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On removable singularities of p-harmonic maps

by

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ABSTRACT. — For the unit ball B_1 in $\mathbb{R}^n$ and a Riemannian manifold M we consider mappings $u: B_1 - \{0\} \to M$ of class

$$C^1(B_1 - \{0\}, M) \cap H^{1,p}(B_1, \mathbb{R}^k)$$

which are stationary points of the p-energy functional

$$\mathbb{E}_1(u) := \int_{B_1} |Du|^p \, dx$$

for some exponent $p \geqq 2$. We shall prove that the point singularity at the origin is removable provided the p-energy $\mathbb{E}_1(u)$ is sufficiently small. There are no *a priori* assumptions on the image of u in M.

Key words : p-harmonic maps, removable singularities, regularity theory, degenerate functionals.

RÉSUMÉ. — On considère la fonctionnelle d'énergie d'ordre p :

$$\mathbb{E}_1(u) := \int_{B_1} |Du|^p \, dx$$

où B_1 est la boule unité de $\mathbb{R}^n$, M est une variété riemannienne, et $u: B_1 - \{0\} \to M$ est de classe $C^1 \cap H^{1,p}$ avec $p \geqq 2$. On montre que si u est un point critique de $\mathbb{E}_1$, la singularité à l'origine disparaît dès que $\mathbb{E}_1(u)$ est assez petit, sans qu'il soit besoin de faire d'hypothèse sur l'image de u dans M.

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1. INTRODUCTION AND STATEMENT OF THE RESULT

In our paper we investigate the regularity problem of p-harmonic maps in higher dimensions. More precisely, we consider the following situation: the parameter domain is the unit ball B_1 in $\mathbb{R}^n$, $n \geqq 2$ (equipped with the flat metric). As target manifold M we have a Riemannian manifold of dimension $m \geqq 1$ which is isometrically embedded in some Euclidian space $\mathbb{R}^k$, $k \geqq m$. We are then interested in mappings $u: B_1 \to M$ of Sobolev class $H^{1,p}(B_1, M)$ being defined as the set of functions u from the linear Sobolev space $H^{1,p}(B_1, \mathbb{R}^k)$ such that $u(x) \in M$ a.e. on B_1. The *p-energy* of $u \in H^{1,p}(B_1, \mathbb{R}^k)$ is defined as

$$\mathbb{E}_1(u) := \int_{B_1} |Du|^p \, dx, \tag{1.1}$$

and u is said to be *weakly p-harmonic* if u is a weak solution of the Euler-Lagrange equations associated to the energy functional (1.1), *i.e.* u satisfies for all $\varphi \in C_0^1(B_1, \mathbb{R}^k)$:

$$\int_{B_1} |Du|^{p-2} (D_\alpha u . D_\alpha \varphi + \varphi . A(u)(D_\alpha u, D_\alpha u)) \, dx = 0, \tag{1.2}$$

where $A(q)(.,.)$ is the second fundamental form of M at q. For exponents $p > 2$ (1.2) is a nonlinear system in the first derivatives and the modulus of ellipticity degenerates at points where the first derivatives of u vanish. If in addition u is also a critical point of (1.1) with respect to compactly supported variations of the parameter domain we say that u is *p-stationary*.

The purpose of the present paper is to prove the following *removable singularity theorem* for p-harmonic maps.

THEOREM. — *Suppose* $u \in C^1(B_1 - \{0\}, M) \cap H^{1,p}(B_1, \mathbb{R}^k)$ *is p-harmonic and* $n \geqq 3$, $2 \leqq p \leqq n$. *If the p-energy* $\mathbb{E}_1(u)$ *of u does not exceed a certain constant* $\varepsilon > 0$ *depending only on n, p, k and the geometry of* M *then u belongs to* $C^{1,\gamma}(B_1, M)$ *for some* $\gamma \in]0, 1[$. *The Hölder exponent* γ *depends also on the absolute data n, k, p and* M *only and is independent of u.*

Remarks. — (i) For minimizers of the p-energy the above theorem is a special case of a more general partial regularity result, *see* [F 1], [F 2], [HL] and [Lu].

(ii) If $p = n$, then the conformal invariance of the n-energy implies that it suffices to assume $\mathbb{E}_1(u) = \int_{B_1} |Du|^n dx < \infty$ to prove our removable singularity theorem.

(iii) The smallness assumption $\mathbb{E}_1(u) \leqq \varepsilon$ is necessary for $2 \leqq p < n$. Indeed, Coron and Gulliver [CGu] proved that the map $u_* : B_1 \to \partial B_1$ defined by $u_*(x) := |x|^{-1} x$ is p-energy minimizing in the class

$$\mathcal{G} := \{ v \in H^{1,p}(B_1, \partial B_1) : v = \text{id on } \partial B_1 \}.$$

Therefore, u_* is a p-stationary map with finite p-energy and isolated singularity at the origin.

(iv) In the quadratic case $p=2$ of (stationary) harmonic mappings several theorems on removable singularities have been proved by various authors; we refer to [Gr], [Li 1], [Li 2], [Sa U], [Sch], [Ta 1], [Ta 2] for a detailed discussion.

(v) The example described in (iii) shows that even for minimizers a linear growth condition of the form

$$\limsup_{x \to 0} |x| \, |Du(x)| < \infty \tag{1.3}$$

is not sufficient to deduce everywhere regularity. As an application of our main theorem we prove in section 4, theorem 4.1, that the origin $x=0$ is not contained in the singular set of a p-harmonic mapping $u \in C^1(B_1 - \{0\}, M)$ provided u satisfies (1.3) as well as the small range condition $\operatorname{Im}(u) \subset \mathbb{B}$ for a regular geodesic ball $\mathbb{B} \subset M$. This result corresponds to the everywhere regularity theorems obtained in [F 1], [F 3] and [F 4] and is optimal as the example in section 4, remark (iii), shows.

(vi) With some minor modifications our main theorem extends to the general Riemannian case of p-harmonic mappings

$$u \in H^{1,p}(\Omega, M) \cap C^1(\Omega - \{x_0\}, M)$$

where Ω denotes an open region contained in some n-dimensional Riemannian manifold and x_0 is a given point in Ω. If

$$\liminf_{\nu \to \infty} r_\nu^{p-n} \int_{\mathbb{B}_{r_\nu}(x_0)} \| Du \|^p \, d\text{vol} = 0$$

holds for a sequence $\{ \mathbb{B}_{r_\nu}(x_0) \}$ of geodesic balls in Ω shrinking to x_0, then x_0 is a regular point of u. ■

In this paper we assume that M is a closed (complete) m-dimensional submanifold of $\mathbb{R}^k$ of class C^3. Since we do not assume that M is compact we additionally require a bound κ on the extrinsic curvature of M which can be expressed in the form

$$|\Pi_q^\perp (q - q')| \leqq \frac{\kappa}{2} |q - q'|^2 \qquad \text{for} \quad q, q' \in M, \tag{1.4}$$

where $\Pi_q^\perp \in \operatorname{Hom}(\mathbb{R}^k, \mathbb{R}^k)$ denotes the orthogonal projection onto the normal space $(\operatorname{Tan}_q M)^\perp$. Condition (1.4) implies that the norm of the second fundamental form A of M in $\mathbb{R}^k$ is bounded by κ and that M has a tubular neighborhood $M_{1/\varkappa}$ of distance $1/\kappa$ in $\mathbb{R}^k$. The nearest point projection π onto M is defined on $M_{1/\varkappa}$ and has Lipschitz constant $\frac{1}{1-\delta}$ on $M_{\delta/\varkappa}$ for $0<\delta<1$. Moreover, we need a bound κ' on the covariant derivative of A, namely

$$\|\nabla A\| \leqq \kappa'. \tag{1.5}$$

For a detailled discussion of the conditions (1.4) and (1.5) we refer to [DS], paragraph 1.

2. A POINTWISE ESTIMATE OF THE GRADIENT

In this section we want to prove the following result:

2.1. THEOREM. – *There exist constants* $\varepsilon_1>0$ *and* C_0 *depending only on* n, p *and the curvature bounds* κ, κ' *such that for any p-harmonic map* $u\in C^1(B_r, M)$ *satisfying the smallness assumption* $r^{p-n}\,\mathbb{E}_r(u)\leqq\varepsilon_1$ *we have*

$$\sup_{B_{r/2}} |Du|^p \leqq C_0\, r^{-n} \int_{B_r} |Du|^p\, dx. \quad \blacksquare \tag{2.1}$$

As a first step in the proof of theorem 2.1 we have the following estimate valid for weakly p-harmonic maps of class C^1 which are defined on an open domain Ω in $\mathbb{R}^n$.

2.2. LEMMA. – *Suppose* $u\in C^1(\Omega, M)$ *is a weakly p-harmonic map. Then* $V[Du] := |Du|^{(p-2)/2}\, Du$ *has weak derivatives which lie in* $L^2_{loc}(\Omega, \mathbb{R}^{nk})$, *and for all* $B_{2r}\subset\subset\Omega$ *we have*

$$\int_{B_r} |DV[Du]|^2\, dx \leqq C_1(n, p, \kappa, \kappa')\, r^{-2}\left(1+\|Du\|^2_{L^\infty(B_{2r})}\right) \int_{B_{2r}} |Du|^p\, dx. \tag{2.2}$$

Proof. – Let $\Delta_{h,i}\, u(x) := \frac{1}{h}[u(x+he_i)-u(x)]$ the difference quotient in the ith direction. Then, for a given $\varphi\in C_0^\infty(B_{3/2\,r})$ with $\varphi\geqq 0$, $\varphi=1$ on

B_r and $|\nabla \varphi| \leqq \frac{4}{r}$ we have

$$(2.3)\quad 0=\int_{B_{2r}} [\Delta_{h,i} F[Du].D(\varphi^2 \Delta_{h,i} u) + \Delta_{h,i}(|Du|^{p-2} A(u)(D_\alpha u, D_\alpha u)).\varphi^2 \Delta_{h,i} u]\,dx$$
$$\geqq \int_{B_{2r}} \varphi^2 \Delta_{h,i} F[Du].D\Delta_{h,i} u\,dx - 2\sup|\nabla\varphi| \int_{B_{2r}} \varphi|\Delta_{h,i} u|.|\Delta_{h,i} F[Du]|\,dx - I$$

where

$$I:=\left|\int_{B_{2r}} \Delta_{h,i}(|Du|^{p-2} A(u)(D_\alpha u, D_\alpha u)).\varphi^2 \Delta_{h,i} u\,dx\right|.$$

Similary to [U], lemma 3.1, we further derive

$$(2.4)\quad \int_{B_{3/2r}} \varphi^2 |\Delta_{h,i} Du|^2 \int_0^1 |Du_\lambda|^{p-2}\,d\lambda\,dx \leqq C_2\left[\|\nabla\varphi\|^2_{L^\infty} \int_{B_{3/2r}} |\Delta_{h,i} u|^2 \int_0^1 |Du_\lambda|^{p-2}\,d\lambda\,dx + I\right]$$

where we have abbreviated $u_\lambda := u+\lambda h \Delta_{h,i} u$. Here we denote by C_1, C_2, ... constants which depend only on n, p, κ and κ'. To treat I we observe that for all $x\in B_{3/2r}$ we have

$$|u(x+he_i)-u(x)| \leqq |h|.\|Du\|_{L^\infty(B_{2r})}.$$

Therefore, choosing $0<h\leqq[2\kappa\|Du\|_{L^\infty}]^{-1}$ we obtain for all $0\leqq\lambda\leqq 1$:

$$(2.5)\quad \operatorname{dist}(u_\lambda(x), M)=\operatorname{dist}((1-\lambda)u(x)+\lambda u(x+he_i), M) \leqq |u(x)-u(x+he_i)| \leqq \frac{1}{2\kappa}.$$

Thus, $u_\lambda(B_{3/2r})\subset M_{1/(2\kappa)}$ and we may use the nearest point projection $\pi:\mathbb{R}^k\supset M_{1/\kappa}\to M$ to define the mappings $\pi_\lambda := \pi\circ u_\lambda : B_{3/2r}\to M$ satisfying $\pi_0(x)=u(x)$ and $\pi_1(x)=u(x+he_i)$ for any $x\in B_{3/2r}$. Next, we compute:

$$\Delta_{h,i}(|Du|^{p-2} A(u)(D_\alpha u, D_\alpha u)) = \frac{1}{h}\int_0^1 \frac{d}{d\lambda}[|Du_\lambda|^{p-2} A(\pi_\lambda)(\Pi_{\pi_\lambda} D_\alpha u_\lambda, \Pi_{\pi_\lambda} D_\alpha u_\lambda)]\,d\lambda$$
$$=(p-2)\int_0^1 |Du_\lambda|^{p-4} Du_\lambda.D(\Delta_{h,i} u)\, A(\pi_\lambda)(\Pi_{\pi_\lambda} D_\alpha u_\lambda, \Pi_{\pi_\lambda} D_\alpha u_\lambda)\,d\lambda$$
$$+\int_0^1 |Du_\lambda|^{p-2}[DA(\pi_\lambda)\, D\pi(u_\lambda)\, \Delta_{h,i} u\, (\Pi_{\pi_\lambda} D_\alpha u_\lambda . \Pi_{\pi_\lambda} D_\alpha u_\lambda)]\,d\lambda$$

$$+2\int_0^1 |Du_\lambda|^{p-2} A(\pi_\lambda)(D\Pi(\pi_\lambda) D\pi(u_\lambda)\Delta_{h,i} u D_\alpha u + \Pi_{\pi_\lambda} D_\alpha \Delta_{h,i} u, \Pi_{\pi_\lambda} D_\alpha u_\lambda)\, d\lambda.$$

Now, from Lip $(\pi)\leqq 2$ on $M_{1/2\varkappa}$ and $|\partial_\xi \Pi_y \eta|\leqq \kappa|\xi|.|\eta|$ for $y\in M$ and ξ, $\eta\in \mathrm{Tan}_y M$ we infer $|(\partial_\lambda \Pi_{\pi_\lambda})\xi|\leqq 2\kappa|\Delta_{h,i} u|.|\xi|$. Using also $|D\pi|\leqq\kappa$ on $M_{1/2\varkappa}$, $\|\nabla A\|\leqq\kappa'$ and Young's inequality we derive the estimate

$$(2.6)\quad |\Delta_{h,i}(|Du|^{p-2} A(u)(D_\alpha u, D_\alpha u)).\Delta_{h,i} u| \leqq \frac{1}{2}\int_0^1 |Du_\lambda|^{p-2}|D\Delta_{h,i} u|^2 d\lambda + C_3\int_0^1 |Du_\lambda|^p |\Delta_{h,i} u|^2 d\lambda.$$

Inserting (2.6) into (2.4) and recalling $|\nabla\varphi|\leqq\frac{4}{r}$ we get

$$(2.7)\quad \int_{B_{3/2r}} \varphi^2 |\Delta_{h,i} Du|^2 \int_0^1 |Du_\lambda|^{p-2} d\lambda\, dx \leqq C_4 r^{-2}(1+\|Du\|^2_{L^\infty(B_{2r})})\int_{B_{3/2r}} |\Delta_{h,i} u|^2 \int_0^1 |Du_\lambda|^{p-2} d\lambda\, dx.$$

Finally, we estimate the left hand side of (2.7) from below. For this we observe that

$$|\Delta_{h,i} V[Du]|^2 \leqq \frac{p^2}{4}|D\Delta_{h,i} u|^2 \int_0^1 |Du_\lambda|^{p-2} d\lambda\, dx,$$

hence

$$\int_{B_r} |\Delta_{h,i} V[Du]|^2 dx \leqq C_5 r^{-2}(1+\|Du\|^2_{L^\infty(B_{2r})}) \times\int_{B_{3/2r}} |\Delta_{h,i} u|^2 \int_0^1 |Du_\lambda|^{p-2} d\lambda\, dx.$$

Passing to the limit, *i.e.* $h\downarrow 0$, we obtain the desired estimate (2.2). ∎

As the second step in the proof of theorem 2.1 we derive for weakly p-harmonic maps of class C^1 an equivalent for the Bochner-Weitzenböck formula for smooth 2-harmonic maps (*see* [EL] for a derivation in the case $p=2$).

2.3. LEMMA. — *There exists a constant* $K<\infty$ *depending only on p and the curvature bounds* κ *and* κ' *such that for any weakly p-harmonic map* $u\in C^1(\Omega, M)$ *we have*

$$(2.8)\qquad \int_\Omega [a_{\alpha\beta}(.,Du)\, D_\beta(|Du|^p)\, D_\alpha\varphi - K\,|Du|^{p+2}\varphi]\, dx \leqq 0$$

for all $\varphi \in C_0^1(\Omega_+)$ with $\varphi \geqq 0$. *Here we use the abbreviations* $\Omega_+ := \{x \in \Omega : |Du| > 0\}$ *and*

$$a_{\alpha\beta}(\,.\,, Du) := \delta_{\alpha\beta} + (p-2)\,\frac{D_\alpha u . D_\beta u}{|Du|^2}\,\chi_{\Omega_+}.$$

Proof. – For $v \in C^\infty(\Omega, \mathbb{R}^k)$ and $\zeta \in C_0^1(\Omega)$ with $\zeta \geqq 0$ we readily verify

$$(2.9)\quad \int_\Omega D_\alpha(|Dv|^{p-2} D_\alpha v) . D_\beta(\zeta D_\beta v)\, dx$$
$$= \int_\Omega D_\beta(|Dv|^{p-2} D_\alpha v)\, D_\alpha(\zeta D_\beta v)\, dx$$
$$= \int_\Omega D_\beta(|Dv|^{p-2} D_\alpha v) . D_\beta v\, D_\alpha \zeta\, dx + \int_\Omega D_\beta(|Dv|^{p-2} D_\alpha v) . D_\alpha D_\beta v\, \zeta\, dx.$$

Since $D_\beta(|Dv|^{p-2} D_\alpha v) = 0$ a. e. on $\{x \in \Omega : |Dv| = 0\}$ the domain of integration in (2.9) may be restricted to $\{x \in \Omega : |Dv| > 0\}$. To estimate the right hand side of (2.9) from below, we use

$$D_\beta(|Dv|^{p-2} D_\alpha v) . D_\alpha v) . D_\beta v = \frac{1}{p}\left[D_\alpha(|Dv|^p) + (p-2)\,\frac{D_\alpha v . D_\beta v}{|Dv|^2}\, D_\beta(|Dv|^p)\right]$$

and

$$D_\beta(|Dv|^{p-2} D_\alpha v) . D_\alpha D_\beta v = |Dv|^{p-2} |D^2 v|^2 + (p-2)\,|Dv|^{p-4} (D_\alpha v . D_\beta D_\alpha v)(D_\gamma v . D_\beta D_\gamma v) \geqq \frac{4}{p+2}\,|DV\,[Dv]|^2.$$

on $\{x \in \Omega : |Dv| > 0\}$ to infer for all $\zeta \in C_0^1(\Omega)$ with $\zeta \geqq 0$

$$(2.10)\quad \int_\Omega D_\alpha(|Dv|^{p-2} D_\alpha v) . D_\beta(\zeta D_\beta v)\, dx \geqq \int_\Omega \left(\frac{1}{p}\, a_{\alpha\beta}(\,.\,, Dv)\, D_\alpha \zeta\, D_\beta(|Dv|^p) + \frac{4}{p+2}\,|DV\,[Dv]|^{p+2} \zeta\right) dx,$$

Now, since Du is continuous on Ω we get from lemma 2.2 $|Du|^t \in H^{1,2}_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}(\Omega_+)$ for any $t \in \mathbb{R}$. In view of

$$Du = |Du|^{1-p/2}\, |Du|^{p/2-1} Du$$

this and lemma 2.2 immediately imply that $D^2 u \in L^2_{\mathrm{loc}}(\Omega_+)$.

Now, let $\varphi \in C_0^1(\Omega_+)$ be a fixed test function with $\varphi \geqq 0$. To prove our lemma we approximate u by a sequence of smooth maps $u_i \in C^\infty(\Omega, \mathbb{R}^k)$ such that $Du_i \to Du$ locally uniformly on Ω and $D^2 u_i \to D^2 u$ in $L^2_{\mathrm{loc}}(\Omega_+)$.

Then, for arbitrary $t \in \mathbb{R}$ we find

$$(2.11) \quad D_\alpha(|Du_i|^t D_\alpha u_i) \to D_\alpha(|Du|^t D_\alpha u) \quad \text{in} \quad L^2_{\text{loc}}(\Omega_+) \text{ as } i \to \infty,$$

and

$$(2.12) \quad D_\alpha(|Du_i|^t) \to D_\alpha(|Du_i|^t) \quad \text{in} \quad L^2_{\text{loc}}(\Omega_+) \text{ as } i \to \infty,$$

and from the locally uniform convergence $Du_i \to Du$ on Ω we also see that

$$(2.13) \quad a_{\alpha\beta}(.\,, Du_i) = \delta_{\alpha\beta} + (p-2)\,\frac{D_\alpha u_i \,.\, D_\beta u_i}{|Du_i|^2} \times \chi_{\{x \in \Omega : |Du_i(x)| > 0\}} \to a_{\alpha\beta}(.\,, Du)$$

on $\Omega_+ = \{x \in \Omega : |Du(x)| > 0\}$. By (2.10) we have for each u_i the inequality

$$(2.14) \quad \int_\Omega D_\alpha(|Du_i|^{p-2} D_\alpha u_i) \,.\, D_\beta(\varphi D_\beta u_i)\, dx \geqq \int_\Omega \left(\frac{1}{p} a_{\alpha\beta}(.\,, Du_i)\, D_\alpha \varphi\, D_\beta(|Du_i|^p) + \frac{4}{p+2} |DV[Du_i]|^2 \varphi \right) dx.$$

From (2.11)-(2.13) we see that (2.14) and the Euler-equation for u imply

$$\int_\Omega \left(\frac{1}{p} a_{\alpha\beta}(.\,, Du)\, D_\alpha \varphi\, D_\beta(|Du|^p) + \frac{4}{p+2} |DV[Du]|^2 \varphi \right) dx \leqq -\int_\Omega \varphi\, D_\beta(A(u)(V_\alpha[Du], V_\alpha[Du])) \,.\, D_\alpha u\, dx \leqq \delta \int_\Omega \varphi |DV[Du]|^2\, dx + \delta^{-1} C_6 \int_\Omega \varphi |Du|^{p+2}\, dx.$$

Here, $\delta > 0$ can be chosen suitable to give for any $\varphi \in C^1_0(\Omega_+)$ with $\varphi \geqq 0$

$$\int_\Omega a_{\alpha\beta}(.\,, Du)\, D_\alpha \varphi\, D_\beta(|Du|^p)\, dx \leqq K \int_\Omega \varphi |Du|^{p+2}\, dx. \quad \blacksquare$$

2.4. LEMMA. – *Inequality* (2.8) *extends to* $\varphi \in C^1_0(\Omega)$ *with* $\varphi \geqq 0$.

Proof. – We first observed that $|Du|^p$ is in the space $H^{1,2}_{\text{loc}}(\Omega) \cap L^\infty_{\text{loc}}$. Hence (2.8) holds for $\varphi \in H^{1,2}_0(\Omega_+)$ with compact support in Ω and $\varphi \geqq 0$. In fact we can find a sequence $\varphi_i \in C^1_0(\Omega_+)$, $\varphi_i \geqq 0$, such that $\varphi_i \to \varphi$ in $H^{1,2}(\Omega)$ with the additional property that the supports of the φ_i are contained in an uniform compact subset of Ω. Passing to the limit $i \to \infty$ we arrive at (2.8) for functions φ as above.

Now, if φ is as in the statement of lemma 2.4 we define for $\varepsilon > 0$

$$\varphi_\varepsilon := \varphi \frac{w}{w_\varepsilon} \in H^{1,2}_0 \cap L^\infty(\Omega_+),$$

with

$$w := |Du|^{p/2}, \qquad w_\varepsilon := \max\{w, \varepsilon\},$$

and use φ_ε as an admissible test function in (2.8). Letting $\varepsilon \to 0$ the proof is completed. □

As a third step in the proof of theorem 2.1 we show that weakly p-harmonic maps of class C_1 are also p-stationary.

2.5. LEMMA. — *Assume that* $u \in C^1(\Omega, M)$ *is weakly p-harmonic. Then*

$$(2.15) \qquad 0 = \int_\Omega (|Du|^p \operatorname{div} X - p |Du|^{p-2} D_\alpha u . D_\beta u D_\alpha X^\beta)\, dx$$

holds for all $X \in C_0^1(\Omega, R^n)$.

Proof. — Since u is of class C^2 on $\Omega_+ := \{x \in \Omega : |Du(x)| > 0\}$ it is easy to check that (2.15) is true for X with compact support in Ω_+. For general X we proceed as follows: We choose a sequence $\eta_i \in C_0^\infty(\Omega_+)$, $0 \leqq \eta_i \leqq 1$, such that $\eta_i \uparrow \chi_{\Omega_+}$. By (2.15) we have

$$\begin{aligned} 0 &= \int_{\Omega_+} (\operatorname{div}(\eta_i X) |Du|^p - p |Du|^{p-2} D_\alpha u . D_\beta u D_\alpha (\eta_i X^\beta))\, dx \\ &= -\int_{\Omega_+} (\nabla |Du|^p . \eta_i X - p D_\alpha (|Du|^{p-2} D_\alpha u . D_\beta u) X^\beta \eta_i)\, dx \\ &\xrightarrow{i \to \infty} -\int_{\Omega_+} (\nabla |Du|^p . X - p D_\alpha (|Du|^{p-2} D_\alpha u . D_\beta u) X^\beta)\, dx \\ &= \int_\Omega (\operatorname{div} X |Du|^p - p |Du|^{p-2} D_\alpha u . D_\beta u D_\alpha X^\beta)\, dx. \end{aligned}$$

Here we make use of the facts (compare lemma 2.2) that

$$|Du|^p \in H^{1,1}_{\mathrm{loc}}(\Omega), \qquad |Du|^{p-2} D_\alpha u . D_\beta u \in H^{1,1}_{\mathrm{loc}}(\Omega),$$

and that the derivatives of these two functions vanish on $\Omega - \Omega_+$. ■

2.6. COROLLARY (Monotonicity formula, *see* [F3], [HL], [P]). — *Let* $u \in H^{1,p}(B_1, M)$ *denote an arbitrary p-stationary map,* $2 \leqq p \leqq n$. *For* $x \in B_1$ *and* $0 < \sigma < \rho \leqq 1 - |x|$ *we have*

$$\rho^{p-n} \mathbb{E}_{x,\rho}(u) - \sigma^{p-n} \mathbb{E}_{x,\sigma}(u) = p \int_{B_\rho(x) - B_\sigma(x)} |x-y|^{p-n} |Du|^{p-2} \left| \frac{\partial u}{\partial r} \right|^2 dy,$$

where $\partial u / \partial r$ *denotes the radial derivative of u with respect to the center x.* ■

Here and in the sequel we abbreviate

$$\mathbb{E}_{x,\rho}(u) := \int_{B_\rho(x)} |Du|^p\, dx$$

and if x is the origin of R^n we write $\mathbb{E}_\rho$ instead of $\mathbb{E}_{x,\rho}$.

Remark. — Corollary 2.6 easily extend to the case of p-harmonic maps of class $C^1(B_1-\{0\}, M)\cap H^{1,p}(B_1, R^k)$ with an isolated singularity at the origin.

We now come to the *proof of Theorem* 2.1 in which we make use of ideas due to R. Schoen [Sch]: We define

$$F(x):=\left(\frac{r}{2}-|x|\right)|Du(x)|$$

and choose $x_0\in\bar{B}_{r/2}$ such that $F(x_0)\geqq F(x)$ for all x in $\bar{B}_{r/2}$. In case $|x_0|=r/2$ the statement of our theorem is obvious. Therefore we may assume

$$\sigma:=\frac{1}{2}\left(\frac{r}{2}-|x_0|\right)>0.$$

This gives

$$\sup_{B_\sigma(x_0)}|Du|=\sup_{B_\sigma(x_0)}F(x)\left(\frac{r}{2}-|x|\right)^{-1}\leqq\sup_{B_{\sigma+|x_0|}(0)}F(x)\left(\frac{r}{2}-|x|\right)^{-1}$$
$$\leqq F(x_0)\left(\frac{r}{2}-\sigma-|x_0|\right)^{-1}=2|Du(x_0)|.$$

We now distinguish two cases.

Case 1. — $|Du(x_0)|<\sigma^{-1}$. Then according to lemma 2.4 we have for all $\varphi\in C_0^1(B_\sigma(x_0))$, $\varphi\geqq 0$, that

$$0\geqq\int(a_{\alpha\beta}D_\alpha|Du|^pD_\beta\varphi-K\varphi|Du|^{p+2})\,dx\geqq\int\left(a_{\alpha\beta}D_\alpha w\,D_\beta\varphi-\frac{4K}{\sigma^2}\varphi w\right)dx,$$

where we have abbreviated $w:=|Du|^p$. Applying [GT], Theorem 8.17, and [Gia], p. 95, we get

$$|Du(x_0)|^p\leqq C_7\,\sigma^{-n}\int_{B_\sigma(x_0)}|Du|^p\,dx$$

with a constant C_7 depending only on n, p, and K. This implies

$$\begin{aligned}\sup_{B_{r/4}(0)} |Du|^p &= \sup_{B_{r/4}(0)} F(x)^p \left(\frac{r}{2} - |x|\right)^{-p} \\ &\leqq 4^p r^{-p} F(x_0)^p \\ &= 4^p r^{-p} \left(\frac{r}{2} - |x_0|\right)^p |Du(x_0)|^p \\ &= 8^p r^{-p} \sigma^p |Du(x_0)|^p \\ &\leqq C_7 8^p r^{-p} \sigma^{p-n} \int_{B_\sigma(x_0)} |Du|^p dx \\ &\leqq C_7 8^p r^{-p} \left(\frac{r}{2}\right)^{p-n} \int_{B_{r/2}(x_0)} |Du|^p dx \\ &\leqq C_8 r^{-n} \int_{B_r(0)} |Du|^p dx.\end{aligned}$$

Case 2. – $|Du(x_0)| \geqq \sigma^{-1}$. Let $\tilde{\sigma} := |Du(x_0)|^{-1} < \sigma$. This implies $|Du(x)| \leqq 2/\tilde{\sigma}$ on the ball $B_{\tilde{\sigma}}(x_0) \subset B_\sigma(x_0)$. Applying again [GT], Theorem 8.17, and [Gia], p. 95, on the ball $B_{\tilde{\sigma}}(x_0)$ we find

$$\tilde{\sigma}^{-p} = |Du(x_0)|^p \leqq C_7 \tilde{\sigma}^{-n} \int_{B_{\tilde{\sigma}}(x_0)} |Du|^p dx,$$

hence (by the monotonicity formula)

$$1 \leqq C_7 \tilde{\sigma}^{p-n} \int_{B_{\tilde{\sigma}}(x_0)} |Du|^p dx \leqq 2^{n-p} C_7 r^{p-n} \int_{B_r(0)} |Du|^p dx.$$

So if we impose the smallness condition

$$r^{p-n} \int_{B_r(0)} |Du|^p dx \leqq \varepsilon_1(n, p, \kappa, \kappa') := \frac{1}{2^{n+1-p} C_7},$$

case 2 can not occur and we have proved (2.1) with a suitable constant C_0. ■

A simple application of theorem 2.1 and the monotonicity formula is the following

2.7. Corollary. – *There exist constants $\varepsilon_2 > 0$ and C_9 depending only on n, p and the curvature bounds κ, κ' such that any p-harmonic map $u \in C^1(B_1 - \{0\}, M)$ with $\mathbb{E}_1(u) \leqq \varepsilon_2$ satisfies for all $0 < |x| \leqq \frac{1}{2}$:*

$$|x|^p |Du(x)|^p \leqq C_9 (2|x|)^{p-n} \int_{B_{2|x|}} |Du|^p dx.$$

Proof. — Using the monotonicity formula we get for $|x| \leqq \frac{1}{2}$:

$$\left(\frac{1}{2}|x|\right)^p \mathbb{E}_{x,|x|/2}(u) \leqq 2^{p-n}\,\mathbb{E}_1(u).$$

Thus, if we impose $\mathbb{E}_1(u) \leqq 2^{p-n}\varepsilon_1$ where ε_1 denotes the constant from theorem 2.1 we may apply theorem 2.1 on the ball $B_{(1/2)|x|}(x)$ and obtain

$$|x|^p |Du(x)|^p \leqq C_0 |x|^p \left(\frac{1}{2}|x|\right)^{-n} \mathbb{E}_{x,|x|/2}(u) \leqq C_9 (2|x|)^{p-n}\,\mathbb{E}_{2|x|}(u). \quad \blacksquare$$

3. THE REGULARITY THEOREM

In this section we give the proof of our removable singularity theorem. To show that a p-harmonic map $u \in C^1(B_1 - \{0\}, M) \cap H^{1,p}(B_1, \mathbb{R}^k)$ with sufficiently small total energy $\mathbb{E}_1(u)$ is Hölder continuous on B_1 it suffices to prove that there exists a radius r with $0 < r \leqq \frac{1}{2}$ and $\alpha \in]0, 1[$ such that for any $x \in B_r$ and $0 < \rho \leqq \frac{1}{2} r$ we have

$$\rho^{p-n}\,\mathbb{E}_{x,\rho}(u) \leqq \text{const. } \rho^{p\alpha}, \tag{3.1}$$

where we have defined

$$\mathbb{E}_{x,\rho}(u) = \int_{B_\rho(x)} |Du|^p\,dx.$$

First we state a discrete version of (3.1).

3.1. Proposition. — *There exist constants* $\varepsilon_0 = \varepsilon_0(n, p, M) > 0$ *and* $\sigma = \sigma(n, p, M) \in]0, 1[$ *such that for any p-harmonic map*

$$u \in C^1(B_1 - \{0\}, M) \cap H^{1,p}(B_1, \mathbb{R}^k)$$

with $\mathbb{E}_1(u) \leqq \varepsilon_0$ *we have*

$$\sigma^{p-n}\,\mathbb{E}_\sigma(u) \leqq \frac{1}{2}\mathbb{E}_1(u).$$

Proof. — We proceed as in [Li2] and prove our proposition by contradiction. For this we assume that the conclusion is false. Then, we may find a sequence of p-harmonic maps $u_i \in C^1(B_1 - \{0\}, M)$ which satisfy

$\mathbb{E}_1(u_i) \leqq i^{-1}$ and

$$\sigma^{p-n}\,\mathbb{E}_\sigma(u_i) \geqq \frac{1}{2}\,\mathbb{E}_1(u_i)$$

for any $\sigma \in]0, 1[$. The *associated normalized sequence*

$$v_i := \frac{u_i = u_{i,1}}{\mathbb{E}_1(u_i)^{1/p}},$$

where $u_{i,1}$ denotes the mean value of u_i over B_1, *i. e.*

$$u_{i,1} := \fint_{B_1} u_i\,dx,$$

satisfies

$$\mathbb{E}_\sigma(v_i) = \frac{\mathbb{E}_\sigma(u_i)}{\mathbb{E}_1(u_i)}, \qquad \mathbb{E}_1(v_i) = 1, \qquad v_{i,1} = 0, \qquad \int_{B_1} |v_i|^p\,dx \leqq c_0,$$

where we have used Poincaré's inequality. By $c_0, c_1, \ldots$ we denote in this section constants which depend only on n, k and p. Then, the weak compactness of $\{v \in H^{1,p}(B_1, \mathbb{R}^k) : \|v\|_{H^{1,p}} \leqq C < \infty\}$ implies that there exists a subsequence (again denoted by v_i) such that $v_i \to v_\infty$ weakly in $H^{1,p}(B_1, \mathbb{R}^k)$. On B_1 we have for all $\varphi \in C_0^1(B_1, \mathbb{R}^k)$:

$$(3.2)\quad \int_{B_1} (|Dv_i|^{p-2}\,D_\alpha v_i . D_\alpha \varphi + \mathbb{E}_1(u_i)^{1/p} \times |Dv_i|^{p-2}\,A(u_i)(D_\alpha v_i, D_\alpha v_i) . \varphi)\,dx = 0.$$

In view of

$$(3.3)\quad \left| \mathbb{E}_1(u_i)^{1/p} \int_{B_1} |Dv_i|^{p-2}\,A(u_i)(D_\alpha v_i, D_\alpha v_i) . \varphi\,dx \right| \leqq \mathbb{E}_1(u_i)^{1/p}\,\kappa\,\|\varphi\|_\infty$$

we find for all $\varphi \in C_0^1(B_1, \mathbb{R}^k)$

$$(3.4)\qquad \lim_{i\to\infty} \int_{B_1} |Dv_i|^{p-2}\,D_\alpha v_i . D_\alpha \varphi\,dx = 0.$$

To prove that v_∞ is weakly p-harmonic on $B_{1/2}$ we argue as follows. By the monotonicity lemma we get for any $0 < |a| \leqq \frac{1}{2}$ and $0 < r \leqq \frac{1}{2}$

$$r^{p-n} \int_{B_r(a)} |Du_i|^p\,dx \leqq 2^{p-n} \int_{B_1} |Du_i|^p\,dx = 2^{p-n}\,\mathbb{E}_1(u_i).$$

Thus, we find $i_0 \in \mathbb{N}$ such that

$$r^{p-n} \int_{B_r(a)} |Du_i|^p \, dx \leqq \varepsilon_1$$

for all $i \geqq i_0$, $0 < |a| \leqq \frac{1}{2}$ and $0 < r \leqq \frac{1}{2}$ where ε_1 denotes the constant from theorem 2.1. Using again theorem 2.1 and the monotonicity formula we get

$$|Du_i(a)|^p \leqq C_0 |a|^{-p} (2|a|)^{p-n} \int_{B_{2|a|}(0)} |Du_i|^p \, dx \leqq C_1 \, \mathbb{E}_1(u_i) |a|^{-p}.$$

In this chapter $C_0, C_1, \ldots$ denote constants depending only on n, p and M. Thus, for any a with $0 < r \leqq |a| \leqq \frac{1}{2}$ we have

$$|Dv_i(a)| \leqq C_2 r^{-1}.$$

Hence, we can pass to a subsequence of v_i (again denoted by v_i) which converges uniformly on $B_{1/2} - B_r$ to v_∞. Using (3.2) for v_i and v_j we find

$$\begin{aligned} \int_{B_1} (|Dv_i|^{p-2} D_\alpha v_i - |Dv_j|^{p-2} D_\alpha v_j) \, . \, D_\alpha \varphi \, dx \\ = \mathbb{E}_1(u_i)^{1/p} \int_{B_1} |Dv_i|^{p-2} A(u_i)(D_\alpha v_i, D_\alpha v_i) \, . \, \varphi \, dx \\ - \mathbb{E}_1(v_j)^{1/p} \int_{B_1} |Dv_j|^{p-2} A(u_j)(D_\alpha v_j, D_\alpha v_j) \, . \, \varphi \, dx. \end{aligned}$$

Choosing $\varphi = \eta^p (v_i - v_j)$ with $\eta \in C_0^1(B_{1/2} - B_r, \mathbb{R})$ we get using the uniform convergence $\| v_i - v_j \|_\infty \to 0$ as $i, j \to \infty$ and $\mathbb{E}_1(u_i) \to 0$ as $i \to \infty$

$$\begin{aligned} \int_{B_{1/2} - B_r} (|Dv_i|^{p-2} D_\alpha v_i - |Dv_j|^{p-2} D_\alpha v_j) \\ \times (D_\alpha v_i - D_\alpha v_j) \eta^p \, dx \to 0, \quad \text{as } i, j \to \infty, \end{aligned}$$

and with [FF], lemma 3.2, we estimate the integral from below and obtain

$$\int_{B_{1/2} - B_r} |Dv_i - Dv_j|^p \eta^p \, dx \to \infty, \quad \text{as } i, j \to \infty. \tag{3.5}$$

Obviously, (3.5) implies the strong convergence $v_i \to v_\infty$ in $H^{1,p}(B_{1/2} - B_r, \mathbb{R}^k)$. To show the strong convergence on $B_{1/2}$ we consider first the case $p < n$. The monotonicity lemma yields for all $0 < r \leqq 1$

$$r^{p-n} \int_{B_r} |Du_i|^p \, dx \leqq \mathbb{E}_1(u_i),$$

which is equivalent to

$$\int_{B_r} |Dv_i|^p \, dx \leqq r^{p-n}. \tag{3.6}$$

Now, let $\varphi \in C_0^1(B_1, \mathbb{R}^k)$. Applying (3.6) we find for any fixed $\delta > 0$ a radius $\rho := \rho(\delta)$ such that for any $0 < r \leqq \rho$ we have

$$\|D\varphi\|_\infty \left| \int_{B_r} (|Dv_i|^{p-1} + |Dv_\infty|^{p-1}) \, dx \right| \leqq \frac{1}{2}\delta. \tag{3.7}$$

Moreover, by the strong convergence $v_i \to v_\infty$ in $H^{1,p}(B_{1/2} - B_r, \mathbb{R}^k)$ we find $i_1 \in \mathbb{N}$ depending only on δ such that

$$\left| \int_{B_{1/2} - B_r} (|Dv_i|^{p-2} D_\alpha v_i - |Dv_\infty|^{p-2} D_\alpha v_\infty) . D_\alpha \varphi \, dx \right| \leqq \frac{1}{2}\delta \tag{3.8}$$

for any $i \geqq i_1(\delta)$. Combining (3.7), (3.8) and (3.4) we get

$$\int_{B_{1/2}} |Dv_\infty|^{p-2} D_\alpha v_\infty . D_\alpha \varphi \, dx = 0, \qquad \forall \varphi \in C_0^1(B_{1/2}, \mathbb{R}^k). \tag{3.9}$$

If $p = n$, we find using $\mathbb{E}(v_i) = 1$, $\mathbb{E}_1(v_\infty) \leqq 1$ and Hölder's inequality

$$\int_{B_\rho} (|Dv_i|^{n-1} + |Dv_\infty|^{n-1}) \, dx \leqq c_1 \rho,$$

which obviously implies (3.7) and we proceed as in the case $p < n$ to deduce (3.9).

Now, since v_∞ is weakly p-harmonic on $B_{1/2}$ we can use the "Uhlenbeck-estimate" [U], theorem 3.2, to infer for all balls $B_r(x) \subset B_{1/2}$

$$\sup_{B_r(x)} |Dv_\infty| \leqq c_2 \left[\fint_{B_r(x)} |Dv_\infty|^p \, dx \right]^{1/p}. \tag{3.10}$$

For $0 < r \leqq \frac{1}{2}$ we easily conclude from (3.10) for all $\sigma \in \left]0, \frac{1}{2} r\right]$

$$\int_{B_\sigma} |Dv_\infty|^p \, dx \leqq c_3 \sigma^n \sup_{B_\sigma} |Dv_\infty|^p \leqq c_4 \left(\frac{\sigma}{r}\right)^n \int_{B_r} |Dv_\infty|^p \, dx. \tag{3.11}$$

Let $\sigma \in \left]0, \frac{1}{2}\right]$, $\mu > 0$ and $p < n$. Applying (3.6) again we find a radius $\rho = \rho(\mu) > 0$ such for any $0 < r \leqq \rho(\mu)$ we have

$$\int_{B_r} (|Dv_\infty|^p + |Dv_i|^p) \, dx \leqq \frac{1}{2}\mu. \tag{3.12}$$

To show (3.12) in the case $p=n$ we argue as follows: On account of the weak convergence $w_i:=Dv_i \to Dv_\infty =: w_\infty$ in L^n the limit $\lim_{i\to\infty} (\mathbb{L}^n \llcorner w_i)(C)$ exists for any set $C \subset B_1$. Moreover the total variation of $\mathbb{L}^n \llcorner w_i$ is finite. Thus, by the Vitali-Hahn-Saks theorem $\{\mathbb{L}^n \llcorner w_i\}_{i\in\mathbb{N}}$ forms a sequence of uniformly absolutely continuous measures, that is, for any $\varepsilon>0$ there exists $\delta>0$ such that $(\mathbb{L}^n \llcorner w_i)(A) \leqq \varepsilon$ for all $i\in\mathbb{N}$ provided $\mathbb{L}^n(A)\leqq\delta$. This obviously proves (3.12) in the case $p=n$.

From the strong convergence $v_i \to v_\infty$ in $H^{1,p}(B_{1/2}-B_r, \mathbb{R}^k)$ we conclude that there exists $i_2=i_2(\mu)\in\mathbb{N}$ such that for all $i\geqq i_2$

$$\left|\int_{B_\sigma - B_r} (|Dv_\infty|^p - |Dv_i|^p)\,dx\right| \leqq \frac{1}{2}\mu. \tag{3.13}$$

Combining (3.12) and (3.13) we see that

$$\sigma^{p-n}\mathbb{E}_\sigma(v_i) \to \sigma^{p-n}\mathbb{E}_\sigma(v_\infty) \quad \text{as } i\to\infty, \qquad \forall\,\sigma\in\left]0, \frac{1}{2}\right].$$

Recalling $\mathbb{E}_1(v_i)=1$ and $\sigma^{p-n}\mathbb{E}_\sigma(v_i)\geqq \frac{1}{2}$ we immediately obtain

$$\sigma^{p-n}\mathbb{E}_\sigma(v_\infty)\geqq\frac{1}{2} \quad\text{and}\quad \mathbb{E}_\sigma(v_\infty)\leqq 1, \qquad \forall\,\sigma\in]0,1].$$

Taking (3.11) into account this implies

$$\frac{1}{2}\leqq\sigma^{p-n}\int_{B_\sigma}|Dv_\infty|^p\,dx \leqq c_5\,\sigma^p\int_{B_{1/2}}|Dv_\infty|^p\,dx\leqq c_6\,\sigma^p,$$

and if we impose $\sigma^p \leqq \frac{1}{2}c_6$ we get a contradiction. ■

Now, we prove our main result. For this suppose that $u\in C^1(B_1-\{0\}, M)\cap H^{1,p}(B_1,\mathbb{R}^k)$ is p-harmonic and satisfies $\mathbb{E}_1(u)\leqq\varepsilon_0$. Then, by proposition 3.1 we find $\sigma\in]0,1[$ such that

$$\sigma^{p-n}\mathbb{E}_\sigma(u)\leqq\frac{1}{2}\mathbb{E}_1(u). \tag{3.14}$$

Since the rescaled function $u_\sigma(x):=u(\sigma x)$ also satisfies the hypothesis of proposition 3.1 we can iterate (3.14) to obtain

$$(\sigma^i)^{p-n}\mathbb{E}_{\sigma^i}(u)\leqq 2^{-i}\,\mathbb{E}_1(u), \qquad \forall\, i\in\mathbb{N}_0.$$

This implies for $0<\rho<1$ (choosing i so that $\sigma^{i+1}\leqq\rho\leqq\sigma^i$)

$$\rho^{p-n}\mathbb{E}_\rho(u)\leqq 2\,\rho^{p\beta}\,\mathbb{E}_1(u), \tag{3.15}$$

where β is defined by

$$\beta := -(\log 2)/(p\log\sigma)$$

Together with corollary 2.7 we get for $0<|x|\leqq\frac{1}{2}$

$$(3.16)\quad |x|^p|Du(x)|^p\leqq C_3(2|x|)^{p-n}\int_{B_{2|x|}(0)}|Du|^p\,dx\leqq C_4\,\mathbb{E}_1(u)|x|^{p\beta}.$$

This implies that $Du\in L^q(B_1)$ for some exponent $q>n$. Therefore by the Sobolev imbedding theorem we get $u\in C^{0,\,1-n/q}(B_1)$ which proves

3.2. THEOREM. – *There exist constants* $\varepsilon=\varepsilon(n, k, p, M)>0$, $C=C(n, k, p, M)<\infty$ *and* $\beta=\beta(n, k, p, M)\in]0, 1[$ *such that each p-harmonic map* $u\in C^1(B_1-\{0\}, M)$ *with* $\mathbb{E}_1(u)\leqq\varepsilon$ *satisfies a Hölder condition*

$$|u(x)-u(y)|\leqq C|x-y|^\beta \quad \textit{for all } x, y\in B_{1/2}.$$

Using the continuity of u we can localize the regularity problem in the target manifold M and we can proceed as in [F1], theorem 7.2, and [F2], theorem 3.2, to get $C^{1,\gamma}$-regularity for some $\gamma\in]0, 1[$.

4. APPLICATIONS : GEOMETRIC CONDITIONS FOR REMOVABLE SINGULARITIES

In this section we consider p-harmonic maps $u: B_1-\{0\}\to\mathbb{B}_r(q)\subset M$ of class $C^1(B_1-\{0\}, M)$ which have an isolated singularity at the origin. Here, $\mathbb{B}_r(q):=\{q'\in M:\mathrm{dist}_M(q', q)\leqq r\}$ denotes a *regular geodesic ball* in M of radius r and center q (*see* [H], p. 3, for the definition). With this notation we state the following result:

4.1. THEOREM. – *Suppose* $u\in C^1(B_1-\{0\}, M)$ *is p-harmonic,* $2\leqq p\leqq n$, *and satisfies the smallness condition*

$$(4.1)\qquad u(B_1-\{0\})\subset\mathbb{B}_r(q)\subset M$$

for some regular geodesic ball $\mathbb{B}_r(q)$ *in* M *as well as*

$$(4.2)\qquad |Du(x)|.|x|\leqq K<\infty,\quad \text{for all } x\neq 0$$

for some constant $K\in]0, \infty[$. *Then the isolated singularity at the origin is removable.*

Remarks. – (i) From [F 1], theorem 7.1, and [F 2], theorem D, we know that for local minimizers in low dimensions $n-1\leqq p<n$ the singular set is discrete and that the behaviour of the derivative near a singular point x_0 is characterised by the linear growth condition

$$(4.3)\qquad \limsup_{x\to x_0}|Du(x)|.|x-x_0|<\infty,$$

so that linear growth is a rather natural hypothesis.

(ii) Using a slightly stronger definition of regular geodesic balls $\mathbb{B}_r(q)$ [requirering the condition $r<\pi/(4\sqrt{\kappa})$, $\kappa \geq 0$ denoting an upper bound for the sectional curvature of M on $\mathbb{B}_r(q)$] it is possible to show everywhere regularity of weakly p-stationary mappings $u \in H^{1,p}(B_1, M)$ with range in $\mathbb{B}_r(q)$ without imposing any growth condition of the form (4.3): the argument uses a partial regularity theorem from [F 3], Theorem 1.1, for weakly p-harmonic mappings $v: B_1 \to M$ saying that under the condition $\operatorname{Im}(v) \subset \mathbb{B}_r(q)$ a point $x \in B_1$ is a regular point if and only if the scaled p-energy of v calculated on small balls centered at x is small enough. If in addition v is also p-stayionary, the nonexistence of nontrivial homogeneous tangent maps shows that the partial regularity criterion holds for all $x \in B_1$. For the details we refer to [F 3], Theorem 1.2.

(iii) The *equator map* $w_*: \mathbb{R}^n \supset B_1 - \{0\} \to S^n$ defined by $w_*(x) := (u_*(x), 0)$ is p-stationary for $n \geq 3$ and $2 \leq p < n$. By direct calculation we also see that $|Dw_*(x)| . |x| = \sqrt{n-1}$ for all $x \neq 0$. Thus, the equator map shows that even in the class of p-stationary mappings with isolated singularities of linear growth the small range condition (4.1) is necessary and sufficient to prove removability of singular points. We conjecture that Theorem 4.1 remains valid without assumption (4.2). Moreover, one should try to calculate an optimal K_0 such that $|Du(x)| . |x| \leq K$, $x \neq 0$, for $K < K_0$ implies $0 \in \operatorname{Reg}(u)$ without imposing further smallness conditions on the range of u. ■

Proof. – According to our main theorem we only have to show that

$$\liminf_{\rho \downarrow 0} \rho^{p-n} \int_{B_\rho} |Du|^p \, dx = 0. \tag{4.4}$$

To prove (4.4) we fix a sequence $\lambda_i \downarrow 0$ of positive numbers and consider the scaled maps $u_i(x) := u(\lambda_i x)$. Then, from (4.2) we get

$$\mathbb{E}_1(u_i) = \lambda_i^p \int_{B_1} |Du(\lambda_i x)|^p \, dx \leq K^p \int_{B_1} |x|^{-p} \, dx \leq c_0 K^p < \infty,$$

where c_0 depends only on n and p as the constants $c_1, c_2, \ldots$ below. Passing to a subsequence we may assume that u_i converges weakly in $H^{1,p}(B_1, \mathbb{R}^k)$ to a map $u_0 \in H^{1,p}(B_1, \mathbb{R}^k)$ and from (4.5) we infer for all $x \neq 0$

$$|Du_i(x)| \leq K |x|^{-1}. \tag{4.5}$$

Hence, we can pass again to a subsequence u_i which converges locally uniformly on $B_1 - \{0\}$ to u_0. Now, for any fixed $\delta > 0$ let $r > 0$ be a radius such that $r^{n-p} < \delta$. Then, we obtain

$$\mathbb{E}_r(u_i) \leq c_0 K^p r^{n-p} \leq c_0 K^p \delta. \tag{4.6}$$

Since u is weakly p-harmonic on B_1 the scaled maps u_i are also weakly p-harmonic on B_1, *i.e.* we have for all $\varphi \in H^{1,p}_{loc} \cap L^\infty(B_1, \mathbb{R}^k)$:

$$(4.7) \quad \int_{B_1} |Du_i|^{p-2} D_\alpha u_i . D_\alpha \varphi \, dx = -\int_{B_1} |Du_i|^{p-2} A(u_i)(D_\alpha u_i, D_\alpha u_i) . \varphi \, dx.$$

For $\psi \in C^1(B_1, \mathbb{R})$ with $0 \leqq \psi \leqq 1$, $\psi = 1$ on $\bar{B}_1 - B_r$, $\operatorname{spt} \psi \subset \bar{B}_1 - B_{r/2}$ we decompose

$$(4.8) \quad \int_{B_1} (|Du_i|^{p-2} D_\alpha u_i - |Du_j|^{p-2} D_\alpha u_j) \times D_\alpha (u_i - u_j) \psi^p \, dx = I_{ij} + I_{ji} + J_{ij} + J_{ji}$$

into a sum of four integrals

$$I_{ij} := \int_{B_1} |Du_i|^{p-2} D_\alpha u_i . D_\alpha (\psi^p (u_i - u_j)) \, dx,$$

$$J_{ij} := -p \int_{B_1} \psi^{p-1} D_\alpha \psi \, |Du_i|^{p-2} D_\alpha u_i . (u_i - u_j) \, dx.$$

Using Hölder's inequality we obtain the estimate

$$|J_{ij}| \leqq \sup_{B_1} |D\psi| \, \mathbb{E}_1(u_i)^{1-1/p} \left[\int_{B_1} |u_i - u_j|^p \, dx \right]^{1/p}.$$

With the help of (4.6) and the curvature bound for the second fundamental form A of M we further derive

$$(4.10) \qquad |I_{ij}| \leqq c_1 \kappa \, \mathbb{E}_1(u_i) \sup_{\operatorname{spt} \psi} |u_i - u_j|.$$

From (4.9), (4.10), the definition of ψ, the L^p-convergence $u_i \to u_0$, the uniform convergence $u_i \to u_0$ on compact subsets of $B_1 - \{0\}$ and [FF], lemma 3.2, we see that (4.8) implies

$$\int_{B_1 - B_r} |D(u_i - u_j)|^p \, dx \to 0, \quad \text{as } i, j \to \infty.$$

Combining this result with (4.6) we obtain

$$\int_{B_1} |D(u_i - u_j)|^p \, dx \leqq c_2 \left(K^p \delta + \int_{B_1 - B_r} |D(u_i - u_j)|^p \, dx \right),$$

and since $\delta > 0$ was arbitrary we get the convergence $u_i \to u_0$ in the $H^{1,p}$-norm on B_1. Thus, u_0 is also p-stationary on B_1 and satisfies $\partial_{rad} u_0 = 0$, which can easily be seen by the use of the monotonicity formula for u_0.

Now, let U_0 be a representation of u_0 with respect to normal coordinates on $\mathbb{B}_r(q)$ centered at q. By virtue of [F 1], [F 4] we have for all $\varphi \in C_0^1(B_1, \mathbb{R})$:

$$\int_{B_1} a(U_0, DU_0)(D_\alpha U_0 . D_\alpha(\varphi U_0) - \Gamma^i_{jk}(U_0) D_\alpha U_0^j D_\alpha U_0^k U_0^i \varphi)\, dx = 0,$$

with $a(U_0, DU_0) := (g_{ik}(U_0) D_\alpha U_0^i D_\alpha U_0^k)^{p/2-1}$. Here g_{ik} denotes the fundamental tensor on $\mathbb{B}_r(q)$ and Γ^i_{jk} are the Christoffel symbols of second kind. We now choose $\varphi(x) := \varphi(|x|)$ (*see* [F 1], [F 4]) and obtain

$$\int_{B_1} a(U_0, DU_0)\, \varphi(|DU_0|^2 - \Gamma^i_{jk}(U_0) D_\alpha U_0^j D_\alpha U_0^k U_0^i)\, dx = 0,$$

and since u_0 takes its values in the regular geodesic ball $\mathbb{B}_r(q)$ the quantity $|DU_0|^2 - \Gamma^i_{jk}(U_0) D_\alpha U_0^j D_\alpha U_0^k U_0^i$ is bounded below by a constant times $g_{jk}(U_0) D_\alpha U_0^j D_\alpha U_0^k$. Thus, for $\varphi \geqq 0$ we obtain

$$\int_{B_1} (g_{jk}(U_0) D_\alpha U_0^j D_\alpha U_0^k)^{p/2}\, \varphi\, dx \leqq 0,$$

and hence $Du_0 = 0$ on B_1. Since $u_i \to u_0$ strongly on B_1 we conclude that $\mathbb{E}_\rho(u_i) \to 0$ as $i \to \infty$ for any $0 < \rho \leqq 1$ which immediatly implies

$$(\lambda_i \rho)^{p-n}\, \mathbb{E}_{\lambda_i \rho}(u) \to 0, \quad \text{as } i \to \infty. \quad \blacksquare$$

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