\mu + \int_\Omega hw dx$.

The second and third Sections deal with applications of this Theorem to the study of two unilateral problems.

SUNTO. — Nella prima parte di quest'articolo viene dimostrato il seguente teorema che costituisce una generalizzazione di risultati ottenuti da H. Brezis e F. E. Browder: Sia $w \in W_0^{m,p}(\Omega)$, $w \geq 0$ q. o. in $\Omega$ e $T \in W^{-m,p'}(\Omega)$, $T = \mu + h$ dove $\mu$ è una misura positiva di Radon e $h \in L^1_{loc}(\Omega)$ è tale che $hw \geq -|\Phi|$ q. o. in $\Omega$ per un certo $\Phi \in L^1(\Omega)$; allora $w$ appartiene a $L^1(\Omega; d\mu)$, $hw$ a $L^1(\Omega)$ e $\langle T, w \rangle = \int_\Omega w d\mu + \int_\Omega hw dx$.

Nella seconda e terza parte tale risultato viene applicato allo studio di due problemi unilaterali.

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RÉSUMÉ. — Dans la première partie de cet article on démontre le théorème suivant, qui généralise des résultats de H. Brezis et F. E. Browder : Soient $w \in W_0^{m,p}(\Omega)$, $w \geq 0$ p. p. dans $\Omega$ et $T \in W^{-m,p'}(\Omega)$, $T = \mu + h$ où $\mu$ est une mesure de Radon positive et où $h \in L^1_{loc}(\Omega)$ est telle que $hw \geq -|\Phi|$ p. p. dans $\Omega$ pour un certain $\Phi \in L^1(\Omega)$; alors $w$ appartient à $L^1(\Omega; d\mu)$, $hw$ appartient à $L^1(\Omega)$ et $\langle T, w \rangle = \int_\Omega w d\mu + \int_\Omega hw dx$.

Dans la deuxième et la troisième parties on applique ce théorème à l'étude de deux problèmes unilatéraux.

## INTRODUCTION

The first Section of the present paper is devoted to prove and to comment the following:

THEOREM. — *Let $\Omega$ be an arbitrary open subset of $\mathbb{R}^N$, $m \in \mathbb{N}$ and $1 < p$, $p' < +\infty$ with $1/p + 1/p' = 1$. Consider $w$ in $W_0^{m,p}(\Omega)$, $w \geq 0$ a.e. in $\Omega$ and $T$ in $W^{-m,p'}(\Omega)$. Assume that $T = \mu + h$, where $\mu$ is a positive Radon measure and $h$ a $L^1_{loc}(\Omega)$ function, and that $h(x)w(x) \geq -|\Phi(x)|$ a.e. $x \in \Omega$ for some $\Phi$ in $L^1(\Omega)$. Then $w$ belongs to $L^1(\Omega; d\mu)$, $hw$ belongs to $L^1(\Omega)$ and*

$$\langle T, w \rangle_{W^{-m,p'}(\Omega), W_0^{m,p}(\Omega)} = \int_\Omega w d\mu + \int_\Omega hw dx.$$

This result extends previous theorems of H. Brezis and F. E. Browder [*6*] who considered the cases where either $\mu \equiv 0$ or $h \equiv 0$. The main tool in order to prove these results is the Hedberg's approximation (in the $W_0^{m,p}(\Omega)$ norm) of a function $u \in W_0^{m,p}(\Omega)$ by a sequence of functions $(u_n)_{n \in \mathbb{N}}$ which belong to $L^\infty(\Omega) \cap W_0^{m,p}(\Omega)$, have compact support in $\Omega$ and satisfy $u_n u \geq 0$, $|u_n| \leq u$ a. e. in $\Omega$ (see [*7*], [*8*] and specially [*9*], Theorem 5).

The second and third Sections of this paper deal with applications of the above Theorem to the study of two unilateral problems.

In Section 2 we consider the strongly nonlinear variational inequality

$$\begin{cases} u \in K_\psi, g(\cdot, u) \in L^1(\Omega), \quad ug(\cdot, u) \in L^1(\Omega) \\ \langle Au, v - u \rangle + \int_\Omega g(\cdot, u)(v - u)dx \geq \langle f, v - u \rangle \\ \forall v \in K_\psi \cap L^\infty(\Omega) \end{cases}$$

and prove the existence of a solution. Here A is a pseudo-monotone operator acting on $W_0^{m,p}(\Omega)$, $f$ lies in $W^{-m,p'}(\Omega)$, $K_\psi = \{v: v \in W_0^{m,p}(\Omega),$

$v \geq \psi$ a. e. in $\Omega \}$ with $\psi \in W_0^{m,p}(\Omega) \cap L^\infty(\Omega)$, and $g$ satisfies the sign condition $sg(x, s) \geq 0$ but no growth restriction with respect to $s$.

The existence of a solution was already proved in [2]. We revisit and simplify here this proof using the Theorem above as an essential tool.

Section 3 is devoted to the quasilinear, second order variational inequality with quadratic growth with respect to the gradient:

$$\begin{cases} u \in K_0, \ g(\cdot, u, \operatorname{grad} u) \in L^1(\Omega), \quad ug(\cdot, u, \operatorname{grad} u) \in L^1(\Omega) \\ \langle Qu, v - u \rangle + \displaystyle\int_\Omega g(\cdot, u, \operatorname{grad} u)(v - u)dx \geq \langle f, v - u \rangle \\ \forall v \in K_0 \cap L^\infty(\Omega) \end{cases}$$

for which we prove the existence of a solution. Here Q is a quasilinear operator, $f$ lies in $H^{-1}(\Omega)$, $K_0 = \{ v : v \in H_0^1(\Omega),\ v \geq 0$ a. e. in $\Omega \}$ and $g$ satisfies the sign condition $sg(x, s, \xi) \geq 0$ as well as the quadratic growth condition with respect to the gradient $|g(x, s, \xi)| \leq b(|s|)(c(x) + |\xi|^2)$.

The existence of a solution for the corresponding equation was proved in [3]. We use here the same techniques and the Theorem above to prove the existence of a solution for the variational inequality with obstacle $\psi = 0$. Another proof of the same result using completely different ideas has been recently given in [1].

NOTATION. — The duality pairing between $W_0^{m,p}(\Omega)$ and $W^{-m,p'}(\Omega)$ will be denoted by $\langle , \rangle$, the space of Radon measures on $\Omega$ by $\mathcal{M}(\Omega)$ and the space of positive Radon measures by $\mathcal{M}^+(\Omega)$. A sequence is said to converge quasi everywhere (denoted by q. e.) in $\Omega$ if it converges at any point of $\Omega$ except on a subset whose $(m, p)$-capacity is zero.

## 1. AN ABSTRACT RESULT OF BREZIS-BROWDER'S TYPE

In this Section we study the following question: let $w$ be an element of $W_0^{m,p}(\Omega)$, and let T be an element of $W^{-m,p'}(\Omega)$ such that $T = \mu + h$, where $\mu$ lies in $\mathcal{M}^+(\Omega)$ and $h$ in $L^1_{loc}(\Omega)$; find sufficient conditions on the data in order for $w$ to belong to $L^1(\Omega; d\mu)$, for $hw$ to belong to $L^1(\Omega)$ and finally to have

$$\langle T, w \rangle = \int_\Omega w d\mu + \int_\Omega hw dx.$$

Let us point out that even if any expression makes sense, it is not obvious that the equality holds true.

This question was solved by H. Brezis and F. E. Browder in [5] and [6] when either $\mu \equiv 0$ or $h \equiv 0$. The case where neither $\mu$ nor $h$ is zero is the goal of the present Section; this case is well suited to the study of variational inequalities with obstacles (see Sections 2 and 3 below).

To be more precise, the question above with either $\mu \equiv 0$ or $h \equiv 0$ was solved by H. Brezis and F. E. Browder in [5] when $m = 1$. They then turn to the general case $m \geq 2$ in [6] using Hedberg's approximation. Since at that time ([7]) Hedberg's approximation seemed to need extra regularity assumptions on $\Omega$, results are stated in [6] only in the cases where $\Omega = \mathbb{R}^N$ (Theorem 1 for $\mu \equiv 0$, Theorem 8 for $h \equiv 0$) or with $\partial\Omega$ locally smooth (Theorem 4 for $\mu \equiv 0$). The regularity assumptions on $\Omega$ in Hedberg's approximation were then removed, first in the case $p > 2-1/N$ in [8], finally in the general case $1 < p < +\infty$ in [9] (see also Addenda 1 and 2 of [5]). For completeness we state here these results in their full generality:

THEOREM 1.1. — ([6]). *Let $\Omega$ be an arbitrary open subset of $\mathbb{R}^N$. If* T *belongs to* $W^{-m,p'}(\Omega) \cap L^1_{loc}(\Omega)$ *and* $w$ *to* $W_0^{m,p}(\Omega)$ *with* $T(x)w(x) \geq -|\Phi(x)|$ a. e. $x \in \Omega$ *for some* $\Phi$ *in* $L^1(\Omega)$, *then* $Tw$ *is an element of* $L^1(\Omega)$ *and*

$$\langle T, w \rangle = \int_\Omega T(x)w(x)dx.$$

THEOREM 1.2. — ([6]). *Let $\Omega$ be a bounded smooth open subset of $\mathbb{R}^N$ when $m \geq 2$. If* T *belongs to* $W^{-m,p'}(\Omega) \cap \mathcal{M}^+(\Omega)$ *and* $w$ *to* $W_0^{m,p}(\Omega)$, *then* $w$ *(or more exactly the quasi continuous representative of* $w$*) is an element of* $L^1(\Omega; dT)$ *and*

$$\langle T, w \rangle = \int_\Omega w dT.$$

Note that there is no regularity assumption $\Omega$ in Theorem 1.1 (see comments above) but that $\Omega$ is assumed to be smooth in Theorem 1.2 ; this assumption is not necessary if $m = 1$, but is essential in the case $m \geq 2$, since there are counterexamples, i. e. functions $u$ which do not belong to $L^1(\Omega; dT)$ when $\Omega$ is not smooth (see Remark 8 of [6]). This relies to the fact that there exist functions which can not be written as differences of nonnegative functions of $W_0^{m,p}(\Omega)$ when $m \geq 2$ and $\Omega$ is not smooth; in contrast one still has $w = w^+ - w^-$ with $w^+$, $w^-$ in $W_0^{1,p}(\Omega)$ when $m = 1$.

Our result is the following:

THEOREM 1.3. — *Let $\Omega$ be an arbitrary open subset of $\mathbb{R}^N$. Let $w$ be an element of $W_0^{m,p}(\Omega)$ such that*

$$w \geq 0 \text{ a. e. in } \Omega$$

*and let* T *be an element of* $W^{-m,p'}(\Omega)$ *such that* $T = \mu + h$, *where* $\mu$ *lies in* $\mathcal{M}^+(\Omega)$ *and* $h$ *in* $L^1_{loc}(\Omega)$; *assume moreover that*

$$hw \geq -|\Phi| \text{ a. e. in } \Omega \text{ for some } \Phi \in L^1(\Omega).$$

*Then* $hw$ *belongs to* $L^1(\Omega)$, $w$ *(or more exactly the quasi continuous representative of* $w$*) to* $L^1(\Omega; d\mu)$ *and one has:*

$$\langle T, w \rangle = \langle \mu + h, w \rangle = \int_\Omega w d\mu + \int_\Omega hw dx.$$

Let us note that in comparison with Theorems 1.1 and 1.2 the assumption $w \geq 0$ a. e. in $\Omega$ of Theorem 1.3 appears as an extra condition; this is partly compensated and justified by the absence of regularity assumption on $\Omega$ (see comment after Theorem 1.2). Finally this condition is satisfied in the applications of Sections 2 and 3.

In the proofs of these Theorems the central role is played by Hedberg's approximation. This result can be stated as follows (see [*7*], [*8*] and specially [*9*] Theorem 5; an expository proof is given is [*6*] Theorem 2 and in [*13*] for the case $\Omega = \mathbb{R}^N$): Let $\Omega$ be an arbitrary open subset of $\mathbb{R}^N$ and $1 < p < +\infty$; for any $w \in W_0^{m,p}(\Omega)$ there exists a sequence $(w_n)_{n\in\mathbb{N}}$ which satisfies:

$$(1.1) \quad \begin{cases} w_n \in W_0^{m,p}(\Omega) \cap L^\infty(\Omega), & \text{supp}(w_n) \text{ compact subset of } \Omega \\ |w_n(x)| \leq \inf(n, |w(x)|) & \text{and} \quad w_n(x)w(x) \geq 0 \text{ q. e. } x \in \Omega \\ w_n \to w \text{ in } W_0^{m,p}(\Omega). \end{cases}$$

*Proof of Theorem 1.3.* — Let $(w_n)_{n\in}$ be the sequence defined by (1.1). Consider for $n$ fixed the mollified sequence $(w_n * \rho_k)_{k\in\mathbb{N}}$ with $\rho_k(x) = k^N\rho(kx)$ where $\rho \in \mathcal{D}(\mathbb{R}^N)$, $\rho \geq 0$, $\int_{\mathbb{R}^N} \rho(x)dx = 1$. For each fixed $n$ and sufficiently large $k$, $w_n * \rho_k$ lies in $\mathcal{D}(\Omega)$ and

$$(1.2) \qquad \langle \mu + h, w_n * \rho_k \rangle = \int_\Omega w_n * \rho_k d\mu + \int_\Omega h(w_n * \rho_k)dx.$$

For any fixed $n$ we have

$$(1.3) \qquad w_n * \rho_k \to w_n \text{ in } W_0^{m,p}(\Omega) \text{ when } k \to +\infty$$

and extracting a subsequence in $k$

$$w_n * \rho_k \to w_n \text{ q. e. (and also a. e.) in } \Omega.$$

On the other hand since $T = \mu + h$ is an element of $W^{-m,p'}(\Omega) \cap \mathcal{M}(\Omega)$, Lemma 2 of [*6*] asserts that $|T|(E) \doteq |\mu + h|(E) = 0$ for any subset E of $\Omega$ whose $(m, p)$-capacity is zero. Then we have

$$\mu(E) \leq |\mu + h|(E) + |h|(E) = 0$$

since $|h|(E) = 0$ for those sets, and thus:

(1.4) $\mu(E) = 0$ if E has zero $(m, p)$-capacity.

Therefore for fixed $n$ and a subsequence $k \to +\infty$

(1.5) $w_n * \rho_k \to w_n$ a. e. and $\mu$-a. e. in $\Omega$.

Since for $n$ fixed there exists a compact subset $Q_n$ of $\Omega$ such that $\operatorname{supp}(w_n * \rho_k) \subset Q_n$ for $k$ sufficiently large, and since

(1.6) $|(w_n * \rho_k)(x)| \leq \|w_n\|_{L^\infty} \quad \forall x \in \Omega \quad \forall k \in \mathbb{N}$,

we can pass to the limit in (1.2) for $k \to +\infty$ and $n$ fixed: we use (1.3) in the left hand side, (1.5), (1.6), $h \in L^1_{loc}(\Omega)$ and Lebesgue's dominated convergence theorem for the right hand side. We obtain

(1.7) $$\langle \mu + h, w_n \rangle = \int_\Omega w_n d\mu + \int_\Omega h w_n dx;$$

note that, in view of (1.1), (1.4), $w_n$ belongs to $L^\infty(\Omega; d\mu)$ and $hw_n$ to $L^1(\Omega)$.

From (1.1) we know that $w_n$ converges to $w$ in $W_0^{m,p}(\Omega)$; by the proof we used before to obtain (1.5) we have

(1.8) $w_n \to w$ a. e. and $\mu$-a. e. in $\Omega$.

On the other hand from $hw \geq -|\Phi|$ and $0 \leq w_n \leq w$ we have

(1.9) $hw_n \geq -|\Phi|$ a. e. in $\Omega$.

Finally, recall that for any $v$ in $W_0^{m,p}(\Omega)$ one has

(1.10) $v \geq 0$ a. e. in $\Omega \Leftrightarrow v \geq 0$ q. e. in $\Omega$.

This equivalence and $w \geq 0$ a. e. in $\Omega$, as well as (1.4) imply

(1.11) $w_n \geq 0$ $\mu$-a. e. in $\Omega$.

Since $\langle \mu + h, w_n \rangle$ is bounded, (1.7) and (1.11) imply $\int_\Omega h w_n dx \leq \text{cst}$; similarly (1.7) and (1.9) imply $\int_\Omega w_n d\mu \leq \text{cst}$. Then in view of (1.8), (1.9) and (1.11), Fatou's lemma yield that $hw$ belongs to $L^1(\Omega)$ and $w$ to $L^1(\Omega; d\mu)$.

Using $0 \leq w \leq w_n$ $\mu$-a. e. in $\Omega$ (recall (1.4)) and $|hw_n| \leq |hw|$ a. e. in $\Omega$, it is now easy to pass to the limit in (1.7): we use the convergence of $w_n$ to $w$ in $W_0^{m,p}(\Omega)$ for the left hand side and Lebesgue's dominated convergence theorem in each term of the right hand side: we obtain

$$\langle \mu + h, w \rangle = \int_\Omega w d\mu + \int_\Omega h w dx,$$

which completes the proof of Theorem 1.3.

REMARK 1.4. — Let us point out that in the case $m = 1$ we can replace the two following hypotheses of Theorem 1.3

$$\begin{cases} w \geq 0 \text{ a. e. in } \Omega \\ hw \geq -|\Phi| \text{ a. e. in } \Omega \text{ for some } \Phi \text{ in } L^1(\Omega) \end{cases}$$

by the unique (and weaker) hypothesis

$$h|w| \geq -|\Phi| \text{ a. e. in } \Omega \text{ for some } \Phi \text{ in } L^1(\Omega)$$

and obtain the same Theorem.

Indeed if $m = 1$ we can write $w = w^+ - w^-$ where $w^+$ and $w^-$ belong to $W_1^{1,p}(\Omega)$. Theorem 1.3 in the former setting can now be applied separately to $w^+$ and $w^-$, which is sufficient to prove the variant.

Note that the use of the decomposition $w = w^+ - w^-$ is confined to the cas $m = 1$ since for $m \geq 2$, $w^+$and $w^-$ do not belong in general to $W_0^{m,p}(\Omega)$.

Note also that in the case $m = 1$, Hedberg's approximation can be replaced in the proof of Theorem 1.3 by some more standard process of approximation: see e. g. the approximation in [5] or use $\varphi_n - (\varphi_n - w^+)^+$ to approximate $w^+$, with $\varphi_n \in \mathcal{D}(\Omega)$, $\varphi_n \geq 0$ and $\varphi_n \to w^+$ in $W_0^{1,p}(\Omega)$.

## 2. AN EXISTENCE RESULT FOR A STRONGLY NONLINEAR VARIATIONAL INEQUALITY

Consider some $f$ in $W^{-m,p'}(\Omega)$ and the convex set $K_\psi$

$$K_\psi = \{ v : v \in W_0^{m,p}(\Omega),\ v \geq \psi \text{ a. e. in } \Omega \} \tag{2.1}$$

where the obstacle $\psi$ is assumed to belong to $W_0^{m,p}(\Omega) \cap L^\infty(\Omega)$.

Consider also a pseudo monotone operator A from $W_0^{m,p}(\Omega)$ in $W^{-m,p'}(\Omega)$. Assume that A maps bounded sets into bounded sets and that A is coercive, i. e. that for some $v_0 \in K_\psi \cap L^\infty(\Omega)$

$$\frac{\langle A(v), v - v_0 \rangle}{\| v \|_{W_0^{m,p}(\Omega)}} \to +\infty \quad \text{as} \quad \| v \|_{W_0^{m,p}(\Omega)} \to +\infty .$$

Examples of such operators are the celebrated Leray-Lions operators (see e. g. [*10*], Chapter 2, or, among others, [*2*], p. 293).

Consider finally a Carathéodory function $g(x, s)$ defined on $\Omega \times \mathbb{R}$ which satisfies:

$$\begin{cases} sg(x, s) \geq 0 \text{ a. e. } x \in \Omega \quad \forall s \in \mathbb{R} \\ h_t \in L^1(\Omega) \text{ for any fixed } t \in \mathbb{R}^+ \quad \text{where} \quad h_t(x) = \sup\limits_{|s| \leq t} |g(x, s)| \end{cases}$$

THEOREM 2.1. — *The variational inequality*

$$(2.2)\quad \begin{cases} u \in K_\psi, \quad g(\cdot, u) \in L^1(\Omega), \quad ug(\cdot, u) \in L^1(\Omega) \\ \langle Au - f, v - u \rangle + \int_\Omega g(\cdot, u)(v - u)dx \geq 0 \\ \forall v \in K_\psi \cap L^\infty(\Omega) \end{cases}$$

*has at least one solution. Moreover if the operator* A *is monotone and g strictly increasing (or if* A *is strictly monotone and g non decreasing) the solution of (2.2) is unique.*

This Theorem was already proved in [*2*]; the case of the equation associated to (2.2) was treated in [*6*] and [*13*]. We revisit here the proof of [*2*]; the first part of the proof closely follows the lines of [*2*] and is just sketched here; the other parts use the abstract Theorem 1.3 above.

*Proof of Theorem 2.1.*

*First part: Approximation and a priori estimates.* — We just sketch the proof; see [*2*], Lemma 1 for more details. Define

$$g_n(x, s) = \begin{cases} \chi_n(x)g(x, s) & \text{if} \quad |g(x, s)| \leq n \\ \chi_n(x)n \dfrac{g(x, s)}{|g(x, s)|} & \text{if} \quad |g(x,s)| > n \end{cases}$$

where $\chi_n(x)$ is the characteristic function of the set $\{x: x \in \Omega, |x| \leq n\}$. Then the approximate problem

$$(2.3)\quad \begin{cases} u_n \in K_\psi \\ \langle Au_n - f, v - u_n \rangle + \int_\Omega g_n(\cdot, u_n)(v - u_n)dx \geq 0 \quad \forall v \in K_\psi \end{cases}$$

has at least one solution (see e. g. [*4*] or [*10*]); using $v = v_0$ as test function in (2.3) allows one to prove that these solutions are bounded in $W_0^{m,p}(\Omega)$ independently of $n$ and that $0 \leq \int_\Omega g_n(\cdot, u_n)u_n\, dx \leq \text{cst}$

Moreover extracting a subsequence (still denoted by $u_n$) such that

$$u_n \rightharpoonup u \quad \text{weakly in } W_0^{m,p}(\Omega) \quad \text{and a. e. in } \Omega,$$

one obtains that $ug(\cdot, u)$ and $g(\cdot, u)$ belong to $L^1(\Omega)$ and that

$$\begin{cases} g_n(\cdot, u_n) \to g(\cdot, u) \quad \text{strongly in } L^1(\Omega) \\ \int_\Omega ug(\cdot, u)dx \leq \liminf_n \int_\Omega u_n g_n(\cdot, u_n)dx. \end{cases}$$

*Second part: Passing to the limit in (2.3).* — Consider $\mu_n$ defined by

$$\mu_n = Au_n - f + g_n(\cdot, u_n).$$

From (2.3) it is clear that $\mu_n$ belongs to $\mathcal{M}^+(\Omega)$. Since A maps bounded sets of $W_0^{m,p}(\Omega)$ into bounded sets of $W^{-m,p'}(\Omega)$, one can always assume that for the same subsequence

$$Au_n \rightharpoonup \chi \quad \text{weakly in } W^{-m,p'}(\Omega)$$

which implies that

$$\mu_n \rightharpoonup \mu \quad \text{in } \mathcal{D}'(\Omega)$$

where

$$\mu = \chi - f + g(\cdot, u).$$

Consider now $w = u - \psi$, $T = \mu + h$, $h = -g(\cdot, u)$. The assumption of Theorem 1.3 are satisfied since $T = \chi - f \in W^{-m,p'}$ and $hw = -g(\cdot, u)(u - \psi)$ belongs to $L^1(\Omega)$ (note that $\psi$ belongs to $W_0^{m,p}(\Omega) \cap L^\infty(\Omega)$). Therefore

$$\text{(2.4)} \quad \begin{cases} u - \psi \in L^1(\Omega; d\mu) \\ \langle \chi - f,\ u - \psi \rangle = \displaystyle\int_\Omega (u - \psi) d\mu - \int_\Omega g(\cdot, u)(u - \psi) dx. \end{cases}$$

Using $v = \psi$ as test function in (2.3) yields

$$\langle Au_n, u_n \rangle \le \langle Au_n, \psi \rangle - \langle f, \psi - u_n \rangle + \int_\Omega g_n(\cdot, u_n)(\psi - u_n) dx$$

which gives, passing to the limit and then using (2.4)

$$\text{(2.5)} \quad \begin{cases} \limsup\limits_n \langle Au_n, u_n \rangle \le \langle \chi, \psi \rangle - \langle f, \psi - u \rangle + \displaystyle\int_\Omega g(\cdot, u)(\psi - u) dx \\ \qquad \le \langle \chi, u \rangle + \displaystyle\int_\Omega (\psi - u)\, d\mu \le \langle \chi, u \rangle; \end{cases}$$

indeed

$$\text{(2.6)} \qquad (\psi - u) \le 0 \quad \mu\text{-a. e.} \quad \text{in } \Omega$$

(use (1.4) and (1.10) to prove (2.6)).

Since A is a pseudo-monotone operator, (2.5) implies that :

$$\chi = Au \quad \text{and} \quad \langle Au_n, u_n \rangle \to \langle Au, u \rangle.$$

It is now easy to pass to the limit in (2.3) for any fixed $v \in K_\psi \cap L^\infty(\Omega)$. The existence of $u$ solution of (2.2) is proved.

*Third part: Complementary system.* — We prove here that for any solution $u$ of (2.2) we have the equality

$$\langle \mathrm{A}u - f, \psi - u \rangle + \int_\Omega g(\cdot, u)(\psi - u)dx = 0. \tag{2.7}$$

Indeed using $v = \psi$ as test function in (2.2) we obtain

$$\langle \mathrm{A}u - f, \psi - u \rangle + \int_\Omega g(\cdot, u)(\psi - u)dx \geq 0.$$

On the other hand $w = u - \psi$, $\mu = \mathrm{A}u - f + g(\cdot, u)$ and $h = -g(\cdot, u)$ satisfy the hypotheses of Theorem 1.3; hence $u - \psi$ belongs to $\mathrm{L}^1(\Omega; d\mu)$ and

$$\langle \mathrm{A}u - f, u - \psi \rangle + \int_\Omega g(\cdot, u)(u - \psi)dx = \int_\Omega (u - \psi)d\mu.$$

Since $u - \psi \geq 0$ $\mu$-a. e. in $\Omega$ (see (2.6)) and $\mu \geqslant 0$, the right hand side of this equality is nonnegative, which implies (2.7).

Note that (2.7) is nothing else than the complementary system corresponding to the variational inequality (2.2), i. e.:

$$\begin{cases} u - \psi \geq 0 \ \mu\text{-a. e. in } \Omega \\ \mu = \mathrm{A}u - f + g(\cdot, u) \geq 0 \quad \text{in } \mathfrak{M}(\Omega) \\ \displaystyle\int_\Omega (u - \psi)d\mu = 0 \end{cases}$$

*Fourth part: Comparison result and uniqueness.* — Consider two solutions $u_1$ and $u_2$ of the variational inequality (2.2) corresponding to two different right hand sides $f_1$ and $f_2$. We will prove that if $g(x, s)$ is non decreasing in $s$ then $g(\cdot, u_1)u_2$ and $g(\cdot, u_2)u_1$ belong to $\mathrm{L}^1(\Omega)$ and that

$$\begin{cases} \langle \mathrm{A}u_1 - \mathrm{A}u_2, u_1 - u_2 \rangle + \displaystyle\int_\Omega (g(\cdot, u_1) - g(\cdot, u_2))(u_1 - u_2)dx \\ \qquad\qquad \leq \langle f_1 - f_2, u_1 - u_2 \rangle \end{cases} \tag{2.8}$$

which clearly implies the uniqueness results stated in Theorem 2.1.

Singe $g$ is non decreasing, we have

$$(g(\cdot, u_1) - g(\cdot, u_2))(u_1 - u_2) \geq 0 \text{ a. e. } \quad \text{in } \Omega.$$

Denoting by I the measurable set $\mathrm{I} = \{ x \in \Omega : u_1(x)u_2(x) \leq 0 \}$ we deduce from this inequality and from $sg(\cdot, s) \geq 0$ that

$$g(\cdot, u_1)u_2 \leq g(\cdot, u_1)u_1 + g(\cdot, u_2)u_2 \quad \text{in } \Omega \backslash \mathrm{I};$$

on the other hand since $sg(\cdot, s) \geq 0$

$$g(\cdot, u_1)u_2 \leq 0 \text{ a. e. } \quad \text{in I}.$$

Thus

$$g(\cdot, u_1)(u_2 - \psi) \leq \Phi \text{ a. e. } \quad \text{in } \Omega \quad \text{for some } \Phi \in L^1(\Omega)$$

since $g(\cdot, u_1)$, $g(\cdot, u_1)u_1$ and $g(\cdot, u_2)u_2$ belong to $L^1(\Omega)$ and $\psi$ to $L^\infty(\Omega)$.

Define now $\mu_1 = Au_1 - f_1 + g(\cdot, u_1)$, $h_1 = -g(\cdot, u_1)$ and $w_1 = u_2 - \psi$. Theorem 1.3 applied to $\mu_1$, $h_1$ and $w_1$ yields

$$(2.9) \quad \begin{cases} u_2 - \psi \in L^1(\Omega; d\mu_1), \qquad g(\cdot, u_1)(u_2 - \psi) \in L^1(\Omega) \\ \langle Au_1 - f_1, u_2 - \psi \rangle + \int_\Omega g(\cdot, u_1)(u_2 - \psi)dx = \int_\Omega (u_2 - \psi) d\mu_1 \geq 0 \end{cases}$$

since $u_2 - \psi \geq 0$ $\mu_1$-a. e. in $\Omega$ and $\mu_1 \geq 0$) (see (2.6)).

Combining (2.9) with the equality (2.7) for $u_1$ and $f_1$ gives

$$\langle Au_1 - f_1, u_2 - u_1 \rangle + \int_\Omega g(\cdot, u_1)(u_2 - u_1) \geq 0.$$

This inequality and the analogous where $(u_1, u_2)$ is replaced by $(u_2, u_1)$ yield (2.8).

## 3. AN EXISTENCE RESULT FOR A QUASILINEAR VARIATIONAL INEQUALITY

In this Section we restrict our attention to the case where $m = 1$, $p = 2$ and where $\Omega$ is a bounded open subset of $\mathbb{R}^N$.

Consider a quasilinear elliptic operator of second order in divergence form

$$Qu = -\operatorname{div}(a(\cdot, u) \operatorname{grad} u)$$

where $a$ is a $N \times N$ matrix whose components are Carathéodory functions $a_{ij}(x, s)$ defined on $\Omega \times \mathbb{R}$ which satisfies

$$\begin{cases} a(x, s)\xi\xi \geq \alpha |\xi|^2 \text{ a. e. } x \in \Omega \quad \forall s \in \mathbb{R}, \quad \forall \xi \in \mathbb{R}^N \\ |a(x, s)| \leq \beta \text{ a. e. } x \in \Omega \quad \forall s \in \mathbb{R} \end{cases}$$

for some real numbers $\alpha$ and $\beta$ with $0 < \alpha < \beta$.

Consider also a Carathéodory function $g(x, s, \xi)$ defined on $\Omega \times \mathbb{R} \times \mathbb{R}^N$ which satisfies:

$$(3.1) \qquad sg(x, s, \xi) \geq 0 \text{ a. e. } x \in \Omega, \quad \forall s \in \mathbb{R}, \quad \forall \xi \in \mathbb{R}^N,$$

$$(3.2) \qquad |g(x, s, \xi)| \leq b(|s|)(c(x) + |\xi|^2)$$

for some continuous nondecreasing function $b: \mathbb{R}^+ \to \mathbb{R}^+$ and some $c \in L^1(\Omega)$, $c \geq 0$.

Finally consider some right hand side $f \in H^{-1}(\Omega)$ and the positive convex cone $K_0$

$$K_0 = \{v : v \in H_0^1(\Omega), v \geq 0 \text{ a. e. in } \Omega\}.$$

THEOREM 3.1. — *The variational inequality*

$$(3.3) \quad \begin{cases} u \in K_0, \quad g(\cdot, u, \operatorname{grad} u) \in L^1(\Omega), \quad ug(\cdot, u, \operatorname{grad} u) \in L^1(\Omega) \\ \langle Qu - f, v - u \rangle + \int_\Omega g(\cdot, u, \operatorname{grad} u)(v - u)dx \geq 0 \\ \forall v \in K_0 \cap L^\infty(\Omega) \end{cases}$$

*has at least one solution.*

Theorem 3.1 extends the result of [3] to the case of the variational inequality with obstacle $\psi = 0$. Note that in [3], sign conditions on $g$ are allowed which are more general than (3.1). Nevertheless we choose here to consider only (3.1) in order to present a simpler proof.

The first and third steps of the proof of Theorem 3.1 follow along the lines of [3]; the second step uses the abstract Theorem 1.3 above.

Note also that a different proof of Theorem 3.1 based on completely different ideas has been recently obtained in [1].

*Proof of Theorem 3.1.*

*First step.* — Define for $\epsilon > 0$ the approximation

$$g_\epsilon(x, s, \xi) = \frac{g(x, s, \xi)}{1 + \epsilon\, |g(x, s, \xi)|}$$

and consider a solution $u_\epsilon$ of

$$(3.4) \quad \begin{cases} u_\epsilon \in K_0 \\ \langle Qu_\epsilon - f, v - u_\epsilon \rangle + \int_\Omega g_\epsilon(\cdot, u_\epsilon, \operatorname{grad} u_\epsilon)(v - u_\epsilon)dx \geq 0 \quad \forall v \in K_0 \end{cases}$$

Since the nonlinearity $g_\epsilon$ is bounded in $L^\infty(\Omega)$ such a solution exists for each $\epsilon > 0$ by a classical result [4].

Using $v = 0$ as test function in (3.4) gives

$$(3.5) \quad \begin{cases} \alpha \| u_\epsilon \|_{H_0^1(\Omega)} \leq \| f \|_{H^{-1}(\Omega)} \\ 0 \leq \int_\Omega u_\epsilon g_\epsilon(\cdot, u_\epsilon, \operatorname{grad} u_\epsilon)dx \leq \frac{1}{\alpha} \| f \|^2_{H^{-1}(\Omega)}. \end{cases}$$

On the other hand define $\mu_\epsilon \in H^{-1}(\Omega)$ by

$$(3.6) \quad \mu_\epsilon = Qu_\epsilon + g_\epsilon(\cdot, u_\epsilon, \operatorname{grad} u_\epsilon) - f;$$

from (3.4) we deduce that $\mu_\epsilon$ is a positive Radon measure. Using (3.2) we have

$$|g_\epsilon(\cdot,u_\epsilon,\operatorname{grad} u_\epsilon)| \le b(1)(c(\cdot)+|\operatorname{grad} u_\epsilon|^2)+u_\epsilon g_\epsilon(\cdot,u_\epsilon,\operatorname{grad} u_\epsilon) \text{ a. e. } \text{ in } \Omega$$

which in view of (3.5) implies that $g_\epsilon(\cdot, u_\epsilon, \operatorname{grad} u_\epsilon)$ is bounded in $L^1(\Omega)$. This in turn implies that the positive measures $\mu_\epsilon$, which are bounded in $L^1(\Omega)+H^{-1}(\Omega)$, are bounded in the sense of measures: indeed, for any compact subset K of $\Omega$ the use of a test function $\varphi_K \in \mathfrak{D}(\Omega)$ with $\varphi_K \ge 0$ in $\Omega$, $\varphi_K = 1$ on Kyields :

$$0 \le \int_K d\mu_\epsilon \le \int_\Omega \varphi_K d\mu_\epsilon = \langle Qu_\epsilon - f, \varphi_K \rangle + \int_\Omega \varphi_K g_\epsilon(\cdot, u_\epsilon, \operatorname{grad} u_\epsilon)dx \le \text{cst},$$

which is the desired result.

Thus there exists $u \in H_0^1(\Omega)$, $u \ge 0$ a. e. in $\Omega$, such that for a subsequence (still denoted by $u_\epsilon$) we have

$$(3.7) \qquad \begin{cases} u_\epsilon \to u & \text{weakly in } H_0^1(\Omega) \text{ and a. e. in } \Omega \\ Qu_\epsilon - f & \text{is bounded in } H^{-1}(\Omega)\cap\mathfrak{M}(\Omega). \end{cases}$$

Note that $Qu_\epsilon - f$ tends weakly to $Qu - f$ in $H^{-1}(\Omega)$ since Q is quasilinear. The compactness result of [*12*] (Theorem 1 and Remark 3) implies that for any $q < 2$

$$Qu_\epsilon - f \to Qu - f \quad \text{strongly in } W_{loc}^{-1,q}(\Omega)$$

We then split $Qu_\epsilon$ in

$$\begin{cases} Qu_\epsilon = Lu_\epsilon + Ru_\epsilon \\ \qquad = -\operatorname{div}(a(\cdot,u)\operatorname{grad} u_\epsilon) - \operatorname{div}((a(\cdot,u_\epsilon)-a(\cdot,u))\operatorname{grad} u_\epsilon); \end{cases}$$

it is easy to see that $Lu_\epsilon$ converges strongly to $Qu-f$ in $W_{loc}^{-1,q}(\Omega)$ for any $q < 2$; Meyers' regularity result [11] now implies that

$$u_\epsilon \to u \quad \text{in} \quad W_{loc}^{1,q}(\Omega) \text{ strongly}$$

which yields

$$(3.8) \qquad \operatorname{grad} u_\epsilon \to \operatorname{grad} u \text{ a. e. in } \Omega.$$

This result combined to (3.5), to the sign condition (3.1) and to Fatou's lemma gives

$$(3.9) \qquad \begin{cases} ug(\cdot, u, \operatorname{grad} u) \in L^1(\Omega) \\ \displaystyle\int_\Omega ug(\cdot, u, \operatorname{grad} u)dx \le \limsup_\epsilon \int_\Omega u_\epsilon g_\epsilon(\cdot, u_\epsilon, \operatorname{grad} u_\epsilon)dx. \end{cases}$$

Since from (3.2)

$$| g(\cdot, u, \operatorname{grad} u) | \leq b(1)(c(\cdot) + |\operatorname{grad} u|^2) + ug(\cdot, u, \operatorname{grad} u) \text{ a. e. in } \Omega$$

we deduce from (3.9) that $g(\cdot, u, \operatorname{grad} u)$ belongs to $L^1(\Omega)$.

Finally using $v = 2u_\epsilon$ as well as $v = 0$ as test function in (3.4) we obtain

$$\langle Qu_\epsilon - f, u_\epsilon \rangle + \int_\Omega u_\epsilon g_\epsilon(\cdot, u_\epsilon, \operatorname{grad} u_\epsilon) = 0.$$

Since the functional

$$v \to \langle Qv, v \rangle = \int_\Omega a(\cdot, v) \operatorname{grad} v \operatorname{grad} v dx$$

is lower semi-continuous for the weak topology of $H_0^1(\Omega)$, we deduce from (3.9) and the last equality that

$$(3.10) \qquad \langle Qu - f, u \rangle + \int_\Omega ug(\cdot, u, \mathit{grad}\, u)dx \leq 0.$$

*Second step.* — Define $\mu \in H^{-1} + L^1(\Omega)$ by

$$\mu = Qu + g(\cdot, u, \operatorname{grad} u) - f.$$

The main difficulty is now to prove (see third step) that

$$(3.11) \qquad \mu \in \mathcal{M}^+(\Omega).$$

Note that (3.11) is not a consequence of (3.6) at this time; indeed we do not know that $g_\epsilon(\cdot, u_\epsilon, \operatorname{grad} u_\epsilon)$ tends to $g(\cdot, u, \operatorname{grad} u)$ that in the sense of distributions, even if we already know that the convergence takes place almost everywhere and that $g(\cdot, u, \operatorname{grad} u)$ belongs to $L^1(\Omega)$.

Before of proving (3.11) let us observe that Theorem 3.1 is easily deduced from (3.11) using the abstract result of Theorem 1.3. Indeed, if (3.11) holds true, the hypotheses of Theorem 1.3 are satisfied by $w = u$, $h = -g(\cdot, u, \operatorname{grad} u)$ and $\mu$; therefore $u$ belongs to $L^1(\Omega; d\mu)$ and

$$\langle Qu - f, u \rangle + \int_\Omega ug(\cdot, u, \operatorname{grad} u)dx = \int_\Omega ud\mu \geq 0,$$

since $u \geq 0$ $\mu$-a. e. (use (1.4) and (1.10) to prove this assertion). Now (3.10) and this inequality imply

$$\langle Qu - f, u \rangle + \int_\Omega ug(\cdot, u, \operatorname{grad} u)dx = 0.$$

On the other hand using again (3.11) and Theorem 1.3 we have that any $v \in K_0 \cap L^\infty(\Omega)$ belongs to $L^1(\Omega; d\mu)$ and satisfies

$$\langle Qu - f, v \rangle + \int_\Omega v g(\cdot, u, \operatorname{grad} u) = \int_\Omega v d\mu \geq 0$$

which proves that $u$ is a solution of the variational inequality (3.3).

*Third step.* — It remains to prove (3.11); the proof follows along the lines of [3], to which we refer for more details and comments. Consider the function

$$v_n^\epsilon = \phi \exp\left\{ - \frac{B(u_\epsilon)}{\alpha} \right\} H\left(\frac{1}{n} u_\epsilon\right)$$

where

$$\begin{cases} \phi \in H_0^1(\Omega) \cap L^\infty(\Omega), \quad \phi \geq 0 \\ B(t) = \displaystyle\int_0^t b(s)ds \quad (b \text{ is introduced in (3.2)}) \\ H: \mathbb{R} \to \mathbb{R}, \quad C^1 \text{ function such that } 0 \leq H \leq 1 \text{ in } \mathbb{R}, \\ H = 1 \text{ in } \left[0, \frac{1}{2}\right], \quad H = 0 \text{ in } [1, +\infty[. \end{cases}$$

Since

$$(3.12) \quad \begin{cases} \operatorname{grad} v_n^\epsilon = \exp\left\{ - \dfrac{B(u_\epsilon)}{\alpha} \right\} H\left(\dfrac{1}{n} u_\epsilon\right) \operatorname{grad} \phi \\ \qquad - \dfrac{1}{\alpha} \phi \exp\left\{ - \dfrac{B(u_\epsilon)}{\alpha} \right\} b(u_\epsilon) H\left(\dfrac{1}{n} u_\epsilon\right) \operatorname{grad} u_\epsilon \\ \qquad + \dfrac{1}{n} \phi \exp\left\{ - \dfrac{B(u_\epsilon)}{\alpha} \right\} H'\left(\dfrac{1}{n} u_\epsilon\right) \operatorname{grad} u_\epsilon \end{cases}$$

it is easy to see that $v_n^\epsilon$ is bounded in $H_0^1(\Omega) \cap L^\infty(\Omega)$ for $n$ fixed and that

$$\begin{cases} v_n^\epsilon \to \phi \exp\left\{ - \dfrac{B(u)}{\alpha} \right\} H\left(\dfrac{1}{n} u\right) \quad \text{for } \epsilon \to 0 \quad (n \text{ fixed}) \\ \text{weakly in } H_0^1(\Omega) \text{ a. e. and weakly } * \text{ in } L^\infty(\Omega). \end{cases}$$

Using the test function $v = u^\epsilon + v_n^\epsilon \in K_0$ in (3.4) we obtain

$$(3.13) \qquad \langle Qu_\epsilon - f, v_n^\epsilon \rangle + \int_\Omega v_n^\epsilon g_\epsilon(\cdot, u_\epsilon, \operatorname{grad} u_\epsilon) dx \geq 0$$

in which we now pass to the limit for $\epsilon$ tendiny to 0 and $n$ fixed.

In view of (3.12) the term

$$\langle Qu_\epsilon, v_n^\epsilon \rangle = \int_\Omega a(\cdot, u_\epsilon) \operatorname{grad} u_\epsilon \operatorname{grad} v_n^\epsilon dx$$

can be splitted in 3 terms corresponding to the 3 differents terms of grad $v_n^\epsilon$. It is easy to pass to the limit in the first one, as well as in the term $\langle f, v_n^\epsilon \rangle$. The third term is estimated by

$$\beta \| \operatorname{grad} u_\epsilon \|_{L^2} \frac{1}{n} \| \phi \|_{L^\infty} \| H' \|_{L^\infty(\mathbb{R})} \| \operatorname{grad} u_\epsilon \|_{L^2}$$

and thus bounded by

$$\frac{1}{n} C_* \| \phi \|_{L^\infty(\Omega)}$$

where $C_*$ does not depend neither on $n$, $\epsilon$ nor on $\phi$. Finally the second term is added to the integral $\int_\Omega v_n^\epsilon g_\epsilon(\cdot, u_\epsilon, \operatorname{grad} u_\epsilon) dx$ and this sum is written as

$$\begin{cases} \displaystyle\int_\Omega \phi \exp\left\{ - \frac{B(u_\epsilon)}{\alpha} \right\} H\left(\frac{1}{n} u_\epsilon\right) l_\epsilon dx \\ \displaystyle l_\epsilon = g_\epsilon(\cdot, u_\epsilon, \operatorname{grad} u_\epsilon) - \frac{b(u_\epsilon)}{\alpha} a(\cdot, u_\epsilon) \operatorname{grad} u_\epsilon \operatorname{grad} u_\epsilon . \end{cases}$$

In view of (3.2) we have

$$l_\epsilon(x) \leq b(u_\epsilon(x)) c(x) \text{ a. e. } x \in \Omega$$

and due to the cut-off function H we obtain

$$\phi \exp\left\{ - \frac{B(u_e)}{\alpha} \right\} H\left(\frac{1}{n} u_\epsilon\right) l_\epsilon \leq \phi b(n) c \text{ a. e. in } \Omega.$$

Application of Fatou's lemma thus yields for fixed $n$:

$$\begin{cases} \displaystyle\int_\Omega \phi \exp\left\{ - \frac{B(u)}{\alpha} \right\} H\left(\frac{1}{n} u\right) \Big[ g(\cdot, u, \operatorname{grad} u) \\ \displaystyle\qquad - \frac{b(u)}{\alpha} a(\cdot, u,) \operatorname{grad} u \operatorname{grad} u \Big] dx \\ \displaystyle\qquad \leq \limsup_{\epsilon \to 0} \left[ \text{« second term »} + \int_\Omega v_n^\epsilon g_\epsilon(\cdot, u_\epsilon, \operatorname{grad} u_\epsilon) dx \right]. \end{cases}$$

Collecting these results together we have proved that passing to the limit in (3.13) gives:

$$(3.14)\quad \begin{cases} \left\langle Qu - f, \phi \exp\left\{-\dfrac{B(u)}{\alpha}\right\} H\left(\dfrac{1}{n}u\right)\right\rangle \\ \qquad + \displaystyle\int_\Omega \phi \exp\left\{-\dfrac{B(u)}{\alpha}\right\} H\left(\dfrac{1}{n}u\right) g(\cdot, u, \operatorname{grad} u)\,dx \\ \qquad + \dfrac{2}{n} C_* \|\phi\|_{L^\infty(\Omega)} \geq 0 \end{cases}$$

for any $\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, $\phi \geq 0$ and any $n \in \mathbb{N}$.

Take now

$$\phi = \exp\left\{+\frac{B(u)}{\alpha}\right\} H\left(\frac{1}{p(n)}u\right)\varphi$$

where $\varphi \in \mathcal{D}(\Omega)$, $\varphi \geq 0$, B and H are defined as before and where $p(n)$ is the number defined by

$$B(p(n)) = \alpha \log \sqrt{n};$$

since B is one to one and at least linearly increasing at infinity, we have $p(n) \to +\infty$ as $n \to +\infty$.

Note that $\phi$ can be used as a test function in (3.14) since $\phi$ belongs to $H_0^1(\Omega) \cap L^\infty(\Omega)$ with $\phi \geq 0$. Moreover

$$\|\phi\|_{L^\infty(\Omega)} \leq \exp\left\{\frac{B(p(n))}{\alpha}\right\} \|\varphi\|_{L^\infty(\Omega)} = \sqrt{n}\, \|\varphi\|_{L^\infty(\Omega)}.$$

Since

$$\begin{cases} \phi \exp\left\{-\dfrac{B(u)}{\alpha}\right\} H\left(\dfrac{1}{n}u\right) = H\left(\dfrac{1}{n}u\right) H\left(\dfrac{1}{p(n)}u\right)\varphi \to \varphi \\ \text{in weakly } H_0^1(\Omega), \text{ a. e. and weakly} * \text{ in } L^\infty(\Omega) \end{cases}$$

it is easy to pass to the limit in (3.14); this gives

$$\langle Qu - f, \varphi \rangle + \int_\Omega \varphi g(\cdot, u, \operatorname{grad} u)\,dx \geq 0$$

for any $\varphi \in \mathcal{D}(\Omega)$, $\varphi \geq 0$, which is exactly the desired result (3.11).

The proof of Theorem 3.1 is now complete.

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