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On the existence of surfaces with prescribed mean curvature and boundary in higher dimensions

by

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ABSTRACT. – Given a Lipschitz continuous function $H : \mathbf{R}^{n+1} \to \mathbf{R}$ we construct hypersurfaces in $\mathbf{R}^{n+1}$ with prescribed mean curvature H which satisfy a Plateau boundary condition provided the mean curvature function H satisfies a certain isoperimetric condition. For $n \leqq 6$ these surfaces are free from interior singularities.

Key words : Geometric Measure Theory, Surfaces of Prescribed Mean Curvature.

RÉSUMÉ. – Étant donné une fonction continue lipschitzienne $H : \mathbf{R}^{n+1} \to \mathbf{R}$, nous construisons des hyper-surfaces dans $\mathbf{R}^{n+1}$ avec une courbure moyenne imposée H qui satisfait une condition frontière de Plateau pourvu que la fonction de courbure moyenne H satisfasse une certaine condition isopérimétrique. Pour $n \leqq 6$, ces surfaces n'ont pas de singularités intérieures.

Classification A.M.S. : 49 Q 20, 53 A 10, 49 Q 10.

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1. INTRODUCTION

The motivation for our work originates from the following problem: Given a C^2 domain G in $\mathbb{R}^{n+1}$, a nontrivial $n-1$ dimensional boundary B in G and a continuous function $H : \bar{G} \to \mathbb{R}$, does there exist an n dimensional hypersurface in G with boundary B and prescribed mean curvature H? In the context of 2 dimensional parametric surfaces in $\mathbb{R}^3$ this problem – know as Plateau's problem for surfaces of prescribed mean curvature – was studied by various authors (*see* [St1] for a list of references). To treat the higher dimensional situation we work in the setting of geometric measure theory, *i. e.*, $B \neq 0$ is a closed rectifiable $n-1$ current with $\operatorname{spt}(B) \subset G$ and $\mathscr{H}^n(\operatorname{spt}(B)) = 0$ and $\mathscr{F}(B; \bar{G})$ – the class of admissible currents – is the set of all locally rectifiable integer multiplicity n currents T in $\mathbb{R}^{n+1}$ with finite mass $M(T)$, support $\operatorname{spt}(T) \subset \bar{G}$ and boundary $\partial T = B$. We say that $T \in \mathscr{F}(B; \bar{G})$ solves the *mean curvature problem for* H *and* B *in* $\bar{G}$, if T safisties

$$(1.1) \qquad \int (Dg \,.\, P_T + H * \mathbf{T} \,.\, g)\, d\|T\| = 0$$

for all $g \in C^1_0(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$ with $\operatorname{spt}(g) \cap \operatorname{spt}(B) = \varnothing$. Here

$$* : \Lambda_n \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$$

denotes the usual isomorphism defined by the standard volume form $dx_1 \wedge \ldots \wedge dx_{n+1}$, $P_T(x) \in \operatorname{Hom}(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$ stands for the orthogonal projection on the subspace associated with the simple n vector $\mathbf{T} \in \Lambda_n \mathbb{R}^{n+1}$ and $Dg \,.\, P_T = \operatorname{trace}(Dg\, P^*_T)$ is the usual inner product in $\operatorname{Hom}(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$. Our convention is that the mean curvature of an oriented hypersurface is the sum of its principal curvatures not the arithmetic mean.

Oriented frontiers (boundaries of sets of finite perimeter) with prescribed mean curvature were treated by Massari [Ma]. However, in this context one cannot solve boundary value problems for hypersurfaces with prescribed mean curvature everywhere in $\mathbb{R}^{n+1}$.

The first successful treatment of the above mean curvature problem in the setting of geometric measure theory and $H \neq 0$ is due to Duzaar and Fuchs [DF1]. They proved that the problem can be solved for H and B in the open ball $\mathbf{B}(R)$ of radius R if

$$(1.2) \qquad \|H\|_{L^\infty(\mathbf{B}(R))} < \frac{n+1}{R}, \qquad |H(x)| < \frac{n}{R} \quad \text{for } x \in \partial \mathbf{B}(R).$$

In [DF2], [DF3] and [DF4] they also showed solvability of the mean curvature problem for H and B in $\mathbb{R}^{n+1}$ provided

$$(1.3) \qquad \|H\|_{L^\infty(\mathbb{R}^{n+1})} < n \sqrt[n]{\frac{\alpha_{n+1}}{2 A_B}},$$

where A_B denotes the area of a mass minimizing current with boundary B and $\alpha_{n+1}=\mathscr{L}^{n+1}(\mathbf{B}(1))$. Assuming (1.3) they also solved the problem in a bounded C^2 domain in $\mathbb{R}^{n+1}$ under an additional assumption relating the mean curvature of ∂G to the prescribed mean curvature H.

The present paper originates from an attempt to generalize the results of Steffen [St1], [St2] to the higher dimensional case. Our main result reads as follows: *Let* G *be a* C^2 *domain in* $\mathbb{R}^{n+1}$ *such that* ∂G *has nonnegative mean curvature* $\mathscr{H}_{\partial G}$. *Suppose furthermore that* B *is a closed rectifiable* $n-1$ *current with* $\operatorname{spt}(B)\subset G$, $\mathscr{F}(B;\bar{G})\neq\varnothing$ *and* $\mathscr{H}^n(\operatorname{spt}(B))=0$ *and that* $H:\bar{G}\to\mathbb{R}$ *is continuous with*

$$(1.4)\qquad |H(x)|\leqq\mathscr{H}_{\partial G}(x)\quad\text{for}\quad x\in\partial G.$$

Then there exists a current $T\in\mathscr{F}(B;\bar{G})$ *with prescribed mean curvature* H *on* $\mathbb{R}^{n+1}\setminus\operatorname{spt}(B)$ *provided one of the following conditions is satisfied:*

1) $\left[\int_G |H|^{n+1}\,d\mathscr{L}^{n+1}\right]^{1/(n+1)}<\frac{1}{\gamma_{n+1}}$

2) $\sup_{t\in\mathbb{R}}\left[\int_{G_t}|H(z,t)|^n\,d\mathscr{L}^n(z)\right]^{1/n}<\frac{1}{\gamma_n}$

3) $\sup_{s>0}[s^{n+1}\mathscr{L}^{n+1}(\{x\in G:|H(x)|\geqq s\})]^{1/(n+1)}<n\sqrt[n+1]{\alpha_{n+1}}$

4) $\sup_{s>o,\,t\in\mathbb{R}}[s^n\mathscr{L}^n(\{z\in G_t:|H(z,s)|\geqq s\})]^{1/n}<(n-1)\sqrt[n]{\alpha_n}.$

(Here $G_t:=\{z\in\mathbb{R}^n:(z,t)\in G\}$ and γ_l denotes the optimal isoperimetric constant $(l+1)\sqrt[l+1]{\alpha_{l+1}}$.)

We now briefly describe the proof of our main theorem and the organization of this paper. In section 2 we define an energy functional $\mathbf{E}_H(T)$ having the property that the first variation $\delta\mathbf{E}_H(T,g)$ with respect to $C_0^1(\mathbb{R}^{n+1},\mathbb{R}^{n+1})$ vector fields g with $\operatorname{spt}(g)\cap\operatorname{spt}(B)=\varnothing$ is given by

$$\delta\mathbf{E}_H(T,g)=\int(Dg.P_T+H*\mathbf{T}.g)\,d\|T\|,$$

and then study the following *obstacle problem:* Minimize the energy functional in $\mathscr{F}(F;\bar{G})$. In theorem 2.1 we prove the existence of an energy minimizing current in $\mathscr{F}(B;\bar{G})$, provided H satisfies a certain *isoperimetric condition* (for the details we refer the reader to section 2). In theorem 3.4 we show for minimizing currents $T\in\mathscr{F}(B;\bar{G})$, that the first variation $\delta\mathbf{E}_H(T,.):C_0^1(\Omega,\mathbb{R}^{n+1})\to\mathbb{R}$, with $\Omega:=\mathbb{R}^{n+1}\setminus\operatorname{spt}(B)$, is represented by integration, *i.e.* we have

$$\delta\mathbf{E}_H(T,g)=\int_\Omega g.\nu\Lambda\,d\|T\|$$

whenever $g \in C_0^1(\Omega, \mathbb{R}^{n+1})$. Here $\Lambda : \Omega \to \mathbb{R}$ is a $\|T\|$ almost unique $\|T\|$ measurable function with support in $\mathrm{spt}(T) \cap \partial G$. Moreover, for $\|T\|$ almost all $x \in \mathrm{spt}(T) \cap \partial G$ we have

$$0 \leqq \Lambda(x) \leqq (H(x) * \mathbf{T} . \nu(x) - \mathscr{H}_{\partial G}(x))^+.$$

Then, in corollary 3.5 we show that (1.4) is a sufficient condition for the vanishing of Λ, so that T has in the distributional sense mean curvature H on $\Omega = \mathbb{R}^{n+1} \setminus \mathrm{spt}(B)$. For $H \in L^\infty(\bar{G})$ a standard monotonicity formula enables us to deduce compactness of spt (T) (*see* corollary 3.7). In section 4 we define locally energy minimizing currents on $\mathbf{B}(a, R)$. Decomposing a locally energy minimizing current locally into a sum of boundaries of sets of finite perimeter we can use the regularity result of Massari [Ma] to conclude the following optimal regularity theorem: *Suppose* $H : \mathbf{B}(a, R) \to \mathbb{R}$ *is locally Lipschitz continuous and* T *is locally energy minimizing on* $\mathbf{B}(a, R)$, *then* Reg(T) *is a* $C^{2,\mu}$ *submanifold in* $\mathbb{R}^{n+1}$, *for every* $0 < \mu < 1$, *on which* T *has mean curvature* H *and locally constant integer multiplicity. Moreover, the singular set* Sing(T) *is empty for* $n \leqq 6$, *locally finite for* $n = 7$ *and of Hausdorff dimension at most* $n - 7$ *for* $n \geqq 8$.

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2. NOTATIONS AND EXISTENCE OF ENERGY MINIMIZING CURRENTS

$\mathbf{B}(a, \rho)$ will denote the open ball with center a and radius ρ; in case $a = 0$ we write $\mathbf{B}(\rho)$ instead of $\mathbf{B}(0, \rho)$. We shall frequently use the formalism of currents, for which we refer to [Fe], Chap. 4 or [Si], Chap. 6. In the following $\mathbb{I}_k(\mathbb{R}^{n+1})$ – *the class of integral k currents* – will be the class of all locally rectifiable integer multiplicity k currents in $\mathbb{R}^{n+1}$ with $\mathbf{M}(T) + \mathbf{M}(\partial T) < \infty$ (such currents would be called "locally integral currents of finite mass and finite boundary mass" in the terminology of [Fe]). For a closed set $K \subset \mathbb{R}^{n+1}$ let $\mathbb{I}_k(K)$ be the class of integral k currents with support in K. The only assumption on the boundary $B \in \mathbb{I}^{n-1}(K)$ (always $\neq 0$) in this section is that B bounds an integral n current $T_0 \in \mathbb{I}_n(K)$. Corresponding to a fixed boundary B we shall use the notation $\mathscr{F}(B; K) := \{T \in \mathbb{I}_n(K) : \partial T = B\}$ for the class of admissible currents with boundary B. For two integral currents S, $T \in \mathbb{I}_n(\mathbb{R}^{n+1})$ with $\partial(T - S) = 0$ let $Q_{S,T}$ be the unique $n+1$ current with finite mass and boundary $S - T$. Since $Q_{S,T}$ is a current in the top dimension there exists an unique integrable function $\theta_{S,T} : \mathbb{R}^{n+1} \to \mathbf{Z}$ such that

$$Q_{S,T}(\varphi) = \langle \mathbf{E}^{n+1} \llcorner \theta_{S,T}; \varphi \rangle = \int_{\mathbb{R}^{n+1}} \theta_{S,T}\, \varphi \qquad \text{for} \quad \varphi \in \mathscr{D}^{n+1}(\mathbb{R}^{n+1})$$

and

$$\mathbf{M}(\mathrm{Q}_{S,T}) = \int_{\mathbb{R}^{n+1}} |\theta_{S,T}|\, d\mathscr{L}^{n+1}.$$

Note that in view of

$$(\mathrm{S}-\mathrm{T})(*g) = (\mathbf{E}^{n+1} \llcorner \theta_{S,T})(d*g) = \int \theta_{S,T} \operatorname{div} g\, d\mathscr{L}^{n+1}$$

the total variation of $\theta_{S,T}$ equals the boundary mass $\mathbf{E}^{n+1} \llcorner \theta_{S,T}$, *i.e.*,

$$\mathbf{M}(\partial(\mathbf{E}^{n+1} \llcorner \theta_{S,T})) = \sup_{\substack{g \in C_0^1(\mathbb{R}^{n+1}, \mathbb{R}^{n+1}),\\ \|g\|_{L^\infty} \leqq 1}} \int \theta_{S,T} \operatorname{div} g\, d\mathscr{L}^{n+1},$$

so that $\theta_{S,T}$ is of bounded variation. The isoperimetric inequality [Fe], 4.5.9, (31), applied to the locally normal current $\mathbf{E}^{n+1} \llcorner \theta_{S,T}$ implies

$$\left[\int_{\mathbb{R}^{n+1}} |\theta_{S,T} - c|\, d\mathscr{L}^{n+1}\right]^{n/(n+1)} \leqq \left[\int_{\mathbb{R}^{n+1}} |\theta_{S,T} - c|^{1+1/n} d\mathscr{L}^{n+1}\right]^{n/(n+1)}$$
$$\leqq \gamma_{n+1} \mathbf{M}(\partial \mathrm{Q}_{S,T})$$

with an unique integer $c \in \mathbf{Z}$. Note that in view of $\theta_{S,T} - c \in \mathbf{Z}$ the first inequality is trivial. Here $\gamma_{n+1} := (n+1)^{-1} \gamma_{n+1}^{-1/(n+1)}$ denotes the *optimal isoperimetric constant*. Combining this with the fact $\theta_{S,T} \in L^1(\mathbb{R}^{n+1})$ we infer that $c=0$, and hence $\theta_{S,T} \in L^1(\mathbb{R}^{n+1}) \cap L^{1+1/n}(\mathbb{R}^{n+1})$ with

$$\left[\int_{\mathbb{R}^{n+1}} |\theta_{S,T}|^{1+1/n} d\mathscr{L}^{n+1}\right]^{n/(n+1)} \leqq \gamma_{n+1} \mathbf{M}(\mathrm{S}-\mathrm{T}). \tag{2.1}$$

Now, let $\mathrm{H} : \mathbb{R}^{n+1} \to \mathbb{R}$ be a measurable function. Then, for two integral n currents S, $\mathrm{T} \in \mathbb{I}_n(\mathbb{R}^{n+1})$ with $\partial(\mathrm{S}-\mathrm{T}) = 0$ and $\theta_{S,T}\mathrm{H} \in L^1(\mathbb{R}^{n+1})$ we define the H-*volume enclosed by* S − T as the quantity

$$\mathbf{V}_\mathrm{H}(\mathrm{S}, \mathrm{T}) := \int_{\mathbb{R}^{n+1}} \theta_{S,T} \mathrm{H}\, d\mathscr{L}^{n+1}.$$

Now, fix some $\mathrm{T}_0 \in \mathscr{F}(\mathrm{B}; \mathrm{K})$. Then, for $\mathrm{T} \in \mathscr{F}(\mathrm{B}; \mathrm{K})$ with $\theta_{\mathrm{T},\mathrm{T}_0} \mathrm{H} \in L^1(\mathbb{R}^{n+1})$ the *energy functional* $\mathbf{E}_\mathrm{H}(\mathrm{T})$ is defined by

$$\mathbf{E}_\mathrm{H}(\mathrm{T}) := \mathbf{M}(\mathrm{T}) + \mathbf{V}_\mathrm{H}(\mathrm{T}, \mathrm{T}_0) = \mathbf{M}(\mathrm{T}) + \int_{\mathbb{R}^{n+1}} \theta_{\mathrm{T},\mathrm{T}_0} \mathrm{H}\, d\mathscr{L}^{n+1}.$$

Motivated by the work of Steffen [St2] we consider a functional μ which associates to every measurable function $\mathrm{H} : \mathbb{R}^{n+1} \to \mathbb{R}$ an extended real number $\mu(\mathrm{H}) \in [0, \infty]$ satisfying the conditions

$$\mu(\mathrm{H}) = \mu(|\mathrm{H}|) \quad \text{and} \quad |\mathrm{H}| \leqq |\tilde{\mathrm{H}}| \;\Rightarrow\; \mu(|\mathrm{H}|) \leqq \mu(|\tilde{\mathrm{H}}|).$$

Examples of functionals μ we actually work with are

$$\mu(\mathrm{H}) := \frac{1}{\gamma_{n+1}}\left[\int |\mathrm{H}|^{n+1}\, d\mathscr{L}^{n+1}\right]^{1/(n+1)}$$

and

$$\mu(\mathrm{H}) := \frac{1}{n} \sup_{t>0}\left[\frac{t^{n+1}}{\alpha_{n+1}} \mathscr{L}^{n+1}(\{x : |\mathrm{H}(x)| \geqq t\})\right]^{1/(n+1)}.$$

THEOREM 2.1. – *If* $\mathrm{K} \subset \tilde{\mathrm{K}} \subset \mathbb{R}^{n+1}$ *are closed sets,* $\mathbb{R}^{n+1} \setminus \tilde{\mathrm{K}}$ *has no component of finite measure and the isoperimetric condition*

$$(2.2) \qquad \left|\int_{\mathrm{A}} \mathrm{H}\, d\mathscr{L}^{n+1}\right| \leqq \mu(\mathrm{H})\, \mathbf{M}(\partial(\mathbf{E}^{n+1} \llcorner \mathrm{A}))$$

holds whenever $\mathrm{A} \subset \tilde{\mathrm{K}}$ *is a set of finite perimeter and* $\mathrm{H} : \tilde{\mathrm{K}} \to \mathbb{R}$ *is measurable, then the following assertions are true:*

(1) *If* $\mathrm{H} : \mathbb{R}^{n+1} \to \mathbb{R}$ *is measurable and* $\mu(\mathrm{H}) < \infty$, *then the* H-*volume* $\mathrm{V}_{\mathrm{H}}(\mathrm{S}, \mathrm{T})$ *is defined for all integral n currents* $\mathrm{S}, \mathrm{T} \in \mathbb{I}_n(\tilde{\mathrm{K}})$ *with* $\partial(\mathrm{S}-\mathrm{T}) = 0$.

(2) *If* $\mathrm{H} : \mathbb{R}^{n+1} \to \mathbb{R}$ *is measurable,* $0 \leqq \mu(\mathrm{H}) < 1$ *and*

$$\inf_{\tilde{\mathrm{H}} \in \mathrm{L}^{\infty}(\mathbb{R}^{n+1})} \mu(\mathrm{H} - \tilde{\mathrm{H}}) = 0,$$

then the variational problem

$$\mathbf{E}_{\mathrm{H}}(\mathrm{T}) \to \min \quad \textit{among all } \mathrm{T} \in \mathscr{F}(\mathrm{B}; \mathrm{K})$$

has a solution, provided B *is a closed rectifiable* $n-1$ *current in* $\mathbb{R}^{n+1}$ *with* $\mathrm{spt}(\mathrm{B}) \subset \mathrm{K}$ *and* $\mathscr{F}(\mathrm{B}; \mathrm{K}) \neq \varnothing$.

Proof. – (1) follows along the lines of [St2], Thm. 1.2, (i). For convenience of the reader we give the proof. First, we show that $\int_{\mathbb{R}^{n+1}} f \mathrm{H}\, d\mathscr{L}^{n+1}$ exists for every function $f \in \mathrm{L}^{1+1/n}(\mathbb{R}^{n+1})$ of bounded variation with $\mathrm{spt}(f) \subset \tilde{\mathrm{K}}$. In case $f \geqq 0$ we find using Fubini's theorem that

$$\int_{\tilde{\mathrm{K}}} f |\mathrm{H}|\, d\mathscr{L}^{n+1} = \int_0^{\infty} \int_{\{f \geqq t\}} |\mathrm{H}(x)|\, d\mathscr{L}^{n+1}(x)\, d\mathscr{L}^1(t).$$

Observing that for $\mathscr{L}^1$ almost all $t > 0$ the sets $\mathrm{A}_t := \{x \in \tilde{\mathrm{K}} : f(x) \geqq t\}$ are sets of finite perimeter (because $f \in \mathrm{L}^{1+1/n}$ is of bounded variation) we apply the isoperimetric condition with A, H replaced by A_t, $|\mathrm{H}|$ to obtain

$$\int_{\tilde{\mathrm{K}}} f |\mathrm{H}|\, d\mathscr{L}^{n+1} \leqq \mu(\mathrm{H}) \int_0^{\infty} \mathbf{M}(\partial(\mathbf{E}^{n+1} \llcorner \mathrm{A}_t))\, d\mathscr{L}^1(t).$$

Noting that

$$\int_0^\infty \mathbf{M}(\partial(\mathbf{E}^{n+1} \llcorner A_t))\, d\mathscr{L}^1(t) = \mathbf{M}(\partial(\mathbf{E}^{n+1} \llcorner f))$$

by [Fe], 4.5.9, (13) we conclude

$$\int_{\tilde{K}} f\,|H|\, d\mathscr{L}^{n+1} \leqq \mu(H)\,\mathbf{M}(\partial(\mathbf{E}^{n+1} \llcorner f)).$$

To treat the general case we observe that if f is of bounded variation, so are $f^+ := \max(f, 0)$ and $f^- := f^+ - f$. Using

$$\mathbf{M}(\partial(\mathbf{E}^{n+1} \llcorner f^+)) + \mathbf{M}(\partial(\mathbf{E}^{n+1} \llcorner f^-)) = \mathbf{M}(\partial(\mathbf{E}^{n+1} \llcorner f))$$

(*see* [Fe], 4.5.9, (13) again) we find

$$(2.3)\qquad \left|\int_{\tilde{K}} f\,H\, d\mathscr{L}^{n+1}\right| \leqq \int_{\tilde{K}} f^+\,|H|\, d\mathscr{L}^{n+1} + \int_{\tilde{K}} f^-\,|H|\, d\mathscr{L}^{n+1} \leqq \mu(H)\,\mathbf{M}(\partial(\mathbf{E}^{n+1} \llcorner f)).$$

We use this to prove the assertions of (1). From the constancy theorem [Fe], 4.1.7 it follows that every integral $n+1$ current $Q \in \mathbb{I}_{n+1}(\mathbb{R}^{n+1})$ with $\operatorname{spt}(\partial Q) \subset \tilde{K}$ has support in $\tilde{K}$, *i.e.*, $Q \in \mathbb{I}_{n+1}(\tilde{K})$. Hence, for two integral n currents $S, T \in \mathbb{I}_n(\tilde{K})$ with $\partial(S-T)=0$ the multiplicity function $\theta_{S,T}$ has support in $\tilde{K}$ [note that $\operatorname{spt}(\theta_{S,T}) = \operatorname{spt}(Q_{S,T}) \subset \tilde{K}$]. Since $\theta_{S,T} \in L^{1+1/n}$ is of bounded variation, we infer from (2.3) the existence of $\mathbf{V}_H(S, T)$.

Now, we come to the *proof of* (2). From the isoperimetric inequality (2.3) and the assumption $0 \leqq \mu(H) < 1$ it follows for all $T \in \mathscr{F}(B; K)$ that

$$(2.4)\qquad \begin{aligned} \mathbf{E}_H(T) &\geqq \mathbf{M}(T) - \left|\int_{\tilde{K}} \theta_{T,T_0}\, H\, d\mathscr{L}^{n+1}\right| \\ &\geqq \mathbf{M}(T) - \mu(H)\,\mathbf{M}(T - T_0) \\ &\geqq (1-\mu(H))\,\mathbf{M}(T) - \mu(H)\,\mathbf{M}(T_0) \\ &\geqq -\mu(H)\,\mathbf{M}(T_0). \end{aligned}$$

Assured by (2.4) that $\mathbf{E}_H : \mathscr{F}(B; K) \to \mathbb{R}$ is bounded below and $\mathscr{F}(B; \bar{G}) \neq \varnothing$ by hypothesis we can define

$$-\infty < \beta := \inf_{S \in \mathscr{F}(B;K)} \mathbf{E}_H(S)$$

and choose a minimizing sequence $\{T_k\} \subset \mathscr{F}(B; K)$, *i.e.*, $\beta = \lim_{k\to\infty} \mathbf{E}_H(T_k)$. W.l.o.g. we may assume that $\mathbf{E}_H(T_k) \leqq \beta + 1$ for all k. Recalling the assumption $0 \leqq \mu(H) < 1$ we deduce from (2.4) the uniform mass bound

$$(2.5)\qquad \mathbf{M}(T_k) \leqq \frac{1 + \beta + \mu(H)\,\mathbf{M}(T_0)}{1 - \mu(H)} =: c_1 < \infty.$$

Applying the compactness theorem for integer multiplicity rectifiable currents we infer the existence of an integral n current $T \in \mathscr{F}(B; K)$ and a subsequence $\{T_{k'}\}$ (w.l.o.g. $k'=k$) such that $T_k \to T$ weakly. Next, we define $Q_k = \mathbf{E}^{n+1} \llcorner \theta_k := Q_{T_k, T_0} \in \mathbb{I}_{n+1}(\tilde{K})$. Then, from (2.1) with $\theta_{S,T}$, $S-T$ replaced by θ_k, $T_k - T_0$ we conclude that

$$\mathbf{M}(Q_k)^{n/(n+1)} \leqq \gamma_{n+1} \mathbf{M}(T_k - T_0) \leqq \gamma_{n+1}(c_1 + \mathbf{M}(T_0)).$$

Hence, by the BV-compactness theorem there exists an integral current $Q = \mathbf{E}^{n+1} \llcorner \theta \in \mathbb{I}_{n+1}(\tilde{K})$ and a subsequence, again denoted by $\{Q_k\}$, such that $\theta_k \to \theta$ in $L^1_{\text{loc}}(\mathbb{R}^{n+1})$. Since we also know that $T_k \to T$ we easily deduce that $Q = \mathbf{E}^{n+1} \llcorner \theta$ is the unique $n+1$ current of finite mass with boundary $T - T_0$.

Recalling the assumption that H can be μ approximated by bounded functions, *i.e.*, $\inf_{\tilde{H} \in L^\infty(\mathbb{R}^{n+1})} \mu(H - \tilde{H}) = 0$, we find a sequence

$$\{H_i\} \subset L^\infty(\mathbb{R}^{n+1})$$

such that $\mu(H - H_i) \to 0$. We may assume that $|H_i| \leqq |H|$ for all i. Then, from the isoperimetric inequality (2.3) we infer that

$$\left| \int_{\tilde{K}} (H - H_i)\, \theta \, d\mathscr{L}^{n+1} \right| \leqq \mu(|H - H_i|)\, \mathbf{M}(\partial(\mathbf{E}^{n+1} \llcorner \theta)) \to 0$$

as $i \to \infty$ uniformly on $\{Q \in \mathbb{I}_{n+1}(\tilde{K}) : \mathbf{M}(\partial Q) \leqq M\}$ for every $M > 0$. This implies in particular that

$$(2.6) \qquad |\mathbf{E}_H(S) - \mathbf{E}_{H_i}(S)| = \left| \int_{\tilde{K}} (H - H_i)\, \theta_{S, T_0}\, d\mathscr{L}^{n+1} \right| \leqq \mu(|H - H_i|)\, \mathbf{M}(S - T_0) \to 0$$

uniformly on $\mathscr{F}(B; K) \cap \{S : \mathbf{M}(S) \leqq M\}$.

Now, let $\rho > 0$. From slicing [Fe], 4.3.6, 4.2.1, with $u(x) = |x|$ we infer for $\mathscr{L}^1$ almost all $r \geqq 0$ that $\langle Q_k; r \rangle \in \mathbb{I}_n(\mathbb{R}^{n+1})$ for all k and that $\operatorname{spt}(\langle Q_r; r \rangle) \subset \partial \mathbf{B}(r) \cap \tilde{K}$. Assured by [Fe], 4.2.1 that

$$\begin{aligned} \int_\rho^{*2\rho} (\mathbf{M}(\langle Q_k; r \rangle) + \mathbf{M}(\langle Q; r \rangle))\, d\mathscr{L}^1(r) &\leqq \mathbf{M}(Q_k \llcorner \{\rho < u < 2\rho\}) + \mathbf{M}(Q \llcorner \{\rho < u < 2\rho\}) \\ &\leqq \mathbf{M}(Q) + \sup_{l \geqq 1} \mathbf{M}(Q_l) =: c_2 < \infty \end{aligned}$$

we obtain using Fatou's lemma

$$\int_\rho^{*2\rho} \left(\mathbf{M}(\langle Q; r \rangle) + \liminf_{k \to \infty} \mathbf{M}(\langle Q_k; r \rangle) \right) d\mathscr{L}^1(r) \leqq c_2,$$

and find a number $\rho<r<2\rho$ which has the property

$$(2.7)\qquad \mathbf{M}(\langle Q; r\rangle)+\liminf_{k\to\infty}\mathbf{M}(\langle Q_k; r\rangle)\leqq\frac{c_2}{\rho}.$$

Once more we replace our sequences by subsequences (depending on r) to assure that for all $k\in\mathbb{N}$

$$(2.8)\qquad \mathbf{M}(\langle Q_k; r\rangle)\leqq\frac{2c_2}{\rho}.$$

For fixed i we decompose

$$\mathbf{E}_{H_i}(T_k)=\mathbf{M}(T_k \llcorner \mathbf{B}(r))+\langle Q_k \llcorner \mathbf{B}(r); \omega_i\rangle +\mathbf{M}(T_k \llcorner \{u\geqq r\})+\langle Q_k \llcorner \{u\geqq r\}; \omega_i\rangle,$$

where ω_i denotes the bounded measurable $n+1$ form $H_i\, dx_1\wedge\ldots\wedge dx_{n+1}$. Observing that $Q_k\in\mathbb{I}_{n+1}(\tilde{K})$ and

$$\partial(Q_k \llcorner \{u\geqq r\})=\partial Q_k \llcorner \{u\geqq r\}-\langle Q_k; r\rangle=(T_k-T_0)\llcorner \{u\geqq r\}-\langle Q_k; r\rangle$$

we can apply the isoperimetric inequality (2.1) to estimate

$$\begin{aligned}|\langle Q_k \llcorner \{u\geqq r\}; \omega_i\rangle|&\leqq\mu(H_i)\mathbf{M}(\partial(Q_k \llcorner \{u\geqq r\}))\\ &\leqq\mu(H_i)[\mathbf{M}((T_k-T_0)\llcorner \{u\geqq r\})+\mathbf{M}(\langle Q_k; r\rangle)].\end{aligned}$$

Since $\mu(H_i)\leqq\mu(H)<1$ it follows that

$$\begin{aligned}\mathbf{E}_{H_i}(T_k)&\geqq\mathbf{M}(T_k \llcorner \mathbf{B}(r))+\langle Q_k \llcorner \mathbf{B}(r); \omega_i\rangle-\mu(H)\mathbf{M}(\langle Q_k; r\rangle)\\ &\quad+(1-\mu(H))\mathbf{M}(T_k \llcorner \{u\geqq r\})-\mu(H)\mathbf{M}(T_0 \llcorner \{u\geqq r\})\\ &\geqq\mathbf{M}(T_k \llcorner \mathbf{B}(r))+\int_{\tilde{K}\cap\mathbf{B}(r)}\theta_k H_i\, d\mathscr{L}^{n+1}-\frac{2c_2}{\rho}-\mathbf{M}(T_0 \llcorner \{u\geqq r\}),\end{aligned}$$

because $\mathbf{M}(\langle Q_k; r\rangle)\leqq 2c_2/\rho$ by (2.8). From the preceding estimate, the lower semicontinuity of mass with respect to weak convergence and the convergence $\theta_k\to\theta$ in $L^1_{loc}(\mathbb{R}^{n+1})$ we get

$$\begin{aligned}\liminf_{k\to\infty}\mathbf{E}_{H_i}(T_k)&\geqq\mathbf{M}(T \llcorner \mathbf{B}(r))+\int_{\tilde{K}\cap\mathbf{B}(r)}\theta H_i\, d\mathscr{L}^{n+1}-\frac{2c_2}{\rho}\\ &\quad-\mathbf{M}(T_0 \llcorner \{u\geqq r\})=\mathbf{E}_{H_i}(T)-\mathbf{M}(T \llcorner \{u\geqq r\})-\mathbf{M}(T_0 \llcorner \{u\geqq r\})\\ &\quad+\langle Q \llcorner \{u\geqq r\}; \omega_i\rangle-\frac{2c_2}{\rho}.\end{aligned}$$

Using once more the isoperimetric inequality (2.3), $\mu(H_i)\leqq\mu(H)$ and (2.7) we find that

$$|\langle Q \llcorner \{u\geqq r\}; \omega_i\rangle|\leqq\mu(H)\left[\mathbf{M}((T-T_0)\llcorner \{u\geqq r\})+\frac{c_2}{\rho}\right].$$

Combining the last two estimates we get

$$(2.9)\quad \liminf_{k\to\infty} \mathbf{E}_{\mathrm{H}_i}(\mathrm{T}_k) \geqq \mathbf{E}_{\mathrm{H}_i}(\mathrm{T}) - 2\,\mathbf{M}(\mathrm{T} \llcorner \{u \geqq r\}) - 2\,\mathbf{M}(\mathrm{T}_0 \llcorner \{u \geqq r\}) - \frac{3c_2}{\rho}.$$

In view of (2.5) we may apply (2.6) with

$$\mathscr{M} := \{\mathrm{T}\} \cup \{\mathrm{T}_k\}_{k\in\mathbb{N}} \subset \mathscr{F}(\mathrm{B};\,\mathrm{K})$$

to obtain

$$|\mathbf{E}_{\mathrm{H}_i}(\mathrm{S}) - \mathbf{E}_{\mathrm{H}}(\mathrm{S})| \leqq \mu(|\mathrm{H}-\mathrm{H}_i|)(c_1 + \mathbf{M}(\mathrm{T}_0)) =: \mu_i \to 0$$

uniformly on $\mathscr{M}$. This implies (note that $\rho < r < 2\rho$)

$$\beta = \liminf_{k\to\infty} \mathbf{E}_{\mathrm{H}}(\mathrm{T}_k) \geqq \mathbf{E}_{\mathrm{H}}(\mathrm{T}) - 2\,\mathbf{M}(\mathrm{T} \llcorner \{u \geqq \rho\}) - 2\,\mathbf{M}(\mathrm{T}_0 \llcorner \{u \geqq \rho\}) - 2\mu_i - \frac{3c_2}{\rho}.$$

Letting $\rho \to \infty$, $i \to \infty$ we then have

$$\mathbf{E}_{\mathrm{H}}(\mathrm{T}) \leqq \beta = \inf_{\mathrm{S}\in\mathscr{F}(\mathrm{B};\mathrm{K})} \mathbf{E}_{\mathrm{H}}(\mathrm{S}),$$

which shows (2) as required. □

Remark 2.2. – From the proof of theorem 2.1 it is obvious that in case $\mathrm{H} \in \mathrm{L}^{\infty}_{\mathrm{loc}}(\mathbb{R}^{n+1})$ with $0 \leqq \mu(\mathrm{H}) < 1$ we need not require μ approximability of H by bounded functions, because the arguments leading to (2.9) remain true, if we replace H_i by H (note that now

$$\int_{\tilde{\mathrm{K}}\cap\mathbf{B}(r)} \theta_k \mathrm{H}\, d\mathscr{L}^{n+1} \to \int_{\tilde{\mathrm{K}}\cap\mathbf{B}(r)} \theta \mathrm{H}\, d\mathscr{L}^{n+1}$$

since $\mathrm{H} \in \mathrm{L}^{\infty}_{\mathrm{loc}}(\mathbb{R}^{n+1})$ and $\theta_k \to \theta$ in $\mathrm{L}^1_{\mathrm{loc}}$).

The following *indecomposability property* of minimizing currents is an immediate consequence of the isoperimetric condition and the assumption $0 \leqq \mu(\mathrm{H}) < 1$.

LEMMA 2.3. – *Suppose that the hypotheses of theorem* 2.1 *part* (2) *are satisfied, in particular* (2.2) *holds with* $\mu(\mathrm{H}) < 1$. *Then, each energy minimizing current* $\mathrm{T} \in \mathscr{F}(\mathrm{B};\,\mathrm{K})$ *is indecomposable, i.e., there exist no closed current* R in K *with*

$$(2.10)\qquad \mathrm{R} \neq 0 \quad \textit{and} \quad \mathbf{M}(\mathrm{T}) = \mathbf{M}(\mathrm{T}-\mathrm{R}) + \mathbf{M}(\mathrm{R}).$$

Proof. – Assuming the statement false we find $\mathrm{R} \neq 0$ with $\partial\mathrm{R} = 0$ and (2.10). Then, the isoperimetric inequality (2.3) applied to R and (2.10) imply

$$\begin{aligned}\mathbf{E}_{\mathrm{H}}(\mathrm{T}) &= \mathbf{M}(\mathrm{T}-\mathrm{R})+\mathbf{M}(\mathrm{R})+\mathrm{V}_{\mathrm{H}}(\mathrm{T}-\mathrm{R},\,\mathrm{T}_0)+\int \theta_{\mathrm{R}}\,\mathrm{H}\,d\mathscr{L}^{n+1}\\ &\geqq \mathbf{E}_{\mathrm{H}}(\mathrm{T}-\mathrm{R})+\mathbf{M}(\mathrm{R})-\left|\int \theta_{\mathrm{R}}\,\mathrm{H}\,d\mathscr{L}^{n+1}\right|\\ &\geqq \mathbf{E}_{\mathrm{H}}(\mathrm{T}-\mathrm{R})+(1-\mu(\mathrm{H}))\,\mathbf{M}(\mathrm{R})>\mathbf{E}_{\mathrm{H}}(\mathrm{T}-\mathrm{R}),\end{aligned}$$

contradicting the minimizing property of T. □

In order to apply our existence theorem to certain functionals μ we recall the following result of Steffen (*see* [St2]).

PROPOSITION 2.4. – *Assume that* μ *is one of the following functionals:*

$$\mu(\mathrm{H}) := \frac{1}{\gamma_{n+1}}\left[\int_{\mathbb{R}^{n+1}} |\mathrm{H}|^{n+1}\, d\mathscr{L}^{n+1}\right]^{1/(n+1)} \tag{2.11}$$

$$\mu(\mathrm{H}) := \frac{1}{n}\sup_{s>0}\left[\frac{s^{n+1}}{\alpha_{n+1}}\mathscr{L}^{n+1}(\{x\in\mathbb{R}^{n+1} : |\mathrm{H}(x)|\geqq s\})\right]^{1/(n+1)} \tag{2.12}$$

$$\mu(\mathrm{H}) := \sup_{t\in\mathbb{R}}\frac{1}{\gamma_n}\left[\int_{\mathbb{R}^n} |\mathrm{H}(z,t)|^n\, d\mathscr{L}^n\right]^{1/n} \tag{2.13}$$

$$\mu(\mathrm{H}) := \frac{1}{n-1}\sup_{t\in\mathbb{R},\, s>0}\left[\frac{s^n}{\alpha_n}\mathscr{L}^n(\{z\in\mathscr{R}^n : |\mathrm{H}(z,t)|\geqq s\})\right]^{1/n}. \tag{2.14}$$

If $\mathrm{H} : \mathbb{R}^{n+1}\to\mathbb{R}$ *is measurable with* $\mu(\mathrm{H})<\infty$, *then the isoperimetric inequality*

$$\left|\int_{\mathrm{A}} \mathrm{H}\, d\mathscr{L}^{n+1}\right| \leqq \mu(\mathrm{H})\,\mathbf{M}(\partial(\mathbf{E}^{n+1} \llcorner \mathrm{A}))$$

holds whenever $\mathrm{A}\subset\mathbb{R}^{n+1}$ *is a set of finite perimeter.*

3. THE VARIATIONAL EQUATION AND APPLICATIONS

In this section we assume that K is the closure of a C^2 domain G in $\mathbb{R}^{n+1}$ and that $\mathrm{H} : \bar{\mathrm{G}}\to\mathbb{R}$ is a continuous function. We extend H to a function on $\mathbb{R}^{n+1}$ by $\mathrm{H}(x)=0$ for $x\notin\bar{\mathrm{G}}$ and assume that the energy functional $\mathbf{E}_{\mathrm{H}}$ is defined on $\mathscr{F}(\mathrm{B};\bar{\mathrm{G}})$. Moreover, we suppose that the variational problem $\mathbf{E}_{\mathrm{H}}(\mathrm{T})\to\min$ among all $\mathrm{T}\in\mathscr{F}(\mathrm{B};\bar{\mathrm{G}})$ has a solution. To derive the variational equation for an energy minimizing current T in $\mathscr{F}(\mathrm{B};\bar{\mathrm{G}})$, we consider variations of the form $\mathrm{T}_t := \Phi_{t\#}\mathrm{T}$ where $\Phi : \mathbb{R}\times\mathbb{R}^{n+1}\to\mathbb{R}^{n+1}$ denotes the flow of a compactly supported C^1 vector-field g with $\operatorname{spt}(g)\cap\operatorname{spt}(\mathrm{B})=\varnothing$. For our obstacle problem it is natural to require that $\operatorname{spt}(\Phi_{t\#}\mathrm{T})\subset\bar{\mathrm{G}}$. This will be the case, if the variational vectorfield satisfies one of the following admissibility conditions:

(1) $g.\nu=0$ on ∂G (ν denoting the inner unit normal vectorfield on ∂G); then $\Phi_{t\#} T \in \mathscr{F}(B; \bar{G})$ for all $t \in \mathbb{R}$.

(2) $g.\nu \geqq 0$ on ∂G; then $\Phi_{t\#} T \in \mathscr{F}(B; \bar{G})$ for all $t \geqq 0$.

The *first variation of mass* is given by (*see* [AW], 4.1)

$$(3.1) \qquad \delta \mathbf{M}(T, g) = \frac{d}{dt} \mathbf{M}(\Phi_{t\#} T)_{|t=0} = \frac{d}{dt}_{|0} \int |(\Lambda_n D\Phi_t) \mathbf{T}| \, d\|T\| = \int Dg . P_{\mathbf{T}} \, d\|T\|.$$

To compute the *first variation of the volume functional* with respect to admissible variational vectorfields we proceed as follows: Let Q be the unique $n+1$ current of finite mass with boundary $T - T_0$ and $Q_t := Q + \Phi_{\#}([\![0, t]\!] \times T)$. Then, the homotopy formula ([Fe], 4.1.9) implies

$$\partial Q_t = \partial Q + \Phi_{t\#} T - T - \Phi_{\#}([\![0, t]\!] \times B) = \Phi_{t\#} T - T_0,$$

because $g_{|\operatorname{spt}(B)} = 0$. In particular, we see that Q_t is the unique $n+1$ current of finite mass with boundary $\Phi_{t\#} T - T_0$. The multiplicity function $\theta_{\Phi_{t\#}T, T_0}$ associated with Q_t is of the form $\theta_{T, T_0} + \theta_{\Phi_{t\#}T, T}$.
In view of $\theta_{T, T_0} H \in L^1(\mathbb{R}^{n+1})$ and $\theta_{\Phi_{t\#}T, T} H \in L^1(\mathbb{R}^{n+1})$ (because $\operatorname{spt}(\theta_{\Phi_{t\#}T, T}) \subset \operatorname{spt} \Phi_{\#}([\![0, t]\!] \times T) \subset \operatorname{spt}(g)$ and $H \in L^{\infty}_{\mathrm{loc}}(\mathbb{R}^{n+1})$) we find for the H-volume enclosed by $\Phi_{t\#} T - T_0$ the formula

$$\mathbf{V}_H(\Phi_{t\#} T, T_0) = \mathbf{V}_H(T, T_0) + \int \theta_{\Phi_{t\#}T, T} H \, d\mathscr{L}^{n+1} = \mathbf{V}_H(T, T_0) + \int \theta_{\Phi_{t\#}T, T} H \varphi \, d\mathscr{L}^{n+1},$$

where φ is a function in $C^0_0(\mathbb{R}^{n+1})$ with $\varphi \equiv 1$ in a neighborhood of $\operatorname{spt}(g)$. Now we approximate $H\varphi$ by a sequence $\{H_i\} \subset C^0_0(\mathbb{R}^{n+1})$ such that

$$\sup_{i \geqq 1} |H_i| \leqq C < \infty \qquad \text{and} \qquad H(x)\varphi(x) = \lim_{i \to \infty} H_i(x)$$

for every $x \in \mathbb{R}^{n+1}$. Then,

$$|\theta_{\Phi_{t\#}T, T} H_i| \leqq C |\theta_{\Phi_{t\#}T, T}| \in L^1(\mathbb{R}^{n+1}).$$

Hence, by the dominated convergence theorem and [Fe], 4.1.9, we obtain

$$\begin{aligned}\int \theta_{\Phi_{t\#}\mathrm{T},\,\mathrm{T}}\,\mathrm{H}\,\varphi\, d\mathscr{L}^{n+1} &= \lim_{i\to\infty}\int \theta_{\Phi_{t\#}\mathrm{T},\,\mathrm{T}}\,\mathrm{H}_i\, d\mathscr{L}^{n+1}\\ &= \lim_{i\to\infty}\langle \Phi_{\#}([\![0,\,t]\!]\times \mathrm{T});\ \omega_i\rangle\\ &= \lim_{i\to\infty}\int\int_0^t f_i(s,\,x)\,ds\,d\|\mathrm{T}\|(x),\end{aligned}$$

where $\omega_i := \mathrm{H}_i\,dx_1\wedge\ldots\wedge dx_{n+1}$ and

$$f_i(s,\,x) := \langle \omega_i(\Phi_s(x));\ \dot{\Phi}_s(x)\wedge(\Lambda_n \mathrm{D}\Phi_s(x))\,\mathbf{T}(x)\rangle.$$

Denoting by f the integrand corresponding to

$$\omega\varphi\,(\omega := \mathrm{H}\,dx_1\wedge\ldots\wedge dx_{n+1})$$

we find $f_i(s,\,x)\to f(s,\,x)$ as $i\to\infty$ for $\|\mathrm{T}\|$ almost all $x\in\mathbb{R}^{n+1}$ and $s\in[0,\,t]$. Furthermore

$$\begin{aligned}|f_i(s,\,x)| &\leqq \mathrm{C}\,\|g\|_{\mathrm{L}^\infty}\Big[\sup_{|v|=1}|\mathrm{D}\Phi_s(x)v|\Big]^n\\ &=: \mathrm{C}\,\|g\|_{\mathrm{L}^\infty}\,\|\mathrm{D}\Phi_s(x)\|^n \leqq \mathrm{C}\,\|g\|_{\mathrm{L}^\infty}\exp[ns\,\|\mathrm{D}g\|_{\mathrm{L}^\infty}].\end{aligned}$$

Using once more the dominated convergence theorem we infer

$$\begin{aligned}\lim_{i\to\infty}\int\int_0^t f_i(s,\,x)\,ds\,d\mathscr{L}^{n+1}(x) &= \int\int_0^t \langle(\varphi\omega)(\Phi_s);\ \dot{\Phi}_s\wedge(\Lambda_n \mathrm{D}\Phi_s)\,\mathbf{T}\rangle\,ds\,d\|\mathrm{T}\|\\ &= \int\int_0^t \langle \omega(\Phi_s);\ \dot{\Phi}_s\wedge(\Lambda_n \mathrm{D}\Phi_s)\,\mathbf{T}\rangle\,ds\,d\|\mathrm{T}\|,\end{aligned}$$

because $\varphi_{|\,\mathrm{spt}\,(g)}\equiv 1$. Therefore

$$(3.2)\quad \mathbf{V}_{\mathrm{H}}(\Phi_{t\#}\mathrm{T},\,\mathrm{T}_0) = \mathbf{V}_{\mathrm{H}}(\mathrm{T},\,\mathrm{T}_0) + \int\int_0^t \langle \omega(\Phi_s);\ \dot{\Phi}_s\wedge(\Lambda_n \mathrm{D}\Phi_s)\,\mathbf{T}\rangle\,ds\,d\|\mathrm{T}\|.$$

If $g\in\mathrm{C}_0^1(\mathbb{R}^{n+1},\,\mathbb{R}^{n+1})$ satisfies the admissibility condition (1) we see that the function $\mathbb{R}\ni s\mapsto\omega(\Phi_s(x))\in\Lambda^{n+1}\mathbb{R}^{n+1}$ is continuous for every $x\in\bar{\mathrm{G}}$. Thus,

$$\frac{1}{t}\int_0^t \langle \omega(\Phi_s(x));\ \dot{\Phi}_s(x)\wedge(\Lambda_n \mathrm{D}\Phi_s(x))\,\mathbf{T}(x)\rangle\,ds \to \langle \omega(x);\ g(x)\wedge\mathbf{T}(x)\rangle$$

as $t\to 0$ for $\|\mathrm{T}\|$ almost all $x\in\mathbb{R}^{n+1}$ (note that $\mathrm{spt}(\mathrm{T})\subset\bar{\mathrm{G}}$), and from (3.2) we obtain

$$\lim_{t\to 0}\frac{1}{t}[\mathbf{V}_{\mathrm{H}}(\Phi_{t\#}\mathrm{T},\,\mathrm{T}_0)-\mathbf{V}_{\mathrm{H}}(\mathrm{T},\,\mathrm{T}_0)] = \int\langle\omega;\ g\wedge\mathbf{T}\rangle\,d\|\mathrm{T}\|,$$

which in view of (3.1) implies

$$\frac{d}{dt}\mathbf{E}_{\mathrm{H}}(\Phi_{t\#}\mathrm{T})_{|t=0}=\int(\mathrm{D}g.\mathrm{P}_{\mathrm{T}}+\mathrm{H}\ast\mathbf{T}.g)\,d\|\mathrm{T}\|. \tag{3.3}$$

If g satisfies the admissibility condition (2), we deduce similarly that

$$\lim_{t\downarrow 0}\frac{1}{t}[\mathbf{E}_{\mathrm{H}}(\Phi_{t\#}\mathrm{T})-\mathbf{E}_{\mathrm{H}}(\mathrm{T})]=\int(\mathrm{D}g.\mathrm{P}_{\mathrm{T}}+\mathrm{H}\ast\mathbf{T}.g)\,d\|\mathrm{T}\|. \tag{3.4}$$

Remark 3.1. – Equation (3.3) [resp. (3.4)] remains true for locally bounded Baire functions $\mathrm{H}:\bar{\mathrm{G}}\to\mathbb{R}$ which are continuous except on a set of vanishing n dimensional Hausdorff measure.

Motivated by (3.3) and (3.4) we define a linear functional

$$\delta\mathbf{E}_{\mathrm{H}}(\mathrm{T},\,.):\mathrm{C}_0^1(\Omega,\mathbb{R}^{n+1})\to\mathbb{R},$$

where $\Omega:=\mathbb{R}^{n+1}\setminus\mathrm{spt}(\mathrm{B})$, called *the first variation of* $\mathbf{E}_{\mathrm{H}}$ *at* T by letting

$$\delta\mathbf{E}_{\mathrm{H}}(\mathrm{T},g):=\int(\mathrm{D}g.\mathrm{P}_{\mathrm{T}}+\mathrm{H}\ast\mathbf{T}.g)\,d\|\mathrm{T}\|.$$

LEMMA 3.2. – *Let* T *be energy minimizing in* $\mathscr{F}(\mathrm{B};\bar{\mathrm{G}})$ *and* $\mathrm{spt}(\mathrm{B})\neq\bar{\mathrm{G}}$. *Then the following statements are true:*

(1) $\delta\mathbf{E}_{\mathrm{H}}(\mathrm{T},g)=0$ *for all* $g\in\mathrm{C}_0^1(\mathbb{R}^{n+1},\mathbb{R}^{n+1})$ *with* $\mathrm{spt}(g)\cap\mathrm{spt}(\mathrm{B})=\varnothing$ *and* $g.\nu=0$ *on* $\partial\mathrm{G}$.

(2) $\delta\mathbf{E}_{\mathrm{H}}(\mathrm{T},g)\geqq 0$ *for all* $g\in\mathrm{C}_0^1(\mathbb{R}^{n+1},\mathbb{R}^{n+1})$ *with* $\mathrm{spt}(g)\cap\mathrm{spt}(\mathrm{B})=\varnothing$ *and* $g.\nu\geqq 0$ *on* $\partial\mathrm{G}$.

Proof. – Since $\mathrm{spt}(\mathrm{B})\neq\bar{\mathrm{G}}$ by hypothesis, the class of admissible vectorfields $g\in\mathrm{C}_0^1(\mathbb{R}^{n+1},\mathbb{R}^{n+1})$ with $\mathrm{spt}(g)\cap\mathrm{spt}(\mathrm{B})=\varnothing$ is nonempty. In case $\mathrm{spt}(g)\cap\mathrm{spt}(\mathrm{T})\neq\varnothing$ the conclusions of (1) [resp. (2)] follow from (3.3) [resp. (3.4)]. □

Now, C^1 vectorfields N on $\mathbb{R}^{n+1}$ with $\mathrm{N}=\nu$ along $\partial\mathrm{G}$ lead us to the following lemma.

LEMMA 3.3. – *Assume* T *is energy minimizing in* $\mathscr{F}(\mathrm{B};\bar{\mathrm{G}})$ *and* $\mathrm{spt}(\mathrm{B})\neq\bar{\mathrm{G}}$. *Then the following statements hold:*

(1) *There exists a positive Radon measure* λ *on* $\Omega:=\mathbb{R}^{n+1}\setminus\mathrm{spt}(\mathrm{B})$ *such that for any* $\varphi\in\mathrm{C}_0^1(\Omega)$:

$$\delta\mathbf{E}_{\mathrm{H}}(\mathrm{T},\varphi\mathrm{N})=\int_{\Omega}\varphi\,d\lambda.$$

(2) λ *is independent of the extension* N *of* ν.

(3) *For* λ *we have the estimate*

$$0\leqq\lambda\leqq\|\mathrm{T}\|\llcorner\mathbf{1}_{\partial\mathrm{G}\cap\Omega}(\mathrm{H}\ast\mathbf{T}.\nu-\mathscr{H}_{\partial\mathrm{G}})$$

Here $\mathscr{H}_{\partial G}$ denotes the mean curvature of ∂G with respect to the interior unit normal ν.

Proof. – Let $\varphi \in C_0^1(\Omega)$, $\varphi \geqq 0$. Then, $g := \varphi N$ is admissible and

$$\delta \mathbf{E}_H(T, \varphi N) \geqq 0$$

by lemma 3.2, (2). Thus, there exists a positive Radon measure on Ω such that

$$\delta \mathbf{E}_H(T, \varphi N) = \int_\Omega \varphi \, d\lambda \quad \text{for all } \varphi \in C_0^1(\Omega). \tag{3.5}$$

Replacing N, λ by $\tilde{N}$, $\tilde{\lambda}$ in (3.5) ($\tilde{N}$ denoting a C^1 extension of ν to $\mathbb{R}^{n+1}$ with $N \neq \tilde{N}$) we find

$$\delta \mathbf{E}_H(T, \varphi(N - \tilde{N})) = \int_\Omega \varphi \, d\lambda - \int_\Omega \varphi \, d\tilde{\lambda}.$$

Since $g := \varphi(N - \tilde{N})$ vanishes on ∂G, we know that $\delta \mathbf{E}_H(T, \varphi(N - \tilde{N})) = 0$ by lemma 3.1, (1). Hence, $\lambda = \tilde{\lambda}$, which proves (2).

Now, we come to the proof of (3). Let φ be an arbitrary nonnegative $C_0^1(\Omega)$ function. Since G is a C^2 domain, there exists a C^2 function $\rho : \mathbb{R}^{n+1} \to \mathbb{R}$ with $\rho = 0$ on ∂G, $\rho < 0$ on $\mathbb{R}^{n+1} \setminus \bar{G}$, $\rho > 0$ on G and $\nabla \rho \neq 0$ on ∂G. Choose $R > 0$ with $\operatorname{spt}(\varphi) \subset \mathbf{B}(R)$ and $\tau_0 > 0$ such that $|\nabla \rho(x)| > 0$ for every $x \in \{|\rho| < 2\tau_0\} \cap \mathbf{B}(2R)$. Next, choose $\vartheta \in C^1(\mathbb{R}, \mathbb{R}_+)$ with $\vartheta(t) = 1$ for $t \leqq 1/2$, $\vartheta(t) = 0$ for $t \geqq 1$ and $\vartheta'(t) \leqq 0$ for all $t \in \mathbb{R}$, let $\vartheta_\varepsilon(t) := \vartheta(t/\varepsilon)$ for $\varepsilon > 0$ and define for $0 < \tau < \tau_0$;

$$\mathscr{N}_\tau := \begin{cases} \vartheta_\tau(\rho) \dfrac{\nabla \rho}{|\nabla \rho|} & \text{on } \mathbf{B}(R) \cap \{|\rho| < \tau\} \\ 0 & \text{on } (\mathbf{B}(R) \cap G) \setminus \{|\rho| < \tau\} \\ \nu & \text{on } \partial G \setminus \mathbf{B}(R). \end{cases}$$

By N_τ denote an arbitrary C^1 extension of $\mathscr{N}_\tau$ to $\mathbb{R}^{n+1}$. Then, for $\|T\|$ almost all $x \in \mathbb{R}^{n+1}$ we have

$$\begin{aligned} P_T(x) . D\mathscr{N}_\tau(x) = (\vartheta_\tau \circ \rho)(x) [\varphi(x) Dn(x) . P_T(x) + \nabla \rho(x) . (P_T(x) n(x))] \\ + (\vartheta'_\tau \circ \rho)(x) |\nabla \rho(x)| |P_T(x) n(x)|^2, \end{aligned}$$

where $n = \nabla \rho / |\nabla \rho|$. In view of $\vartheta'_\tau \leqq 0$, we get

$$\int_\Omega \varphi \, d\lambda \leqq \int \vartheta_\tau(\rho) [\varphi(Dn . P_T + H \ast T . n) + \nabla \varphi . (P_T n)] \, d\|T\|.$$

Since the left hand side of the preceding inequality is independent of τ, we can pass to the limit $\tau \downarrow 0$. Noting that $\vartheta_\tau(\rho(x)) \to \mathbf{1}_{\{\rho \leqq 0\}}(x)$ as $\tau \downarrow 0$

and $\operatorname{spt}(T)\subset\bar{G}$, we obtain with the dominated convergence theorem

$$(3.6)\qquad \int_\Omega \varphi\, d\lambda \leqq \int \mathbf{1}_{\partial G}[\varphi(Dn.P_T + H * T.n) + \nabla\varphi.(P_T n)]\, d\|T\|.$$

Using the fact that $\operatorname{Tan}(\|T\|, x) = T_x\partial G$ for $\|T\|$ almost all $x\in\operatorname{spt}(T)\cap\partial G$ we find $P_T(x)\,n(x)=0$ and

$$[Dn(x).P_T(x)]\,n(x) = -\mathscr{H}_{\partial G}(x)$$

a.e. on $\operatorname{spt}(T)\cap\partial G$ and hence, (3.6) becomes

$$\int_\Omega \varphi\, d\lambda \leqq \int \mathbf{1}_{\partial G}\,\varphi\,(H * T.\nu - \mathscr{H}_{\partial G})\, d\|T\|,$$

which implies the desired estimate for λ. □

From lemma 3.3 we next derive a variational equation for vectorfields $g\in C_0^1(\Omega, \mathbb{R}^{n+1})$. For this we fix some C^1 extension N of ν and cover $\operatorname{spt}(g)$ by finitely many open sets U with $U\cap\operatorname{spt}(B)=\varnothing$. In case $U\cap\partial G\neq\varnothing$ we assume that U is carrying an orthogonal frame $X_1, \ldots, X_n$, N of C^1 vectorfields. Choosing C^1 functions ψ_U with compact support in U such that $\sum_U \psi_U = 1$ on $\operatorname{spt}(g)$ we obtain $g=\sum_U \psi_U g$. Now, if $U\cap\partial G=\varnothing$, we have $(\psi_U g).\nu=0$ on ∂G, so that lemma 3.2, (1) implies

$$(3.7)\qquad \delta E_H(T, \psi_U g) = 0.$$

In case $U\cap\partial G\neq\varnothing$ we multiply the corresponding vectorfields $X_1, \ldots, X_n$ by cut off functions to obtain C^1 vectorfields with compact support in U which coincide with X_i on $\operatorname{spt}(\psi_U)$. Since $X_1(x), \ldots, X_n(x)$, $N(x)$ form an orthogonal basis of $\mathbb{R}^{n+1}$ if $\psi_U(x)\neq 0$, we have

$$\psi_U g = \varphi_{n+1} N + \sum_{i=1}^{n} \varphi_i X_i$$

with $\varphi_i\in C_0^1(U)$ (note that $\operatorname{spt}(\varphi_i)\subset\operatorname{spt}(\psi_U)$). For $i=1, \ldots, n$ we apply lemma 3.2, (1) with g replaced by $\varphi_i X_i\in C_0^1(U; \mathbb{R}^{n+1})$ to obtain

$$(3.8)\qquad \delta E_H(T, \varphi_i X_i) = 0.$$

Using lemma 3.3, (2) and the fact that $\varphi_{n+1}=\psi_U g.\nu$ on ∂G we conclude that

$$\delta E_H(T, \varphi_{n+1} N) = \int_\Omega \psi_U g.\nu\, d\lambda.$$

From this, (3.7) and (3.8) we immediately obtain the following theorem, which is well known (*see* [DF1], Thm. 4.1) in case of C^2 domains $G\subset\mathbb{R}^{n+1}$ with compact closure $\bar{G}$.

THEOREM 3.4. – *Suppose* $G \subset \mathbb{R}^{n+1}$ *is a* C^2 *domain,* $H ; \bar{G} \to \mathbb{R}$ *is continuous,* $B \in \mathbb{I}_{n-1}(G)$ *and* $\operatorname{spt}(B) \neq \bar{G}$. *Suppose further that* T *is energy minimizing in the class* $\mathscr{F}(B; \bar{G})$. *Then,*

$$\delta E_H(T, g) = \int (Dg . P_T + H * \mathbf{T} . g)\, d\|T\| = \int_\Omega g . \nu\, d\lambda \tag{3.9}$$

holds whenever g *is a* C^1 *vectorfield on* $\mathbb{R}^{n+1}$ *having compact support in* $\Omega = \mathbb{R}^{n+1} \setminus \operatorname{spt}(B)$ (λ *denoting the unique Radon measure from lemma* 3.2 *on* Ω).

From lemma 3.3, (3) it easily follows that $H(x) * \mathbf{T}(x) . \nu(x) \geqq \mathscr{H}_{\partial G}(x)$ holds for $\|T\|$ almost all $x \in \operatorname{spt}(T) \cap \partial G \cap \Omega$, so that

$$0 \leqq \lambda \leqq \|T\| \llcorner \mathbf{1}_{\partial G \cap \Omega} (H * \mathbf{T} . \nu - \mathscr{H}_{\partial G})^+$$

on $\mathbb{R}^{n+1} \setminus \operatorname{spt}(B)$. Evidently, λ is absolutely continuous with respect to $\|T\|$. The theory of symmetrical derivation (*see* [Fe], 2.8.19, 2.9) shows the following: there exists a real valued positive $\|T\|$ measurable function $\Lambda : \Omega \to \mathbb{R}_+$ with support in $\operatorname{spt}(T) \cap \partial G$ satisfying

$$0 \leqq \Lambda(x) \leqq (H(x) * \mathbf{T}(x) . \nu(x) - \mathscr{H}_{\partial G}(x))^+$$

for $\|T\|$ almost all $x \in \operatorname{spt}(T) \cap \partial G \cap \Omega$, such that $\lambda = \|T\| \llcorner \Lambda$ on Ω. This implies in view of (3.9)

$$\int (Dg . P_T + H * \mathbf{T} . g)\, d\|T\| = \int_\Omega g . \nu \Lambda\, d\|T\|, \tag{3.10}$$

whenever $g \in C_0^1(\Omega, \mathbb{R}^{n+1})$.

COROLLARY 3.5. – *Let* G *be a* C^2 *domain* in $\mathbb{R}^{n+1}$ *such that*

$$|H(x)| \leqq \mathscr{H}_{\partial G}(x) \qquad \textit{for} \quad x \in \partial G.$$

Suppose further, that the boundary $B \in \mathbb{I}_{n-1}(\bar{G})$ *satisfies* $\operatorname{spt}(B) \neq \bar{G}$. *Then, any energy minimizing current* $T \in \mathscr{F}(B; \bar{G})$ *has prescribed mean curvature* H *in* $\Omega = \mathbb{R}^{n+1} \setminus \operatorname{spt}(B)$, *i.e., we have*

$$\int (Dg . P_T + H * \mathbf{T} . g)\, d\|T\| = 0, \tag{3.11}$$

whenever $g \in C_0^1(\Omega, \mathbb{R}^{n+1})$.

Proof. – Since $|H(x)| \leqq \mathscr{H}_{\partial G}(x)$ on ∂G, we have

$$H(x) * \mathbf{T}(x) . \nu(x) - \mathscr{H}_{\partial G}(x) \leqq 0$$

for $\|T\|$ almost all $x \in \operatorname{spt}(T) \cap \partial G$. Hence, $(H * \mathbf{T} . \nu - \mathscr{H}_{\partial G})^+ = 0$ a.e. on $\operatorname{spt}(T) \cap \partial G$ and $\lambda = 0$ as required. □

Remark 3.6. – If we assume $\mathscr{H}^n(\operatorname{spt}(B)) = 0$ (and $B \neq 0$) instead of $\operatorname{spt}(B) \neq \bar{G}$, then each energy minimizing current $T \in \mathscr{F}(B; \bar{G})$ has support disjoint to spt(B) $\|T\|$ almost everywhere. In particular, T has $\|T\|$ almost everywhere prescribed mean curvature H. However, there are situations with $\mathcal{H}^n(\mathrm{spt}(B)) \neq 0$, even with $\mathcal{H}^{n+1}(\mathrm{spt}(B)) \neq 0$, in which the support of an energy minimizing current T is not completely contained in spt(B).

COROLLARY 3.7. – *If* spt(B) *is compact and* $H \in L^\infty(\bar{G})$, *then also each energy minimizing current* T *in* $\mathcal{F}(B; \bar{G})$ *has compact support. If* spt(B) *is unbounded, then* $\mathrm{dist}(x, \mathrm{spt}(B)) \to 0$ *as* $|x| \to \infty$, $x \in \mathrm{spt}(T)$.

Proof. – From (3.11) we see that T has bounded mean curvature vector $\mathbf{H} := H * \mathbf{T}$. Therefore, by [AW], 5.3, we obtain for $a \in \mathrm{spt}(T)$ that the function

$$r^{-n}\mathbf{M}(T \llcorner \mathbf{B}(a, r)) \exp[\|H\|_{L^\infty} r]$$

is nondecreasing in r on $0 < r < \mathrm{dist}(a, \mathrm{spt}(B))$. Furthermore, the upper semi-continuity of the density and the monotonicity formula imply

$$1 \leqq \Theta^n(\|T\|, a) = \lim_{\rho \downarrow 0} \frac{\mathbf{M}(T \llcorner \mathbf{B}(a, \rho))}{\alpha_n \rho^n} \leqq \frac{\mathbf{M}(T \llcorner \mathbf{B}(a, r))}{\alpha_n r^n} \exp[\|H\|_{L^\infty} r],$$

so that

$$\mathbf{M}(T \llcorner \mathbf{B}(a, r)) \geqq \alpha_n r^n \exp[-\|H\|_{L^\infty} r], \tag{3.12}$$

whenever $0 < r < \mathrm{dist}(a, \mathrm{spt}(B))$. We use this to prove the assertions of the corollary. Assuming the contrary, we find $r > 0$ and infinitely many disjoint balls $\mathbf{B}(a_k, r)$ with $a_k \in \mathrm{spt}(T) \setminus \mathrm{spt}(B)$ and $\mathbf{B}(a_k, r) \cap \mathrm{spt}(B) = \emptyset$. Applying (3.12) with a replaced by a_k we deduce that

$$\mathbf{M}(T \llcorner \mathbf{B}(a_k, r)) \geqq \alpha_n r^n \exp[-\|H\|_{L^\infty} r]$$

for every $k \in \mathbb{N}$, contradicting $\mathbf{M}(T) < \infty$. □

By the results of section 2 and 3 we can solve the Plateau problem for a boundary current $B \in \mathbb{I}_{n-1}(\bar{G})$ and a continuous prescribed mean curvature function $H : \bar{G} \to \mathbb{R}$ in a domain G in $\mathbb{R}^{n+1}$, if a certain isoperimetric inequality holds and $|H(x)| \leqq \mathcal{H}_{\partial G}(x)$ on ∂G.

THEOREM 3.8. – *Let* G *be a* C^2 *domain in* $\mathbb{R}^{n+1}$ *with* ∂G *having nonnegative mean curvature* $\mathcal{H}_{\partial G}$ *and let* $H : \bar{G} \to \mathbb{R}$ *be a continuous function with*

$$|H(x)| \leqq \mathcal{H}_{\partial G}(x) \quad \textit{for} \quad x \in \partial G \tag{3.13}$$

(*no condition if* $G = \mathbb{R}^{n+1}$). *Suppose further, that the boundary* $0 \neq B \in \mathbb{I}_{n-1}(\bar{G})$ *satisfies* $\mathrm{spt}(B) \neq \bar{G}$ *and* $\mathcal{F}(B; \bar{G}) \neq \emptyset$. *Then, there exists a current* $T \in \mathcal{F}(B; \bar{G})$ *with prescribed mean curvature* H *in* $\mathbb{R}^{n+1} \setminus \mathrm{spt}(B)$ *provided one of the following conditions is satisfied*

$$\left[\int_G |H|^{n+1} d\mathcal{L}^{n+1}\right]^{1/(n+1)} < \frac{1}{\gamma_{n+1}} \tag{3.14}$$

$$(3.15)\qquad \sup_{t\in\mathbb{R}}\left[\int_{G_t}|H(z,t)|^n\,d\mathcal{L}^n(z)\right]^{1/n}<\frac{1}{\gamma_n}$$

$$(3.16)\qquad \sup_{s>0}[s^{n+1}\mathcal{L}^{n+1}(\{x\in G : |H(x)|\geqq s\})]^{1/(n+1)}<n\sqrt[n+1]{\alpha_{n+1}}$$

$$(3.17)\qquad \sup_{s>0,\,t\in\mathbb{R}}[s^n\mathcal{L}^n(\{z\in G_t : |H(z,s)|\geqq s\})]^{1/n}<(n-1)\sqrt[n]{\alpha_n},$$

where $G_t := \{z\in\mathbb{R}^n : (z,t)\in G\}$.

Proof. – We extend H to a function on $\mathbb{R}^{n+1}$ by $H(x)=0$ for $x\notin\bar{G}$. Then using proposition 2.4 and theorem 2.1 (with $\tilde{K}=\mathbb{R}^{n+1}$, $K=\bar{G}$ there) we find a solution $T\in\mathcal{F}(B;\bar{G})$ of the variational problem

$$E_H(S)\to\min\quad\text{among all } S\in\mathcal{F}(B;\bar{G}).$$

Since $|H(x)|\leqq\mathcal{H}_{\partial G}(x)$ on ∂G we can apply corollary 3.5 to infer that each energy minimizing current T in $\mathcal{F}(B;\bar{G})$ has prescribed mean curvature H on $\mathbb{R}^{n+1}\setminus\operatorname{spt}(B)$. □

From theorem 3.8 we immediately obtain

COROLLARY 3.9. – *Suppose* G, ∂G, $\mathcal{H}_{\partial G}$ *and* B *are as in theorem* 3.8, $H:\bar{G}\to\mathbb{R}$ *is continuous and*

$$|H(x)|\leqq\mathcal{H}_{\partial G}(x)\qquad\textit{for}\quad x\in\partial G.$$

Then, there exists a current $T\in\mathcal{F}(B;\bar{G})$ *with prescribed mean curvature* H *in* $\mathbb{R}^{n+1}\setminus\operatorname{spt}(B)$, *if one of the following conditions holds:*

$$(3.18)\qquad \sup_G|H|<(n+1)\left(\frac{\alpha_{n+1}}{\mathcal{L}^{n+1}(G)}\right)^{1/(n+1)}$$

$$(3.19)\qquad \sup_{G_t}|H(.,t)|\leqq cn\left(\frac{\alpha_n}{\mathcal{L}^n(G_t)}\right)^{1/n}\quad\textit{for all } t\in\mathbb{R},$$

where $0\leqq c<1$.

In particular, if G is the unit ball $\mathbf{B}=\mathbf{B}(1)$ in $\mathbb{R}^{n+1}$, (3.18) and (3.13) are satisfied, if

$$\sup_{\mathbf{B}}|H|<n+1,\qquad |H(x)|\leqq n\qquad\text{for}\quad x\in\partial\mathbf{B}.$$

Hence, the result of [DF1] is contained as a special case. If G is the cylinder $\mathbf{C}:=\mathbf{B}^n(0,1)\times\mathbb{R}$ in $\mathbb{R}^{n+1}$, then (3.19) and (3.13) are implied by

$$\sup_{\mathbf{C}}|H|<n,\qquad |H(x)|\leqq n-1\qquad\text{for}\quad x\in\partial\mathbf{C}.$$

It should be noted that results of this type are well known for parametric 2 dimensional surfaces of prescribed mean curvature H in $\mathbb{R}^3$. The reader is referred to [Hi], [GS1], [GS2], [St1] and [St2].

4. REGULARITY OF MINIMIZERS

From Allard's work [AW] and corollary 3.5 we know that energy minimizings currents T in $\mathscr{F}(B; \bar{G})$ are smooth, *i.e.*, locally represented by an oriented n dimensional C^1 submanifold with constant integer multiplicity, on a dense and relatively open subset of $\operatorname{spt}(T)\setminus\operatorname{spt}(B)$. In this section we are going to prove the optimal regularity theorem for energy minimizing currents $T \in \mathscr{F}(B; \bar{G})$ with $\operatorname{spt}(T) \subset G$. The proof is based on the regularity theory initiated by Federer [Fe1] for mass minimizing currents in codimension one.

Our general assumptions in this section are the following: $H : \mathbf{B}(R) \to \mathbb{R}$ is a locally Lipschitz continuous function and $\omega : \mathbf{B}(R) \to \Lambda^{n+1}\mathbb{R}^{n+1}$ denotes the $n+1$ form $H\, dx_1 \wedge \ldots \wedge dx_{n+1}$. Moreover, $T \in \mathscr{D}_n(\mathbf{B}(R))$ is a locally rectifiable integer multiplicity n current with $0 < \mathbf{M}(T) < \infty$ and $\partial T = 0$ in $\mathbf{B}(R)$ such that

$$\mathbf{M}_W(T) \leqq \mathbf{M}_W(T+X) + \langle Q_X; \omega \rangle \tag{4.1}$$

for any open set $W \subset\subset \mathbf{B}(R)$ and any rectifiable integer multiplicity current $X \in \mathscr{D}_n(\mathbf{B}(R))$ with $\partial X = 0$ and $\operatorname{spt}(X) \subset W$ (Q_X denoting the unique $n+1$ current with finite mass and boundary X). Currents T which satisfy (4.1) are called *locally energy minimizing*. To describe the regularity theory for locally energy minimizing currents we first recall the following definition.

DEFINITION 4.1. – Let $U \subset \mathbb{R}^{n+1}$ be open and $T \in \mathscr{D}_n(U)$ a locally rectifiable integer multiplicity n current. Then:

1) Reg(T) – the regular set of T – is the set of all points $a \in \operatorname{spt}(T)$ with the property: there exists an open ball $\mathbf{B}(a, \rho)$, an integer $m \in \mathbb{N}$ and an oriented n dimensional submanifold $M \subset \mathbb{R}^{n+1}$ such that

$$T \llcorner \mathbf{B}(a, \rho) = m[\![M \cap \mathbf{B}(a, \rho)]\!].$$

2) By Sing (T) – the singular set of T – we denote the relatively closed complement of Reg(T) in $\operatorname{spt}(T)\setminus\operatorname{spt}(\partial T)$.

We are now in a position to state the following *regularity theorem*.

THEOREM 4.2. – *Suppose* $H : \mathbf{B}(R) \to \mathbb{R}$ *is locally Lipschitz continuous,* $T \in \mathscr{D}_n(\mathbf{B}(R))$ *is a locally rectifiable integer multiplicity current with* $\mathbf{M}(T) < \infty$ *and* $\partial T = 0$ *in* $\mathbf{B}(R)$ *such that*

$$\mathbf{M}_W(T) \leqq \mathbf{M}_W(T+X) + \langle Q_X; \omega \rangle$$

for every open set $W \subset\subset \mathbf{B}(R)$ *and any rectifiable integer multiplicity current* $X \in \mathscr{D}_n(\mathbf{B}(R))$ *with* $\partial X = 0$ *and* $\operatorname{spt}(X) \subset W$. *Then,* Reg(T) *is a* $C^{2,\mu}$ *submanifold, for every* $0 < \mu < 1$, *on which* T *has mean curvature* H *and locally constant multiplicity. Moreover,* Sing(T) *is empty for* $n \leqq 6$, *locally finite in*

$\mathbf{B}(\mathrm{R})$ *if* $n=7$ *and of Hausdorff dimension at most* $n-7$ *in case* $n>7$, *i.e.*

$$\mathcal{H}^{n-7+\delta}(\operatorname{Sing}(\mathrm{T}))=0 \quad \textit{for all } \delta>0.$$

Proof. – First, choose $0<\varepsilon<\mathrm{R}$ (arbitrary close to R) such that $\mathrm{T} \llcorner \mathbf{B}(\varepsilon) \in \mathbb{I}_n(\mathbb{R}^{n+1})$. Since $\partial(\mathrm{T} \llcorner \mathbf{B}(\varepsilon)) \in \mathbb{I}_{n-1}(\mathbb{R}^{n+1})$ and $\operatorname{spt} \partial(\mathrm{T} \llcorner \mathbf{B}(\varepsilon)) \subset \partial \mathbf{B}(\varepsilon)$, there exists $\Xi \in \mathbb{I}_n(\mathbb{R}^{n+1})$ with $\operatorname{spt}(\Xi) \subset \partial \mathbf{B}(\varepsilon)$ and $\partial \Xi=\partial(\mathrm{T} \llcorner \mathbf{B}(\varepsilon))$. Next, let $\mathrm{S}:=\mathrm{T} \llcorner \mathbf{B}(\varepsilon)-\Xi$. From the decomposition theorem ([Fe], 4.5.17] we infer that

$$\mathrm{S}=\sum_{i \in \mathbf{Z}} \partial(\mathbf{E}^{n+1} \llcorner \mathrm{U}_i), \qquad \|\mathrm{S}\|=\sum_{i \in \mathbf{Z}} \|\partial(\mathbf{E}^{n+1} \llcorner \mathrm{U}_i)\|,$$

where $\mathrm{U}_i \subset \mathrm{U}_{i-1}$ is a sequence of $\mathscr{L}^{n+1}$ measurable sets of finite perimeter. Let $\mathrm{T}_i:=[\partial(\mathbf{E}^{n+1} \llcorner \mathrm{U}_i)] \llcorner \mathbf{B}(\varepsilon)$. Then, $\operatorname{spt}(\partial \mathrm{T}_i) \subset \partial \mathbf{B}(\varepsilon)$ and

$$(4.2) \qquad \mathrm{T} \llcorner \mathbf{B}(\varepsilon)=\sum_{i \in \mathbf{Z}} \mathrm{T}_i, \qquad \|\mathrm{T} \llcorner \mathbf{B}(\varepsilon)\|=\sum_{i \in \mathbf{Z}} \|\mathrm{T}_i\|.$$

From (4.1) and the decomposition (4.2) it easiliy follows that for each $i \in \mathbf{Z}$ the corresponding current T_i is locally energy minimizing on $\mathbf{B}(\varepsilon)$. This implies in particular that

$$(4.3) \quad \mathbf{M}_{\mathbf{B}(\varepsilon)}(\partial(\mathbf{E}^{n+1} \llcorner \mathrm{U}_i))+\int_{\mathbf{B}(\varepsilon) \cap \mathrm{U}_i} \mathrm{H}\, d\mathscr{L}^{n+1} \leqq \mathbf{M}_{\mathbf{B}(\varepsilon)}(\partial(\mathbf{E}^{n+1} \llcorner \mathrm{B}))+\int_{\mathbf{B}(\varepsilon) \cap \mathrm{B}} \mathrm{H}\, d\mathscr{L}^{n+1}$$

for every Borel set $\mathrm{B} \subset \mathbb{R}^{n+1}$ such that $(\mathrm{B} \setminus \mathrm{U}_i) \cup (\mathrm{U}_i \setminus \mathrm{B})$ has compact closure in $\mathbf{B}(\varepsilon)$. In view of Massari's regularity theorem [Ma] we find an open subset $\mathrm{O}_i \subset \mathbf{B}(\varepsilon)$ such that $\operatorname{spt}(\mathrm{T}_i) \cap \mathrm{O}_i$ is an n dimensional $\mathrm{C}^{1,\mu}$ submanifold of $\mathbb{R}^{n+1}$ for all $0<\mu<1$. Moreover, $\mathbf{B}(\varepsilon) \setminus \mathrm{O}_i$ is empty for $n \leqq 6$, finite for $n=7$ and has Hausdorff dimension at most $n-7$ for $n \geqq 7$, *i.e.*, we have

$$\mathcal{H}^{n-7+\delta}(\mathbf{B}(\varepsilon) \setminus \mathrm{O}_i)=0 \quad \text{for all } \delta>0.$$

Now, for $a \in \operatorname{spt}(\mathrm{T}_i)$ and $0<\rho<\varepsilon-|a|$ the mass estimate (3.12) applied to T_i implies that

$$\mathbf{M}(\mathrm{T}_i \llcorner \mathbf{B}(a, \rho)) \geqq \alpha_n \rho^n \exp[-\rho \|\mathrm{H}\|_{\mathrm{L}^\infty(\mathbf{B}(\varepsilon))}];$$

consequently the set $\Delta(a):=\{i \in \mathbf{Z} : a \in \operatorname{spt}(\mathrm{T}_i)\}$ is finite and

$$\alpha(a):=\{\operatorname{dist}(a, \operatorname{spt}(\mathrm{T}_i)) : i \notin \Delta(a)\}>0.$$

Let

$$\mathrm{O}:=\mathbf{B}(\varepsilon) \setminus \bigcup_{i \in \mathbf{Z}} (\operatorname{spt}(\mathrm{T}_i) \setminus \mathrm{O}_i).$$

Then

$$\mathbf{B}(\varepsilon) \setminus \mathrm{O} \subset \bigcup_{i \in \mathbf{Z}} (\operatorname{spt}(\mathrm{T}_i) \setminus \mathrm{O}_i)$$

and

$$\mathscr{H}^{n-7+\delta}(\mathbf{B}(\varepsilon)\setminus \mathrm{O})=0$$

whenever $\delta>0$. Now, we show that O is open and that $\mathrm{spt}(\mathrm{T})\cap \mathrm{O}$ is an n dimensional C^1 submanifold of $\mathbb{R}^{n+1}$. Let $a\in \mathrm{spt}(\mathrm{T})\cap \mathrm{O}$. Then, we have $a\in \mathrm{Reg}(\mathrm{T}_i)$ for every $i\in\Delta(a)$. Using the inclusion $\mathrm{U}_i\subset \mathrm{U}_j$ for $i>j$ and $\partial \mathrm{Tan}(\mathrm{U}_i, a)=\mathrm{Tan}(\mathrm{spt}(\mathrm{T}_i), a)$ we obtain $\mathrm{Tan}(\mathrm{spt}(\mathrm{T}_i), a)=\mathrm{Tan}(\mathrm{spt}(\mathrm{T}_j), a)$ whenever i, $j\in\Delta(a)$. Assume that $a=(0, 0)\in\mathbb{R}^n\times\mathbb{R}$, $\mathrm{Tan}(\mathrm{spt}(\mathrm{T}_i), a)=\mathbb{R}^n\times\{0\}$ and

$$\mathrm{spt}(\mathrm{T}_i)=\mathrm{graph}(u_i) \quad \text{for all } i\in\Delta(a)$$

in a neighborhood of a, where $u_i:\mathbb{R}^n\supset\mathbf{B}^n(r)\to\mathbb{R}$ are C^1 functions with $u_i(0)=0$ and $\nabla u_i(0)=0$ (note that Massari's theorem implies $u_i\in\mathrm{C}^{1,\mu}$ for all $0<\mu<1$). Moreover, u_i is a weak solution of the non-parametric mean curvature equation on $\mathbf{B}^n(r)$, *i.e.*,

$$\mathrm{div}\frac{\nabla u_i}{\sqrt{1+|\nabla u_i|^2}}=\mathrm{H}(\,.\,, u_i).$$

In view of the inclusion $\mathrm{U}_i\subset U_j$, $i>j$, we either have $u_i\geqq u_j$ or $u_i\leqq u_j$. Thus, we may assume that $v:=u_i-u_j\geqq 0$ for all i, $j\in\Delta(a)$ with $i>j$. Furthermore, v is a weak solution of the uniformly elliptic equation

$$\frac{\partial}{\partial x_l}\left(a^{lk}(x)\frac{\partial v}{\partial x_k}\right)+cv=0$$

where

$$a^{lk}(x):=\int_0^1\frac{1}{\sqrt{1+|\nabla u_t|^2}}\left(\delta^{lk}-\frac{\partial_l u_t\,\partial_k u_t}{1+|\nabla u_t|^2}\right)dt$$

and

$$c(x):=\begin{cases}\dfrac{\mathrm{H}(x, u_i(x))-\mathrm{H}(x, u_j(x))}{u_i(x)-u_j(x)} & \text{if } u_i(x)\neq u_j(x),\\ 0 & \text{otherwise,}\end{cases}$$

and $u_t:=(1-t)u_i+tu_j$. Notice that $c\in\mathrm{L}^\infty(\mathbf{B}(\varepsilon))$, because H is locally Lipschitz continuous on $\mathbf{B}(\mathrm{R})$. The Haranck inequality ([GT], Thm. 8.20) implies $v=0$ in $\mathbf{B}^n(r)$, *i.e.*, $u_i=u_j$ for all i, $j\in\Delta(a)$, and hence, $\mathrm{spt}(\mathrm{T}_i)=\mathrm{spt}(\mathrm{T}_j)$ in a neighborhood of $a\,(\Rightarrow \mathrm{spt}(\mathrm{T})=\mathrm{graph}(u_i)$ for some $i\in\Delta(a))$ and $a\in\mathrm{Reg}(\mathrm{T})$. The usual elliptic regularity theory (e.g. [Mo]) then yields that $\mathrm{spt}(\mathrm{T})$ is a $\mathrm{C}^{2,\mu}$ submanifold of $\mathbb{R}^{n+1}$ in a neighborhood of a for all $0<\mu<1$. □

In order to apply the regularity theorem to solutions $\mathrm{T}\in\mathscr{F}(\mathrm{B};\bar{\mathrm{G}})$ of variational problems of type $\mathbf{E}_\mathrm{H}(\,.\,)\to\min$ on $\mathscr{F}(\mathrm{B};\bar{\mathrm{G}})$ ($\bar{\mathrm{G}}$ denoting the closure of a C^2 domain G in $\mathbb{R}^{n+1}$, $H : \bar{G} \to \mathbb{R}$ a locally Lipschitz continuous bounded function) one clearly tries to exhibit geometric conditions on H and ∂G which guarantee that spt(T) lies in the interior of $\bar{G}$, e.g. $\mathrm{spt}(T) \subset G$. Then, theorem 4.2 can be applied to minimizers $T \in \mathscr{F}(B; \bar{G})$ on suitably small balls $\mathbf{B}(a, R)$ with $a \in \mathrm{spt}(T) \setminus \mathrm{spt}(B)$. To give the precise statement we have to introduce some additional notations. For $x \in \bar{G}$ denote by $d(x)$ the distance from x to ∂G. Since ∂G is of class C^2 there exists an open neighborhood $\mathcal{O}$ of ∂G in $\mathbb{R}^{n+1}$ such that the nearest point projection π onto ∂G is defined on $\mathcal{O}$ by $\pi(x) \in \partial G$, $|\pi(x) - x| = d(x)$ and is of class C^1. For $r > 0$ we define

$$f_r(x) := \begin{cases} \pi(x) + r\nu(x) & \text{if} \quad x \in \mathcal{O}, \quad d(x) \leqq r \\ x & \text{otherwise.} \end{cases}$$

Then, a simple adaptation of the proof of [DF1], lem. 7.2 yields the following proposition.

PROPOSITION 4.3. — *Suppose* G, ∂G, H, *and* f_r *are as above* $\mathrm{spt}(B) \subset G$, $T \in \mathscr{F}(B; \bar{G})$, spt(T) *is compact. Assume also that*

$$|H(x)| < \mathscr{H}_{\partial G}(x) \qquad \textit{for} \quad x \in \partial G.$$

Then, there exists $r_0 > 0$ *such that* $f_{r\#} T \in \mathscr{F}(B; \bar{G})$ *and*

$$\mathbf{E}_H(f_{r\#} T) < \mathbf{E}_H(T)$$

whenever $0 < r < r_0$ *and* dist(spt(T), ∂G) $< r$. *In particular* T *cannot be energy minimizing in* $\mathscr{F}(B; \bar{G})$, *if* $\mathrm{spt}(T) \cap \partial G \neq \varnothing$.

Results of this type are well known for 2 dimensional parametric surfaces of prescribed mean curvature in $\mathbb{R}^3$ (*see* [GS1], [St1]).

From corollary 3.7, proposition 4.3 and theorems 3.8, 4.2 we immediately deduce the following final result.

THEOREM 4.4. — *Assume in addition to the hypotheses of theorem* 3.8 (*in particular one of the conditions* (3.14)-(3.17) *is satisfied*) *that* spt(B) *is a compact subset of* G *and* $H : \bar{G} \to \mathbb{R}$ *is locally Lipschitz continuous and bounded with*

$$|H(x)| < \mathscr{H}_{\partial G}(x) \qquad \textit{for} \quad x \in \partial G.$$

Then, the Plateau problem for H *and* B *admits a solution* $T \in \mathscr{F}(B; \bar{G})$ *with compact support in* G *and each solution* T *has the following property:* $\mathrm{spt}(T) \setminus (\mathrm{spt}(B) \cup \mathrm{Sing}(T))$ *is a* $C^{2,\mu}$ *submanifold of* $\mathbb{R}^{n+1}$, *for every* $0 < \mu < 1$, *on which* T *has mean curvature* H *and locally constant multiplicity; the relatively closed set* Sing(T) *in* $\mathrm{spt}(T) \setminus \mathrm{spt}(B)$ *is empty for* $n \leqq 6$, *locally finite for* $n = 7$ *and of Hausdorff dimension at most* $n - 7$ *for* $n \geqq 8$.

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