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Long-time behavior for a regularized scalar conservation law in the absence of genuine nonlinearity

by

H. F. WEINBERGER

School of Mathematics,
University of Minnesota, U.S.A.

To my friend Avron Douglis on his seventieth birthday

ABSTRACT. — It is shown that as time approaches infinity, the solution of the initial value problem for a regularized one-dimensional scalar conservation law converges along rays to the solution of a certain Riemann problem for the hyperbolic conservation law, even when this conservation law is not genuinely nonlinear.

Key words : Conservation law, Long-time behavior, Regularization.

RÉSUMÉ. — On démontre que quand le temps devient très grand, la solution du problème de Cauchy pour une loi de conservation scalaire régularisée à une dimension converge le long des rayons vers la solution d'un problème de Riemann pour la loi de conservation hyperbolique, même quand cette loi n'est pas véritablement non linéaire.

Classification A.M.S. : 35 K 15, 35 L 65.

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1. INTRODUCTION

The scalar equation

$$\frac{\partial u}{\partial t}+\frac{\partial \varphi(u)}{\partial x}=\varepsilon \frac{\partial^2 u}{\partial x^2} \tag{1.1}$$

has been studied for a long time. Among other things, it serves as a model of the regularizing effect of a small viscosity on a hyperbolic conservation law.

The equation (1.1) with $\varphi(u)=\frac{1}{2}u^2$ was introduced as an approximation to the equations of fluid flow by H. Bateman [1] and by J. M. Burgers ([5], [6]). E. Hopf [9] and J. D. Cole [7] independently showed that the solution of the initial value problem for this particular equation can be reduced to the well-known solution of the initial value problem for the heat equation.

In an important series of papers O. A. Oleinik and her students ([10], [11], [12], [16], [17], [18], [20]) showed the well-posedness of the initial value problem for the equation (1.1), as well as the continuity in ε at $\varepsilon=0$ of the solution.

When the function $\varphi(u)$ is convex, A. M. Il'in and O. A. Oleinik ([10], [11]) obtained farreaching results about the large-time behavior of the solution. They showed that if the initial values

$$u(x, 0)=u_0(x)$$

have the limits u_- at $-\infty$ and u_+ at $+\infty$, if $u_->u_+$, and if u_0-u_- and u_0-u_+ are integrable near $-\infty$ and near $+\infty$, respectively, then the solution converges uniformly to a travelling wave solution $\tilde{u}(x-ct)$ as t goes to infinity. They also showed that if $u_->u_+$, then the solution converges uniformly to a simple wave of the form $\mathrm{H}(x/t)$. The condition that $\varphi''\neq 0$ is the condition that, in the terminology of Lax ([13], [14]), the equation obtained by setting $\varepsilon=0$ in (1.1) is genuinely nonlinear.

In this work we study an equation of the form

$$\frac{\partial u}{\partial t}+\frac{\partial \varphi(u)}{\partial x}=\varepsilon \frac{\partial}{\partial x}\left(a(u)\frac{\partial u}{\partial x}\right) \tag{1.2}$$

with $a(u)$ strictly positive and $\varphi(u)$ not necessarily convex.

The equation (1.2) with $\varepsilon=0$ and φ not convex was introduced by S. E. Buckley and M. C. Leverett [4] as a model for the one-dimensional convection-dominated displacement of oil by water in a porous medium. The equation 1.2) arises in this problem when the capillary terms are retained, and there has been a great deal of interest in its numerical treatment. (*See*, e. g., [21].) The equation (1.2) also arises as a model for a stable monotone finite difference approximation to the equation with $\varepsilon=0$. (*See* [8].)

In this work we shall obtain some results about the large-time behavior of the solution of the initial value problem for (1.2) when φ'' is continuous and has only isolated zeros. We shall show that several of the important features of the Il'in-Oleinik results can be extended to this case, although some are, of course, lost.

In Section 2 we show that as t approaches infinity, the solution is essentially bounded by the limits superior and inferior at infinity of the initial data, so that all initial values outside these limits are diffused away.

The main result of this work is proved in Section 3. We show that if the initial function $u_0(x)$ has limits at $\pm\infty$, and if u is the solution of (1.2) with $u(x, 0)=u_0(x)$, then for all but finitely many values of ξ the function $u(t, \xi t)$ approaches $V(\xi)$, where $V(x/t)$ is that solution of a certain Riemann problem which satisfies the Condition E of Oleinik [19]. This Riemann problem is obtained by setting ε equal to zero and taking for initial values the constants $u_-=u_0(-\infty)$ for $x<0$ and $u_+=u_0(+\infty)$ for $x>0$. The function V is monotone and piecewise continuous, and Theorem 3.3 states that the convergence is uniform in ξ and ε when ξ is restricted to any closed interval where V is continuous, and ε remains bounded. Theorem 3.1 states that, while the function $u(t, \xi t)$ need not converge at a point of discontinuity of $V(\xi)$, it is essentially bounded by the right and left limits of V at this point when t is large.

Our results are proved by applying the comparison lemma for parabolic equations to solutions of the equation (1.2). We do not use the auxiliary function introduced by Il'in and Oleinik, and consequently we do not need to assume the integrability of the functions u_0-u_- and u_0-u_+. The convergence along rays which we obtain is, of course, considerably weaker than the uniform convergence obtained by Il'in and Oleinik. In fact, Il'in and Oleinik gave an example which shows that one cannot expect uniform convergence without their integrability condition, even when φ is convex.

A. S. Kalashnikov [12] proved under essentially the conditions on φ which we use that as ε decreases to zero, the solution of the initial value problem for (1.1) approaches the weak solution which satisfies Condition E of the corresponding problem with $\varepsilon=0$, uniformly on finite t-intervals. Therefore, since our results are uniform in ε, they also apply to the solution of the latter problem. Our results for this case are, of course, much weaker than the hyperbolic convergence results of Liu [15] when u_0 has constant values outside a bounded interval.

As additional motivation for this work, we observe that a natural way to study the behavior of the solution of an initial value problem for the multidimensional regularized scalar conservation law

$$\frac{\partial u}{\partial t} - \operatorname{div} \mathbf{f}(u) = \varepsilon \sum \frac{\partial}{\partial x_i}\left(a_{ij}(u) \frac{\partial u}{\partial x_j}\right) \tag{1.3}$$

is to look at a comparison solution $v(\mathbf{e}\cdot\mathbf{x})$ which depends only on the single variable $\mathbf{e}\cdot\mathbf{x}$ with $\mathbf{e}$ a fixed unit vector. The function v must then satisfy the equation (1.2) with the function $\varphi(u)$ replaced by $\mathbf{e}\cdot\mathbf{f}(u)$ and $a(u)$ replaced by $\sum a_{ij}(u) e_i e_j$. Even if all the components of the vector-valued function $\mathbf{f}$ are convex, there will, in general, be vectors $\mathbf{e}$ for which $\mathbf{e}\cdot\mathbf{f}$ is not convex, so that a knowledge of the behavior of solutions of (1.1) for nonconvex φ is useful in the treatment of the multidimensional problem.

It is easily seen that a comparison principle applies to solutions of the equation (1.3), so that particular solutions of the form $v(\mathbf{e}\cdot\mathbf{x})$ serve to bound other solutions. In particular, our results show that if for some unit vector $\mathbf{e}$ the function $\mathbf{e}\cdot\mathbf{f}''$ is continuous and has only isolated zeros, and if the initial values $u(\mathbf{x}, 0)$ have limits $u_\pm$ as $\mathbf{e}\cdot\mathbf{x}$ tends to $\pm\infty$, then for all but finitely many values of ξ the restriction of the solution $u(\mathbf{x}, t)$ to the plane $\mathbf{e}\cdot\mathbf{x}=\xi t$ converges as $t\to\infty$ to a constant $V(\xi)$, where $V(\mathbf{e}\cdot\mathbf{x}/t)$ is the solution of a one-dimensional Riemann problem which satisfies Condition E. Bauman and Phillips [2] have given an example which shows that one cannot hope to find uniform convergence in this case, even when $\mathbf{e}\cdot\mathbf{f}$ is convex and $u(\mathbf{x}, 0)$ has constant values when $\mathbf{e}\cdot\mathbf{x}$ lies outside a bounded interval.

Bauman and Phillips [3] have also obtained a convergence result analogous to that of Liu [15] for the equation (1.3) with $\varepsilon=0$.

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2. ASYMPTOTIC BOUNDS FOR THE SOLUTION

In this Section we shall show that if the function φ'' is continuous and has only isolated zeros, then for large t the solution of the initial-value problem

$$\begin{gathered} \frac{\partial u}{\partial t} + \frac{\partial \varphi(u)}{\partial x} = \varepsilon \frac{\partial}{\partial x}\left(a(u) \frac{\partial u}{\partial x}\right) \\ u(0, x) = u_0(x) \end{gathered} \tag{2.1}$$

is essentially bounded by the limits inferior and superior at $|x| = \infty$ of the initial data. We assume throughout this work that $\varphi(u)$ and $a(u)$ are twice continuously differentiable and that $a(u)$ is strictly positive.

We shall prove this result by means of several lemmas. The first of these is the well-known comparison principle, which will be a principal tool throughout this paper.

LEMMA 2.1 (comparison principle). — *Suppose that φ'' is continuous. Let $v(t, x)$ satisfy the differential inequality*

$$\frac{\partial v}{\partial t} + \frac{\partial \varphi(v)}{\partial x} - \frac{\partial}{\partial x}\left(a(v)\frac{\partial v}{\partial x}\right) \leqq 0,$$

and let

$$\frac{\partial w}{\partial t} + \frac{\partial \varphi(w)}{\partial x} - \frac{\partial}{\partial x}\left(a(w)\frac{\partial w}{\partial x}\right) \geqq 0.$$

Then $v(0, x) \leqq w(0, x)$ implies that $v(t, x) \leqq w(t, x)$ fot $t \geqq 0$.

Proof. — We note that the difference $w-v$ satisfies the equation

$$\frac{\partial(w-v)}{\partial t} + \varphi'(w)\frac{\partial(w-v)}{\partial x} = a(w)\frac{\partial^2(w-v)}{\partial x^2} + a'(w)\frac{\partial}{\partial x}(w+v)\frac{\partial}{\partial x}(w-v) + q(x)(w-v),$$

where $q(x)$ depends upon φ'', a', and a'' evaluated at points between $w(x)$ and $v(x)$ and upon the x-derivatives of v and w. The generalized maximum principle for parabolic equations [22], Chapter 3, Theorem 7, Remark (ii), then yields the result.

Remark. — *It is easily verified that the conclusion of the Lemma is still true if the differential inequalities are valid except on finitely many arcs of the form $x = x(t)$, provided v and w are continuous, their second x-derivatives remain bounded, and the quantity $\partial(w-v)/\partial x$ has a nonnegative jump across each of these arcs.*

Since any constant is a solution of the equation, the comparison principle implies that

$$\inf u_0 \leqq u(t, x) \leqq \sup u_0. \tag{2.2}$$

In particular, it follows that the solution u is not affected if the function φ is altered outside this interval.

An important class of comparison functions is the set of travelling wave solutions. These are solutions of the form $v(t, x) = \tilde{u}(x-ct)$, where $\tilde{u}$ is a function of one variable y, and c is a constant. If we substitute this function in the differential equation, we find that

$$\frac{d}{dy}[-c\tilde{u}+\varphi(\tilde{u})]=\varepsilon\frac{d}{dy}\left(a(\tilde{u})\frac{d\tilde{u}}{dy}\right).$$

Therefore

$$\varepsilon\, a(\tilde{u})\frac{d\tilde{u}}{dy}=\varphi(\tilde{u})-c\tilde{u}-\mathrm{K},$$

where K is a constant of integration. This first-order equation has the implicit solution

$$y=y_0+\varepsilon\int_b^{\tilde{u}}\frac{a(v)\,dv}{\varphi(v)-cv-\mathrm{K}}. \tag{2.3}$$

Once the parameters c and K are chosen, one chooses some interval over which the function $\varphi(v)-cv-\mathrm{K}$ does not vanish and fixes a point b in this interval. Changing b within the interval is equivalent to changing the parameter y_0, so that the set of solutions is essentially parameterized by c, K, and y_0, while the choice of b can be thought of as a choice of one of a discrete set of intervals.

We note in particular that if the line through two points $(r, \varphi(r))$ and $(s, \varphi(s))$ of the graph of φ lies below this graph in the interval (r, s), and if we choose

$$c=\frac{\varphi(s)-\varphi(r)}{s-r}$$

and

$$\mathrm{K}=\frac{s\,\varphi(r)-r\,\varphi(s)}{s-r},$$

then the function $\varphi(v)-cv-\mathrm{K}$ is positive in the interval (r, s) and vanishes at least linearly at its endpoints. Therefore, if b is chosen to lie in this interval, then the travelling wave function $\tilde{u}$ defined by the integral (2.3) increases from the value r at $-\infty$ to s at $+\infty$. Similarly, if the graph of φ lies below the secant line in (r, s), one obtains a travelling wave which decreases from s to r as y goes from $-\infty$ to ∞.

LEMMA 2.2. – *If φ'' is continuous and if the initial function u_0 is bounded, then for each fixed t the solution $u(t, x)$ of the problem* (2.1) *has the properties*

$$\limsup_{x\to\pm\infty} u(t, x)\leq \limsup_{x\to\pm\infty} u_0(x)$$

and

$$\liminf_{x \to \pm\infty} u(t, x) \geqq \liminf_{x \to \pm\infty} u_0(x),$$

uniformly in $0 < \varepsilon \leqq \varepsilon_0$.

Proof. — Suppose that u_0 has the limit superior u_- as $x \to -\infty$, and that $|u_0| \leqq M$. Choose any $\mu \in (u_-, M)$ and $M' > M$. Next choose c so small that

$$\varphi(\mu) + c(v - \mu) < \varphi(v) \qquad \text{for} \quad \mu < v \leqq M'.$$

We let $K = \varphi(\mu) - c\mu$ and define a travelling wave solution $\tilde{u}(x - ct)$ by means of (2.3) with $b = M$. In order to ensure that $\tilde{u}(y)$ is defined for all y, we modify the function φ above M' if necessary, so that $\varphi(v) - cv - K$ becomes zero at some point v_0 above M'. Since $\mu > u_-$, we can choose the parameter y_0 so small that $u_0(x) \leqq \mu$ for $x \leqq y_0$. Then for $x \leqq y_0$ we have $\tilde{u}(x) \geqq \mu \geqq u_0(x)$, while for $x \geqq y_0$, $\tilde{u}(x) \geqq M \geqq u_0(x)$. Thus $\tilde{u}(x) \geqq u_0(x)$, so that the comparison principle yields the inequality

$$u(t, x) \leqq \tilde{u}(x - ct).$$

Since the right-hand side has the limit μ as x goes to $-\infty$, we see that

$$\limsup_{t \to -\infty} u(t, x) \leqq \mu.$$

Since μ can be chosen arbitrarily close to u_-, it follows that

$$\limsup_{x \to -\infty} u(t, x) \leqq \limsup_{x \to -\infty} u_0(x).$$

The other parts of the Lemma can be obtained by observing that the functions $-u(t, x)$, $u(t, -x)$, and $-u(t, -x)$ all satisfy differential equations of the same form, and hence satisfy the same inequality.

LEMMA 2.3. — *Suppose that* $u_0(x) \leqq M$, *that*

$$\limsup_{|x| \to \infty} u_0(x) \leqq \mu,$$

and that φ'' *is continuous and does not vanish in the interval* (μ, M). *Then*

$$\lim_{t \to \infty} \sup_{x} u(t, x) \leqq \mu,$$

uniformly in ε *for* $0 < \varepsilon \leqq \varepsilon_0$.

Proof. — We observe that the function $u(t, -x)$ satisfies a partial differential equation of the form (2.1) but with φ replaced by $-\varphi$. Therefore, we shall assume without loss of generality that $\varphi''(u) < 0$ in the interval (μ, M). By the mean value theorem there is a point M'_1 in the interval (μ, M) such that $\varphi'(M'_1)$ is equal to the slope of the line segment from $(\mu, \varphi(\mu))$ to $(M, \varphi(M))$. We then define the sequence M'_i by saying that $\varphi'(M'_{i+1})$ is equal to the slope of the line segment from $(\mu, \varphi(\mu))$ to $(M'_i, \varphi(M'_i))$. Since $\varphi''<0$, this sequence is easily seen to decrease to μ.

We now pick a fixed number $\hat{\mu}$ in the interval (μ, M). Then there is an integer k such that $M'_k<\hat{\mu}$. We now choose a finite sequence $M_1, M_2, \ldots, M_k$ such that

$$M'_1<M_1<M,$$
$$M'_i<M_i<M'_{i-1} \quad \text{for} \quad i=2, \ldots, k,$$

and

$$M_k<\hat{\mu}. \tag{2.4}$$

We observe that since $\varphi''<0$, $\varphi'(M_1)$ is less than the slope of the line segment from $(\mu, \varphi(\mu))$ to $(M, \varphi(M))$. Choose $\bar{\mu}\in(\mu, M_k)$ and close to μ and $\bar{M}>M$ and close to M so that the line segment L from $(\bar{\mu}, \varphi(\bar{\mu}))$ to $(\bar{M}, \varphi(\bar{M}))$ lies below the graph of φ between these points and that its slope is greater than $\varphi'(M_1)$. By continuity there is an

$$m_1<M_1$$

such that

$$\varphi'(m_1)<c\equiv\frac{\varphi(\bar{M})-\varphi(\bar{\mu})}{\bar{M}-\bar{\mu}}. \tag{2.5}$$

We now define the travelling wave solution $\tilde{u}(x-ct)$ by (2.3) with c given by (2.5), $b=M$, K determined so that the denominator in the integrand vanishes at $v=\bar{\mu}$ and $v=\bar{M}$, and with y_0 chosen so that

$$u_0(x)\leqq\bar{\mu} \quad \text{for} \quad x\leqq y_0.$$

Since the line segment L lies below the graph, the integrand in (2.3) is positive. As in the proof of the preceding Lemma, the above inequality implies that $\tilde{u}(x)\geqq u_0(x)$, so that Lemma 2.1 shows that

$$u(t, x)\leqq\tilde{u}(x-ct) \tag{2.6}$$

for $t\geqq 0$.

We also define a travelling wave solution $\tilde{u}^*(x-\varphi'(m_1)t)$ by the formula (2.3) with $c=\varphi'(m_1)$, $K=K^*$ chosen so that the denominator in the integrand vanishes (to second order) at m_1, and $b=M$. The parameter $y_0=y_0^*$ is now chosen so that $u_0\leqq m_1$ for $x\geqq y_0$. In order to be sure that the function u^* is defined for all y, we modify the function $\varphi(v)$ for $v>\bar{M}$, if necessary, in such a way that the denominator in the integral becomes zero at some $v_0>\bar{M}$. Since the line segment lies above the graph, the function $\tilde{u}^*(y)$ decreases from $v_0>M$ to $m_1>\mu$ as y increases from $-\infty$ to ∞. As before, our the choice $y_0=y_0^*$ implies that $\tilde{u}^*(x)\geqq u_0(x)$, and hence that

$$u(t, x) \leqq \tilde{u}^*(x - \varphi'(m_1)\, t). \tag{2.7}$$

We observe that by construction $\tilde{u}(-\infty) = \mu_1$ while $\tilde{u}^*(\infty) = m_1$. Because both of these numbers are below M_1, because of the inequality (2.5) on the speeds of the two waves, and because $\tilde{u}$ is increasing and $\tilde{u}^*$ is decreasing, we see that there is a $T_1 \geqq 0$ such that the inequality

$$[c - \varphi'(m_1)]\, T_1 + y_0 - y_0^* \\ - \varepsilon \int_{M_1}^{M} \left[\frac{1}{\varphi(v) - cv - K} - \frac{1}{\varphi(v) - \varphi'(m_1)\, v - K^*} \right] dv \geqq 0$$

is satisfied, and that, together with (2.6) and (2.7), this inequality implies that

$$u(T_1, x) \leqq M_1.$$

Moreover, a fixed T_1 makes this inequality valid for all ε on any bounded interval.

We see from Lemma 2.2 that

$$\limsup_{|x| \to \infty} u(T_1, x) \leqq \mu.$$

Thus if we think of $u(T_1, x)$ as the initial values for a new problem starting at $t = T_1$, these values satisfy the conditions of the Lemma, but with M replaced by the smaller value M_1. We repeat this argument k times to find a T_k which is independent of ε such that

$$u(T_k, x) \leqq M_k.$$

Since $M_k < \hat{\mu}$, we see from Lemma 2.1 that $u(t, x) < \hat{\mu}$ for $t \geqq T_k$. Because $\hat{\mu}$ can be chosen arbitrarily close to μ and because the maximum principle shows that $\sup_x u(t, x)$ is nonincreasing, the Lemma is established.

A slight modification of the proof leads to a somewhat stronger result.

LEMMA 2.4. – *Suppose that* $u_0(x) \leqq M$, *that*

$$\lim_{|x| \to \infty} u_0(x) \leqq \mu,$$

and that

(*a*) *the line segment* L *from* $(\mu, \varphi(\mu))$ *to* $(M, \varphi(M))$ *does not intersect the graph of* φ *at an interior point and is not tangent to this graph at the right end point;*

(*b*) *There is an* $m_1 \in (\mu, M)$ *such that* $\varphi'' \neq 0$ *on the interval* $(\mu, m_1]$ *and, if c denotes the slope of* L,

$$\varphi''(m_1)\, [\varphi'(m_1) - c] > 0.$$

Then

$$\limsup_{t \to \infty} \sup_x u(t, x) \leqq \mu.$$

Proof. — Choose $\hat{\mu} \in (\mu, m_1)$. We again assume without loss of generality that $\varphi''(m_1) < 0$. By continuity there is an $M_1 > m_1$ such that $\varphi'' < 0$ on the interval (μ, M_1), and we can find a $\bar{\mu} \in (\mu, \hat{\mu})$ and an $\bar{M} > M$ such that the slope of the line segment from $(\bar{\mu}, \varphi(\bar{\mu}))$ to $(M, \varphi(\bar{M})$ is still greater than $\varphi'(m_1)$. We can carry out the first step of the proof of Lemma 2.3 to find a T_1 independent of ε such that $u(T_1, x) \leqq M_1$. By Lemma 2.2,

$$\limsup_{|x| \to \infty} u(T_1, x) \leqq \mu.$$

Since $\varphi'' < 0$ on the interval $[\mu, M_1]$, Lemma 2.3 shows that there is a T_k undependent of ε such that $u \leqq \hat{\mu}$ for $t \geqq T_k$. Since $\hat{\mu}$ is arbitrarily close to μ, this proves the Lemma.

We are now ready to prove the principal result of this Section.

THEOREM 2.1. — *Suppose that the function* φ'' *is continuous and that its zeros are isolated. If* u *is a bounded solution of the initial value problem* (2.1), *then*

$$\lim_{t \to \infty} \sup_x u(t, x) \leqq \limsup_{|x| \to \infty} u_0(x)$$

and

$$\lim_{t \to \infty} \inf_x u(t, x) \geqq \liminf_{|x| \to \infty} u_0(x),$$

uniformly in ε *on any bounded interval* $(0, \varepsilon_0]$.

Proof. — Let $M = \sup u_0$ and

$$\mu = \limsup_{|x| \to \infty} u_0(x).$$

If φ'' does not vanish in the interval (μ, M), the result follows immediately from Lemma 2.3. If there is a single zero of φ'' in this interval, say at μ_1, then Lemma 2.3 shows that for any $M_1 > \mu_1$ there is a T_1 such that $u \leqq M_1$ for $t > T_1$. A simple continuity argument shows that if M_1 is sufficiently close to μ_1, then the hypotheses of Lemma 2.4 with M replaced by M_1 are satisfied and the result follows.

Finally, if there are several zeros $\mu_1 > \mu_2 > \ldots > \mu_r$ in the interval (μ, M), we use Lemma 2.3 to reduce the bound on u to a number M_1 so close to μ_1 that Lemma 2.4 can be applied to the interval (μ_2, M_1). This Lemma then reduces u below a number M_2 so close to μ_2 that Lemma 2.4 can be applied again. After r applications of Lemma 2.4 one reaches a bound $\hat{\mu}$ which is arbitrarily close to μ, which proves the first inequality of the Lemma.

The second inequality is proved by applying the first one to the function $-u(t, x)$.

3. LONG-TIME BEHAVIOR OF THE SOLUTION

In this Section we shall be concerned with the behavior of the solution of the initial value problem (2.1) for large values of t. We shall assume that the initial function u_0 has limits at both plus and minus infinity:

$$\lim_{x \to -\infty} u_0 = u_-,$$

$$\lim_{x \to +\infty} u_0 = u_+.$$

By Lemma 2.2 the solution $u(t, x)$ has the same limits for each value of t.

Theorem 2.1 shows that if $u_+ = u_-$, then u converges uniformly to this constant. Therefore, we only need to investigate the case in which $u_+ \neq u_-$. Because replacing the variable x by $-x$ gives a partial differential equation of the same form, we may assume without loss of generality that

$$u_+ > u_-.$$

We shall show that for large values of t, the solution $u(t, x)$ is close to that weak solution of the Riemann problem

$$\text{(3.1)} \qquad \begin{gathered} \frac{\partial v}{\partial t} + \frac{\partial \varphi(v)}{\partial x} = 0, \\ v(0, x) = \begin{cases} u_- & \text{for } x < 0, \\ u_+ & \text{for } x > 0. \end{cases} \end{gathered}$$

which satisfies the entropy condition E of Oleinik [19]. This problem is obtained by setting $\varepsilon = 0$ in (2.1) and by using only the limits at $\pm\infty$ of the initial function u_0. It was shown by Oleinik that there is exactly one such solution of this problem.

In order construct a solution of this problem, we let

$$\Gamma = \{(v, z) : z = \psi(v)\}$$

be the lower boundary of the convex hull of the set

$$\{(v, z) : u_- \leqq v \leqq u_+,\ z \geqq \varphi(v)\}.$$

If φ satisfies the conditions of Theorem 2.1, then ψ is continuously differentiable, ψ' is nondecreasing, and there are points

$$u_- \leqq v_1 \leqq w_1 < v_2 < \ldots \leqq w_N \leqq u_+,$$

such that $\psi=\varphi$ and φ' is strictly increasing in each interval $[v_j, w_j]$, and ψ is linear and bounded above by φ outside these intervals.

We define the function V (ξ) by the formula

$$\text{(3.2)} \qquad V(\xi)=\begin{cases} u_- & \text{for } \xi<\psi'(u_-), \\ v & \text{for } \xi=\psi'(v), \quad u_-\leqq v\leqq u_+, \\ u_+ & \text{for } \xi>\psi'(u_+). \end{cases}$$

That is, V is the inverse function of the nondecreasing function ψ', with constant extensions outside its interval of definition. Note that when ξ is equal to the slope of one of the linear segments of ψ, V takes on all values on the interval $[w_j, v_{j+1}]$ which is the projection of the segment. We shall think of V as a piecewise continuous function with jumps at these slopes.

It is easily seen that the function $v(t, x)=V(x/t)$ is a weak solution of the Riemann problem (3.1). That is, v satisfies the differential equation at those points of continuity where $\psi'\neq 0$, and also satisfies the Rankine-Hugoniot relations

$$\frac{dx}{dt}=\frac{\varphi(v(t, x+0))-\varphi(v(t, x-0))}{v(t, x+0)-v(t, x-0)}$$

along each line where v is discontinuous. Because the lines in the (v, z)-plane which correspond to these jumps lie below the graph of φ, a remark of Keyfitz [23] shows that this solution satisfies the entropy condition E of O. A. Oleinik [19]. As Oleinik proved, it is the only weak solution which satisfies this condition.

We begin with a partial convergence result.

THEOREM 3.1. – *Let the conditions of Theorem* 2.1 *be satisfied. Then for every real number* ξ

$$\limsup_{t\to\infty} \max_{x\leqq\xi t} u(t, x)\leqq V(\xi+0),$$
$$\liminf_{t\to\infty} \min_{x\geqq\xi t} u(t, x)\geqq V(\xi-0),$$

uniformly in ε *for* $0<\varepsilon\leqq\varepsilon_0$.

Proof. – If $\xi\geqq\psi'(u_+)$, the first line follows from Theorem 2.1.

If $u_-\leqq\xi<\psi'(u_+)$, then $V(\xi+0)$ is the left endpoint of an interval on which $\psi=\varphi$ and $\varphi''>0$. For any v_1 in this interval choose $c\in(\xi, \varphi'(v_1))$. Then the line of slope c through the point $(v_1, \varphi(v_1))$ lies below the graph of φ for $v_1<v\leqq u_+$.

If $\xi<\psi'(u_-)$, choose $c\in(\xi, \psi'(u_-))$ and any $v_1>u_-=V(\xi+0)$ so close to u_- that the line of slope c through the point $(v_1, \varphi(v_1))$ lies below the graph of φ for $v_1<v\leqq u_+$.

In either of these cases, then, we can construct a travelling wave $\tilde{u}(x-ct)$ with the value v_1 at $-\infty$ and a value $\bar{M}$ larger than u_+ at $+\infty$. By Theorem 2.1 there is a T such that $u(T, x)<\bar{M}$. As before, we can find a translation y_0 such that $\tilde{u}(x-c\mathrm{T})>u(\mathrm{T}, x)$ and hence $\tilde{u}(x-ct)\geqq u(t, x)$ for $t\geqq \mathrm{T}$. Thus

$$\max_{x\leqq \xi t} u(t, x)\leqq \tilde{u}((\xi-c)t),$$

and since $\xi<c$, the right-hand side approaches v_1 as t goes to infinity. Since v_1 is arbitrarily close to $\mathrm{V}(\xi+0)$, we obtain the first line of (3.2). The second line is proved in the same manner.

We note that this theorem implies that if ξ is a point of continuity of V, then $u(t, \xi t)$ converges to $\mathrm{V}(\xi)$. We shall strengthen this result.

By combining Theorem 3.1 with Theorem 2.1, we obtain the following result, which gives limits on the asymptotic speeds with which changes can reach infinity.

THEOREM 3.2. — *Let the conditions of Theorem* 2.1 *be satisfied, and define the constants*

$$c_{\pm}^{*}=\psi'(u_{\pm}).$$

Then

$$\lim_{t\to\infty}\max_{x\leqq ct}|u(t, x)-u_{-}|=0 \qquad \text{for} \quad c<c_{-}^{*}$$

and

$$\lim_{t\to\infty}\max_{x\geqq ct}|u(t, x)-u_{+}|=0 \qquad \text{for} \quad c>c_{+}^{*},$$

uniformly in ε *for* $0<\varepsilon<\varepsilon_0$.

LEMMA 3.1. — *Let the hypotheses of Theorem* 2.1 *be satisfied. Suppose that the closed interval* $[r, s]$ *with* $u_{-}<\mathrm{V}(r)\leqq \mathrm{V}(s)<u_{+}$ *does not contain any point of discontinuity of* $\mathrm{V}(\xi)$, *and that* φ'' *is continuously differentiable and positive on an open interval which contains* $[\mathrm{V}(r), \mathrm{V}(s)]$. *Then*

$$\max_{rt\leqq x\leqq st}|u(t, x)-\mathrm{V}(x/t)|=O((\varepsilon/t)^{1/2})$$

as t goes to infinity.

Proof. — By hypothesis, there are A and B such that

$$u_{-}<\mathrm{A}<\mathrm{V}(r)\leqq \mathrm{V}(s)<\mathrm{B}<u_{+}$$

and φ'' is positive and continuously differentiable on the interval [A, B]. Therefore we can define the inverse function $\mathrm{H}(\xi)$ of φ' on this interval:

$$\mathrm{H}(\varphi'(v))=v \qquad \text{for} \quad \mathrm{A}\leqq v\leqq \mathrm{B} \tag{3.3}$$

Since $\mathrm{V}(s)<u_{+}$ and V is continuous at s, there is a $\mathrm{B}_1\in(\mathrm{V}(s), \mathrm{B})$ such that the tangent line to the graph of φ at $v=\mathrm{B}_1$ lies strictly below the graph of φ in the interval $(\mathrm{B}_1, u_{+}]$. As above, we modify φ strictly to the right of some $\nu>u_{+}$ so that this tangent line intersects the new graph at a point above ν. We define the corresponding travelling wave $\hat{u}(x-ct)$ by (2.3) with $y_0=0$, $c=\varphi'(B_1)$, and $b=u_+$.

We suppose for the moment that

$$\varphi(A)+(u_- - A)\varphi'(A)>\varphi(u_-), \tag{3.4}$$

so that the tangent line to the graph of φ at A intersects this graph at a point of the open interval (u_-, A). Then by continuity there are numbers $\mu\in(u_-, A)$ and $A_1\in(A, V(r))$ such that the interior of the line segment from $(\mu, \varphi(\mu))$ to $(A_1, \varphi(A_1))$ lies below the graph of φ, while it is tangent to this graph at the right end point. We choose $\tilde{c}\in(\varphi'(A), \varphi'(A_1))$. Then the line of slope $\tilde{c}$ through $(\mu, \varphi(\mu))$ lies below the graph of φ in the interval $(\mu, u_+]$. We can therefore define another travelling wave $\tilde{u}(x-\tilde{c}t)$ by the formula (2.3) with c replaced by $\tilde{c}$, $b=A$, and $y_0=0$. This wave increases from μ to a value above $\nu>u_+$.

We shall patch together the simple wave H and the travelling waves $\hat{u}$ and $\tilde{u}$ to obtain an upper bound for the solution u of (2.1). We begin by determining functions $\sigma(t)$ and $\hat{x}(t)$ with the property that when t is sufficiently large, the functions $H\left(\frac{x}{t}-\sigma(t)\right)$ and $\hat{u}(x-ct)$ and their x-derivatives coincide at $x=\hat{x}(t)$. That is,

$$\hat{u}(\hat{x}(t)-ct)=H\left(\frac{\hat{x}(t)}{t}-\sigma(t)\right)\equiv U(t), \tag{3.5}$$

and

$$\hat{u}'(\hat{x}-ct)=\frac{1}{t}H'\left(\frac{\hat{x}}{t}-\sigma\right). \tag{3.6}$$

We differentiate the first relation with respect to t and use the second relation to find that

$$\sigma'(t)=-\frac{\hat{x}-ct}{t^2}. \tag{3.7}$$

In order to estimate the quantity on the right, we use the relations (3.3) and (3.5) to find that

$$\frac{\hat{x}}{t}-\sigma=\varphi'(U). \tag{3.8}$$

Thus we need to evaluate U. We differentiate the formulas (2.3) and (3.3) and use (3.6) to find the condition

$$\varphi''(U)[\varphi(U)-cU-K]=\frac{\varepsilon a(U)}{t}. \tag{3.9}$$

When t is large, the right-hand side of this expression is near zero. Moreover, U must lie in the intersection of the ranges of $\hat{u}$ and H, which is the interval $(B_1, B]$. The left-hand side of (3.9) is positive in this interval but vanishes at B_1. By expanding the left-hand side around this value, we see that

$$U = B_1 + \sqrt{\frac{2\varepsilon a(B_1)}{\varphi''(B_1)^2 t}} + O\left(\frac{\varepsilon}{t}\right).$$

We substitute this expression into (3.8) and recall that $\varphi'(B_1)=c$ to see that

$$\hat{x} = ct + t\sigma + \sqrt{2\varepsilon a(B_1)t} + O(\varepsilon). \tag{3.10}$$

Thus (3.7) becomes

$$(t\sigma)' = -(2\varepsilon a(B_1)/t)^{1/2} + O(\varepsilon/t). \tag{3.11}$$

By integrating this expression, we find that

$$\sigma = -2\sqrt{\frac{2\varepsilon a(B_1)}{t}} + O\left(\frac{\varepsilon \log t}{t}\right) \tag{3.12}$$

and

$$\sigma' = \sqrt{\frac{2\varepsilon a(B_1)}{t^3}} + O\left(\frac{\varepsilon \log t}{t^2}\right).$$

We now notice that when $x = t[\varphi'(A) + \sigma(t)]$, $H\left(\frac{x}{t} - \sigma\right) = A$, while, since $\tilde{c} > \varphi'(A)$, $\tilde{u}(x - \tilde{c}t)$ approaches $\mu < A$ as t goes to infinity. Similarly we see that because $\tilde{c} < \varphi'(A_1)$, when $x = t[\varphi'(A_1) + \sigma(t)]$, $H = A_1 < \tilde{u}$ for large values of t. Moreover, the x-derivative of $\tilde{u}$ is $1/\varepsilon$ times a fixed function of $\tilde{u}$, while that of the function H is of order $1/t$. We conclude that for large t there is a unique function $\tilde{x}(t)$ such that

$$\tilde{u}(\tilde{x}(t) - \tilde{c}t) = H\left(\frac{\tilde{x}(t)}{t} - \sigma(t)\right).$$

Moreover, the representation (2.3) of $\tilde{u}$ shows that

$$\tilde{x}(t) = [\tilde{c} + O(\varepsilon/t)]\, t. \tag{3.13}$$

The implicit function theorem shows that $\tilde{x}(t)$ is smooth.

We now define the patched function

$$W(t, x) = \begin{cases} \tilde{u}(x - \tilde{c}t) & \text{for } x \leqq \tilde{x}(t) \\ H\left(\frac{x}{t} - \sigma(t)\right) & \text{for } \tilde{x}(t) \leqq x \leqq \hat{x}(t). \\ \hat{u}(x - \hat{c}t) & \text{for } x \geqq \hat{x}(t). \end{cases} \tag{3.14}$$

This function is defined for all x when t is sufficiently large.

An easy computation shows that the function $h(t, x)=\mathrm{H}((x-t\sigma)/t)$ satisfies the equation

$$\frac{\partial h}{\partial t}+\frac{\partial \varphi(h)}{\partial x}-\varepsilon \frac{\partial}{\partial x}\left(a(h) \frac{\partial h}{\partial x}\right)=-\mathrm{H}' t^{-1}(t\sigma)'-\varepsilon t^{-2}(a(\mathrm{H})\,\mathrm{H}')'.$$

We differentiate the relation (3.3) twice to see that H′ and H″/H′ are bounded. Thus we see from (3.11) that there is a T_0 such that when $t \geqq \mathrm{T}_0$ and $\tilde{x}(t)<x<\hat{x}(t)$,

$$\frac{\partial \mathrm{W}}{\partial t}+\frac{\partial \varphi(\mathrm{W})}{\partial x}-\varepsilon \frac{\partial^2 \mathrm{W}}{\partial x^2} \geqq 0. \tag{3.15}$$

Because $\tilde{u}$ and $\hat{u}$ are solution of the partial differential equation in (2.1), this inequality is also verified for $x<\tilde{x}(t)$ and $x>\hat{x}(t)$.

Thus the function W is defined and satisfies the inequality (3.15) for $t \geqq \mathrm{T}_0$ except on the curves $x=\tilde{x}(t)$ and $x=\hat{x}(t)$. It is continuous and its x-derivative has a negative jump across the first curve, while by construction W is continuously differentiable across the second curve.

We now choose a time $\hat{\mathrm{T}}>\mathrm{T}_0$ which is so large that $u(\hat{\mathrm{T}}, x) \leqq \nu$, and recall that the limit of W at $x=\infty$ is greater than ν. Since the limit of u at $x=-\infty$ is $u_-<\mu$, there is a translation b so that

$$u(\hat{\mathrm{T}}, x) \leqq \mathrm{W}(\hat{\mathrm{T}}, x+b).$$

We see that the functions $u(t+\hat{\mathrm{T}}, x)$ and $\mathrm{W}(t+\hat{\mathrm{T}}, x+b)$ satisfy the conditions in the Remark after Lemma 2.1, and we conclude that

$$u(t, x) \leqq \mathrm{W}(t, x+b) \quad \text{for} \quad t \geqq \hat{\mathrm{T}}. \tag{3.16}$$

Since $\tilde{c}=\varphi'(\mathrm{A}_1)<r \leqq s<\varphi'(\mathrm{B}_1)=\hat{c}$, the forms (3.10), (3.12), and (3.13) show that when t is sufficiently large, $\tilde{x}(t)<rt \leqq st<\hat{x}(t)$. Moreover, it is clear from the definitions (3.2) and (3.3) that on the interval $[r, s]$ we have $\mathrm{H}(\xi)=\mathrm{V}(\xi)$. Thus the upper bound immediately yields the inequality

$$u(t, x) \leqq \mathrm{V}(x/t)+O((\varepsilon/t)^{1/2}) \quad \text{for} \quad rt \leqq x \leqq st. \tag{3.17}$$

We have derived the upper bound under the additional hypothesis that the tangent line to the graph of φ at $v=\mathrm{A}$ intersects this graph at a point between u_- and A. If this is not the case, we introduce a modified three times continuously differentiable function $\tilde{\varphi}$ with the properties that

$$\tilde{\varphi}(v)=\varphi(v) \quad \text{for} \quad v \geqq \mathrm{V}(r),$$

that there exists a $\mathrm{C} \in (u_-, \mathrm{V}(r))$ such that

$$\tilde{\varphi}''(v)\begin{cases} <0 & \text{for} \quad u_-<v<\mathrm{C} \\ >0 & \text{for} \quad \mathrm{C}<v \leqq \mathrm{B}, \end{cases}$$

and that

$$\tilde{\varphi}'(v) \leqq \varphi'(v) \quad \text{for} \quad v \geqq u_-. \tag{3.18}$$

That is, $\tilde{\varphi}''$ is made so large over most of the interval $(u_-, V(r))$ that the inequality (3.18) is valid, but a little "hook" is put on near $v=u_-$. Then there are numbers $\mu<\tilde{A}<\tilde{A}_1$ in the interval $(u_-, V(r+0))$ such that $\varphi''>0$ on the interval $[\tilde{A}, B]$ and the line segment from $(\mu, \tilde{\varphi}(\mu))$ to $(\tilde{A}_1, \varphi(\tilde{A}_1))$ lies below the graph of $\tilde{\varphi}$ in its interior and is tangent to this graph at its right end point. We now construct the travelling wave $\hat{u}$ as before, but we define the travelling wave $\tilde{u}$ corresponding to the line segment from μ to $\tilde{A}_1$ with φ replaced by $\tilde{\varphi}$. We also define H as the inverse function of φ' on the interval $[\tilde{A}, B]$. We patch these three functions together in the same way as before to define the function W.

As above we find that W satisfies the differential inequality (3.15), but with φ replaced $\tilde{\varphi}$. Since W is increasing and $\varphi' \geqq \tilde{\varphi}'$, the differential inequality (3.15) for φ follows immediately. Thus we again obtain the upper bound (3.16).

Since $\tilde{\varphi}$ coincides with φ for $v \geqq V(r)$, we again see that $H=V$ on the interval $[r, s]$, and the upper bound (3.17) follows as before.

We now apply the same reasoning to the function $-u(t, -x)$ to obtain a lower bound of the same form. We combine these two bounds to find the statement of the Lemma.

Remark. – *We note that a time interval* T_0 *must elapse to bring the solution down to a neighborhood of the interval* $[u_-, u_+]$. *Thus, while Theorem* 3.2 *gives the right order of convergence uniformly in* ε *for* $\varepsilon>0$, *one must be careful if one wishes to fix t and let* ε *approach zero.*

We can combine the above theorems to obtain the following result.

THEOREM 3.3. – *Let* φ'' *be continuously differentiable and let its zeros be isolated. If the closed interval* $[r, s]$ *with* $r \geqq -\infty$ *and* $s \leqq \infty$ *contains no point of discontinuity of* V, *then*

$$\lim_{t \to \infty} \max_{rt \leqq x \leqq st} |u(t, x) - V(x/t)| = 0,$$

uniformly in ε *for* $0<\varepsilon \leqq \varepsilon_0$.

Proof. – Surround each of the finite set of points of the interval $[V(r), V(s)]$ where either $\varphi''=0$ or $v=u_\pm$ by an open interval (s_j, r_j) which is so short that

$$V(r_j) - V(s_j) < \delta/2. \tag{3.19}$$

On each of the intermediate intervals $[r_j, s_{j+1}]$ the function φ'' is positive. By continuity, the same is true on a larger interval $[A_j, B_j]$ where $A_j<r_j<s_{j+1}<B_j$. We now choose a new function $\tilde{\varphi}$ which is three times continuously differentiable in the interval $[A_j, B_j]$ and which has the

properties

$$\tilde{\varphi}(v) = \varphi(v) \quad \text{for} \quad v \geqq B_j,$$

and

$$\tilde{\varphi}' \leqq \varphi' \quad \text{for all } v.$$

Moreover, we choose $\tilde{\varphi}$ so that

$$(3.20) \quad |\tilde{\varphi}'(v) - \varphi'(v)| < \frac{1}{2}\delta \min_{V(A_j) \leqq u \leqq V(B_j)} \varphi''(u) \quad \text{for} \quad A_j \leqq v \leqq B_j.$$

The proof of Lemma 3.1 now shows that

$$\max_{r_j t \leqq x \leqq s_{j+1} t} [u(t, x) - \tilde{H}(x/t)] \leqq O((\varepsilon/t)^{1/2}),$$

where $\tilde{H}$ is the inverse function of $\tilde{\varphi}'$. Since V is the inverse function of φ' on the interval $[r, s]$, an obvious estimate and (3.20) yield the inequality

$$|\tilde{H}(\xi) - V(\xi)| \leqq \frac{1}{2}\delta \quad \text{for} \quad r_j \leqq \xi \leqq s_{j+} L.$$

Thus

$$\max_{r_j t \leqq x \leqq s_{j+1} t} [u(t, x) - V(x/t)] \leqq \delta/2 + O((\varepsilon/t)^{1/2}).$$

By applying this argument to the function $-u(t, -x)$ we obtain an analogous lower bound. It follows from these two bounds that if T is sufficiently large, then

$$\max_{r_j t \leqq x \leqq s_{j+1} t} |u(t, x) - V(x/t)| < \delta \quad \text{for} \quad t \geqq T$$

for all the intermediate intervals.

On the other hand, the inequalities (3.19) together with Theorem 3.1 show that if T is sufficiently large, the above inequalities are also valid on the intervals $s_j t \leqq x \leqq r_j t$. If the interval (r, s) contains one of the end points $u_{\pm}$, Theorem 3.2 gives the same result for the balance of the interval. Since $\delta > 0$ is arbitrary, we have proved the statement of the Theorem.

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