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G-convergence of monotone operators

by

Valeria CHIADO' PIAT, Gianni DAL MASO and Anneliese DEFRANCESCHI

SISSA, Strada Costiera 11, 34014 Trieste, Italy

ABSTRACT. — A general notion of G-convergence for sequences of maximal monotone operators of the form $\mathscr{A}_h u = -\operatorname{div}(a_h(x, Du))$ is introduced in terms of the asymptotic behavior, as $h \to +\infty$, of the solutions u_h to the equations $\mathscr{A}_h u_h = f_h$ and of their momenta $a_h(x, Du_h)$. The main results of the paper are the local character of the G-convergence and the G-compactness of some classes of nonlinear monotone operators.

Key words : G-convergence, monotone operators, nonlinear elliptic equations.

RÉSUMÉ. — On présente une notion générale de G-convergence pour des opérateurs maximaux monotones sous forme divergence. On démontre le caractère local de la G-convergence et de la G-compacité pour certaines classes d'opérateurs de ce type.

Mots clés : G-convergence, opérateurs monotones, équations elliptiques non linéaires.

INTRODUCTION

The aim of this paper is to study a general notion of G-convergence for nonlinear monotone operators $\mathscr{A} : H_0^{1,p}(\Omega) \to H^{-1,q}(\Omega)$ of the form

$$\mathscr{A} u = -\operatorname{div}(a(x, Du)), \tag{0.1}$$

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where Ω is a bounded open subset of $\mathbf{R}^n$, $1<p<+\infty$, and $1/p+1/q=1$. We assume that the (possibly multivalued) map a: $\Omega\times\mathbf{R}^n\to\mathbf{R}^n$ which occurs in (0.1) is measurable on $\Omega\times\mathbf{R}^n$, is maximal monotone on $\mathbf{R}^n$ for almost every $x\in\Omega$, and satisfies suitable coerciveness and boundedness conditions (*see* Section 2). The class of all these maps will be denoted by $M_\Omega(\mathbf{R}^n)$.

The main examples of maps of the class $M_\Omega(\mathbf{R}^n)$ have the form

$$a(x,\xi)=\partial_\xi\psi(x,\xi), \tag{0.2}$$

where ∂_ξ denotes the subdifferential with respect to ξ and $\psi:\Omega\times\mathbf{R}^n\to[0,+\infty[$ is measurable in (x,ξ), convex in ξ, and satisfies the inequalities

$$c_1|\xi|^p\leqq\psi(x,\xi)\leqq c_2(1+|\xi|^p)$$

for suitable constants $0<c_1\leqq c_2$. In this case the operator (0.1) is the subdifferential of the functional

$$\Psi(u)=\int_\Omega\psi(x,\mathrm{D}u)\,dx \tag{0.3}$$

and the notion of G-convergence of the operators (0.1) can be studied in connection with the notion of Γ-convergence of the corresponding functionals (0.3) (*see* [1], [17], [3]).

Let us return to the general case of maps of the class $M_\Omega(\mathbf{R}^n)$ for which the representation (0.2) is not always possible. Let (a_h) be a sequence in $M_\Omega(\mathbf{R}^n)$ and let $a\in M_\Omega(\mathbf{R}^n)$. To introduce the notion of G-convergence in $M_\Omega(\mathbf{R}^n)$ we begin with the simpler case where a_h and a are single-valued and strictly monotone on $\mathbf{R}^n$. We then say that (a_h) G-converges to a if, for every $f\in H^{-1,q}(\Omega)$ and for every sequence (f_h) converging to f strongly in $H^{-1,q}(\Omega)$, the solutions u_h of the equations

$$\begin{cases} -\operatorname{div}(a_h(x,\mathrm{D}u_h))=f_h & \text{on } \Omega,\\ u_h\in H_0^{1,p}(\Omega), \end{cases} \tag{0.4}$$

satisfy the following conditions:

$$u_h\to u,\quad \text{weakly in } H^{1,p}(\Omega),$$
$$a_h(x,\mathrm{D}u_h)\to a(x,\mathrm{D}u)\quad \text{weakly in } (L^q(\Omega))^n,$$

where u is the solution of the equation

$$\begin{cases} -\operatorname{div}(a(x,\mathrm{D}u))=f & \text{on } \Omega,\\ u\in H_0^{1,p}(\Omega). \end{cases} \tag{0.5}$$

If we drop the hypothesis that a_h and a are single-valued and strictly monotone, then the definition of G-convergence is more delicate, due to the non-uniqueness of the solutions of the equations (0.4) and (0.5).

In the general case we say that (a_h) G-convergences to a if for every increasing sequence of integers $\tau(h)$, for every $f \in H^{-1,q}(\Omega)$, for every sequence (f_h) converging to f strongly in $H^{-1,q}(\Omega)$, for every sequence (u_h) of solutions of the equations

$$(0.6)\qquad \left\{\begin{array}{l} -\operatorname{div}(a_{\tau(h)}(x, Du_h)) \ni f_h \quad \text{on } \Omega, \\ u_h \in H_0^{1,p}(\Omega), \end{array}\right.$$

and for every sequence (g_h) in $(L^q(\Omega))^n$ with

$$g_h(x) \in a_{\tau(h)}(x, Du_h(x)) \quad \text{a.e. in } \Omega \qquad \text{and} \qquad -\operatorname{div} g_h = f_h \quad \text{in } \Omega,$$

there exists an increasing sequence of integers $\sigma(h)$ such that

$$u_{\sigma(h)} \to u, \quad \text{weakly in } H^{1,p}(\Omega)$$

and

$$g_{\sigma(h)} \to g, \quad \text{weakly in } (L^q(\Omega))^n,$$

where u is a solution of the equation

$$(0.7)\qquad \left\{\begin{array}{l} -\operatorname{div}(a(x, Du)) \ni f \quad \text{on } \Omega, \\ u \in H_0^{1,p}(\Omega), \end{array}\right.$$

and

$$g(x) \in a(x, Du(x)) \quad \text{a.e. in } \Omega.$$

Let us emphasize that the notion of G-convergence in $M_\Omega(\mathbf{R}^n)$ is independent of the particular boundary condition chosen in the definition, in the sense that, given $\varphi \in H^{1,p}(\Omega)$, we can replace $H_0^{1,p}(\Omega)$ by $\varphi + H_0^{1,p}(\Omega)$ in (0.4), (0.5), (0.6), (0.7) without changing the G-convergent sequences and their limits.

The main result of this paper is the compactness of the class $M_\Omega(\mathbf{R}^n)$ with respect to G-convergence. Moreover we prove the following localization property: if (a_h) G-converges to a, (b_h) G-converges to b, and $a_h(x, .) = b_h(x, .)$ for almost every x in an open subset Ω' of Ω, then $a(x, .) = b(x, .)$ for almost every $x \in \Omega'$.

Finally we determine some subsets of $M_\Omega(\mathbf{R}^n)$ which are closed under G-convergence. This allows us to prove in a unified way the compactness, with respect to G-convergence, of all general classes of linear or nonlinear operators of the form (0.1) which have been considered in the literature.

The notion of G-convergence for second order linear elliptic operators was studied by E. De Giorgi and S. Spagnolo in the symmetric case (*see* [24], [25], [26], [12]), and then extended to the non-symmetric case by F. Murat and L. Tartar under the name of H-convergence (*see* [27], [28], and [18]). We refer to [5] and [23] for the related problem of the homogenization of elliptic equations and to [30] for the extension of the notion of G-convergence to higher order linear elliptic operators.

The properties of the G-convergence for quasilinear elliptic operators were studied by L. Boccardo, Th. Gallouet, and F. Murat in [7], [8], and [6].

The first results in the nonlinear case (0.1), with $p=2$, are due to F. Murat and L. Tartar, who studied (in [20]) the properties of the G-convergence in a suitable class of monotone operators of the form (0.1), assuming that the maps a are uniformly Lipschitz continuous and uniformly strictly monotone on $\mathbf{R}^n$. The corresponding homogenization results were studied by L. Tartar in [27] and H. Attouch in [2].

A similar theory of G-convergence for more general classes of uniformly equicontinuous strictly monotone operators was developed by U. E. Raitum in the case $2 \leqq p < +\infty$ (*see* [22]). For the corresponding homogenization results we refer to [13] and [14].

We remark that, in order to include the case (0.2), we do not assume the maps of our class $M_\Omega(\mathbf{R}^n)$ to be continuous or strictly monotone on $\mathbf{R}^n$, and this requires a deep change in the proof of the compactness of $M_\Omega(\mathbf{R}^n)$ under G-convergence. While all proofs in the quoted papers are based essentially on a density argument, which is made possible by the continuity of the operators $\mathscr{A}$ or of the inverse operators $\mathscr{A}^{-1}$, our proof relies on a theorem by F. Hiai and H. Umegaki concerning the representation of every closed decomposable subset of L^p as the set of all measurable selections of a suitable multivalued map (*see* [15]).

1. MULTIVALUED FUNCTIONS

In this section we fix the notation and recall some results concerning multivalued functions and their measurability. Furthermore, we summarize the main theorems for multivalued monotone operators on Banach spaces which will be applied in this paper.

If x, y are elements of a set X, by $[x, y]$ we denote the ordered pair formed by x and y, whereas (x, y) denotes the scalar product of x and y, provided X is a Hilbert space.

Multivalued functions. – Let X and Y be two sets. A *multivalued function* F from X to Y is a map that associates with any $x \in X$ a subset Fx of Y. The subsets Fx are called the *images* or *values* of F. The sets

$$D(F)=\{x \in X : Fx \neq \varnothing\} \quad \text{and} \quad G(F)=\{[x, y] \in X \times Y : y \in Fx\}$$

are called the *domain* of F and the *graph* of F, respectively. The *range* of F is, by definition, the set

$$R(F)=\bigcup_{x \in X} Fx.$$

If for every $x \in X$ the set Fx contains exactly one element of Y, we say that F is single-valued.

In general, we shall identify every multivalued function F with its graph in $X \times Y$. The *inverse* F^{-1} of the multivalued map F from X to Y is the multivalued function from Y to X defined by $x \in F^{-1} y$ if and only if $y \in F x$; in other words, F^{-1} is the multivalued function, whose graph is symmetric to the graph of F.

MEASURABLE MULTIVALUED FUNCTIONS. – Let $(X, \mathscr{T})$ be a measurable space, and let $F : X \to \mathbf{R}^n$ be a multivalued function from the space X to the family of non-empty subsets of the space $\mathbf{R}^n$. For every $B \subseteq \mathbf{R}^n$ the inverse image of B under F is denoted by

$$F^{-1}(B) = \{ x \in X : B \cap F x \neq \varnothing \}.$$

We shall consider the following measurability conditions:

(1.1) for each Borel set $B \subset \mathbf{R}^n$, $F^{-1}(B) \in \mathscr{T}$;

(1.2) for each closed set $C \subset \mathbf{R}^n$, $F^{-1}(C) \in \mathscr{T}$;

(1.3) for each open set $U \subset \mathbf{R}^n$, $F^{-1}(U) \in \mathscr{T}$;

(1.4) there exists a sequence (σ_h) of measurable selections such that $Fx = \mathrm{cl}\{\sigma_h(x) : h \in \mathbf{N}\}$ for each x (a selection of F is a map $\sigma : X \to \mathbf{R}^n$ such that $\sigma(x) \in F x$ for every x);

(1.5) $G(F) \in \mathscr{T} \otimes \mathscr{B}(\mathbf{R}^n)$, where $\mathscr{B}(\mathbf{R}^n)$ is the σ-field of all Borel subsets of $\mathbf{R}^n$.

We say that a multivalued function $F : X \to \mathbf{R}^n$ is *measurable* [with respect to $\mathscr{T}$ and $\mathscr{B}(\mathbf{R}^n)$] if (1.2) is verified. Let us state a theorem which links this definition of measurability of a multivalued function F to the other conditions on F listed above.

THEOREM 1.1. – *Let $(X, \mathscr{T})$ be a measurable space. Let $F : X \to \mathbf{R}^n$ be a multivalued function with non-empty closed values. Then the following conditions hold:*

(i) (1.1) ⇒ (1.2) ⇔ (1.3) ⇔ (1.4) ⇒ (1.5);

(ii) *If there exists a complete σ-finite measure μ defined on $\mathscr{T}$, then all conditions* (1.1)-(1.5) *are equivalent.*

The proof of the above theorem can be found in [11], Chapter III, Section 2. A useful tool for problems of this type is given by the projection theorem below (*see* [11], Theorem III.23).

THEOREM 1.2. – *Let $(X, \mathscr{T}, \mu)$ be a measurable space, where μ is a complete σ-finite measure defined on $\mathscr{T}$. If G belongs to $\mathscr{T} \otimes \mathscr{B}(\mathbf{R}^n)$, then the projection $\mathrm{pr}_x G$ belongs to $\mathscr{T}$.*

The next theorem states the equivalence between conditions (1.2) and (1.5) for certain multivalued functions even if the measure space is not complete.

THEOREM 1.3. — *Let* $(X, \mathcal{T}, \mu)$ *be a measurable space, where* μ *is a complete* σ*-finite measure defined on* $\mathcal{T}$. *Let* $F: X \to \mathbf{R}^n \times \mathbf{R}^m$ *be a multivalued function with non-empty closed values. Let* $H: X \times \mathbf{R}^n \to \mathbf{R}^m$ *be the multivalued function defined by*

$$(1.6)\qquad H(x, \xi) = \{\eta \in \mathbf{R}^m : [\xi, \eta] \in F x\}.$$

Then the following conditions are equivalent:

(i) F *is measurable with respect to* $\mathcal{T}$ *and* $\mathcal{B}(\mathbf{R}^n) \otimes \mathcal{B}(\mathbf{R}^m)$;

(ii) $G(F) \in \mathcal{T} \otimes \mathcal{B}(\mathbf{R}^n) \otimes \mathcal{B}(\mathbf{R}^m)$;

(iii) H *is measurable with respect to* $\mathcal{T} \otimes \mathcal{B}(\mathbf{R}^n)$ *and* $\mathcal{B}(\mathbf{R}^m)$;

(iv) $G(H) \in \mathcal{T} \otimes \mathcal{B}(\mathbf{R}^n) \otimes \mathcal{B}(\mathbf{R}^m)$.

Proof. — By Theorem 1.1 (ii) we have that (i)$\Leftrightarrow$(ii). Moreover, Theorem 1.1 (i) guarantees that (iii)$\Rightarrow$(iv). Since $G(F) = G(H)$, we obtain easily that (ii)$\Leftrightarrow$(iv). To conclude the proof of the theorem we shall show that (ii)$\Rightarrow$(iii). To this aim it is enough to prove that (ii) yields $H^{-1}(C) \in \mathcal{T} \otimes \mathcal{B}(\mathbf{R}^n)$ for every compact subset C of $\mathbf{R}^m$. Let us fix a compact set $C \subseteq \mathbf{R}^m$. By taking (1.6) into account we have that

$$(1.7)\qquad H^{-1}(C) = \{[x, \xi] \in X \times \mathbf{R}^n : \exists\, \eta \in \mathbf{R}^m : [\xi, \eta] \in F x \cap (\mathbf{R}^n \times C)\}.$$

Let B denote the set of all $x \in X$ such that $F x \cap (\mathbf{R}^n \times C)$ is non-empty. By (ii) and the projection Theorem 1.2 it follows that $B \in \mathcal{T}$. If Φ is the multivalued function from X to $\mathbf{R}^n \times \mathbf{R}^m$ defined by $\Phi x = F x \cap (\mathbf{R}^n \times C)$, then $D(\Phi) = B$ and (1.7) becomes

$$(1.8)\qquad H^{-1}(C) = \{[x, \xi] \in X \times \mathbf{R}^n : \exists\, \eta \in \mathbf{R}^m : [\xi, \eta] \in \Phi x\}.$$

Since $G(\Phi) = G(F) \cap (X \times \mathbf{R}^n \times C) \in \mathcal{T} \otimes \mathcal{B}(\mathbf{R}^n) \otimes \mathcal{B}(\mathbf{R}^m)$, by Theorem 1.1 there exists a sequence $[\varphi_h, g_h]$ of measurable functions from B to $\mathbf{R}^n \times \mathbf{R}^m$ such that

$$(1.9)\qquad \Phi x = \mathrm{cl}\{[\varphi_h(x), g_h(x)] : h \in \mathbf{N}\}$$

for every $x \in B$. By taking (1.9) into account let us define the set

$$(1.10)\qquad M = \{[x, \xi] \in X \times \mathbf{R}^n : x \in B,\ \xi \in \mathrm{cl}\{\varphi_h(x) : h \in \mathbf{N}\}\}.$$

We shall prove that $M = H^{-1}(C)$. The inclusion $H^{-1}(C) \subseteq M$ follows easily from (1.8), (1.9), and (1.10). To prove that $M \subseteq H^{-1}(C)$, let us fix $[x, \xi] \in M$. By definition there exists a subsequence $(\varphi_{\sigma(h)})$ of (φ_h) such that $(\varphi_{\sigma(h)}(x))$ converges to ξ. Moreover, the corresponding sequence $(g_{\sigma(h)}(x))$ belongs to the compact set C. Hence, by passing, if necessary, to a subsequence we may assume that $(g_{\sigma(h)}(x))$ converges to some $\eta \in \mathbf{R}^m$. By (1.9) we have $[\xi, \eta] \in \Phi x$, hence $[x, \xi] \in H^{-1}(C)$, which concludes the proof of the equality $M = H^{-1}(C)$. Since

$$M = \{[x, \xi] \in X \times \mathbf{R}^n : x \in B,\ \inf_{h \in \mathbf{N}} |\xi - \varphi_h(x)| = 0\},$$

we have that $M \in \mathscr{T} \otimes \mathscr{B}(\mathbf{R}^n)$ and the proof of the theorem is accomplished. ■

Finally, let us give a more general theorem for the existence of a measurable selection of a multivalued function due to Aumann and von Neumann (*see* [11], Theorem III.22).

THEOREM 1.4. — *Let* $(X, \mathscr{T})$ *be a measurable space and let* F *be a multivalued function from* X *to* $\mathbf{R}^n$ *with non-empty values. If the graph* G(F) *belongs to* $\mathscr{T} \otimes \mathscr{B}(\mathbf{R}^n)$ *and there exists a complete* σ*-finite measure defined on* $\mathscr{T}$, *then* F *has a measurable selection.*

MAXIMAL MONOTONE OPERATORS. — Our present aim is to remind the definition and some basic properties of multivalued maximal monotone operators in Banach spaces.

Let X be a Banach space and let X^* be its topological dual. By $\langle\,,\rangle$ we denote the duality pairing between X^* and X.

DEFINITION 1.5. — A subset $A \subseteqq X \times X^*$ is called *monotone* (resp. *strictly monotone*) if

$$\langle y_1 - y_2, x_1 - x_2 \rangle \geqq 0 \qquad (\text{resp.} > 0)$$

for any $[x_1, y_1] \in A$, $[x_2, y_2] \in A$.

DEFINITION 1.6. — A monotone subset $A \subseteqq X \times X^*$ is called *maximal monotone* if it is not properly contained in any other monotone subset of $X \times X^*$, *i.e.* for every $[x, y] \in X \times X^*$ such that

$$\langle y - \eta, x - \xi \rangle \geqq 0, \qquad \forall [\xi, \eta] \in A$$

it follows that $[x, y] \in A$.

We say that a multivalued operator $F : X \to X^*$ is *monotone* (resp. *maximal monotone*) if its graph is a monotone (resp. maximal monotone) subset of $X \times X^*$.

REMARK 1.7. — Since the monotonicity is invariant under transposition of the domain and the range of a map, F is (maximal) monotone if and only if F^{-1} has this property.

Let us note that if F is a (multivalued) maximal monotone operator on X, then for any $x \in D(F)$ the image Fx is a closed convex subset of X^* (*see*, for example, [21], Chapter III.2).

Before giving the statement of the next theorem, which will be heavily applied in Sections 2 and 5, we recall the definition of the concept of upper-semicontinuous multivalued operator.

DEFINITION 1.8. — Let S_1 and S_2 be two topological spaces, and let F be a multivalued function of S_1 into S_2. Then F is said to be *upper-semicontinuous* if for every $s_0 \in S_1$ and for every open neighborhood V of

Fs_0 in S_2 there exists a neighborhood U of s_0 in S_1 such that $Fs \subseteq V$ for every $s \in U$.

The following result provides a useful criterion for maximal monotonicity (*see* [10], Theorem (3.18)).

THEOREM 1.9. — *Let* X *be a Banach space and let* X^* *be its dual. Let* F *be a multivalued monotone operator of* X *into* X^*. *Suppose that for each* x *in* X, Fx *is a non-empty weak* closed convex subset of* X^* *and that for each line segment in* X, F *is an upper-semicontinuous multivalued operator from the line segment to* X^*, *with* X^* *given its weak* topology. Then* F *is maximal monotone.*

Finally, we state a surjectivity result for a class of multivalued monotone operators which is of crucial importance in the proof of our theorems in Sections 2 and 4.

THEOREM 1.10. — *Let* X *be a reflexive Banach space and let* X^* *be its dual. Let* F *be a multivalued maximal monotone operator from* X *to* X^*. *If* F *is coercive, then* $R(F) = X^*$.

We remind that the (multivalued) operator $F: X \to X^*$ is called *coercive* if

$$\lim_{\|x\| \to \infty} \frac{\langle Fx, x \rangle}{\|x\|} = +\infty.$$

The proof of Theorem 1.10 can be found in [21], Chapter III, Theorem 2.10.

2. MULTIVALUED MONOTONE OPERATORS IN SOBOLEV SPACES

In this section we study a class of multivalued monotone operators on Sobolev spaces of the type $-\operatorname{div}(a(x, Du))$.

Throughout the paper we denote by p a fixed real number, $1 < p < +\infty$, and by q its dual exponent, $1/p + 1/q = 1$. Moreover we fix a bounded open subset Ω of $\mathbf{R}^n$, two non-negative functions $m_1, m_2 \in L^1(\Omega)$ and two constants $c_1 > 0$, $c_2 > 0$. By $\mathscr{L}(\Omega)$ we denote the σ-field of all Lebesgue measurable subsets of Ω, and by $\mathscr{B}(\mathbf{R}^n)$ the σ-field of all Borel subsets of $\mathbf{R}^n$. The Euclidean norm and the scalar product in $\mathbf{R}^n$ are denoted by $|\,.\,|$ and $(.\,,\,.)$, respectively.

DEFINITION 2.1. — By $M_\Omega(\mathbf{R}^n)$ we denote the class of all multivalued functions $a: \Omega \times \mathbf{R}^n \to \mathbf{R}^n$ with closed values which satisfy the following conditions:

(i) for $a.e.\ x \in \Omega$ the multivalued function $a(x, .): \mathbf{R}^n \to \mathbf{R}^n$ is maximal monotone;

(ii) a is measurable with respect to $\mathscr{L}(\Omega) \otimes \mathscr{B}(\mathbf{R}^n)$ and $\mathscr{B}(\mathbf{R}^n)$, *i.e.*

$$a^{-1}(\mathrm{C}) = \{[x, \xi] \in \Omega \times \mathbf{R}^n : a(x, \xi) \cap \mathrm{C} \neq \varnothing\} \in \mathscr{L}(\Omega) \otimes \mathscr{B}(\mathbf{R}^n)$$

for every closed set $\mathrm{C} \subseteq \mathbf{R}^n$;

(iii) the estimates

$$|\eta|^q \leqq m_1(x) + c_1(\eta, \xi), \tag{2.1}$$

$$|\xi|^p \leqq m_2(x) + c_2(\eta, \xi) \tag{2.2}$$

hold for $a.e.\ x \in \Omega$, for every $\xi \in \mathbf{R}^n$, and $\eta \in a(x, \xi)$.

REMARK 2.2. – Conditions (2.1) and (2.2) imply that there exist two functions $m_3 \in \mathrm{L}^q(\Omega)$, $m_4 \in \mathrm{L}^1(\Omega)$ and two constants $c_3 > 0$, $c_4 > 0$ such that

$$|\eta| \leqq m_3(x) + c_3 |\xi|^{p-1}, \tag{2.3}$$

$$(\eta, \xi) \geqq m_4(x) + c_4 |\xi|^p \tag{2.4}$$

for $a.e.\ x \in \Omega$, for every $\xi \in \mathbf{R}^n$, and $\eta \in a(x, \xi)$. Conversely, if a satisfies (2.3) and (2.4), then (2.1) and (2.2) hold for suitable m_1, m_2, c_1, c_2.

REMARK 2.3. – For $a.e.\ x \in \Omega$ and for every $\xi \in \mathbf{R}^n$ the set $a(x, \xi)$ is closed and convex in $\mathbf{R}^n$ by (i) (*see*, for instance, [21], Section III.2.3). Moreover, (ii) and Theorem 1.1 (i) imply that the graph of a belongs to $\mathscr{L}(\Omega) \otimes \mathscr{B}(\mathbf{R}^n) \otimes \mathscr{B}(\mathbf{R}^n)$. By (2.3), for $a.e.\ x \in \Omega$ the maximal monotone operator $a(x, .)$ is locally bounded, hence $a^{-1}(x, .)$ is surjective (*see* [21], III.4.2). This implies that $a(x, \xi) \neq \varnothing$ for $a.e.\ x \in \Omega$ and for every $\xi \in \mathbf{R}^n$.

Given $a \in \mathrm{M}_\Omega(\mathbf{R}^n)$, $f \in \mathrm{H}^{-1, q}(\Omega)$, and $\varphi \in \mathrm{H}^{1, p}(\Omega)$ we consider the Dirichlet boundary value problem

$$\begin{cases} -\operatorname{div}(a(x, \mathrm{D}u)) \ni f & \text{on } \Omega, \\ u \in \mathrm{H}^{1, p}_\varphi(\Omega), \end{cases} \tag{2.5}$$

where $\mathrm{H}^{1, p}_\varphi(\Omega) = \{u \in \mathrm{H}^{1, p}(\Omega) : u - \varphi \in \mathrm{H}^{1, p}_0(\Omega)\}$.

To study the solutions of (2.5), and in particular their dependence on f and a, we shall give some equivalent formulations of this problem which are used in the sequel.

DEFINITION 2.4. – Let $\varphi \in \mathrm{H}^{1, p}(\Omega)$. By $\mathrm{M}(\mathrm{H}^{1, p}_\varphi)$ [resp. $\mathrm{M}(\mathrm{H}^{1, p})$] we denote the class of all multivalued operators $\mathrm{A} : \mathrm{H}^{1, p}_\varphi(\Omega) \to (\mathrm{L}^q(\Omega))^n$ [resp. $\mathrm{A} : \mathrm{H}^{1, p}(\Omega) \to (\mathrm{L}^q(\Omega))^n$] satisfying the following conditions:

(i) if $u_i \in \mathrm{H}^{1, p}_\varphi(\Omega)$ [resp. $\mathrm{H}^{1, p}(\Omega)$] and $g_i \in \mathrm{A}\, u_i$, $i = 1, 2$, then

$$(\mathrm{D}u_1 - \mathrm{D}u_2, g_1 - g_2) \geqq 0 \quad a.e. \text{ on } \Omega;$$

(ii) the estimates

$$|g|^q \leqq m_1 + c_1(Du, g) \quad a.e. \text{ on } \Omega, \tag{2.6}$$

$$|Du|^p \leqq m_2 + c_2(Du, g) \quad a.e. \text{ on } \Omega, \tag{2.7}$$

hold for every $u \in H_\varphi^{1,p}(\Omega)$ [resp. $u \in H^{1,p}(\Omega)$] and $g \in A u$.

By $\mathcal{M}(H_\varphi^{1,p})$ [resp. $\mathcal{M}(H^{1,p})$] we denote the class of all multivalued operators $\mathcal{A}: H_\varphi^{1,p}(\Omega) \to H^{-1,q}(\Omega)$ [resp. $\mathcal{A}: H^{1,p}(\Omega) \to H^{-1,q}(\Omega)$] of the form

$$\mathcal{A} u = \{-\operatorname{div} g : g \in A u\}, \tag{2.8}$$

with $A \in M(H_\varphi^{1,p})$ [resp. $A \in M(H^{1,p})$].

REMARK 2.5. – In the case $\varphi = 0$ the operators of the class $\mathcal{M}(H_0^{1,p})$ are monotone according to Definition 1.5 in consequence of (i). If $\mathcal{A} \in \mathcal{M}(H_0^{1,p})$ is maximal monotone, then $D(\mathcal{A}) = H_0^{1,p}(\Omega)$. Indeed $\mathcal{A}$ is locally bounded by (2.6), hence $\mathcal{A}^{-1}$ is surjective (*see* [21], III.4.2).

DEFINITION 2.6. – Let $\varphi \in H^{1,p}(\Omega)$. To every $a \in M_\Omega(\mathbf{R}^n)$ we associate the operators $A \in M(H^{1,p})$ and $\mathcal{A} \in \mathcal{M}(H^{1,p})$ defined by

$$A u = \{g \in (L^q(\Omega))^n : g(x) \in a(x, Du(x)) \text{ for } a.e.\ x \in \Omega\},$$
$$\mathcal{A} u = \{-\operatorname{div} g : g \in A u\}.$$

Their restrictions to $H_\varphi^{1,p}(\Omega)$ belong to $M(H_\varphi^{1,p})$ and $\mathcal{M}(H_\varphi^{1,p})$ and will be denoted by A^φ and $\mathcal{A}^\varphi$, respectively.

By taking these definitions into account, problem (2.5) becomes then equivalent to the following one: given $f \in H^{-1,q}(\Omega)$, find $u \in H_\varphi^{1,p}(\Omega)$ such that

$$\begin{cases} f \in \mathcal{A}^\varphi u, \\ u \in H_\varphi^{1,p}(\Omega), \end{cases} \tag{2.9}$$

or equivalently, find $u \in H_\varphi^{1,p}(\Omega)$ and $g \in (L^q(\Omega))^n$ such that

$$\begin{cases} g \in A^\varphi u, \\ -\operatorname{div} g = f, \\ u \in H_\varphi^{1,p}(\Omega). \end{cases} \tag{2.10}$$

Let us denote by I the (single-valued) monotone operator from $L^p(\Omega)$ to $L^q(\Omega)$ defined by $I u = |u|^{p-2} u$. The next theorem is more than needed for solving problem (2.9) in the case $\varphi = 0$, but it is used in its generality in Section 6.

THEOREM 2.7. – *Let $\mathcal{A}^0$ be the operator in $\mathcal{M}(H_0^{1,p})$ associated to a function $a \in M_\Omega(\mathbf{R}^n)$ in the case $\varphi = 0$ (Definition 2.6). Then*

(i) *$\mathcal{A}^0$ is maximal monotone;*

(ii) *$R(\mathcal{A}^0 + \lambda I) = H^{-1,q}(\Omega)$ for every $\lambda \geqq 0$.*

Proof. — Let us start with the proof of (i). To this aim we show that the operator $\mathscr{A}^0$ satisfies the assumptions of Theorem 1.9.

(*a*) For every $u \in H_0^{1,p}(\Omega)$, we have $\mathscr{A}^0 u \neq \varnothing$. To prove this assertion let us fix $u \in H_0^{1,p}(\Omega)$. By Remark 2.3 the set $a(x, Du(x))$ is non-empty, closed, and convex in $\mathbf{R}^n$ for *a. e.* $x \in \Omega$. Therefore, by taking Theorem 1.1 into account we conclude that there exists a measurable function $g : \Omega \to \mathbf{R}^n$ such that $g(x) \in a(x, Du(x))$ for *a. e.* $x \in \Omega$. Finally, the estimate (2.3) yields $g \in (L^q(\Omega))^n$, which concludes the proof of (*a*).

(*b*) For every $u \in H_0^{1,p}(\Omega)$, $\mathscr{A}^0 u$ is a convex subset of $H^{-1,q}(\Omega)$. This follows easily from the fact that $a(x, Du(x))$ is a convex subset of $\mathbf{R}^n$ for *a. e.* $x \in \Omega$ (Remark 2.3).

(*c*) For every $u \in H_0^{1,p}(\Omega)$, $\mathscr{A}^0 u$ is a weakly closed subset of $H^{-1,q}(\Omega)$ and the multivalued operator $\mathscr{A}^0$ is upper-semicontinuous from the strong topology of $H_0^{1,p}(\Omega)$ to the weak topology of $H^{-1,q}(\Omega)$. By the boundedness condition (2.3), to prove this assertion it is enough to show that, if (u_h) converges to u strongly in $H_0^{1,p}(\Omega)$, (f_h) converges to f weakly in $H^{-1,q}(\Omega)$, and $f_h \in \mathscr{A}^0 u_h$ for every $h \in \mathrm{N}$, then $f \in \mathscr{A}^0 u$. Under these assumptions on f_h, f, u_h, u, the boundedness condition (2.3) guarantees the existence of a sequence of functions $g_h \in (L^q(\Omega))^n$ and of a function $g \in (L^q(\Omega))^n$ such that (up to a subsequence) (g_h) converges to g weakly in $(L^q(\Omega))^n$, $g_h(x) \in a(x, Du_h(x))$ for *a. e.* $x \in \Omega$, $-\operatorname{div} g_h = f_h$, and $-\operatorname{div} g = f$. Therefore, it remains to verify that $g(x) \in a(x, Du(x))$ for *a. e.* $x \in \Omega$. If we show that the set

$$M = \{ x \in \Omega : \exists\, \xi \in \mathbf{R}^n,\ \exists\, \eta \in a(x, \xi) : (g(x) - \eta, Du(x) - \xi) < 0 \}$$

has Lebesgue measure zero, then the maximal monotonicity of a yields $g(x) \in a(x, Du(x))$ *a. e.* on Ω, which concludes the proof of (*c*). To prove that $|M| = 0$, let us write $M = \{ x \in \Omega : Gx \neq \varnothing \}$, where

$$Gx = \{ [\xi, \eta] \in \mathbf{R}^n \times \mathbf{R}^n : \eta \in a(x, \xi),\ (g(x) - \eta,\ Du(x) - \xi) < 0 \}.$$

By Remark 2.3 the graph of G belongs to $\mathscr{L}(\Omega) \otimes \mathscr{B}(\mathbf{R}^n) \otimes \mathscr{B}(\mathbf{R}^n)$, thus $M \in \mathscr{L}(\Omega)$ by the projection Theorem 1.2. By the Aumann-von Neumann Theorem 1.4 there exists a measurable selection $[\xi, \eta]$ of G defined on M. Therefore $\eta(x) \in a(x, \xi(x))$ and

$$(g(x) - \eta(x), Du(x) - \xi(x)) < 0 \tag{2.11}$$

for every $x \in M$. On the other hand, the monotonicity assumption on a implies that

$$(g_h(x) - \eta(x), Du_h(x) - \xi(x)) \geqq 0, \quad a.e. \text{ on } M \tag{2.12}$$

for every $h \in \mathrm{N}$. If $|M| > 0$, there exists a measurable subset M' of M with $|M'| > 0$ such that $[\xi(x), \eta(x)]$ is bounded on M'. By integrating (2.12) on M' and by passing to the limit as $h \to +\infty$, we obtain

$$\int_{M'} (g(x)-\eta(x), Du(x)-\xi(x))\,dx \geqq 0,$$

which contradicts (2.11) being $|M'|>0$. Therefore we have to conclude that $|M|=0$. This proves (*c*) and completes the proof of (i).

Proof of (ii). By (i) we have that $\mathscr{A}^0$ is maximal monotone. Since $D(\mathscr{A}^0)=D(I)=H_0^{1,p}(\Omega)$, and I is maximal monotone on $H_0^{1,p}(\Omega)$, the operator $\mathscr{A}^0+\lambda I$ is maximal monotone for every $\lambda \geqq 0$ (*see* [21], III.3.6). By (2.2) it is also coercive and therefore $R(\mathscr{A}^0+\lambda I)=H^{-1,q}(\Omega)$ by Theorem 1.10.

REMARK 2.8. — Problem (2.9) has a solution for every $\varphi \in H^{1,p}(\Omega)$. Indeed, let us define the multivalued function $a_\varphi(x,\xi)=a(x,\xi+D\varphi(x))$ which still belongs to the class $M_\Omega(\mathbf{R}^n)$. If $\mathscr{A}_\varphi^0$ denotes the operator in $\mathscr{M}(H_0^{1,p})$ associated to the function a_φ by Definition 2.6, it follows easily that $\mathscr{A}^\varphi(u+\varphi)=\mathscr{A}_\varphi^0 u$ for every $u \in H_0^{1,p}(\Omega)$. Since by Theorem 2.7 (ii) we have that $R(\mathscr{A}_\varphi^0)=H^{-1,q}(\Omega)$, our assertion follows immediately.

Finally, the following result is a useful tool to check the maximality of certain monotone operators on $H_0^{1,p}(\Omega)$.

LEMMA 2.9. — *Let* $\mathscr{A}$ *be a (multivalued) monotone operator from* $H_0^{1,p}(\Omega)$ *into* $H^{-1,q}(\Omega)$, *let* $\lambda>0$, *and let* I *be the (single-valued) function from* $L^p(\Omega)$ *to* $L^q(\Omega)$ *defined by* $Iu=|u|^{p-2}u$. *If* $R(\mathscr{A}+\lambda I)=H^{-1,q}(\Omega)$, *then* $\mathscr{A}$ *is maximal monotone.*

Proof. — Let $\mathscr{B}: H_0^{1,p}(\Omega) \to H^{-1,q}(\Omega)$ be a (multivalued) monotone operator such that $\mathscr{A} \subseteqq \mathscr{B}$. The proof will be accomplished if we show that $\mathscr{B} \subseteqq \mathscr{A}$. Let $f \in \mathscr{B}u$. It is clear that

$$f+\lambda I u \in \mathscr{B}u+\lambda I u. \tag{2.13}$$

On the other hand, since $R(\mathscr{A}+\lambda I)=H^{-1,q}(\Omega)$ there exists $v \in H_0^{1,p}(\Omega)$ such that $f+\lambda I u \in \mathscr{A}v+\lambda I v$. Then the assumption $\mathscr{A} \subseteqq \mathscr{B}$ implies

$$f+\lambda I u \in \mathscr{B}v+\lambda I v. \tag{2.14}$$

By taking (2.13) and (2.14) into account, the strict monotonicity of the operator $\mathscr{B}+\lambda I$ yields $v=u$ *a. e.* on Ω. Thus, $f+\lambda I u \in \mathscr{A}u+\lambda I u$, or equivalently, $f \in \mathscr{A}u$, which concludes the proof of the lemma.

3. G-CONVERGENCE OF MONOTONE OPERATORS

In this section we introduce a notion of convergence in the class of multivalued functions $M_\Omega(\mathbf{R}^n)$ which permits a satisfactory analysis of the perturbations of Dirichlet problems of the form (2.5).

The convergence considered here is defined in terms of a general concept of set-convergence named Kuratowski convergence (*see* [16], Section 29) which can be formulated in abstract terms in an arbitrary topological space (X, τ) as follows.

DEFINITION 3.1. — Let (E_h) be a sequence of subsets of X. We define the *sequential lower limit* and the *sequential upper limit of* (E_h) by

$$(3.1)\quad K_{seq}(\tau)\text{-}\liminf_{h\to\infty} E_h=\{u\in X:\exists\, u_h\overset{\tau}{\to}u,\ \exists\, k\in\mathbf{N},\ \forall\, h\geqq k: u_h\in E_h\},$$

and

$$(3.2)\quad K_{seq}(\tau)\text{-}\limsup_{h\to\infty} E_h$$
$$=\{u\in X:\exists\,\sigma(h)\to+\infty,\exists\, u_h\overset{\tau}{\to}u,\forall\, h\in\mathbf{N}: u_h\in E_{\sigma(h)}\}.$$

Then, we say that the sequence (E_h) $K_{seq}(\tau)$-*converges to a set* E *in* X if

$$(3.3)\qquad K_{seq}(\tau)\text{-}\liminf_{h\to\infty} E_h=K_{seq}(\tau)\text{-}\limsup_{h\to\infty} E_h=E$$

and in this case we write $K_{seq}(\tau)\text{-}\lim_{h\to\infty} E_h=E$.

REMARK 3.2. — From the definitions above it follows immediately that

$$K_{seq}(\tau)\text{-}\liminf_{h\to\infty} E_h\subseteqq K_{seq}(\tau)\text{-}\limsup_{h\to\infty} E_h.$$

Therefore (E_h) $K_{seq}(\tau)$-converges to E if and only if

$$E\subseteqq K_{seq}(\tau)\text{-}\liminf_{h\to\infty} E_h\qquad\text{and}\qquad K_{seq}(\tau)\text{-}\limsup_{h\to\infty} E_h\subseteqq E.$$

REMARK 3.3. — It is easy to prove that

$$K_{seq}(\tau)\text{-}\limsup_{h\to\infty} E_h\subseteqq E$$

if and only if every subsequence $(E_{\sigma(h)})$ of (E_h) has a further subsequence $(E_{\sigma(\tau(h))})$ such that

$$K_{seq}(\tau)\text{-}\liminf_{h\to\infty} E_{\sigma(\tau(h))}\subseteqq E.$$

This notion of set-convergence has been particularized to obtain the graph-convergence of sequences of maximal monotone operators on reflexive Banach spaces (*see* [3], Definition 3.58), which is useful for handling convergence problems for the stationary and evolution equations associated to such operators.

To study perturbations of Dirichlet problems of the form (2.5) we introduce here a stronger notion of convergence.

We denote by w the weak topology on $H^{1,p}(\Omega)$. If σ_1 denotes the weak topology of $(L^q(\Omega))^n$ and σ_2 the topology on $(L^q(\Omega))^n$ induced by the pseudo-metric $d(g_1, g_2)=\|\operatorname{div} g_1 - \operatorname{div} g_2\|_{H^{-1,q}}$, we denote by σ the weakest topology on $(L^q(\Omega))^n$ which is stronger than σ_1 and σ_2. In other words, (g_h) converges to g in σ if and only if (g_h) converges to g weakly in $(L^q(\Omega))^n$ and $(-\operatorname{div} g_h)$ converges to $-\operatorname{div} g$ strongly in $H^{-1,q}(\Omega)$.

The connection between w and σ is explained by the following lemma, which will be frequently used in the sequel.

LEMMA 3.4. – *Let (u_h) be a sequence converging to u weakly in $H^{1,p}(\Omega)$, and let (g_h) be a sequence in $(L^q(\Omega))^n$ converging to g in the topology σ. Then*

$$\int_\Omega g_h Du_h \varphi \, dx \to \int_\Omega g \, Du \, \varphi \, dx$$

for every $\varphi \in C_0^\infty(\Omega)$.

Proof. – The lemma is a simple case of compensated compactness (*see* [19], [29]). It can be proved by observing that

$$\int_\Omega g_h Du_h \varphi \, dx = \langle -\operatorname{div} g_h, u_h \varphi \rangle - \int_\Omega g_h u_h D\varphi \, dx$$

for every $\varphi \in C_0^\infty(\Omega)$. ■

Having in mind the usual identification of a multivalued map with its graph we give the following definition.

DEFINITION 3.5. – We say that a sequence (a_h) in $M_\Omega(\mathbf{R}^n)$ *G-converges to* $a \in M_\Omega(\mathbf{R}^n)$ if

$$K_{seq}(w \times \sigma)\text{-}\limsup_{h \to \infty} A_h^0 \subseteqq A^0,$$

where A_h^0 and A^0 are the operators in $M(H_0^{1,p})$ associated to a_h and a by Definition 2.6 in the case $\varphi = 0$.

REMARK 3.6. – The condition $K_{seq}(w \times \sigma)\text{-}\limsup_{h \to \infty} A_h^0 \subseteqq A^0$ in the above definition determines uniquely the G-limit a, as we shall prove in Corollary 5.9.

REMARK 3.7. – Using Remarks 3.2 and 3.3 it is easy to prove that the G-convergence satisfies the following axioms:

(i) axiom of the constant sequences: if $a_h = a$ for every $h \in \mathbf{N}$, then (a_h) G-converges to a;

(ii) axiom of the subsequences: if (a_h) G-converges to a, and $(a_{\sigma(h)})$ is a subsequence of (a_h), then $(a_{\sigma(h)})$ G-converges to a;

In the sequel we enunciate some results regarding the G-convergence on the class $M_\Omega(\mathbf{R}^n)$ and make some comments connecting these results to our investigation on convergence of solutions to sequences of Dirichlet problems of type (2.5). We shall prove the following Theorem in Section 6.

THEOREM 3.8. – *Let* $\varphi \in H^{1,p}(\Omega)$, *let* (a_h) *be a sequence in* $M_\Omega(\mathbf{R}^n)$ *and let* $a \in M_\Omega(\mathbf{R}^n)$. *Then the following conditions are equivalent:*

(i) (a_h) G-*converges to* a,

(ii) $K_{seq}(w \times \sigma)\text{-}\limsup_{h \to \infty} A_h \subseteqq A$,

(ii) $K_{seq}(w \times \sigma)\text{-}\limsup_{h \to \infty} A_h^\varphi \subseteqq A^\varphi$,

where A_h, A *are the operators in* $M(H^{1,p})$ *associated to* a_h *and* a *by Definition* 2.6 *and* A_h^φ, A^φ *are the corresponding operators in* $M(H_\varphi^{1,p})$.

REMARK 3.9. – It follows immediately from the boundedness hypothesis (2.6) that the inclusion

$$K_{seq}(w \times \sigma)\text{-}\limsup_{h \to \infty} A_h \subseteqq A \tag{3.4}$$

is equivalent to the following condition: if $\sigma(h)$ is a sequence of integers, (f_h) is a sequence in $H^{-1,q}(\Omega)$, and (u_h) is a sequence of local solutions in $H^{1,p}(\Omega)$ of the equations

$$-\operatorname{div}(a_{\sigma(h)}(x, Du_h)) \ni f_h \quad \text{on } \Omega$$

with

$$\begin{gathered} \sigma(h) \to +\infty, \\ u_h \to u \quad \text{weakly in } H^{1,p}(\Omega), \\ f_h \to f \quad \text{strongly in } H^{-1,q}(\Omega), \end{gathered}$$

then u is a solution to the equation

$$-\operatorname{div} a(x, Du) \ni f \quad \text{on } \Omega,$$

and for every sequence (g_h) in $(L^q(\Omega))^n$, with

$$g_h(x) \in a_{\sigma(h)}(x, Du_h(x)) \quad a.e. \text{ in } \Omega \qquad \text{and} \qquad -\operatorname{div} g_h = f_h \quad \text{in } \Omega,$$

there exists a subsequence $(g_{\tau(h)})$ such that

$$g_{\tau(h)} \to g \quad \text{weakly in } (L^q(\Omega))^n$$

and

$$g(x) \in a(x, Du(x)) \quad a.e. \text{ in } \Omega.$$

REMARK 3.10. – The inclusion

$$K_{seq}(w \times \sigma)\text{-}\limsup_{h \to \infty} A_h^\varphi \subseteqq A^\varphi \tag{3.5}$$

is equivalent to the following condition: for every increasing sequence of integers $\tau(h)$, for every $f \in H^{-1,q}(\Omega)$, for every sequence (f_h) converging to f strongly in $H^{-1,q}(\Omega)$, for every sequence (u_h) of solutions of the equations

$$\begin{cases} -\operatorname{div}(a_{\tau(h)}(x, Du_h)) \ni f_h & \text{on } \Omega, \\ u_h \in H^{1,p}_\varphi(\Omega), \end{cases}$$

and for every sequence (g_h) in $(L^q(\Omega))^n$ with

$$g_h(x) \in a_{\tau(h)}(x, Du_h(x)) \quad a.e. \text{ in } \Omega \qquad \text{and} \qquad -\operatorname{div} g_h = f_h \quad \text{in } \Omega,$$

there exists an increasing sequence of integers $\sigma(h) \to +\infty$ such that

$$u_{\sigma(h)} \to u \quad \text{weakly in } H^{1,p}(\Omega)$$

and

$$g_{\sigma(h)} \to g \quad \text{weakly in } (L^q(\Omega))^n,$$

where u is a solution of the equation

$$\begin{cases} -\operatorname{div}(a(x, Du)) \ni f & \text{on } \Omega, \\ u \in H^{1,p}_\varphi(\Omega), \end{cases} \tag{3.6}$$

and

$$g(x) \in a(x, Du) \quad a.e. \text{ in } \Omega. \tag{3.7}$$

In fact, assume (3.5) and suppose that $\tau(h) f, f_h, u_h, g_h$ satisfy the above assumptions. By the coerciveness condition (2.7) the sequence (u_h) is bounded in $H^{1,p}(\Omega)$ and therefore (g_h) is bounded in $(L^q(\Omega))^n$ by the growth condition (2.6). Thus, there exists a subsequence $[u_{\sigma(h)}, g_{\sigma(h)}]$ of $[u_h, g_h]$ which converges to $[u, g]$ weakly in $H^{1,p}(\Omega) \times (L^q(\Omega))^n$. This implies that $(-\operatorname{div} g_h)$ converges to $-\operatorname{div} g$ weakly in $H^{-1,q}(\Omega)$, hence $f = -\operatorname{div} g$. Therefore $[u_{\sigma(h)}, g_{\sigma(h)}]$ converges to $[u, g]$ in the topology $w \times \sigma$ and the assumption (3.5) implies $g \in A^\varphi u$, hence (3.7). This yields that u is a solution of (3.6), being $f = -\operatorname{div} g$.

The converse implication is trivial.

The following result, which will be proved in Section 6, shows the relationship between our definition of G-convergence and that one considered by Ambrosetti and Sbordone in [1].

Let us denote by ρ the strong topology in $H^{-1,q}(\Omega)$.

THEOREM 3.11. – *Let $\varphi \in H^{1,p}(\Omega)$. Let (a_h) be a sequence in $M_\Omega(\mathbf{R}^n)$ which G-converges to $a \in M_\Omega(\mathbf{R}^n)$. Then*

(i) $K_{seq}(w \times \rho)\text{-}\lim_{h \to \infty} \mathscr{A}_h = \mathscr{A}$,

(ii) $K_{seq}(w \times \rho)\text{-}\lim_{h \to \infty} \mathscr{A}^\varphi_h = \mathscr{A}^\varphi$,

where $\mathscr{A}_h$, $\mathscr{A}$ are the operators in $\mathscr{M}(H^{1,p})$ associated to a_h and a by Definition 2.6 and $\mathscr{A}^\varphi_h$, $\mathscr{A}^\varphi$ are the corresponding operators in $\mathscr{M}(H^{1,p}_\varphi)$.

REMARK 3.12. − The condition

$$\text{(3.8)} \qquad \mathrm{K}_{\mathrm{seq}}(w \times \rho)\text{-}\lim_{h \to \infty} \mathscr{A}_h^{\varphi} = \mathscr{A}^{\varphi}$$

can be expressed in terms of convergence of solutions of differential equations. More precisely, (3.8) holds if and only if both the following conditions (*a*) and (*b*) are satisfied:

(*a*) if (f_h) converges to f strongly in $\mathrm{H}^{-1,q}(\Omega)$, (u_h) converges to u weakly in $\mathrm{H}^{1,p}(\Omega)$, and u_h satisfies the equation

$$\text{(3.9)} \qquad \begin{cases} -\operatorname{div}(a_h(x, \mathrm{D}u_h)) \ni f_h & \text{on } \Omega, \\ u_h \in \mathrm{H}_{\varphi}^{1,p}(\Omega), \end{cases}$$

for infinitely many $h \in \mathbf{N}$, then u is a solution to

$$\text{(3.10)} \qquad \begin{cases} -\operatorname{div}(a(x, \mathrm{D}u)) \ni f & \text{on } \Omega, \\ u \in \mathrm{H}_{\varphi}^{1,p}(\Omega); \end{cases}$$

(*b*) if $f \in \mathrm{H}^{-1,q}(\Omega)$ and u is a solution to (3.10), then there exist (f_h) coverging to f strongly in $\mathrm{H}^{-1,q}(\Omega)$ and (u_h) converging to u weakly in $\mathrm{H}^{1,p}(\Omega)$ such that u_h satisfies the equation (3.9) for every $h \in \mathbf{N}$.

REMARK 3.13. − Conditions (i) and (ii) in Theorem 3.11 do not imply that (a_h) G-converges to a. The reason lies in the fact that a is not uniquely determined by the associated operator $\mathscr{A}$ as the following example shows.

Assume $n=3$, and let $\varphi \in \mathrm{C}_0^{\infty}(\Omega)$. Let us define

$$a(x, \xi) = \xi$$

and

$$b(x, \xi) = \xi + \mathrm{D}\varphi(x) \times \xi,$$

where $\times$ denotes the external product in $\mathbf{R}^3$. It is easy to see that a and b belongs to the class $\mathrm{M}_{\Omega}(\mathbf{R}^3)$ with $p=2$, $m_1=m_2=0$, $c_1=(1+\max_{\Omega}|\mathrm{D}\varphi|^2)$, and $c_2=1$.

Since

$$\int_{\Omega} ((\mathrm{D}\varphi \times \mathrm{D}u), \mathrm{D}v)\, dx = 0 \quad \text{for every } u, v \in \mathrm{H}^{1,2}(\Omega),$$

it follows that

$$\text{(3.11)} \qquad \int_{\Omega} (a(x, \mathrm{D}u), \mathrm{D}v)\, dx = \int_{\Omega} (b(x, \mathrm{D}u), \mathrm{D}v)\, dx \quad \text{for every } u, v \in \mathrm{H}^{1,2}(\Omega).$$

This implies that the operators in $\mathscr{M}(\mathrm{H}^{1,2})$ associated to a and b according to Definition 2.6 coincide.

4. A COMPACTNESS THEOREM

The main purpose of this section is to prove the following compactness result for the G-convergence on the class of multivalued functions $M_\Omega(\mathbf{R}^n)$.

THEOREM 4.1. – *Let* (a_h) *be a sequence in* $M_\Omega(\mathbf{R}^n)$. *Then there exists a subsequence* $(a_{\sigma(h)})$ *of* (a_h) *which* G-*converges to a function a of the class* $M_\Omega(\mathbf{R}^n)$.

Without any difficulty Theorem 4.1 comes out from the next two theorems and from the definition of G-convergence.

THEOREM 4.2. – *Let* (a_h) *be a sequence in* $M_\Omega(\mathbf{R}^n)$, *and let* (A_h^0) *be the sequence of operators in* $M(H_0^{1,p})$ *associated to* (a_h) *by Definition* 2.6. *Then there exist a subsequence* $(A^0_{\sigma(h)})$ of (A_h^0) *and an operator* $B \in M(H_0^{1,p})$ *such that*

$$K_{seq}(w \times \sigma)\text{-}\lim_{h \to \infty} A^0_{\sigma(h)} = B.$$

Moreover $D(B) = H_0^{1,p}(\Omega)$.

THEOREM 4.3. – *Let* $B \in M(H_0^{1,p})$ *with* $D(B) \supseteqq C_0^\infty(\Omega)$. *Then there exists a unique function* $a \in M_\Omega(\mathbf{R}^n)$ *such that* $B \subseteqq A^0$, *where* A^0 *denotes the operator in* $M(H_0^{1,p})$ *associated to a by Definition* 2.6.

The proofs of these theorems are quite technical and will be divided in several steps. We devote this section to the proof of Theorem 4.2, whereas Theorem 4.3 will be proved in the next section.

The following proposition is the first step of the proof of Theorem 4.2.

PROPOSITION 4.4. – *Let* (B_h) *be a sequence of operators of the class* $M(H_0^{1,p})$. *Then there exists a subsequence* $(B_{\sigma(h)})$ *which* $K_{seq}(w \times \sigma)$-*converges to an operator* $B \in M(H_0^{1,p})$.

Proof. – On every separable reflexive Banach space X there exists a metric d such that for every sequence (x_h) in X the following conditions are equivalent:

(4.1) $$x_h \to x \quad \text{weakly in X};$$

(4.2) $$(x_h) \text{ is norm-bounded in X and } d(x_h, x) \to 0.$$

By τ_1 we denote the topology induced by a metric on $H_0^{1,p}(\Omega)$ which satisfies (4.1) and (4.2). By τ_2 we denote the topology on $(L^q(\Omega))^n$ induced by the metric

$$d_2(g_1, g_2) = d(g_1, g_2) + \|\operatorname{div} g_1 - \operatorname{div} g_1\|_{H^{-1,q}(\Omega)},$$

where d is a metric on $(L^q(\Omega))^n$ which satisfies (4.1) and (4.2).

Since $\tau_1 \times \tau_2$ has a countable base, by the Kuratowski compactness theorem (*see* [16], Section 29, Theorem VIII) there exists a subsequence of

(B_h), still denoted by (B_h), which $K_{seq}(\tau_1 \times \tau_2)$-converges to a set $B \subseteq H_0^{1,p}(\Omega) \times (L^q(\Omega))^n$.

By Remark 3.2, to prove that (B_h) $K_{seq}(w \times \sigma)$-converges to B, it is enough to show that

$$K_{seq}(w \times \sigma)\text{-}\limsup_{h \to \infty} B_h \subseteq B, \tag{4.3}$$

and

$$B \subseteq K_{seq}(w \times \sigma)\text{-}\liminf_{h \to \infty} B_h. \tag{4.4}$$

Let us verify (4.3). Let $[u, g] \in K_{seq}(w \times \sigma)\text{-}\limsup_{h \to \infty} B_h$. Then, there exist $\sigma(h) \to +\infty$ and $[u_h, g_h]$ converging to $[u, g]$ in the topology $w \times \sigma$ such that $[u_h, g_h] \in B_{\sigma(h)}$ for every $h \in \mathbf{N}$. By (4.1) and (4.2) we get immediately that $[u_h, g_h]$ converges to $[u, g]$ in $\tau_1 \times \tau_2$ and we conclude that $[u, g] \in B$.

Let us prove (4.4). Let $[u, g] \in B$. Then there exists a sequence $[u_h, g_h]$ which converges to $[u, g]$ in $\tau_1 \times \tau_2$ such that $[u_h, g_h] \in B_h$ for h large enough. Since $(\operatorname{div} g_h)$ is bounded in $H^{-1,q}(\Omega)$, condition (2.7) implies that (u_h) is bounded in $H_0^{1,p}(\Omega)$, hence (g_h) is bounded in $(L^q(\Omega))^n$ by (2.6). Then the equivalence between (4.1) and (4.2) yields that (u_h) converges to u weakly in $H_0^{1,p}(\Omega)$ and (g_h) converges to g weakly in $(L^q(\Omega))^n$. Since $(-\operatorname{div} g_h)$ converges to $-\operatorname{div} g$ strongly in $H^{-1,q}(\Omega)$, we conclude that $[u_h, g_h]$ converges to $[u, g]$ in the topology $w \times \sigma$, which implies (4.4).

Finally, let us prove that the operator B belongs to the class $M(H_0^{1,p})$. We verify here only condition (i) of Definition 2.4. The boundedness and coerciveness conditions (2.6) and (2.7) can be proved in the same way. Let us fix $u^i \in H_0^{1,p}(\Omega)$ and $g^i \in B u^i$, $i=1,2$. By (4.4) there exists a sequence $[u_h^i, g_h^i]$ converging to $[u^i, g^i]$ in the topology $w \times \sigma$ in the topology $w \times \sigma$ such that $[u_h^i, g_h^i] \in B_h$ for h large enough. Since $B_h \in M(H_0^{1,p})$, we have

$$\int_\Omega (Du_h^1 - Du_h^2, g_h^1 - g_h^2)\,\varphi\,dx \geqq 0$$

for every $\varphi \in C_0^\infty(\Omega)$, $\varphi \geqq 0$ on Ω. By Lemma 3.4 it follows that

$$\int_\Omega (Du^1 - Du^2, g^1 - g^2)\,\varphi\,dx \geqq 0$$

for every $\varphi \in C_0^\infty(\Omega)$, $\varphi \geqq 0$ on Ω. This implies that

$$(D u^1 - D u^2, g^1 - g^2) \geqq 0 \quad a.e. \text{ on } \Omega,$$

hence B satisfies condition (i) of Definition 2.4. ■

The second step to achieve the proof of Theorem 4.2 is based on the next proposition.

PROPOSITION 4.5. — *Let* (B_h) *be a sequence of operators in* $M(H_0^{1,p})$ *and* $(\mathcal{B}_h)$ *be the corresponding sequence in* $\mathcal{M}(H_0^{1,p})$ *according to* (2.8). *Assume that*

$$B = K_{seq}(w \times \sigma)\text{-}\lim_{h\to\infty} B_h.$$

Then

$$\mathcal{B} = K_{seq}(w \times \rho)\text{-}\lim_{h\to\infty} \mathcal{B}_h,$$

where $\mathcal{B}$ *is the operator of the class* $\mathcal{M}(H_0^{1,p})$ *associated to* $B \in M(H_0^{1,p})$ *according to* (2.8) *and* ρ *denotes the strong topology of* $H^{-1,q}(\Omega)$.

Proof. — The inclusion $\mathcal{B} \subseteq K_{seq}(w \times \rho)\text{-}\liminf_{h\to\infty} \mathcal{B}_h$ is trivial. To prove the inclusion $K_{seq}(w \times \rho)\text{-}\limsup_{h\to\infty} \mathcal{B}_h \subseteq \mathcal{B}$, let us fix $[u, f] \in K_{seq}(w \times \rho)\text{-}\limsup_{h\to\infty} \mathcal{B}_h$. By (3.2) there exist $\sigma(h) \to +\infty$, and a sequence $[u_h, f_h]$ converging to $[u, f]$ in $w \times \rho$ such that $[u_h, f_h] \in \mathcal{B}_{\sigma(h)}$ for every $h \in \mathbf{N}$. By Definition 2.6 this implies that there exists $g_h \in B_{\sigma(h)} u_h$ such that $-\operatorname{div} g_h = f_h$. By (2.6) we have

$$\int_\Omega |g_h|^q \, dx \leqq c \left[1 + \int_\Omega |Du_h|^p \, dx \right]$$

for a suitable constant c, which implies that the sequence (g_h) is bounded in $(L^q(\Omega))^n$. Thus there exists a subsequence $(g_{\tau(h)})$ converging weakly in $(L^q(\Omega))^n$ to a function g, which yields that $(-\operatorname{div} g_{\tau(h)})$ converges to $-\operatorname{div} g$ weakly in $H^{-1,q}(\Omega)$. Since, by assumption, (f_h) converges to f strongly in $H^{-1,q}(\Omega)$, we conclude that $f = -\operatorname{div} g$. Therefore, $[u_{\tau(h)}, g_{\tau(h)}]$ converges to $[u, g]$ in the topology $w \times \sigma$ and $[u_{\tau(h)}, g_{\tau(h)}] \in B_{\sigma(\tau(h))}$. Thus $[u, g] \in B$ and $[u, f] \in \mathcal{B}$. ■

We are now able to prove Theorem 4.2.

PROOF OF THEOREM 4.2. — By Proposition 4.4 there exist a subsequence $(A^0_{\sigma(h)})$ of (A^0_h) and an operator B belonging to $M(H_0^{1,p})$ such that

$$K_{seq}(w \times \sigma)\text{-}\lim_{h\to\infty} A^0_{\sigma(h)} = B. \tag{4.5}$$

Let us prove that $D(B) = H_0^{1,p}(\Omega)$. Since the K-convergence is stable with respect to continuous perturbations, Proposition 4.5 together with (4.5) implies that for every $\lambda \geqq 0$, we have

$$K_{seq}(w \times \rho)\text{-}\lim_{h\to\infty} (\mathcal{A}^0_{\sigma(h)} + \lambda I) = \mathcal{B} + \lambda I, \tag{4.6}$$

where $\mathcal{B}$ is the operator in $\mathcal{M}(H_0^{1,p})$ associated to B according to (2.8). Let us prove that $R(\mathcal{B} + \lambda I) = H^{-1,q}(\Omega)$. Let $f \in H^{-1,q}(\Omega)$. By

Theorem 2.7 (ii), for every $h \in \mathbf{N}$ there exists $u_h \in \mathrm{H}_0^{1,p}(\Omega)$ such that

$$\mathscr{A}^0_{\sigma(h)} u_h + \lambda \mathrm{I} u_h \ni f.$$

By (2.7) the sequence (u_h) is bounded in $\mathrm{H}_0^{1,p}(\Omega)$, thus it contains a subsequence which converges to a function u weakly in $\mathrm{H}_0^{1,p}(\Omega)$. By (4.6) we have

$$\mathscr{B} u + \lambda \mathrm{I} u \ni f,$$

which gives $\mathrm{R}(\mathscr{B} + \lambda \mathrm{I}) = \mathrm{H}^{-1,q}(\Omega)$.

By Lemma 2.9 the operator $\mathscr{B}$, hence $\mathscr{B}^{-1}$, is maximal monotone. By (2.6) the operator $\mathscr{B}^{-1}$ is coercive on $\mathrm{H}^{-1,q}(\Omega)$. Therefore Theorem 1.10 implies that $\mathrm{R}(\mathscr{B}^{-1}) = \mathrm{H}_0^{1,p}(\Omega)$, which is equivalent to $\mathrm{D}(\mathscr{B}) = \mathrm{H}_0^{1,p}(\Omega)$. This yields $\mathrm{D}(\mathrm{B}) = \mathrm{H}_0^{1,p}(\Omega)$ and concludes the proof of the theorem. ■

5. A REPRESENTATION THEOREM

The main goal of this section is the proof of the following theorem, which contains Theorem 4.3 of Section 4.

THEOREM 5.1. — *Let* $\mathrm{B} \in \mathrm{M}(\mathrm{H}^{1,p})$ *with* $\mathrm{D}(\mathrm{B}) \supseteq \mathrm{C}_0^\infty(\Omega)$. *Then there exists a unique multivalued function* $a \in \mathrm{M}_\Omega(\mathbf{R}^n)$ *such that* $\mathrm{B} \subseteq \mathrm{A}$, *where* A *denotes the operator in* $\mathrm{M}(\mathrm{H}^{1,p})$ *associated to a by Definition* 2.6.

The following representation theorem for maximal monotone operators in the class $\mathscr{M}(\mathrm{H}_0^{1,p})$ is an easy consequence of Theorem 5.1 and Remark 2.5.

THEOREM 5.2. — *Any maximal monotone operator in* $\mathscr{M}(\mathrm{H}_0^{1,p})$ *is associated to a function* $a \in \mathrm{M}_\Omega(\mathbf{R}^n)$ *according to Definition* 2.6.

Before starting with the proof of Theorem 5.1 we shall introduce some notions and results related to measurable multivalued functions.

By $\mathscr{F}$ we denote the family of all measurable multivalued functions $\mathrm{F}: \Omega \to \mathbf{R}^n \times \mathbf{R}^n$ with non-empty closed values, and for every $\mathrm{F} \in \mathscr{F}$ we indicate by $\mathscr{S}_\mathrm{F}^{p,q}$ the set of all $(\mathrm{L}^p(\Omega))^n \times (\mathrm{L}^q(\Omega))^n$-selections of F, *i.e.*

$$\mathscr{S}_\mathrm{F}^{p,q} = \{ f \in (\mathrm{L}^p(\Omega))^n \times (\mathrm{L}^q(\Omega))^n : f(x) \in \mathrm{F}x \ a.e. \text{ on } \Omega \}.$$

Then the following results hold (*see*, for instance, [15], Lemma 1.1 and Corollary 1.2).

LEMMA 5.3 (Castaign representation). — *Let* $\mathrm{F} \in \mathscr{F}$. *If* $\mathscr{S}_\mathrm{F}^{p,q}$ *is non-empty, then there exists a sequence of functions* (f_n) *belonging to* $\mathscr{S}_\mathrm{F}^{p,q}$ *such that* $\mathrm{F}x = \mathrm{cl}\{ f_h(x) : h \in \mathbf{N} \}$ *for all* $x \in \Omega$.

LEMMA 5.4. — *Let* $\mathrm{F}_1, \mathrm{F}_2 \in \mathscr{F}$. If $\mathscr{S}_{\mathrm{F}_1}^{p,q} = \mathscr{S}_{\mathrm{F}_2}^{p,q} \neq \emptyset$, *then* $\mathrm{F}_1 x = \mathrm{F}_2 x$ *a.e. on* Ω.

Let M be a set of single-valued measurable functions $f:\Omega\to\mathbf{R}^n\times\mathbf{R}^n$. We call M *decomposable* [with respect to $\mathscr{L}(\Omega)$], if $f_1, f_2\in M$ and $U\in\mathscr{L}(\Omega)$ imply $1_U f_1 + 1_{\Omega\setminus U} f_2\in M$, where 1_U and $1_{\Omega\setminus U}$ indicate the characteristic functions of U and of $\Omega\setminus U$, respectively. The following theorem gives a characterization of the closed decomposable subsets of $(L^p(\Omega))^n\times(L^q(\Omega))^n$ (for the proof *see* [15], Theorem 3.1).

THEOREM 5.5. – *Let* M *be a non-empty closed subset of* $(L^p(\Omega))^n\times(L^q(\Omega))^n$. *Then* M *is decomposable if and only if there exists* $F\in\mathscr{F}$ *such that* $M=\mathscr{S}_F^{p,q}$.

PROOF OF THEOREM 5.1. – Let E be the subset of $(L^p(\Omega))^n\times(L^q(\Omega))^n$ defined by

$$E=\{[Du, g]\in(L^p(\Omega))^n\times(L^q(\Omega))^n : u\in H^{1,p}(\Omega),\ g\in B u\}. \tag{5.1}$$

Then, E is non-empty and satisfies the following monotonicity condition:

(5.2) if $[\varphi_1, g_1], [\varphi_2, g_2]\in E$, then $(\varphi_1-\varphi_2, g_1-g_2)\geqq 0$ *a.e.* on Ω.

Moreover for every $[\varphi, g]\in E$ we have

$$|g|^q\leqq m_1+c_1(\varphi, g) \quad a.e. \text{ on } \Omega, \tag{5.3}$$

$$|\varphi|^p\leqq m_2+c_2(\varphi, g) \quad a.e. \text{ on } \Omega. \tag{5.4}$$

Let dec E be the smallest decomposable set containing E. It is easy to prove that $[\varphi, g]\in\operatorname{dec}E$ if and only if there exists a finite Borel partition $(\Omega_i)_{i\in I}$ of Ω and a finite family $([\varphi_i, g_i])_{i\in I}$ of elements of E such that $[\varphi, g]=[\varphi_i, g_i]$ *a.e.* on Ω_i. Therefore, dec E is non-empty and (5.2), (5.3), (5.4) hold with E replaced by dec E.

Besides dec E, let us consider also the set

$$\tilde{E}=\operatorname{cl}_{s\times w}(\operatorname{dec}E), \tag{5.5}$$

defined as the closure of dec E in $(L^p(\Omega))^n\times(L^q(\Omega))^n$, with $(L^p(\Omega))^n$ endowed with its strong topology and $(L^q(\Omega))^n$ endowed with its weak topology. The next proposition, whose proof will be given later, summarizes the main properties of $\tilde{E}$.

PROPOSITION 5.6. – *Let* $\tilde{E}$ *be the set defined by* (5.5). *Then the following properties hold:*

(*a*) *for every* $[\varphi, g]\in\tilde{E}$ *there exists a sequence* $[\varphi_h, g_h]\in\operatorname{dec}E$ *such that* (φ_h) *converges to* φ *strongly in* $(L^p(\Omega))^n$ *and* (g_h) *converges to* g *weakly in* $(L^q(\Omega))^n$;

(*b*) $\tilde{E}$ *is decomposable and* (5.2), (5.3), (5.4) *hold with* E *replaced by* $\tilde{E}$;

(*c*) $\tilde{E}$ *is maximal monotone.*

PROOF OF THEOREM 5.1 (Continuation). – Since $\tilde{E}$ is a non-empty, closed, and decomposable subset of $(L^p(\Omega))^n\times(L^q(\Omega))^n$, by Theorem 5.5

there exists a measurable multivalued function $F : \Omega \to \mathbf{R}^n \times \mathbf{R}^n$ with non-empty closed values such that

(5.6) $\tilde{E} = \{ [\varphi, g] \in (L^p(\Omega))^n \times (L^q(\Omega))^n : [\varphi(x), g(x)] \in F x \text{ for } a.e.\, x \in \Omega \}.$

Let us define the multivalued function $a : \Omega \times \mathbf{R}^n \to \mathbf{R}^n$ by

$$a(x, \xi) = \{ \eta \in \mathbf{R}^n : [\xi, \eta] \in F x \}. \tag{5.7}$$

We shall prove in Lemma 5.7 that a belongs to the class $M_\Omega(\mathbf{R}^n)$. By (5.1), (5.6), and (5.7) we have $B \subseteq A$, where A denotes the operator in $M(H^{1,p})$ associated to a by Definition 2.6. The uniqueness of a will be proved in Proposition 5.8. ■

PROOF OF PROPOSITION 5.6. — Let us start with (*a*). Let $[\varphi_0, g_0] \in \tilde{E}$, and let $\mathcal{U}_1$ be the ball in $(L^p(\Omega))^n$ with center φ_0 and radius 1. Since (5.3) holds for dec E, there exists a constant $R = R(c_1, m_1, \varphi_0)$ such that, if $[\varphi, g] \in \operatorname{dec} E$ and $\varphi \in \mathcal{U}_1$, then $g \in \mathcal{B}_R$, where $\mathcal{B}_R$ denotes the ball in $(L^q(\Omega))^n$ with center 0 and radius R. We may also assume that $g_0 \in \mathcal{B}_R$. Therefore,

$$\operatorname{dec} E \cap (\mathcal{U} \times (\mathcal{V} \cap \mathcal{B}_R)) = \operatorname{dec} E \cap (\mathcal{U} \times \mathcal{V}) \neq \varnothing \tag{5.8}$$

for every neighborhood $\mathcal{V}$ of g_0 in the weak topology of $(L^q(\Omega))^n$ and for every neighborhood $\mathcal{U}$ of φ_0 in the strong topology of $(L^p(\Omega))^n$ such that $\mathcal{U} \subseteq \mathcal{U}_1$.

Since the weak topology is metrizable on $\mathcal{B}_R$, there exists a countable base $(\mathcal{V}_h)$ for the neighborhood system of g_0 in $\mathcal{B}_R$ endowed with the weak topology of $(L^q(\Omega))^n$. We may also assume that $\mathcal{V}_{h+1} \subseteq \mathcal{V}_h$ for every $h \in \mathbf{N}$. Let us denote by $\mathcal{U}_h$ the ball in $(L^p(\Omega))^n$ with center φ_0 and radius $1/h$. By (5.8) the sets $\operatorname{dec} E \cap (\mathcal{U}_h \times \mathcal{V}_h)$ are non-empty, thus for every $h \in \mathbf{N}$ we may pick up $[\varphi_h, g_h] \in \operatorname{dec} E$ such that $\varphi_h \in \mathcal{U}_h$ and $g_h \in \mathcal{V}_h$. This yields that (φ_h) converges to φ_0 strongly in $(L^p(\Omega))^n$ and (g_h) converges to g_0 weakly in $(L^q(\Omega))^n$, concluding the proof of (*a*).

By applying (*a*), we obtain easily property (*b*) of $\tilde{E}$ from the analogous property of dec E.

Finally, let us prove (*c*). To this aim we apply Theorem 1.9 to $\tilde{E}$. We prove first that for every $\varphi \in (L^p(\Omega))^n$, the set $\tilde{E}(\varphi)$ is non-empty. In the case $\varphi \in (L^p(\Omega))^n$, φ piecewise constant and with compact support on Ω, the proof follows easily from the assumption $D(B) \supseteq C_0^\infty(\Omega)$ and the definition of dec E. The general case can be obtained by approximation of $\varphi \in (L^p(\Omega))^n$ in the strong topology of $(L^p(\Omega))^n$ with functions (φ_h) of the previous type. In fact, from above it follows that there exists $g_h \in (L^q(\Omega))^n$ such that $g_h \in \tilde{E}(\varphi_h)$. Then, the estimate (5.3) for $\tilde{E}$ [proved in (*b*)] implies that (g_h) is bounded in $(L^q(\Omega))^n$. By passing, if necessary, to a subsequence, (g_h) converges to function g in the weak topology of $(L^q(\Omega))^n$ and g lies in $\tilde{E}(\varphi)$; the first assumption of Theorem 1.9 is so guaranteed. It is clear that for every $\varphi \in (L^p(\Omega))^n$ the set $\tilde{E}(\varphi)$ is decomposable and weakly closed in $(L^q(\Omega))^n$. Let us prove that $\tilde{E}(\varphi)$ is convex. Fix $g_1, g_2 \in \tilde{E}(\varphi)$ and $t \in (0, 1)$. There exists a sequence (U_h) of subsets of Ω such that $1|_{U_h} \to t$ and $1|_{\Omega \setminus U_h} \to (1-t)$ in the weak* topology of $L^\infty(\Omega)$. Since $\tilde{E}(\varphi)$ is decomposable we have $1|_{U_h} g_1 + 1|_{\Omega \setminus U_h} g_2 \in \tilde{E}(\varphi)$. Since $\tilde{E}(\varphi)$ is weakly closed in $(L^q(\Omega))^n$, taking the limit as $h \to +\infty$, we obtain $tg_1 + (1-t) g_2 \in \tilde{E}(\varphi)$, which proves that $\tilde{E}(\varphi)$ is convex. Finally, let us prove that $\tilde{E}$ is upper semi-continuous from $(L^p(\Omega))^n$, with the strong topology, into $(L^q(\Omega))^n$, with the weak topology. Fix $\varphi \in (L^p(\Omega))^n$, and let $\mathscr{V}$ be an open neighborhood of $\tilde{E}(\varphi)$ in the weak topology of $(L^q(\Omega))^n$. We claim that for every sequence (φ_h) converging to φ strongly in $(L^p(\Omega))^n$ there exists $k \in \mathbf{N}$ such that $\tilde{E}(\varphi_h) \subseteq \mathscr{V}$ for every $h \geq k$. Assume the contrary. Then there exists a subsequence $(\varphi_{\sigma(h)})$ of (φ_h) and a sequence (g_h) such that $g_h \in \tilde{E}(\varphi_{\sigma(h)})$ and $g_h \notin \mathscr{V}$ for every $h \in \mathbf{N}$. By the estimate (5.3) for $\tilde{E}$ [proved in (*b*)] the sequence (g_h) is bounded in $(L^q(\Omega))^n$, thus there exists a subsequence, $(g_{\tau(h)})$ of (g_h) which converges weakly in $(L^q(\Omega))^n$ to a function g. Since $[\varphi_{\sigma(\tau(h))}, g_{\tau(h)}] \in \tilde{E}$ for every $h \in \mathbf{N}$ we have $g \in \tilde{E}(\varphi)$, hence $g \in \mathscr{V}$. But the last fact requires that $g_h \in \mathscr{V}$ for h large enough, which contradicts our assumption. This implies that $\tilde{E}$ is upper-semicontinuous and concludes the proof of (*c*). ■

LEMMA 5.7. – *The function a defined by* (5.7) *belongs to* $M_\Omega(\mathbf{R}^n)$.

Proof of Lemma 5.7. – The measurability of a follows immediately from the measurability of F and from Theorem 1.3. Moreover, the property (5.2) for $\tilde{E}$ and the Castaign representation of F (Lemma 5.3) imply that Fx is monotone for *a.e.* $x \in \Omega$. We come now to the maximal monotonicity of a. By (5.7) it is enough to show that the set M defined by

$$M = \{ x \in \Omega : \exists [\xi, \eta] \in \mathbf{R}^n \times \mathbf{R}^n : [\xi, \eta] \notin Fx \text{ and } (\xi - \xi', \eta - \eta') \geq 0, \forall [\xi', \eta'] \in Fx \}$$

has Lebesgue measure zero. To this aim let us write $M = \{ x \in \Omega : \Phi x \neq \varnothing \}$, where

$$\Phi x = \{ [\xi, \eta] \in \mathbf{R}^n \times \mathbf{R}^n : [\xi, \eta] \notin Fx \text{ and } (\xi - \xi', \eta - \eta') \geq 0, \forall [\xi', \eta'] \in Fx \}.$$

Since $F \in \mathscr{F}$ and $\tilde{E} = \mathscr{S}_F^{p,q} \neq \varnothing$, by Lemma 5.3 there exists a sequence $[\varphi_h, g_h] \in (L^p(\Omega))^n \times (L^q(\Omega))^n$ such that

$$\begin{aligned}\Phi x &= \{ [\xi, \eta] \in \mathbf{R}^n \times \mathbf{R}^n : [\xi, \eta] \notin Fx \text{ and } (\xi - \varphi_h(x), \eta - g_h(x)) \geq 0, \forall h \in \mathbf{N} \} \\ &= \bigcap_{h \in \mathbf{N}} \{ [\xi, \eta] \in \mathbf{R}^n \times \mathbf{R}^n : [\xi, \eta] \notin Fx \text{ and } (\xi - \varphi_h(x), \eta - g_h(x)) \geq 0 \}.\end{aligned}$$

Since φ_h, g_h are measurable and F is measurable, it follows easily that the graph of Φ belongs to $\mathscr{L}(\Omega) \otimes \mathscr{B}(\mathbf{R}^n) \otimes \mathscr{B}(\mathbf{R}^n)$, thus $M \in \mathscr{L}(\Omega)$ by the projection Theorem 1.2. By the Aumann-von Neumann Theorem 1.4

there exists a measurable selection $[\varphi_0, g_0]$ of Φ defined on M. Therefore, for every $x \in M$ we have

$$(5.9) \qquad [\varphi_0(x), g_0(x)] \notin F x,$$

and

$$(5.10) \qquad (\varphi_0(x)-\xi, g_0(x)-\eta) \geqq 0 \quad \text{for every } [\xi, \eta] \in F x.$$

If $|M|>0$, there exists a measurable subset M′ of M with $|M'|>0$ such that $[\varphi_0(x), g_0(x)]$ is bounded on M′. Given $[\varphi_*, g_*] \in \tilde{E}$ we define the functions

$$\bar{\varphi}(x)=\begin{cases} \varphi_0(x) & \text{if } x \in M', \\ \varphi_*(x) & \text{if } x \in \Omega \setminus M', \end{cases}$$

and

$$\bar{g}(x)=\begin{cases} g_0(x) & \text{if } x \in M', \\ g_*(x) & \text{if } x \in \Omega \setminus M'. \end{cases}$$

Then $[\bar{\varphi}, \bar{g}] \in (L^p(\Omega))^n \times (L^q(\Omega))^n$. By (5.10) and by property (5.2) of $\tilde{E}$ we have that

$$\int_\Omega (\bar{\varphi}-\varphi, \bar{g}-g)\,dx = \int_{M'} (\varphi_0-\varphi, g_0-g)\,dx + \int_{\Omega \setminus M'} (\varphi_*-\varphi, g_*-g)\,dx \geqq 0$$

for every $[\varphi, g] \in \tilde{E}$. Since $\tilde{E}$ is maximal monotone [Proposition 5.6 (*c*)], the above inequality yields $[\bar{\varphi}, \bar{g}] \in \tilde{E}$,or equivalently, $[\bar{\varphi}(x), \bar{g}(x)] \in F x$ *a.e.* on Ω. But this implies that $[\varphi_0(x), g_0(x)] \in F x$ for *a.e.* $x \in M'$, which contradicts (5.9) being $|M'|>0$. Therefore, we have to conclude that the set M has Lebesgue measure zero, which guarantees that $a(x, .)$ is maximal monotone for *a.e.* $x \in \Omega$. To conclude that $a \in M_\Omega(\mathbf{R}^n)$ it remains to verify that a satisfies (2.1) and (2.2), but this is an easy consequence of Lemma 5.3 and of properties (5.3) and (5.4) for $\tilde{E}$ [Proposition 5.6 (*b*)]. ■

The following proposition will be crucial in the proof of the localization property considered in the next section.

PROPOSITION 5.8. – *Let* $C \in M(H^{1,p})$ *with* $D(C) \supseteq \psi + C_0^\infty(\Omega)$ *for a given* $\psi \in H^{1,p}(\Omega)$. *Let a and b be two functions of the class* $M_\Omega(\mathbf{R}^n)$ *and let* A *and* B *be the corresponding operators of the class* $M(H^{1,p})$. *If* $C \subseteq A$ *and* $C \subseteq B$, *then* $a(x, \xi)=b(x, \xi)$ *for a.e.* $x \in \Omega$ *and for every* $\xi \in \mathbf{R}^n$.

Proof. – It is enough to prove the proposition when $\psi=0$, since the general case can be obtained easily by translation (Remark 2.8).

Let E be the subset of $(L^p(\Omega))^n \times (L^q(\Omega))^n$ defined as in (5.1) with **B** replaced by C and let

$$E_a=\{[\varphi, g] \in (L^p(\Omega))^n \times (L^q(\Omega))^n : g(x) \in a(x, \varphi(x)) \ a.e. \text{ on } \Omega\}.$$

It is clear that $C \subseteqq A$ implies $E \subseteqq E_a$. Since E_a is decomposable we have $\operatorname{dec} E \subseteqq E_a$. Since E_a is maximal monotone (*see* [9], Example 2.3.3), it is sequentially closed in $(L^p(\Omega))^n \times (L^q(\Omega))^n$ with $(L^p(\Omega))^n$ endowed with its strong topology and $(L^q(\Omega))^n$ endowed with its weak topology. Therefore, $\tilde{E} \subseteqq E_a$, hence $\tilde{E} = E_a$ by the maximal monotonicity of $\tilde{E}$ [Proposition 5.6 (*c*)].

Analogously, we obtain $\tilde{E} = E_b$, therefore Lemma 5.4 implies that $a(x, \xi) = b(x, \xi)$ for *a.e.* $x \in \Omega$ and for every $\xi \in \mathbf{R}^n$. ■

The following corollary proves the uniqueness of the G-limit.

COROLLARY 5.9. – *Let* $\varphi \in H^{1,p}(\Omega)$, *let* (a_h) *be a sequence of functions of the class* $M_\Omega(\mathbf{R}^n)$, *and let*

$$C = K_{\text{seq}}(w \times \sigma)\text{-}\limsup_{h \to \infty} A_h^\varphi,$$

where A_h^φ *are the operators in* $M(H_\varphi^{1,p})$ *associated to* a_h *by Definition* 2.6. *Let a and b be two functions of the class* $M_\Omega(\mathbf{R}^n)$ *and let* A^φ *and* B^φ *be the corresponding operators of the class* $M(H_\varphi^{1,p})$. *If* $C \subseteqq A^\varphi$ *and* $C \subseteqq B^\varphi$, *then* $a(x, \xi) = b(x, \xi)$ *for a.e.* $x \in \Omega$ *and for every* $\xi \in \mathbf{R}^n$.

Proof. – It is enough to prove the corollary when $\varphi = 0$, since the general case can be obtained easily by translation (Remark 2.8).

Assume that $C \subseteqq A^0$ and $C \subseteqq B^0$. Since $A^0 \in M(H_0^{1,p})$ we have immediately $C \in M(H_0^{1,p})$, and by Theorem 4.2 we get $D(C) = H_0^{1,p}(\Omega)$. The conclusion follows now from Proposition 5.8. ■

6. LOCALIZATION AND BOUNDARY CONDITIONS

In the first part of this section we prove the local character of the G-convergence in the class $M_\Omega(\mathbf{R}^n)$. In the second part we study the convergence of solutions to non-homogeneous Dirichlet problems of the form (2.5).

Let Ω' be an open subset of Ω. Besides the topologies w and σ on $H^{1,p}(\Omega)$ and $(L^q(\Omega))^n$ introduced in Section 3, we consider the topologies w' and σ' defined analogously on $H^{1,p}(\Omega')$ and $(L^q(\Omega'))^n$. For every $a \in M_\Omega(\mathbf{R}^n)$ we denote by a' the function of $M_{\Omega'}(\mathbf{R}^n)$ defined by

$$a' = a\big|_{\Omega' \times \mathbf{R}^n}. \tag{6.1}$$

Then the following localization property holds.

THEOREM 6.1. – *Let* (a_h) *be a sequence in* $M_\Omega(\mathbf{R}^n)$ *which* G-*converges to a in* $M_\Omega(\mathbf{R}^n)$. *Then* (a'_h) G-*converges to* a' *in* $M_{\Omega'}(\mathbf{R}^n)$.

This theorem is an easy consequence of the next result.

THEOREM 6.2. — *Let* (a_h) *be a sequence in* $M_\Omega(\mathbf{R}^n)$ *which G-converges to* $a \in M_\Omega(\mathbf{R}^n)$. *Then*

$$\text{(6.2)} \qquad K_{seq}(w' \times \sigma')\text{-}\limsup_{h \to \infty} A'_h \subseteq A',$$

where A'_h *and* A' *are the operators in* $M(H^{1,p}(\Omega'))$ *associated to* a'_h *and* a' *by Definition* 2.6.

Proof. — By Remark 3.3 it is enough to show that for every subsequence $(a_{\sigma(h)})$ of (a_h) there exists a further subsequence $(a_{\sigma(\tau(h))})$ such that

$$\text{(6.3)} \qquad K_{seq}(w' \times \sigma')\text{-}\liminf_{h \to \infty} A'_{\sigma(\tau(h))} \subseteq A'.$$

By the definition of G-convergence and by Theorem 4.2 for every subsequence $(a_{\sigma(h)})$ of (a_h) there exists a further subsequence $(a_{\sigma(\tau(h))})$ such that

$$\text{(6.4)} \qquad C = K_{seq}(w \times \sigma)\text{-}\lim_{h \to \infty} A^0_{\sigma(\tau(h))} \subseteq A^0,$$

where $A^0_{\sigma(\tau(h))}$ and A^0 are the operators of $M(H^{1,p}_0(\Omega))$ associated to $a_{\sigma(\tau(h))}$ and a by Definition 2.6. This implies that

$$\text{(6.5)} \qquad D(K_{seq}(w' \times \sigma')\text{-}\liminf_{h \to \infty} A'_{\sigma(\tau(h))}) \supseteq C^\infty_0(\Omega').$$

Indeed, let $u' \in C^\infty_0(\Omega')$ and let $u \in C^\infty_0(\Omega)$ such that $u|_{\Omega'} = u'$. Since $D(C) = H^{1,p}_0(\Omega)$ (Theorem 4.2), there exists $g \in (L^q(\Omega))^n$ such that $[u, g] \in C$. Thus, there exists a sequence $[u_h, g_h]$ converging to $[u, g]$ in the topology $w \times \sigma$ such that $[u_h, g_h] \in A^0_{\sigma(\tau(h))}$ for every $h \in \mathbf{N}$. It is clear that $[u_h|_{\Omega'}, g_h|_{\Omega'}]$ converges to $[u|_{\Omega'}, g|_{\Omega'}]$ in the topology $w' \times \sigma'$; therefore $[u', g|_{\Omega'}] \in K_{seq}(w' \times \sigma')\text{-}\liminf_{h \to \infty} A'_{\sigma(\tau(h))}$, which proves (6.5).

Proceeding as in proof of Proposition 4.4 we can also show that $(K_{seq}(w' \times \sigma')\text{-}\liminf_{h \to \infty} A'_{\sigma(\tau(h))}) \in M(H^{1,p}(\Omega'))$. Therefore, by Theorem 5.1 there exists $b' \in M_{\Omega'}(\mathbf{R}^n)$ such that

$$\text{(6.6)} \qquad K_{seq}(w' \times \sigma')\text{-}\liminf_{h \to \infty} A'_{\sigma(\tau(h))} \subseteq B',$$

where B' denotes the operator of $M(H^{1,p}(\Omega'))$ associated to b' by Definition 2.6. We define $C' = \{[u|_{\Omega'}, g|_{\Omega'}] : [u, g] \in C\}$. It is clear that $C' \in M(H^{1,p}(\Omega'))$ and $D(C') \supseteq C^\infty_0(\Omega')$, being $D(C) = H^{1,p}_0(\Omega)$. By (6.4) we have

$$\text{(6.7)} \qquad C' \subseteq A'.$$

Moreover, without any difficulty it can be shown that

$$\text{(6.8)} \qquad C' \subseteq K_{seq}(w' \times \sigma')\text{-}\liminf_{h \to \infty} A'_{\sigma(\tau(h))} \subseteq B'.$$

By taking (6.7) and (6.8) into account, Proposition 5.8 guarantees that $a'=b'$. Therefore $A'=B'$ and (6.6) implies (6.3). ■

The following corollary is an easy consequence of Theorem 6.2.

COROLLARY 6.3. – *Let* (a_h) *and* (b_h) *be sequences in* $M_\Omega(\mathbf{R}^n)$ *which G-converge to a and b, respectively. If* $a'_h=b'_h$, *then* $a'=b'$.

Let $(\Omega^i)_{i\in I}$ be a family of open subsets of Ω such that $|\Omega\setminus\bigcup_{i\in I}\Omega^i|=0$. For every $a\in M_\Omega(\mathbf{R}^n)$ we denote by a^i the restriction of a to $\Omega^i\times\mathbf{R}^n$. The next corollary follows immediately from the compactness Theorem 4.1 and Corollary 6.3.

COROLLARY 6.4. – *A sequence* (a_h) *in* $M_\Omega(\mathbf{R}^n)$ *G-converges to* $a\in M_\Omega(\mathbf{R}^n)$ *if and only if* (a_h^i) *G-converges to* a^i *in* $M_{\Omega^i}(\mathbf{R}^n)$ *for every* $i\in I$.

We now prove the results stated in Section 3 regarding the convergence of solutions to non-homogeneous Dirichlet problems.

Proof of Theorem 3.8. – (i) ⇒ (ii). It follows from Theorem 6.2 with $\Omega'=\Omega$.

(ii) ⇒ (iii). – Let $[u, g]\in K_{\text{seq}}(w\times\sigma)\text{-}\limsup_{h\to\infty} A_h^\varphi$. By (3.2) there exist a sequence of integers $\sigma(h)\to+\infty$, and a sequence $[u_h, g_h]$ converging to $[u, g]$ in the topology $w\times\sigma$ such that $[u_h, g_h]\in A^\varphi_{\sigma(h)}\subseteq A_{\sigma(h)}$ for every $h\in\mathbf{N}$, hence $[u, g]\in A$ by (ii). Since clearly $u-\varphi\in H_0^{1,p}(\Omega)$, we have $[u, g]\in A^\varphi$, which gives (iii).

(iii) ⇒ (i). The compactness Theorem 4.1 implies that for every subsequence $(a_{\sigma(h)})$ of (a_h) there exist a further subsequence $(a_{\sigma(\tau(h))})$ of $(a_{\sigma(h)})$ and a function $b\in M_\Omega(\mathbf{R}^n)$ such that $(a_{\sigma(\tau(h))})$ G-converges to b. Since (i) implies (iii), we get

$$K_{\text{seq}}(w\times\sigma)\text{-}\limsup_{h\to\infty} A^\varphi_{\sigma(\tau(h))}\subseteq B^\varphi, \tag{6.9}$$

where B^φ is the operator of $M(H_\varphi^{1,p})$ associated to b.

On the other hand, by assumpation we have

$$K_{\text{seq}}(w\times\sigma)\text{-}\limsup_{h\to\infty} A^\varphi_{\sigma(\tau(h))}\subseteq A^\varphi. \tag{6.10}$$

By Corollary 5.9 we deduce that $a(x, \xi)=b(x, \xi)$ for *a.e.* $x\in\Omega$ and for every $\xi\in\mathbf{R}^n$. The proof can now be completed by using the Urysohn axiom (Remark 3.7). ■

We conclude this section by giving the proof of Theorem 3.11.

Proof of Theorem 3.11. – Let us prove (ii). To this aim we show first that the G-convergence of the sequence (a_h) to the function a in $M_\Omega(\mathbf{R}^n)$ implies that

$$K_{\text{seq}}(w\times\rho)\text{-}\lim_{h\to\infty}\mathscr{A}_h^0=\mathscr{A}^0. \tag{6.11}$$

By the definition of G-convergence and by Theorem 4.2 for every subsequence $(a_{\sigma(h)})$ of (a_h) there exists a further subsequence $(a_{\sigma(\tau(h))})$ such that

$$B = K_{seq}(w \times \sigma)\text{-}\lim_{h\to\infty} A^0_{\sigma(\tau(h))} \subseteq A^0.$$

By Proposition 4.5 this implies that

$$\mathscr{B} = K_{seq}(w \times \rho)\text{-}\lim_{h\to\infty} \mathscr{A}^0_{\sigma(\tau(h))} \subseteq \mathscr{A}^0. \tag{6.12}$$

Since $R(\mathscr{A}^0_{\sigma(\tau(h))} + \lambda I) = H^{-1,q}(\Omega)$ for every $\lambda \geq 0$ [Theorem 2.7 (ii)], it follows that $R(\mathscr{B} + \lambda I) = H^{-1,q}(\Omega)$ (*see* the proof of Theorem 4.2), hence $\mathscr{B}$ is maximal monotone (Lemma 2.9). Therefore, by the monotonicity of $\mathscr{A}^0$ the inclusion (6.12) implies that

$$K_{seq}(w \times \rho)\text{-}\lim_{h\to\infty} \mathscr{A}^0_{\sigma(\tau h))} = \mathscr{A}^0$$

and (6.11) follows from the Urysohn property of the K-convergence.

To prove (ii) in the general case $\varphi \in H^{1,p}(\Omega)$, for every $A \in M(H^{1,p})$ we consider the operator A^0_φ of the class $M(H^{1,p}_0)$ defined by

$$A^0_\varphi v = A^\varphi(v + \varphi) \quad \text{for every } v \in H^{1,p}_0(\Omega),$$

and the operator $\mathscr{A}^0_\varphi$ of $\mathscr{M}(H^{1,p}_0)$ associated to A^0_φ by (2.8). By Theorem 3.8 the G-convergence of the sequence (a_h) to the function a in $M_\Omega(\mathbf{R}^n)$ can be expressed by

$$K_{seq}(w \times \sigma)\text{-}\limsup_{h\to\infty} A^\varphi_h \subseteq A^\varphi,$$

which implies that

$$K_{seq}(w \times \sigma)\text{-}\limsup_{h\to\infty} (A_h)^0_\varphi \subseteq A^0_\varphi. \tag{6.13}$$

Since $(A_h)^0_\varphi$, A^0_φ are operators of $M(H^{1,p}_0)$, the inclusion (6.13) implies, as already seen, that

$$K_{seq}(w \times \rho)\text{-}\lim_{h\to\infty} (\mathscr{A}_h)^\varphi_\varphi = \mathscr{A}^\varphi_\varphi,$$

which gives immediately (ii).

Proof of (i). By Theorem 3.8 the G-convergence of (a_h) to a in $M_\Omega(\mathbf{R}^n)$ implies that

$$K_{seq}(w \times \sigma)\text{-}\limsup_{h\to\infty} A_h \subseteq A.$$

Arguing as in the proof of Proposition 4.5 we obtain

$$K_{seq}(w \times \sigma)\text{-}\limsup_{h\to\infty} \mathscr{A}_h \subseteq \mathscr{A}.$$

By (ii) it follows that

$$\mathscr{A}^{\varphi} = \mathrm{K}_{\mathrm{seq}}(w \times \rho)\text{-}\lim_{h \to \infty} \mathscr{A}_h^{\varphi} \subseteqq \mathrm{K}_{\mathrm{seq}}(w \times \rho)\text{-}\liminf_{h \to \infty} \mathscr{A}_h$$

$$\subseteqq \mathrm{K}_{\mathrm{seq}}(w \times \rho)\text{-}\limsup_{h \to \infty} \mathscr{A}_h \subseteqq \mathscr{A}$$

for every $\varphi \in \mathrm{H}^{-1,q}(\Omega)$, which yields (i). ■

7. SOME G-CLOSED CLASSES OF OPERATORS

In this section we consider some subsets of $\mathrm{M}_{\Omega}(\mathbf{R}^n)$, which are closed under G-convergence. These classes are obtained by imposing to the operator a some additional conditions of uniform equicontinuity or strict monotonicity.

DEFINITION 7.1. – Given a non-negative function $m \in \mathrm{L}^1(\Omega)$ and two constants α and c, with $0 \leqq \alpha \leqq (p/2) \wedge (p-1)$ and $c > 0$, we denote by $\mathrm{U} = \mathrm{U}(\alpha, c, m)$ the class of all operators $a \in \mathrm{M}_{\Omega}(\mathbf{R}^n)$ such that

$$m(x) + (\eta_1, \xi_1) + (\eta_2, \xi_2) \geqq 0 \tag{7.1}$$

and

$$|\eta_1 - \eta_2| \leqq c\, \Phi^{(p-1-\alpha)/p} (\eta_1 - \eta_2, \xi_1 - \xi_2)^{\alpha/p} \tag{7.2}$$

for *a. e.* $x \in \Omega$, for every $\xi_1, \xi_2 \in \mathbf{R}^n$ and $\eta_1 \in a(x, \xi_1)$, $\eta_2 \in a(x, \xi_2)$, where $\Phi = \Phi(x, \xi_1, \xi_2, \eta_1, \eta_2)$ denotes the left hand side of (7.1).

DEFINITION 7.2. – Given a non-negative function $m \in \mathrm{L}^1(\Omega)$ and two constants β and c, with $p \vee 2 \leqq \beta < +\infty$ and $c > 0$, we denote by $\mathrm{S} = \mathrm{S}(\beta, c, m)$ the class of all operators $a \in \mathrm{M}_{\Omega}(\mathbf{R}^n)$ such that

$$m(x) + (\eta_1, \xi_1) + (\eta_2, \xi_2) \geqq 0$$

and

$$(\eta_1 - \eta_2, \xi_1 - \xi_2) \geqq c\, \Phi^{(p-\beta)/p} |\xi_1 - \xi_2|^{\beta} \tag{7.3}$$

for *a. e.* $x \in \Omega$, for every $\xi_1, \xi_2 \in \mathbf{R}^n$ and $\eta_1 \in a(x, \xi_1)$, $\eta_2 \in a(x, \xi_2)$, where $\Phi = \Phi(x, \xi_1, \xi_2, \eta_1, \eta_2)$ denotes the left hand side of (7.1).

REMARK 7.3. – Conditions (2.1) and (2.2) imply that there exists a non-negative function $m \in \mathrm{L}^1(\Omega)$ such that (7.1) holds for every $a \in \mathrm{M}_{\Omega}(\mathbf{R}^n)$.

Moreover, by (7.2) every function a of the class U is single-valued.

REMARK 7.4. – By using the estimates (2.1) and (2.2) it is easy to see that, if $0 \leqq \alpha' \leqq \alpha \leqq (p/2) \wedge (p-1)$, then $\mathrm{U}(\alpha, c, m) \subseteqq \mathrm{U}(\alpha', c', m')$ for suitable c' and m'. In the same way it can be proved that, if $p \vee 2 \leqq \beta \leqq \beta' < +\infty$, then $S(\beta, c, m) \subseteq S(\beta', c', m')$ for suitable c' and m'.

The model example of operator of the classes U and S is given by

$$a(x, \xi) = b(x) |\xi|^{p-2} \xi.$$

Indeed, if $0 < b_1 \leqq b(x) \leqq b_2 < +\infty$ for every $x \in \Omega$, then

$$a \in U\left(\frac{p}{2} \wedge (p-1), c', m'\right) \cap S(p \vee 2, c'', m'')$$

for suitable c', c'', m' and m''.

Before proving that the classes U and S are closed under G-convergence we compare them with some other classes of monotone operators which are not closed, but are defined in a simpler way.

DEFINITION 7.5. — Given a non-negative function $m \in L^p(\Omega)$ and two constants α and c, with $0 \leqq \alpha \leqq 1 \wedge (p-1)$ and $c > 0$, we denote by $U^* = U^*(\alpha, c, m)$ the class of all single-valued operators $a \in M_\Omega(\mathbf{R}^n)$ such that

$$(7.4) \qquad |a(x, \xi_1) - a(x, \xi_2)| \leqq c(m(x) + |\xi_1| + |\xi_2|)^{p-1-\alpha} |\xi_1 - \xi_2|^\alpha$$

for *a. e.* $x \in \Omega$ and for every $\xi_1, \xi_2 \in \mathbf{R}^n$.

DEFINITION 7.6. — Given a non-negative function $m \in L^p(\Omega)$ and two constants β and c, with $p \vee 2 \leqq \beta < +\infty$ and $c > 0$, we denote by $S^* = S^*(\beta, c, m)$ the class of all operators $a \in M_\Omega(\mathbf{R}^n)$ such that

$$(7.5) \qquad (\eta_1 - \eta_2, \xi_1 - \xi_2) \geqq c(m(x) + |\xi_1| + |\xi_2|)^{p-\beta} |\xi_1 - \xi_2|^\beta$$

for *a. e.* $x \in \Omega$, for every $\xi_1, \xi_2 \in \mathbf{R}^n$ and $\eta_1 \in a(x, \xi_1)$, $\eta_2 \in a(x, \xi_2)$.

REMARK 7.7. — From (2.3) we obtain that

$$(7.6) \qquad U(\alpha, c, m) \subseteq U^*\left(\frac{\alpha}{p-\alpha}, c', m'\right)$$

for suitable c' and m'. Conversely, given c', c'', m', and m'', from (2.4) it follows that

$$(7.7) \qquad U^*(\alpha, c', m') \cap S^*(\beta, c'', m'') \subseteq U\left(\frac{\alpha p}{\beta}, c, m\right)$$

for suitable c and m. Moreover, given c and m, we have

$$(7.8) \qquad S(\beta, c, m) \subseteq S^*(\beta, c', m')$$

$$(7.9) \qquad S^*(\beta, c, m) \subseteq S(\beta, c'', m'')$$

for suitable c', c'', m', and m''.

In particular, if $2 \leqq p < +\infty$, (7.6) and (7.7) imply

$$\mathrm{U}\left(\frac{p}{2}, c, m\right) \subseteq \mathrm{U}^*(1, c', m') \tag{7.10}$$

$$\mathrm{U}^*(1, c', m') \cap \mathrm{S}^*(p, c'', m'') \subseteq \mathrm{U}(1, c, m). \tag{7.11}$$

Finally, if $1 < p \leqq 2$, (7.6) and (7.7) yield

$$\mathrm{U}(p-1, c, m) \subseteq \mathrm{U}^*(p-1, c', m') \tag{7.12}$$

$$\mathrm{U}^*(p-1, c', m') \cap \mathrm{S}^*(2, c'', m'') \subseteq \mathrm{U}\left(\frac{(p-1)p}{2}, c, m\right). \tag{7.13}$$

The following lemma is crucial in the proof of the closedness of the classes U and S.

LEMMA 7.8. – *Let γ and δ be two non-negative constants with $\gamma + \delta \leqq 1$. Let ψ, ζ, θ be functions in $L^1(\Omega)$ and let (ψ_h), (ζ_h), (θ_h) be sequences in $L^1(\Omega)$ converging to ψ, ζ, θ in the weak sense of distributions. Suppose that $\zeta_h \geqq 0$, $\theta_h \geqq 0$, and*

$$|\psi_h| \leqq (\zeta_h)^\gamma (\theta_h)^\delta \quad \text{a.e. in } \Omega. \tag{7.14}$$

Then

$$|\psi| \leqq \zeta^\gamma \theta^\delta \quad \text{a.e. in } \Omega. \tag{7.15}$$

Proof. – Let $\varepsilon = 1 - \gamma - \delta$. By (7.14), for every $\varphi \in C_0^\infty(\Omega)$, $\varphi \geqq 0$ we have

$$\int_\Omega |\psi_h| \varphi\, dx \leqq \left(\int_\Omega \zeta_h \varphi\, dx\right)^\gamma \left(\int_\Omega \theta_h \varphi\, dx\right)^\delta \left(\int_\Omega \varphi\, dx\right)^\varepsilon. \tag{7.16}$$

Since $(\psi_h \varphi)$ converges to $\psi\varphi$ in the weak sense of distributions we obtain

$$\int_\Omega |\psi| \varphi\, dx \leqq \liminf_{h \to \infty} \int_\Omega |\psi_h| \varphi\, dx.$$

Therefore, by taking the limit in (7.16) as $h \to +\infty$ we get

$$\int_\Omega |\psi| \varphi\, dx \leqq \left(\int_\Omega \zeta\varphi\, dx\right)^\gamma \left(\int_\Omega \theta\varphi\, dx\right)^\delta \left(\int_\Omega \varphi\, dx\right)^\varepsilon \tag{7.17}$$

for every $\varphi \in C_0^\infty(\Omega)$, $\varphi \geqq 0$. By standard approximation argument we obtain (7.17) for every $\varphi \in L^\infty(\Omega)$, $\varphi \geqq 0$. In particular, we have

$$\int_{B_\rho(x)} |\psi| \varphi\, dx \leqq \left(\int_{B_\rho(x)} \zeta\varphi\, dx\right)^\gamma \left(\int_{B_\rho(x)} \theta\varphi\, dx\right)^\delta \left(\int_{B_\rho(x)} \varphi\, dx\right)^\varepsilon$$

for every $x \in \Omega$ and $\rho \geqq 0$ small enough and this implies (7.15) by the Lebesgue derivation theorem.

THEOREM 7.9. – *The classes U and S are closed under G-convergence.*

Proof. — Let us fix α, c, and m as in Definition 7.1. Let (a_h) be a sequence in $U(\alpha, c, m)$ which G-converges to a function $a \in M_\Omega(\mathbf{R}^n)$. We have to prove that $a \in U(\alpha, c, m)$. By hypothesis we have

$$K_{seq}(w \times \sigma)\text{-}\limsup_{h \to \infty} A_h^0 \subseteqq A^0,$$

where A_h^0 and A^0 are the operators of $M(H_0^{1,p})$ associated to a_h, a by Definition 2.6. By Theorem 4.2 there exists a subsequence of (a_h), still denoted by (a_h), such that

$$B = K_{seq}(w \times \sigma)\text{-}\lim_{h \to \infty} A_h^0 \subseteqq A^0. \tag{7.18}$$

As in the proof of Theorem 5.1 we introduce the set E defined by

$$E = \{ [Du, g] \in (L^p(\Omega))^n \times (L^q(\Omega))^n : g \in B u \}$$

and we denote by dec E the smallest decomposable subset of $(L^p(\Omega))^n \times (L^q(\Omega))^n$ containing E. Moreover, we consider the set

$$\tilde{E} = \mathrm{cl}_{s \times w}(\mathrm{dec}\, E),$$

defined as the closure of dec E in $(L^p(\Omega))^n \times (L^q(\Omega))^n$, with $(L^p(\Omega))^n$ endowed with its strong topology and $(L^q(\Omega))^n$ endowed with its weak topology. As in the proof of Proposition 5.8 it follows that

$$\tilde{E} = \{ [\varphi, g] \in (L^p(\Omega))^n \times (L^q(\Omega))^n : g(x) \in a(x, \varphi(x)) \ a.e. \text{ in } \Omega \}. \tag{7.19}$$

We are now in a position to prove (7.1) and (7.2). This will be done in three steps.

STEP 1. — If $[u^1, g^1], [u^2, g^2] \in B$, then

$$|g^1 - g^2| \leqq c\, \zeta^{(p-1-\alpha)/p} (g^1 - g^2, Du^1 - Du^2)^{\alpha/p} \tag{7.20}$$

a. e. on Ω, where

$$\zeta = m + (g^1, Du^1) + (g^2, Du^2) \geqq 0.$$

STEP 2. — If $[\varphi^1, g^1], [\varphi^2, g^2] \in \tilde{E}$, then

$$|g^1 - g^2| \leqq c\, \omega^{(p-1-\alpha)/p} (g^1 - g^2, \varphi^1 - \varphi^2)^{\alpha/p} \tag{7.21}$$

a. e. on Ω, where

$$\omega = m + (g^1, \varphi^1) + (g^2, \varphi^2) \geqq 0.$$

STEP 3. — The inequalities (7.1) and (7.2) hold for *a. e.* $x \in \Omega$, for every $\xi^1, \xi^2 \in \mathbf{R}^n$ and $\eta^1 \in a(x, \xi^1)$, $\eta^2 \in a(x, \xi^2)$.

Proof of Step 1. — Let $[u^i, g^i] \in B$, $i = 1,2$. By (7.18) there exists a sequence $[u_h^i, g_h^i]$ converging to $[u^i, g^i]$ in the topology $w \times \sigma$, such that $[u_h^i, g_h^i] \in A_h^0$ for every $h \in \mathbf{N}$. Since $a_h \in U(\alpha, c, m)$ we have

$$|g_h^1 - g_h^2| \leqq c\, \zeta_h^{(p-1-\alpha)/p} (g_h^1 - g_h^2, Du_h^1 - Du_h^2)^{\alpha/p},$$

where

$$\zeta_h = m + (g_h^1, Du_h^1) + (g_h^2, Du_h^2) \geqq 0.$$

Let us define

$$\psi_h = g_h^1 - g_h^2, \qquad \psi = g^1 - g^2,$$
$$\theta_h = (g_h^1 - g_h^2, Du_h^1 - Du_h^2), \qquad \theta = (g^1 - g^2, Du^1 - Du^2).$$

By Lemma 3.4 (ζ_h) converges to ζ and (θ_h) converges to θ weakly in the sense of distributions. Therefore $\zeta \geqq 0$ *a. e.* in Ω and Lemma 7.8 yields

$$|\psi| \leqq \zeta^{(p-1-\alpha)/p} \theta^{\alpha/p} \quad a. e. \text{ in } \Omega,$$

proving (7.20). ●

Proof of Step 2. – The result of Step 1 can be reformulated by saying that (7.21) holds for $[\varphi^i, g^i] \in E$. The characterization of dec E mentioned in the proof of Theorem 5.1 implies (7.21) for $[\varphi^i, g^i] \in \operatorname{dec} E$. Let us prove the same property for $\tilde{E}$. Let $[\varphi^i, g^i] \in \tilde{E}$, $i = 1,2$. By Proposition 5.6 (*a*) there exists a sequence $\varphi_h^i, g_h^i] \in \operatorname{dec} E$ such that (φ_h^i) converges to φ^i strongly in $(L^p(\Omega))^n$ and (g_h^i) converges to g^i weakly in $(L^q(\Omega))^n$. Since (7.21) holds on dec E, we have

$$|g_h^1 - g_h^2| \leqq c\, \omega_h^{(p-1-\alpha)/p} (g_h^1 - g_h^2, \varphi_h^1 - \varphi_h^2)^{\alpha/p},$$

where

$$\omega_h = m + (g_h^1, \varphi_h^1) + (g_h^2, \varphi_h^2) \geqq 0.$$

By applying Lemma 7.8 to

$$\psi_h = g_h^1 - g_h^2 \qquad \text{and} \qquad \theta_h = (g_h^1 - g_h^2, \varphi_h^1 - \varphi_h^2)$$

we obtain (7.21) for $[\varphi^1, g^1]$ and $[\varphi^2, g^2]$. Moreover $\omega \geqq 0$ *a. e.* in Ω, being $\omega_h \geqq 0$ *a. e.* in Ω for $h \in \mathbf{N}$. ●

Proof of Step 3. – Let us denote by M the set of all $x \in \Omega$ such that (7.2) is not satisfied for some $\xi^1, \xi^2, \eta^1, \eta^2$ with $\eta^1 \in a(x, \xi^1)$, $\eta^2 \in a(x, \xi^2)$. We have to prove that $|M| = 0$. To this aim we write $M = \{x \in \Omega : Gx \neq \varnothing\}$, where $G : \Omega \to (\mathbf{R}^n)^4$ is the multivalued function defined by

$$Gx = \{[\xi^1, \xi^2, \eta^1, \eta^2] : |\eta_1 - \eta_2| > c\, \Phi^{(p-1-\alpha)/p} (\eta_1 - \eta_2, \xi_1 - \xi_2)^{\alpha/p}, \eta^i \in a(x, \xi^i), i = 1,2\},$$

where $\Phi = m + (\eta^1, \xi^1) + (\eta^2, \xi^2)$. By Remark 2.3 the graph of G belongs to $\mathscr{L}(\Omega) \otimes \mathscr{B}(\mathbf{R}^n)^4$, thus $M \in \mathscr{L}(\Omega)$ by the projection Theorem 1.2. By the Aumann-von Neumann Theorem 1.4 there exists a measurable selection $[\varphi_0^1, \varphi_0^2, g_0^1, g_0^2]$ of G defined on M. Therefore, for every $x \in M$ we have

$$|g_0^1 - g_0^2| > c [m + (g_0^1, \varphi_0^1) + (g_0^2, \varphi_0^2)]^{(p-1-\alpha)/p} (g_0^1 - g_0^2, \varphi_0^1 - \varphi_0^2)^{\alpha/p} \tag{7.22}$$

and

$$g_0^i(x) \in a(x, \varphi_0^i(x)), \qquad i = 1,2. \tag{7.23}$$

If $|M|>0$, there exists a measurable subset M' of M with positive measure such that $[\varphi_0^1, \varphi_0^2, g_0^1, g_0^2]$ is bounded on M'. By Step (*a*) in the proof of Theorem 2.7 there exists $g_* \in (L^q(\Omega))^n$ such that

$$g_*(x) \in a(x, 0) \quad a.e. \text{ in } \Omega. \tag{7.24}$$

For $i=1,2$, we define

$$\varphi^i(x) = \begin{cases} \varphi_0^i(x) & \text{if } x \in M', \\ 0 & \text{if } x \in \Omega \setminus M', \end{cases} \tag{7.25}$$

$$g^i(x) = \begin{cases} g_0^i(x) & \text{if } x \in M', \\ g_*(x) & \text{if } x \in \Omega \setminus M'. \end{cases} \tag{7.26}$$

Then $[\varphi^i, g^i] \in (L^p(\Omega))^n \times (L^q(\Omega))^n$ and $g^i(x) \in a(x, \varphi^i(x))$ *a. e.* in Ω by (7.23) and (7.24).

Therefore $[\varphi^i, g^i] \in \tilde{E}$ by (7.19), hence

$$|g_0^1 - g_0^2| \leq c[m + (g_0^1, \varphi_0^1) + (g_0^2, \varphi_0^2)]^{(p-1-\alpha)/p} (g_0^1 - g_0^2, \varphi_0^1 - \varphi_0^2)^{\alpha/p} \quad a.e. \text{ in } M'$$

by Step 2. This contradicts (7.22) being $|M'|>0$. Therefore, we have to conclude that M has Lebesgue measure 0, which proves that (7.2) holds for *a. e.* $x \in \Omega$.

The proof of (7.1) is analogous, therefore the class $U(\alpha, c, m)$ is closed with respect to G-convergence. •

To prove that the class $S(\beta, c, m)$, $p \vee 2 \leq \beta < +\infty$, is closed, we note that (7.3) is equivalent to

$$|\xi_1 - \xi_2| \leq c\, \Phi^{(\beta-p)/\beta p} (\eta_1 - \eta_2, \xi_1 - \xi_2)^{1/\beta}$$

and the proof can be concluded as in the case $U(\alpha, c, m)$. ■

Theorem 7.9 and Remark 7.7 allow us to obtain some compactness results concerning the class U^* and S^*.

COROLLARY 7.10. – *Assume $p=2$. Given two non-negative functions $m', m'' \in L^2(\Omega)$ and two constans $c'>0$, $c''>0$, there exist two non-negative functions $\tilde{m}', \tilde{m}'' \in L^2(\Omega)$ and two constants $\tilde{c}'>0$, $\tilde{c}''>0$ with the following property: if*

$$a_h \in U^*(1, c', m') \cap S^*(2, c'', m'')$$

for every $h \in N$, then there exists a subsequence $(a_{\sigma(h)})$ of (a_h) which G-converges to a function

$$a \in U^*(1, \tilde{c}', \tilde{m}') \cap S^*(2, \tilde{c}'', \tilde{m}'').$$

The same result was obtained by different methods by F. Murat and L. Tartar *in* [20].

COROLLARY 7.11. – *Assume $1<p\leq 2$. Given two non-negative functions $m', m'' \in L^p(\Omega)$ and two constants $c'>0$, $c''>0$, there exist two non-negative* *functions* $\tilde{m}'$, $\tilde{m}'' \in L^p(\Omega)$ *and two constants* $\tilde{c}'>0$, $\tilde{c}''>0$ *with the following property: if*

$$a_h \in U^*(p-1, c', m') \cap S^*(2, c'', m'')$$

for every $h \in \mathbf{N}$, *then there exists a subsequence* $(a_{\sigma(h)})$ *of* (a_h) *which* G-*converges to a function*

$$a \in U^*\left(\frac{p-1}{3-p}, \tilde{c}', \tilde{m}') \cap S^*(2, \tilde{c}'', \tilde{m}''\right).$$

A similar result was obtained by N. Fusco and G. Moscariello in the case of the homogeneization (*see* [13], [14]).

Corollary 7.12. – *Assume* $2 \leqq p < +\infty$ *and* $0 \leqq \alpha \leqq 1$. *Given two non-negative functions* m', $m'' \in L^p(\Omega)$ *and two constants* $c'>0$, $c''>0$, *there exist two non-negative functions* $\tilde{m}'$, $\tilde{m}'' \in L^p(\Omega)$ *and two constants* $\tilde{c}'>0$, $\tilde{c}''>0$ *with the following property: if*

$$a_h \in U^*(\alpha, c', m') \cap S^*(p, c'', m'')$$

for every $h \in \mathbf{N}$, *then there exists a subsequence* $(a_{\sigma(h)})$ *of* (a_h) *which* G-*converges to a function*

$$a \in U^*\left(\frac{\alpha}{p-\alpha}, \tilde{c}', \tilde{m}'\right) \cap S^*(p, \tilde{c}'', \tilde{m}'').$$

Compare this result with those obtained by U. E. Raitum in [22]. We refer also to [13], [14] for the case $\alpha = 1$.

Definition 7.13. – By $L(c_1, c_2)$ we denote the class of all operators $a: \Omega \times \mathbf{R}^n \to \mathbf{R}^n$ of the form

$$a(x, \xi) = a(x)\xi \quad \text{for } a.e.\ x \in \Omega, \text{for every } \xi \in \mathbf{R}^n,$$

where $a(x) = (a_{ij}(x))$ is a $n \times n$-matrix of bounded measurable functions such that

$$|a(x)\xi|^2 \leqq c_1 (a(x)\xi, \xi), \tag{7.27}$$

$$|\xi|^2 \leqq c_2 (a(x)\xi, \xi) \tag{7.28}$$

for *a. e.* $x \in \Omega$ and for every $\xi \in \mathbf{R}^n$.

By $L_{sym}(c_1, c_2)$ we denote the class of all operators of $L(c_1, c_2)$ corresponding to a symmetric matrix $(a_{ij}(x))$.

Remark 7.14. – It is easy to see that $L(c_1, c_2)$ is the set of all operators $a \in M_\Omega(\mathbf{R}^n)$, with $p = 2$, such that for *a.e.* $x \in \Omega$ the multivalued function $a(x, .)$ is linear, *i.e.* its graph is a linear subspace of $\mathbf{R}^n \times \mathbf{R}^n$.

Theorem 7.15. – *The classes* $L(c_1, c_2)$ *and* $L_{sym}(c_1, c_2)$ *are closed under* G-*convergence.*

Proof. — We give a sketch of the proof only for $L(c_1, c_2)$, the case of $L_{\text{sym}}(c_1, c_2)$ being analogous. By arguing as in the proof of Theorem 7.9, for which we refer for the notation, the result will be achieved in three steps.

STEP 1. — B is a linear subspace of $H_0^{1,2}(\Omega) \times (L^2(\Omega))^n$.

STEP 2. — $\tilde{E}$ is a linear subspace of $(L^2(\Omega))^n \times (L^2(\Omega))^n$.

STEP 3. — For *a. e.* $x \in \Omega$, the multivalued function $a(x, .)$ is linear.
The proof of each step is completely analogous to the proof of the corresponding step in Theorem 7.9, and is therefore omitted. ■

The compactness under G-convergence of the class $L(c_1, c_2)$ was proved by different methods by F. Murat and L. Tartar in [18] and [27]. The symmetric case was studied earlier by S. Spagnolo and E. De Giorgi (*see* [24], [25], [26], [12]).

REFERENCES

[1] A. AMBROSETTI and C. SBORDONE, Γ-convergenza e G-convergenza per problemi non lineari di tipo ellittico, *Boll. Un. Mat. Ital.*, **13**, 1976, pp. 352-362.

[2] H. ATTOUCH, Introduction à l'homogénisation d'inéquations variationnelles, *Rend. Sem. Mat. Univ. Politec. Torino*, Vol. **40**, 1982, pp. 1-23.

[3] H. ATTOUCH, Variational convergence for functions and operators, Pitman, London, 1984.

[4] V. BARBU and Th. PRECUPANU, *Convexity and optimization in Banach spaces*, Editura Academiei, Bucuresti, 1978.

[5] A. BENSOUSSAN, J.-L. LIONS and G. PAPANICOLAOU, *Asymptotic analysis for periodic structures*, North Holland, Amsterdam, 1978.

[6] L. BOCCARDO and Th. GALLOUET, Homogenization with jumping nonlinearities, *Ann. Mat. Pura Appl.*, Vol. **138**, 1984, pp. 211-221.

[7] L. BOCCARDO and F. MURAT, Remarques sur l'homogénéisation de certains problèmes quasi linéaires, *Portugal. Math.*, Vol. **41**, 1982, pp. 535-562.

[8] L. BOCCARDO and F. MURAT, Homogénéisation de problèmes quasi linéaires, *Studio di problemi-limite della analisi funzionale* (Bressanone, 1981), 13-51, Pitagora editrice, Bologna, 1982.

[9] H. BREZIS, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, North-Holland, Amsterdam, 1973.

[10] F. E. BROWDER, Nonlinear operators and nonlinear equations of evolution in Banach spaces, *Proceedings of Symposia Pure Math.*, 18, American Mathematical Society, 1976.

[11] C. CASTAING and M. VALADIER, Convex analysis and measurable multifunctions, *Lec. Notes Math.*, **580**, Springer-Verlag, Berlin, 1977.

[12] E. DE GIORGI and S. SPAGNOLO, Sulla convergenza degli integrali dell'energia per operatori ellittici del secondo ordine, *Boll. Un. Mat. Ital.*, Vol. **8**, 1973, pp. 391-411.

[13] N. FUSCO and G. MOSCARIELLO, On the homogenization of quasilinear divergence structure operators, *Ann. Mat. Pura Appl.*, Vol. **146**, 1987, pp. 1-13.

[14] N. FUSCO and G. MOSCARIELLO, *Further results on the homogenization of quasilinear operators*, Ricerche Mat. 35, 1986, pp. 231-246.

[15] F. HIAI and H. UMEGAKI, Integrals, conditional expectations, and martingales of multivalued functions, *J. Multivariate Anal.*, **7**, 1977, pp. 149-182.
[16] K. KURATOWSKI, *Topology*, Academic Press, New York, 1968.
[17] M. MATZEU, Su un tipo di continuità dell'operatore subdifferenziale, *Boll. Un. Mat. Ital.*, Vol. **14-B**, 1977, pp. 480-490.
[18] F. MURAT, H-convergence, *Séminaire d'Analyse Fonctionnelle et Numérique*, Université d'Alger, 1977.
[19] F. MURAT, Compacité par compensation, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, Vol. **5**, 1978, pp. 489-507.
[20] F. MURAT and L. TARTAR, *Homogénéisation d'opérateurs monotones*, Manuscript, 1981.
[21] D. PASCALI and S. SBURLAN, *Nonlinear mappings of monotone type*, Editura Academiei, Bucuresti, 1978.
[22] U. E. RAITUM, On the G-Convergence of quasilinear elliptic operators with unbounded coefficients, *Soviet Math. Dokl.*, **24**, 1981, pp. 472-476.
[23] E. SANCHEZ-PALENCIA, Non homogeneous media and vibration theory, *Lec. Notes Phys.*, **127**, Springer-Verlag, Berlin, 1980.
[24] S. SPAGNOLO, Sul limite delle soluzioni di problemi di Cauchy relativi all'equazione del calore, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, Vol. **21**, 1967, pp. 657-699.
[25] S. SPAGNOLO, Sulla convergenza di soluzioni di equazioni paraboliche ed ellittiche, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, Vol. **22**, 1968, pp. 577-597.
[26] S. SPAGNOLO, Convergence in energy for elliptic operators, *Proc. Third Symp. Numer. Solut. Partial Diff. Equat.* (College Park, 1975), pp. 469-498, Academic Press, San Diego, 1976.
[27] L. TARTAR, *Cours Peccot*, Collège de France, Paris, 1977.
[28] L. TARTAR, Quelques remarques sur l'homogénéisation, *Proceedings of the Japan-France seminar* 1976 *"Functional analysis and numerical analysis"*, pp. 469-482, Japan Society for the Promotion of Science, 1978.
[29] L. TARTAR, Compensated compactness and applications to partial differential equations, Nonlinear analysis & mechanics, Heriot-Watt symposium, Vol. **IV**, *Res. Notes Math.*, Vol. **39**, pp. 136-211, Pitman, London, 1979.
[30] V. V. ZHIKOV, S. M. KOZLOV, O. A. OLEINIK and KHA T'EN NGOAN, Averaging and G-convergence of differential operators, *Russian Math. Surveys*, Vol. **34**, 1979, pp. 69-147.

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