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An eigenvalue problem for the complex Monge-Ampère operator in pseudoconvex domains

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ABSTRACT. — A problem of existence and unicity of eigenvalues and corresponding eigenfunctions concerning the complex Monge-Ampère operator det $(\partial^2 u/\partial z_j \partial \bar{z}_k)$ and right-hand side of the forme $F(z, u)$ is studied in a bounded strictly pseudoconvex domain of $\mathbb{C}^n$.

RÉSUMÉ. — Un problème d'existence et d'unicité de valeurs propres et de fonctions correspondantes concernant l'opérateur de Monge-Ampère complexe det$(\partial^2 u/\partial z_j \partial \bar{z}_k)$ à second membre de la forme $F(z, u)$ est étudié dans un domaine borné strictement pseudoconvexe de $\mathbb{C}^n$.

Mots clés : Valeurs propres, fonctions propres, opérateur complexe de Monge-Ampère, point fixe.

Classification A.M.S. : 35 G 30, 35 J 65, 32 F 15, 35 P 15.

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1. INTRODUCTION

The aim of this paper is the study of the following eigenvalue problem (e.v.p.) for the complex Monge-Ampère operator in an arbitrary bounded strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$, *i.e.* to find out a couple (γ, u), where γ is a strictly positive constant, u is a realvalued function of the complex variable $z=(z_1, z_2, \ldots, z_n)$, $u \in C^2(\Omega) \cap C(\bar{\Omega})$ and (γ, u) is a solution of the boundary value problem (b.v.p.):

$$(1) \qquad \begin{cases} \det(u_{i\bar{j}}) = \gamma F(z, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u \text{ is strictly plurisubharmonic in } \Omega. \end{cases}$$

In a previous work [7], under the basic assumption that $F|z|, u)$ is a quasi-homogeneous function of order k, $k \leqq n$, w.r.t. the last variable [*see* $(\mathrm{iii})_1$ below], as well as under some monotony conditions and minimal smoothness of F, we were able to prove that, for every $r>0$, there exists an unique couple (γ, u), solution of (1), such that u admitts a prescribed norm, *i.e.* $\|u\|_{C^0(\bar{B})} = \sup_{\bar{B}} |u| = r$, in the case when the considered domain is a ball in $\mathbb{C}^n$. In such a situation, the unique solution u belongs to $C^2(\bar{B})$ and it turns out to be a radially-symmetric function. For example, the results of [7] hold for a right-hand side $F=(-u)^k$, $0<k \leqq n$.

Let us note that for $F=(-u)^n$, the problem (1) was completely investigated by P. L. Lions [10] in the case of the real Monge-Ampère operator considered in a bounded strictly convex domain $\Omega \subset \mathbb{R}^n$. Using some results on the Hamilton-Jacobi-Bellman equation, established in earlier works, the autor succeded to prove the existence of an unique eigenvalue $\gamma_1>0$ and an unique (up to multiplication by a constant) eigenfunction $u_1 \in C^\infty(\Omega) \cap C^{1,1}(\bar{\Omega})$. As it was pointed out in this work, the results remain valid without change in the case of complex Monge-Ampère operator. The method used in [10] is different from this one we used in [7], where Rutman's technique for linear positive operators was adjusted to the non-linear case.

In the present paper, using the above mentioned result of [10], we will show that under similar assumptions as in [7], the whole machinery of [7] holds not only for a ball, but also for an arbitrary pseudoconvex domain Ω in $\mathbb{C}^n$.

In paragraph 2 we state the main results and some auxiliary propositions which will be necessary for the proofs. The paragraph 3 deals with the uniqueness and the multiplicity of the eigenvalue and finally, in paragraph 4, we prove the existence of an eigenvalue and an eigenfunction for the problem (1).

2. STATEMENT OF THE RESULTS

As the right-hand side $F(p, q)$, where $p \in \bar{\Omega}$ and $q \leqq 0$, is concerned, we need the following assumptions:

(i) positivity: $F(p, q) > 0$ for $p \in \bar{\Omega}$ and $q < 0$;

$(\text{ii})_1$ monotonicity: $F(p, q)$ is a nonincreasing function w.r.t. q for every fixed $p \in \bar{\Omega}$;

$(\text{ii})_2$ monotonicity: $F(tp, q)$ is a nonincreasing function w.r.t. $t \geqq 0$ for every fixed $p \in \bar{\Omega}$ and $q < 0$;

$(\text{iii})_1$ quasi-homogeneity: there exists a $k > 0$ such that

$$F(p, tq) \geqq t^k F(p, q)$$

for every $t \in [0, 1]$, $p \in \bar{\Omega}$ and $q < 0$;

$(\text{iii})_2$ homogeneity: there exists a $k > 0$ such that $F(p, tq) = t^k F(p, q)$ for every $t \in [0, 1]$, $p \in \bar{\Omega}$ and $q < 0$.

Thus, the following theorems will be proved:

THEOREM 1 (Uniqueness). – *Let Ω be a bounded domain in $\mathbb{C}^n$ and* $F \in C^0(\bar{\Omega} \times]-\infty, 0])$ *satisfies the conditions* (i), $(\text{ii})_2$ *and* $(\text{iii})_1$ *for $k < n$. Then, for every $r > 0$, there exists* at most *one couple* (γ, u) *with* $\gamma > 0$, $u \in C^2(\Omega) \cap C(\bar{\Omega})$ *which is solution of* (1) *such that* $\|u\|_{C^0(\bar{\Omega})} = r$.

THEOREM 2 (Existence). – *Let Ω be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with a C^∞-smooth boundary and* $F \in C^\infty(\bar{\Omega} \times]-\infty, 0[) \cap C^0(\bar{\Omega} \times]-\infty, 0])$ *satisfies the conditions* (i), $(\text{ii})_1$ *and* $(\text{iii})_1$. *Then*

1° if $k < n$, for every $r > 0$ and for each integer $l \geqq 3$, there exists *at least* one couple (γ, u) with $\gamma > 0$, $u \in C^\infty(\Omega) \cap C^{l-1, 1}(\bar{\Omega})$ which is a solution of (1) such that $\|u\|_{C^l(\bar{\Omega})} = r$.

2° if $k = n$, for every $r > 0$ and for each integer $l \geqq 3$, there exists *at least* one couple (γ, u) with $\gamma > 0$, $u \in C^{l-1, l}(\bar{\Omega})$ which is a solution of (1) such that $\|u\|_{C^l(\bar{\Omega})} = r$.

As an immediate consequence of Theorem 1 and Theorem 2 we get the following existence and uniqueness result:

THEOREM 3. – *Let Ω be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with a C^∞-smooth boundary and* $F \in C^\infty(\bar{\Omega} \times]-\infty, 0[) \cap C^0(\bar{\Omega} \times]-\infty, 0])$ *satisfies the conditions* (i), $(\text{ii})_1$, $(\text{ii})_2$ *and* $(\text{iii})_2$ *for $k < n$. Then, for every $r > 0$, there exists an* unique *couple* (γ, u) *with* $\gamma > 0$, $u \in C^\infty(\bar{\Omega})$ *which is a solution of* (1) *such that* $\|u\|_{C^0(\bar{\Omega})} = r$.

Remarks. – 1. In fact we will prove a slightly more general result than in Theorem 1. More precisely, under the assumptions of Theorem 1, when $k \leqq n$, we will show that the e.v.p. (1) has an unique eigenvalue $\gamma > 0$ and this eigenvalue has a multiplicity of order 1 only if $k < n$.

2. The results of theorems 1, 2 and 3 hold for a right-hand side $F = (-u)^k$, $0 < k < n$.

3. For $k=n$, using Theorem 2 and the unicity theorem of P.-L. Lions [10], Theorem 1, we obtain the following slightly more general regularity result than in [10]: the e.v.p. (1) with $F=(-u)^n$ has an unique eigenvalue $\gamma_1>0$ and an unique up to scaling eigenfunction $u\in C^\infty(\bar{\Omega})$.

A large part of the ingredients of the proofs consists of some facts of the cone theory in a Banach space. Let us now recall those which will be useful for our purpose.

PROPOSITION 1 [6], p. 242. — *Let* E *be a Banach space with a cone* K. *If* $u, v\in K$ *and* $v-\gamma u\notin K$ *for some positive constant* γ, *then* $v-tu\in K$ *implies* $t<\gamma$.

THEOREM 4 (E. Rothe) [6], p. 244. — *Let* T *be a positive compact operator in the Banach space* E *with a cone* K. *Then, for every* $r>0$, T *has* at least *one eigenfunction* $v_r\in K$, $\|v_r\|=r$, *corresponding to the positive eigenvalue* λ_r, *if*

$$v\in K,\ \inf_{\|v\|=r}\|Tv\|>0.$$

On the other hand, we will need the following lemmas.

LEMMA 1. — *Suppose* Ω *is a bounded strictly pseudoconvex domain in* $\mathbb{C}^n$ *with a* C^∞*-smooth boundary. Then there exist a positive constant* C *and a strictly plurisubharmonic in* Ω *function* $w\in C^2(\Omega)\cap C^0(\bar{\Omega})$ *which satisfies the inequality*

$$\det(w_{i\bar{j}})\leqq C^n(-w)^n \quad \text{in } \Omega \tag{2}$$

and $w=0$ *on* $\partial\Omega$.

This lemma follows easily from the result of P.-L. Lions [10] which asserts that the problem "$\det(u_1)_{i\bar{j}}=\lambda_1^n(-u_1)^n$ in Ω, $u_1=0$ on $\partial\Omega$" possesses at least one solution, where $\lambda_1>0$ and $u_1\in C^\infty(\Omega)\cap C^{1,1}(\bar{\Omega})$. The proof in [10] was given for the real Monge-Ampère operator, but it also holds without change for the complex case.

Finally, we will make use of the following comparison principle obtained by L. Caffarelli, J. J. Kohn, L. Nirenberg and J. Spruck:

LEMMA 2 [4], p. 215, lemma 1.1. — *If* $\Omega\subset\mathbb{C}^n$ *is a bounded domain and if,* $v, w\in C^\infty(\bar{\Omega})$ *are plurisubharmonic, with* v *strictly plurisubharmonic and such that*

$$\det(v_{i\bar{j}})\geqq\det(w_{i\bar{j}})$$

and $v\leqq w$ *on* $\partial\Omega$, *then* $v\leqq w$ *in* $\bar{\Omega}$.

3. UNIQUENESS AND MULTIPLICITY OF THE EIGENVALUE

First we will prove the uniqueness of the eigenvalue if $k \leqq n$. Without loss of generality we can assume that Ω contains the origin of $\mathbb{C}^n$.

Proof of Theorem 1. — Suppose there exist two different eigenvalues λ, $\mu\,(\lambda > \mu)$ with corresponding strictly plurisubharmonic eigenfunctions u and $\tilde{v}$ resp.

Let $a > 1$ be a sufficiently close to 1 positive constant such that $\lambda > \mu a^{2n}$. Now consider the domain $\Omega_a = \{ z \in \mathbb{C}^n,\ az \in \Omega \}$ and the function $v(z) = \tilde{v}(az)$, which is strictly plurisubharmonic in Ω_a, admitts a prescribed norm $\| v \|_{C^0(\bar{\Omega})} = r$ and satisfies the e.v.p.:

$$\left\{ \begin{aligned} \det(v_{i\bar{j}}) &= \mu a^{2n} F(az, v) \quad \text{in } \Omega_a \\ v &= 0 \quad \text{in } \partial\Omega_a. \end{aligned} \right. \tag{3}$$

Introduce the function $w = v/u$. It is clear that $w > 0$ in Ω_a, $w \in C^2(\Omega_a) \cap C^0(\bar{\Omega}_a)$ and $w = 0$ on $\partial\Omega_a$.

If we suppose that w attains its maximum at the interior point $z_0 \in \Omega_a$, then we will have the estimate:

$$w(z_0) \geqq \frac{v(z_1)}{u(z_1)} = \frac{\| v \|_{C^0(\bar{\Omega}_a)}}{|u(z_1)|} = \frac{r}{|u(z_1)|} \geqq 1, \tag{4}$$

where $z_1 \in \Omega_a$ is the point in which v attains its maximum. Taking into account (4), $(\text{ii})_2$, $(\text{iii})_1$ and the fact that $w_i(z_0) = w_{\bar{j}}(z_0) = 0$ at the maximum point of w, we get successively the following inequalities:

$$\begin{aligned} \mu a^{2n} F(az_0, v(z_0)) &= \det(v_{i\bar{j}}) \\ &= \det(uw_{i\bar{j}} + u_i w_{\bar{j}} + u_{\bar{j}} w_i + w u_{i\bar{j}}) \\ &= \det(uw_{i\bar{j}} + w u_{i\bar{j}}) \\ &\geqq w^n(z_0) \det(u_{i\bar{j}}) \\ &= \lambda w^n(z_0) F(z_0, u(z_0)) \\ &= \lambda w^n(z_0) F\left(z_0, \frac{v(z_0)}{w(z_0)}\right) \\ &\geqq \lambda w^{n-k}(z_0) F(z_0, v(z_0)) \\ &\geqq \lambda w^{n-k}(z_0) F(az_0, v(z_0)). \end{aligned}$$

Note that in the above calculations we also made use of the inequality

$$\det(uw_{i\bar{j}} + w u_{i\bar{j}}) \geqq \det(w u_{i\bar{j}})$$

at the point z_0 where the matrix $(w_{i\bar{j}})$ is non-negative [I. Gel'fand — Lectures on linear algebra, GMTTL, Leningrad, 1951 (Russian), Ch. II.12.2, Theorem 4].

Since v is strictly plurisubharmonic, then $v(z_0) < 0$, and from (i), we can deduce that $F(az_0, v(z_0)) > 0$. Consequently, $w^{n-k}(z_0) \leqq \mu a^{2n}/\lambda < 1$ which contradicts (4).

Thus w attains its maximum on $\partial\Omega$, *i.e.* $w\equiv 0$ and hence $v\equiv 0$ in Ω which is impossible.

Let us now prove that the unique eigenvalue $\gamma>0$ has a multiplicity of order one when $k<n$. For this purpose, suppose u and $\tilde{v}$ are two different eigenfunctions corresponding to the same γ, *i.e.* there exists a point $z_1\in\Omega$ in which $\tilde{v}(v_1)>u(z_1)$.

Let $b<1$ be a positive constant sufficiently close to 1 so that the function $v(z)=b^{2n/(k-n)}\tilde{v}(bz)$ satisfies the inequality $v(z_1)>u(z_1)$. Consider the auxiliary function $w=u/v$ in Ω. Thus, $w\in C^2(\Omega)\cap C^0(\bar{\Omega})$, $w\geqq 0$ and $w(z_1)>1$.

If we suppose that w attains its maximum at the interior point $z_2\in\Omega$, then we can deduce the inequality

$$\|w\|_{C^0(\bar{\Omega})}=w(z_2)\geqq w(z_1)>1. \tag{5}$$

By means of the required properties of F and (3), simple computations give us the following inequalities valid at the point z_2:

$$\begin{aligned}
\gamma F(z_2, u(z_2)) &= \det(u_{i\bar{j}})\\
&= \det(w_{i\bar{j}}v+w_i v_{\bar{j}}+w_{\bar{j}}v_i+wv_{i\bar{j}})\\
&= \det(w_{i\bar{j}}v+wv_{i\bar{j}})\\
&\geqq \det(wv_{i\bar{j}})=w^n(z_2)\det(v_{i\bar{j}})\\
&= \gamma\, b^{2n^2/(k-n)+2n}\, w^n(z_2)\, F(bz_2, \tilde{v}(bz_2))\\
&\geqq \gamma\, b^{2nk/(k-n)}\, w^n(z_2)\, F(z_2, b^{2n/(k-n)}v(z_2))\\
&\geqq \gamma\, w^n(z_2)\, F\left(z_2, \frac{u(z_2)}{w(z_2)}\right)\\
&\geqq \gamma\, w^{n-k}(z_2)\, F(z_2, u(z_2)).
\end{aligned}$$

Since u is strictly plurisubharmonic, it follows that $u(z_2)<0$ and $F(z_2, u(z_2))>0$. Consequently, $w(z_2)\leqq 1$, which contradicts (5). Hence w attains its maximum on the boundary $\partial\Omega$, *i.e.* $w\equiv 0$ and thus $w\equiv 0$ in Ω which is impossible because of the choice of the eigenfunction. Thus Theorem 1 is proved.

4. EXISTENCE OF AN EIGENVALUE AND AN EIGENFUNCTION

We start with the *proof of Theorem* 2: Let ε be a fixed positive constant. Consider the regularized problem:

$$\begin{aligned}
&\det(u_{i\bar{j}})=\gamma F(z, u-\varepsilon) \quad \text{in } \Omega,\\
&u=0 \quad \text{on } \partial\Omega,\\
&u \text{ is strictly plurisubharmonic in } \Omega.
\end{aligned} \tag{6}$$

We will prove that for every $\varepsilon>0$ and every $r>0$ there is at least one solution (γ, u) of (6), where γ is a positive constant and $u\in C^\infty(\bar{\Omega})$,

$\| u \|_{C^l(\bar{\Omega})} = r$. Here, l is an integer and γ and u depend of ε and r, but for convenience the indices ε and r will be omitted.

Consider the Banach space $C^l(\bar{\Omega})$ with the cone K of all non-positive functions. Define the operator T acting in K as follows: for every $v \in K$, $T[v] = u$ is a function belonging to the class $C^{l+1}(\bar{\Omega})$, being the unique solution of the b.v.p.:

$$\det(u_{i\bar{j}}) = F(z, v - \varepsilon) \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{7}$$

$$u \text{ is strictly plurisubharmonic in } \bar{\Omega}.$$

Since u is strictly plurisubharmonic, then $u < 0$ and consequently $T[v] \in K$. From the regularity of u it is easy to check that T is a continuous compact operator for every $\varepsilon > 0$. Introduce the constant

$$R^n = \min_{z \in \bar{\Omega}} F(z, -2r)$$

It is clear that $R = R(r) > 0$.

Without loss of generality we may assume that the function w of Lemma 1 has a prescribed norm, namely $\| w \|_{C^0(\bar{\Omega})} = 2r$, since constant times w also satisfies (2). Finally, assume $\varepsilon \leq 2r$. Thus, using (ii_1), $(iii)_1$ and (2) we get:

$$\begin{aligned} \det(u_{i\bar{j}}) &= F(z, v - \varepsilon) \geq F(z, -\varepsilon) \\ &\leq \left(\frac{\varepsilon}{2r}\right)^k F(z, -2r) \geq \left(\frac{\varepsilon}{2r}\right)^k R^n \\ &= \left(\frac{\varepsilon}{2r}\right)^k \frac{R^n}{C^n} C^n \left(-\frac{w}{2r}\right)^n \\ &\geq \left(\frac{\varepsilon}{2r}\right)^k \frac{R^n}{C^n} \det\left[\left(\frac{w}{2r}\right)_{i\bar{j}}\right] \\ &= \det\left[\left(\frac{\varepsilon}{2r}\right)^{k/n} \left(\frac{R w}{2Cr}\right)_{i\bar{j}}\right]. \end{aligned}$$

Since $u = w = 0$ on $\partial\Omega$, applying Lemma 2, we can deduce that

$$u \leq \left(\frac{\varepsilon}{2r}\right)^{k/n} \left(\frac{R w}{2Cr}\right) \quad \text{in } \Omega. \tag{8}$$

Consequently:

$$\begin{aligned} \inf_{v \in K,\ \| v \|_{C^l(\bar{\Omega})} = r} \| T[v] \|_{C^l(\bar{\Omega})} &= \inf \| u \|_{C^l(\bar{\Omega})} \\ &\geq \| u \|_{C^0(\bar{\Omega})} \geq \left(\frac{\varepsilon}{2r}\right)^{k/n} \left(\frac{R}{2Cr}\right) \| w \|_{C^0(\bar{\Omega})} = \left(\frac{\varepsilon}{2r}\right)^{k/n} \left(\frac{R}{C}\right) > 0. \end{aligned}$$

Thus we are able to apply Theorem 4 which implies that the operator T has a positive eigenvalue $\lambda^{\varepsilon, r}$ and an eigenfunction $u^{\varepsilon, r} \in K$, such that

$\| u^{\varepsilon, r} \|_{C^l(\bar{\Omega})} = r$ and, of course

$$T[u^{\varepsilon, r}] = \lambda^{\varepsilon, r} u^{\varepsilon, r}.$$

By virtue of the properties of T it follows that $u^{\varepsilon, r}$ is a strictly plurisubharmonic function in $\bar{\Omega}$. For convenience we will omit the indices ε and r and we will denote $\lambda^{\varepsilon, r} = \gamma^n$. Hence (γ, u) is a solution of the b.v.p.:

$$\det(u_{i\bar{j}}) = \gamma^n F(z, u-\varepsilon) \quad \text{in } \Omega, \qquad u=0 \quad \text{on } \partial\Omega.$$

Since $u \in C^{l+1}(\bar{\Omega})$ and for every $\varepsilon > 0$, the above equation is uniformly elliptic in $\bar{\Omega}$, using the regularity theory, we get $u \in C^\infty(\bar{\Omega})$.

Our next step is to deduce a majoration of γ by a constant independant of ε. To do this we will apply Proposition 1 several times.

Since $u(z_3) - w(z_3) = u(z_3) + \| w \|_{C^0(\bar{\Omega})} \geqq 2r - r > 0$ at the maximum point z_3 of w [note that $\| w \|_{C^0(\bar{\Omega})} = -w(z_3)$], then $u - w \notin K$. From (8) and Proposition 1 we get the estimate:

$$\left(\frac{\varepsilon}{2r}\right)^{k/n} \left(\frac{R}{2Cr}\right) < 1. \tag{9}$$

Now, using (8), (9), $(ii)_1$, $(iii)_1$ and Lemma 1 we can deduce the inequalities:

$$\begin{aligned} \det(u_{i\bar{j}}) &= \gamma^n F(z, u-\varepsilon) \geqq \gamma^n F(z, u) \\ &\geqq \gamma^n F\left(z, \left(\frac{\varepsilon}{2r}\right)^{k/n} \left(\frac{Rw}{2Cr}\right)\right) \\ &\geqq \gamma^n \left(\frac{\varepsilon}{2r}\right)^{k^2/n} \left(\frac{R}{2Cr}\right)^k F(z, w) \\ &\geqq \gamma^n \left(\frac{\varepsilon}{2r}\right)^{k^2/n} \left(\frac{R}{2Cr}\right)^k \left(-\frac{w}{2r}\right)^k F(z, -2r) \\ &\geqq \gamma^n \left(\frac{\varepsilon}{2r}\right)^{k^2/n} \left(\frac{R}{2Cr}\right)^k \left(-\frac{w}{2r}\right)^n R^n \\ &\geqq \gamma^n \left(\frac{\varepsilon}{2r}\right)^{k^2/n} \left(\frac{R}{2Cr}\right)^{k+n} \det(w_{i\bar{j}}), \end{aligned}$$

that is:

$$\det(u_{i\bar{j}}) \geqq \det\left(\left[\left(\frac{R\gamma}{2Cr}\right)\left(\frac{\varepsilon}{2r}\right)^{k^2/n^2} \left(\frac{R}{2Cr}\right)^{k/n} w\right]_{i\bar{j}}\right).$$

By means of the comparision principle (Lemma 2), one has:

$$u \leqq \gamma \left(\frac{R}{2Cr}\right)^{1+(k/n)} \left(\frac{\varepsilon}{2r}\right)^{k^2/n^2} w \quad \text{in } \Omega \tag{10}$$

from which, taking into account Proposition 1, one get

$$\gamma \left(\frac{R}{2Cr}\right)^{1+(k/n)} \left(\frac{\varepsilon}{2r}\right)^{k^2/n^2} < 1.$$

Now, repeat this procedure p times. We obtain

$$(11) \qquad u \leqq \gamma^{1+(k/n)+\ldots+(k/n)^{p-1}} \left(\frac{\mathrm{R}}{2\,\mathrm{C}\,r}\right)^{1+(k/n)+\ldots+(k/n)^{p}} \left(\frac{\varepsilon}{2\,r}\right)^{(k/n)^{p+1}} w$$

in Ω, and

$$(12) \qquad \gamma^{1+(k/n)+\ldots+(k/n)^{p-1}} \left(\frac{\mathrm{R}}{2\,\mathrm{C}\,r}\right)^{1+(k/n)+\ldots+(k/n)^{p}} \left(\frac{\varepsilon}{2\,r}\right)^{(k/n)^{p+1}} < 1.$$

In a first case, when $k=n$, inequality (12) reduces to

$$\gamma^{p} \left(\frac{\mathrm{R}}{2\,\mathrm{C}\,r}\right)^{p+1} \left(\frac{\varepsilon}{2\,r}\right) < 1$$

which holds for every integer p and for fixed ε and r only if

$$\frac{\gamma^{\mathrm{R}}}{2\,\mathrm{C}\,r} \leqq 1.$$

In the second possible case, when $k<n$, letting $p \to +\infty$ in (12), we deduce

$$\left(\frac{\gamma^{\mathrm{R}}}{2\,\mathrm{C}\,r}\right)^{n/(n-k)} \leqq 1.$$

So, in both cases, γ satisfies the estimate:

$$(13) \qquad \gamma \leqq \frac{2\,\mathrm{C}\,r}{\mathrm{R}} = \frac{2\,\mathrm{C}\,r}{\min\limits_{z \in \bar{\Omega}} \mathrm{F}(z, -2\,r)},$$

where the right-hand side is a constant independent of the choice of ε.

In this way, because of (13) and the equality $\|u\|_{\mathrm{C}^{l}(\bar{\Omega})} = r$, one can substract a sequence $\{\varepsilon_m\}_{m=1}^{\infty}$ such that $\varepsilon_m \to 0$, with $\gamma^{\varepsilon_m, r} \to \gamma^{r}$, $u^{\varepsilon_m, r} \to u^{r}$, where $0 \leqq \gamma^{r} \leqq \dfrac{2\,\mathrm{C}\,r}{\mathrm{R}}$, $u^{r} \in \mathrm{C}^{l-1,1}(\bar{\Omega})$, $\|u\|_{\mathrm{C}^{l}(\bar{\Omega})} = r$ and (γ^{r}, u^{r}) is a solution of

$$(14) \qquad \begin{cases} \det(u^{r}_{i\bar{j}}) = (\gamma^{r})^{n}\, \mathrm{F}(z, u^{r}) & \text{in } \Omega \\ u^{r} = 0 & \text{on } \partial\Omega, \end{cases}$$

for each $r>0$.

It is clear that $\gamma^{r}>0$ since u^{r} is a non-trivial solution of (14). Thus, the point 2° of Theorem 2 was proved.

Let us return to the case $k<n$. Letting $p \to \infty$ for fixed $\varepsilon>0$ and then letting $\varepsilon \to 0$ in (11), we have

$$u^{r} \leqq \left(\frac{\gamma^{r}\,\mathrm{R}}{2\,\mathrm{C}\,r}\right)^{n/(k-n)} w \quad \text{in } \Omega.$$

Consequently,

$$\det(u^r_{i\bar{j}}) = (\gamma^r)^n F(z, u^r) \geqq (\gamma^r)^n F\left(z, \left(\frac{\gamma^r R}{2Cr}\right) w\right) \geqq A(\Omega') > 0,$$

where $A(\Omega')$ is a constant and Ω' an arbitrary compact subdomain of Ω. Thus u^r is a strictly plurisubharmonic function in Ω. Hence the equation (14) is uniformly elliptic on each compact domain Ω', $\bar{\Omega}' \subset \Omega$. The general elliptic regularity theory allows to conclude that $u^r \in C^\infty(\Omega) \cap C^{l-1,1}(\bar{\Omega})$. Thus Theorem 2 is proved.

Proof of Theorem 3. — From Theorem 2 we know that for every $r > 0$ and for every integer $l \geqq 3$ there exists at least one couple $(\gamma^{r,l}, u^{r,l})$, where $\gamma^{r,l} > 0$ and $u^{r,l} \in C^\infty(\Omega) \cap C^{l-1,1}(\bar{\Omega})$, which is a solution of (1) and such that $\|u^{r,l}\|_{C^l(\bar{\Omega})} = r$.

Since $\tilde{u} = \dfrac{r_0 u^{r,l}}{\|u^{r,l}\|_{C^0(\bar{\Omega})}}$ is a strictly plurisubharmonic function, which is solution of the b.v.p.:

$$\begin{cases} \det(\tilde{u}_{i\bar{j}}) = \gamma^{r,l} \left(\dfrac{r_0}{\|u^{r,l}\|_{C^0(\bar{\Omega})}}\right)^{n-k} F(z, \tilde{u}) \\ \tilde{u} = 0 \quad \text{on } \partial\Omega, \end{cases}$$

with prescribed norm $\|\tilde{u}\|_{C^0(\bar{\Omega})} = r_0$, then, from Theorem 1, it follows that, denoting

$$\gamma^0 = \gamma^{r,l} \left(\frac{r_0}{\|u^{r,l}\|_{C^0(\bar{\Omega})}}\right)^{n-k},$$

and

$$u^0 = \frac{r_0 u^{r,l}}{\|u^{r,l}\|_{C^0(\bar{\Omega})}},$$

for every $r > 0$ and every integer $l \geqq 3$, we succeded to find a positive constant γ^0 and a function $u^0 \in C^\infty(\bar{\Omega})$, strictly plurisubharmonic in Ω, such that (γ^0, u^0) is solution of (1). Thus Theorem 3 is proved.

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