Ann. Inst. Henri Poincaré,

Vol. 10, n° 2, 1993, p. 131-189. *Analyse non linéaire*

Blow-up behaviour of one-dimensional semilinear parabolic equations

by

M. A. HERRERO and J. J. L. VELÂZQUEZ

Departamento de Matemática Aplicada,
Facultad de Matemáticas,
Universidad Complutense,
28040 Madrid, Spain

ABSTRACT. – Consider the Cauchy problem

$$u_t - u_{xx} - F(u) = 0; \qquad x \in \mathbb{R}, \quad t > 0$$
$$u(x, 0) = u_0(x); \qquad x \in \mathbb{R}$$

where $u_0(x)$ is continuous, nonnegative and bounded, and $F(u) = u^p$ with $p > 1$, or $F(u) = e^u$. Assume that u blows up at $x = 0$ and $t = T > 0$. In this paper we shall describe the various possible asymptotic behaviours of $u(x, t)$ as $(x, t) \to (0, T)$. Moreover, we shall show that if $u_0(x)$ has a single maximum at $x = 0$ and is symmetric, $u_0(x) = u_0(-x)$ for $x > 0$, there holds

1) If $F(u) = u^p$ with $p > 1$, then

$$\lim_{t \uparrow T} u(\xi((T-t)|\log(T-t)|)^{1/2}, t) \times (T-t)^{1/(p-1)} = (p-1)^{-(1/(p-1))} \left[1 + \frac{(p-1)\xi^2}{4p}\right]^{-(1/(p-1))}$$

uniformly on compact sets $|\xi| \leqq R$ with $R > 0$,

2) If $F(u) = e^u$, then

$$\lim_{t \uparrow T} (u(\xi((T-t)|\log(T-t)|)^{1/2}, t) + \log(T-t)) = -\log\left[1 + \frac{\xi^2}{4}\right]$$

uniformly on compact sets $|\xi| \leqq R$ with $R > 0$.

Key words : Semilinear diffusion equations, blow-up, asymptotic behaviour of solutions.

Classification A.M.S. : 35 B 40, 35 K 55, 35 K 57.

Annales de l'Institut Henri Poincaré - Analyse non linéaire - 0294-1449
Vol. 10/93/02/131/59/$7.90/

RÉSUMÉ. — On considère le problème de Cauchy

$$u_t - u_{xx} - F(u) = 0; \qquad x \in \mathbb{R}, \quad t > 0$$
$$u(x, 0) = u_0(x); \qquad x \in \mathbb{R}$$

où $u_0(x)$ est une fonction continue, non négative et bornée, et $F(u) = u^p$ avec $p > 1$ ou $F(u) = e^u$. Nous supposons que u explose au point $x = 0$ en temps $T > 0$. Dans ce travail, nous obtenons tous les comportements asymptotiques possibles de la solution $u(x, t)$ quand $(x, t) \to (0, T)$.

1. INTRODUCTION AND DESCRIPTION OF RESULTS

This paper deals with the initial value problem

$$u_t = u_{xx} + F(u) \quad \text{when } x \in \mathbb{R}, \quad t > 0 \tag{1.1}$$
$$u(x, 0) = u_0(x) \quad \text{when } x \in \mathbb{R} \tag{1.2}$$

where $u_0(x)$ is continuous, nonnegative and bounded, and

$$F(u) = u^p, \quad p > 1, \qquad \text{or} \qquad F(u) = e^u \tag{1.3}$$

Local (in time) existence of solutions of (1.1), (1.2), which are positive for any $t > 0$, follows at once from standard results. It is well known, however, that with such a choice of $F(u)$, solutions may develop singularities in finite time. For instance, if $F(u) = u^p$ with $1 < p \leqq 3$, for any nontrivial solution of (1.1), (1.2) there exists a finite time T such that

$$\limsup_{t \uparrow T} \left(\sup_{x \in \mathbb{R}} u(x, t) \right) = +\infty$$

We then say that $u(x, t)$ blows-up in a finite time T, which is called the blow-up time of u. When $p \geqq 3$, this fact occurs if $u_0(x)$ is large enough, but there also exist global nontrivial solutions, these last originating from small enough initial values (*cf.* [Fu], [AW]. . .).

For solutions exhibiting blow-up, a natural question is the manner in which this phenomenon takes place. For instance, it is interesting to determine the nature of the blow-up set (*i.e.*, the set of points where singularities appear). In particular, for Cauchy, Cauchy-Dirichlet or Cauchy-Neumann problems associated to (1.1), (1.3), several authors have discussed conditions under which blow-up occurs at a single point (*cf.* for instance [W], [MW], [GP], [FM]. . .).

Another question which has deserved attention over the past years is that of determining the asymptotic behaviour of solutions as the blow-up time is approached; *cf.* for instance [W], [GP], [GK1], [GK2], [GK3], [FM], [BBE], [L]... Let us just mention here a few results which will be used in the sequel. Assume that $u(x, t)$ is a solution of (1.1), (1.2) which blows-up at $x=0$ and $t=T$. We then have

(i) If $F(u)=u^p$ with $p>1$,

$$\lim_{t \uparrow T} u(x(T-t)^{1/2}, t)(T-t)^{1/(p-1)}=(p-1)^{-(1/(p-1))}. \qquad (1.4a)$$

uniformly on compact sets $|x| \leqq R$ with $R>0$ *cf.* [GP], [GK1], [GK2], [GK3]),

(ii) If $F(u)=e^u$,

$$\lim_{t \uparrow T} (u(x(T-t)^{1/2}, t)+\ln(T-t))=0 \qquad (1.4b)$$

uniformly on compact sets $|x| \leqq R$ with $R>0$. When (1.1) is considered in bounded domains with homogeneous Dirichlet conditions, estimate (1.4*b*) has been obtained in [BBE] under suitable assumptions on initial values. Actually (1.4*b*) can be obtained in our case by adapting the arguments in [Li]. To this end, two bounds are needed. The first one is

$$u(x, t)+\log(T-t) \leqq C \qquad (1.5a)$$

for any $x \in \mathbb{R}$ and $t<T$, C being some positive constant. If $u_0(x)$ is, say, monotone decreasing for $x>0$, symmetric with respect to the origin and with a single maximum at $x=0$, the above inequality can be obtained arguing as in [GP], Thm. 4.2. Furthermore, one also needs to show that

$$|u_x| \leqq C(T-t)^{-(1/2)-(1/(p-1))} \qquad (1.5b)$$

for any $x \in \mathbb{R}$ and $t<T$, and some $C>0$. This in turn can be obtained by adapting the arguments in [GK1], Proposition 1. The reader will notice that $(p-1)^{-(1/(p-1))}$ $(T-t)^{-(1/(p-1))}$ and $(-\log(T-t))$ are explicit solutions of (1.1) with the previous choices of F(u).

A more precise information on the behaviour near the blow-up time (including higher-order expansions for solutions) can be obtained in a formal, non rigorous may, by means of singular perturbation techniques, as for instance in [D], [GHV].

Recently, a further step has been obtained in [B]. In that paper, the author considers (1.1) with $F(u)=u^u$ in a bounded interval with zero side conditions, and proves that there exist initial values $u_0(x)$ such that the corresponding solutions behave near the blow-up time exactly as suggested by formal perturbation methods. An interesting question left open in [B] is how to know a priori whether a given initial value u_0 will actually determine a solution belonging to the class obtained in that article.

In this work we consider some cases where the results suggested by perturbation theory can be made rigorous. For instance, we show

THEOREM 1. — *Assume that the solution* $u(x, t)$ *of* (1.1), (1.2) *with* $F(u)=u^p$, $p>1$, *blows up in a finite time* $T>0$ *at the point* $x=0$. *Then if* $u_0(x)$ *is symmetric with respect to the origin and has a single maximum at* $x=0$, *there holds.*

$$\lim_{t\uparrow T} u(\xi((T-t)|\log(T-t)|)^{1/2}, t)(T-t)^{1/(p-1)} = (p-1)^{-(1/(p-1))}\left[1+\frac{(p-1)\xi^2}{4p}\right]^{-(1/(p-1))} \quad (1.6)$$

uniformly on compact sets $|\xi|\leqq R$ *with* $R>0$.

We recall that sufficient conditions for blow-up at a point can be found for instance in [MW], [GP].

Concerning the exponential case, we have

THEOREM 2. — *Assume that the solution* $u(x, t)$ *of* (1.1), (1.2) *with* $F(u)=e^u$ *blow up in a finite time* $T>0$, *at the point* $x=0$. *Then, if* $u_0(x)$ *is symmetric with respect to the origin, and has a single maximum at* $x=0$, *there holds*

$$\lim_{t\uparrow T} (u(\xi((T-t)|)^{1/2}, t)+\log(T-t)) = -\log\left(1+\frac{\xi^2}{4}\right) \quad (1.7)$$

uniformly on compact sets $|\xi|\leqq R$ *with* $R>0$.

Let us point out that (1.6), (1.7) were formally obtained in [GHV] by means of the method of matched asymptotic expansions. Theorems 1 and 2 can be proved basically in the same way, so that from now on we shall concentrate on the first result, to sketch then briefly the modifications required to cover the situation where $F(u)=e^u$. Our approach is deeply influenced by dynamical systems theory, a viewpoint already used by several authors to describe asymptotics of solutions near singular points in a variety of problems (*cf.* for instance [A], [CMV]). Since the proof under consideration is rather long and technical, we shall first sketch its main points, and leave the details to the following Sections.

To begin with, we notice that the blow-up time T may be normalized by setting $T=1$ without loss of generality. This convention will be assumed henceforth. We then perform the well known change of variables (*cf.* [GP], [GK1], . . .)

$$\left.\begin{aligned} u(x, t)=(1-t)^{-(1/(p-1))}\varphi(y, \tau) \qquad \text{where } y=\frac{x}{(1-t)^{1/2}}, \\ \tau=-\log(1-t) \end{aligned}\right\} \quad (1.8)$$

In this way, φ satisfies

$$\varphi_\tau=\varphi_{yy}-\frac{y}{2}\varphi_y-\frac{1}{p-1}\varphi+\varphi^p=\varphi_{yy}-\frac{y}{2}\varphi_y+\varphi+f_1(\varphi) \quad (1.9a)$$

where

$$f_1(\varphi)=\varphi^p-\frac{p}{p-1}\varphi \qquad (1.9\,b)$$

We now linearize about the stationary solution of (1.9) given by $\bar{\varphi}(y)=(p-1)^{-(1/(p-1))}$ by setting

$$\varphi(y,\tau)=(p-1)^{-(1/(p-1))}+\psi(y,\tau) \qquad (1.10)$$

Then $\psi(y,\tau)$ solves

$$\psi_\tau=\psi_{yy}-\frac{y}{2}\psi_y+\psi+f(\psi) \qquad (1.11\,a)$$

where

$$f(\psi)=((p-1)^{-(1/(p-1))}+\psi)^p-(p-1)^{-(p/(p-1))}-\frac{p}{p-1}\psi, \qquad (1.11\,b)$$

so that $f(s)=O(s^2)$ as $s\to 0$. Here and henceforth, we shall freely use the customary asymptotic notations $o(\ )$, $O(\ )$, $\ll$, $\simeq$, etc.

We now describe our functional framework. For $1\leqq q<+\infty$, and any integer $k\geqq 1$, we define the spaces

$$L^q_w(\mathbb{R})=\left\{g\in L^q_{\text{loc}}(\mathbb{R}):\int_{\mathbb{R}}|g(s)|^q e^{-(s^2/4)}\,ds<+\infty\right\},$$

$$H^k_w(\mathbb{R})=\left\{g\in L^2_{\text{loc}}(\mathbb{R}):\text{for any } j\in[O,k],\ g^{(j)}\in L^2_{\text{loc}}(\mathbb{R})\right.$$
$$\left.\text{and}\int_{\mathbb{R}}|g^{(j)}(s)|^2 e^{-(s^2/4)}\,ds<+\infty\right\}.$$

It is readily seen that $L^2_w(\mathbb{R})$ [resp. $L^q_w(\mathbb{R})$, $1\leqq q<\infty$, $q\neq 2$] is a Hilbert space (resp. a Banach space) when endowed with the norm

$$\left.\begin{aligned}&\|g\|^2_{2,w}\equiv\langle g,g\rangle=\int_{\mathbb{R}}(g(s))^2 e^{-(s^2/4)}\,ds<+\infty\\&\left[\text{resp. }\|g\|^q_{q,w}\equiv\int_{\mathbb{R}}|g(s)|^q e^{-(s^2/4)}\,ds<+\infty\right]\end{aligned}\right\} \qquad (1.12)$$

Clearly, for $k\geqq 1$, $H^k_w(\mathbb{R})$ can be given a structure of Hilbert space in a straightforward way. Since the L^2_w-norm will be repeatidly used in the sequel, from now on we shall drop the subscripts (2, w) in (1.12) for the sake of simplicity.

It is then natural to consider (1.11) as a dynamical system in $L^2_w(\mathbb{R})$, since it can be written in the form

$$\psi_\tau=A\psi+f(\psi) \qquad (1.13)$$

where

$$A\psi=\psi_{yy}-\frac{y}{2}\psi_y+\psi;\qquad D(A)=H^2_w(\mathbb{R}) \tag{1.14}$$

is a self-adjoint operator in $L^2_w(\mathbb{R})$, having eigenvalues $\lambda_n=1-\frac{n}{2}$; $n=0, 1, 2, \ldots$ with eigenfunctions $H_n(y)$ given by

$$\left.\begin{array}{c}H_n(y)=c_n\tilde{H}_n(y/2),\text{ where } c_n=(2^{n/2}(4\pi)^{1/4}(n!)^{1/2})^{-1},\\ \text{and } \tilde{H}_n(y) \text{ is the standard } n^{\text{th}}\text{-Hermite polynomial,}\\ \text{so that } \|H_n\|=1 \text{ for any } n.\end{array}\right\} \tag{1.15}$$

To proceed further, we make use of previous results [*cf.* (1.4*a*)] to notice that

$$\left.\begin{array}{c}\psi \text{ is bounded for large } \tau, \text{ and } \psi(y,\tau)\to 0 \text{ as } \tau\to\infty,\\ \text{uniformly on compact sets } |y|\leqq R \text{ with } R>0.\\ \text{In particular, } \|\psi(.,\tau)\|_{q,w}\to 0 \text{ as } \tau\to\infty \text{ for any } q\in[1,\infty)\end{array}\right\} \tag{1.16}$$

We now write

$$\psi(y,\tau)=\sum_{k=0}^{\infty}a_k(\tau)H_k(y) \tag{1.17}$$

where, in view of (1.11), the Fourier coefficients $\{a_k(\tau)\}$ satisfy

$$\left.\begin{array}{c}\dot{a}_k(\tau)=\left(1-\frac{k}{2}\right)a_k(\tau)+\langle f(\psi), H_k\rangle;\\ k=0, 1, 2, \ldots\end{array}\right\} \tag{1.18}$$

The proof of Theorem 1 begins by analyzing the asymptotics of $\psi(y,\tau)$ as $\tau\to\infty$. By analogy with classical ODE theory, we expect that one of the modes will eventually dominate in (1.17), *i.e.*

$$\psi(y,\tau)\simeq a_j(\tau)H_j(y) \text{ as } \tau\to\infty,\quad \text{for some } j=0, 1, 2, \ldots \tag{1.19}$$

A first major step consists in showing that, if $\lim_{\tau\to\infty}\|\psi(.,\tau)\|=0$, the index j in (1.19) must be larger or equal than two. Namely, we prove in Section 4 below that

$$\lim_{\tau\to\infty}\frac{|a_0(\tau)|+|a_1(\tau)|}{\|\psi(.,\tau)\|}=0 \tag{1.20}$$

We then describe in Section 5 all the possible behaviours of $\psi(y,\tau)$ as $\tau\to\infty$, which were previously conjected in [GHV]. Since this result is of inedependent interest, we write it in detail here

THEOREM 3. – *Assume that* $F(u)=u^p$ *with* $p>1$, $u_0(x)$ *is continuous nonnegative and bounded, and the solution of* (1.1), (1.2) *blows up at* $x=0$,

$t=1$. *Then one of the following cases occurs*

$$\psi(.,\tau)\equiv 0 \quad \textit{for any} \quad \tau>0, \tag{1.21 a}$$

or

$$\psi(.,\tau)+\frac{(4\pi)^{1/4}(p-1)^{-(1/(p-1))}}{\sqrt{2}\,p}\cdot\frac{H_2(.)}{\tau}=o\left(\frac{1}{\tau}\right) \textit{ as } \tau\to\infty, \tag{1.21 b}$$

or

$$\left.\begin{array}{l}\textit{There exist } m\geqq 3 \textit{ and } C\neq 0 \textit{ such that}\\ \psi(.,\tau)-C\,e^{(1-(m/2))\tau}H_m(.)=o(e^{(1-(m/2))\tau}) \quad \textit{as } \tau\to\infty\\ \textit{where convergence takes place in } H^1_w \textit{ as well}\\ \textit{in } C^{k,\alpha}_{loc}, \textit{for any } k\geqq 1 \textit{ and } \alpha\in(0,1).\end{array}\right\} \tag{1.21 c}$$

The fact that $(1.21\,b)$ holds in our case is a consequence of our assumptions on the initial value. Indeed, $u_0(x)$ has a single maximum at $x=0$, and is symmetric with respect to the origin, whence so does $u(x,t)$. Therefore, m must be an even number $m\geqq 2$, and for $m\geqq 4$, $H_m(y)$ has exactly $\left(\frac{m}{2}\right)$ maxima. Since the number of maxima cannot increase in time, this rules out $(1.21\,c)$.

As to $(1.21\,a)$, this condition can be shown to imply $u_0(x)=(p-1)^{-(1/(p-1))}$. It is to be noticed, however, that flat profiles as those in $(1.21\,c)$ actually exist, and shall be discussed elsewhere, (*cf.* [HV1]). We have thus obtained

$$\psi(y,\tau)=-\frac{(4\pi)^{1/4}(p-1)^{-(1/(p-1))}}{\sqrt{2}\,p}\cdot\frac{H_2(y)}{\tau}+o\left(\frac{1}{\tau}\right) \quad \text{as } \tau\to\infty \tag{1.22}$$

uniformly on compact sets $|y|\leqq R$ with $R>0$, and (1.19) holds in such a way with $j=2$. We remark that to derive $(1.21\,b)$ or (1.22), we proceed by integrating asymptotically (1.18) when $k=2$, a rather delicate case. Indeed, higher order terms are then to be taken into account, thus yielding the factor $\left(\frac{1}{\tau}\right)$ in (1.22).

The last crucial step in the proof of Theorem 1 consists in extending the convergence result in (1.22) to larger regions of the form $|x|\leqq C(1-t)^{1/2}|\ln(1-t)|^{1/2}$ with $C>0$. This is done in Section 6, by means of a further change of variables. Instead of dealing with $\psi(y,\tau)$ given in (1.8), (1.10), we now set

$$G=\varphi^{-(1/(p-1))}-(p-1) \tag{1.23}$$

so that G solves

$$G_\tau=G_{yy}-\frac{y}{2}G_y+G-\frac{p}{p-1}\cdot\frac{G_y^2}{(p-1)+G}\equiv AG+L(y,\tau) \tag{1.24}$$

We are now able to obtain a suitable bound for $L(\xi\sqrt{\tau}, \tau)$ as $\tau \to \infty$ when $|\xi|$ stays bounded (*cf.* Lemma 6.2). This fact is instrumental in showing that

$$\lim_{\tau \to \infty} G(\xi\sqrt{\tau}, \tau) = \frac{(p-1)^2}{4p}\xi^2 \tag{1.25}$$

uniformly when $|\xi| \leqq R$ with $R > 0$. Taking into account (1.23), (1.10) and (1.8), Theorem 1 follows now from (1.25).

At this time, the reader may wonder whether the new change of variables (1.23) is actually necessary to obtain (1.6). As a matter of fact, we do not know how to derive such result from equations (1.9) or (1.11) alone. The reason is that the terms $f_1(\varphi)$, $f(\psi)$ are not of lower order than the linear parts of their respective equations on sets

$$|x| \sim (1-t)^{1/2} |\ln(1-t)|^{1/2}, \quad t \uparrow 1,$$

as can be seen by using the asymptotic formula (1.6) on (1.9), (1.11). On the other hand, the term $L(y, \tau)$ in (1.24) is indeed of lower order than G_τ and AG in such sets. Therefore, an asymptotic analysis of (1.24) can be performed by considering such equations as a small perturbation of the linear equation $G_\tau = AG$, on the region under consideration.

We have briefly described the main results of this paper, which are contained in Sections 4-6 below. The arguments in these Sections are easily adapted to deal with the exponential case; a sketch of the modifications required to this end is to be found in Section 7. In particular, Theorem 3 is to be replaced now by

THEOREM 4. – *Assume that $F(u) = e^u$, $u_0(x)$ is continuous nonnegative and bounded, and the solution of* (1.1), (1.2) *blows up at $x = 0$, $t = 1$. Let $\zeta(y, \tau) = \ln(1-t) + u(x, t)$, where y, τ are as in* (1.8). *Then one of the following cases occurs*

$$\zeta(\cdot, \tau) \equiv 0 \quad \textit{for any } \tau > 0, \tag{1.26a}$$

$$\zeta(\cdot, \tau) + \frac{(4\pi)^{1/4}}{\sqrt{2}} \cdot \frac{H_2(\cdot)}{\tau} = o\left(\frac{1}{\tau}\right) \quad \textit{as } \tau \to \infty, \tag{1.26b}$$

There exist $m \geqq 3$ and $C \neq 0$ such that

$$\zeta(\cdot, \tau) = C e^{(1-(m/2))\tau} H_m(\cdot) + o(e^{(1-(m/2))\tau}) \quad \textit{as } \tau \to \infty \tag{1.26c}$$

where convergence takes place in H_w^1 as well in $C_{loc}^{k,\alpha}$, for any $k \geqq 1$ and $\alpha \in (0, 1)$.

Finally, several auxiliary tools are used to obtain our asymptotic results. From these, some are already well known and are recalled in the Appendixes at the end of the paper for convenience. Some others, however, had to be implemented to fulfill the previously described steps, and make the content of Section 2 and 3 below.

2. AN A PRIORI ESTIMATE FOR THE HEAT EQUATION

Let $S(t)$ be the linear semigroup corresponding to the heat equation in the strip $S=[0, 1) \times \mathbb{R}$. Take now $u_0(x) \in L^1_{\mathrm{loc}}(\mathbb{R})$ satisfying suitable growth conditions as $|x| \to \infty$, so that $(S(t) u_0(x))$ makes sense in S. Set $s_+ = \max\{s, 0\}$. We then have

LEMMA 2.1. – *For any r, q with $r>1$ and $q>1$, there exists $C=C(r, q)$ such that, for $\left(\dfrac{r-q}{r-1}\right)_+ < t < 1$,*

$$\left[\int_{\mathbb{R}} |S(t) u_0(x)|^r \exp\left(-\frac{x^2}{4(1-t)}\right) dx\right]^{1/r} \leqq \frac{C(1-t)^{1/2r}}{t^{1/2q}(q-r+t(r-1))^{1/2r}} \left[\int_{\mathbb{R}} |u_0(x)|^q \exp\left(-\frac{x^2}{4}\right) dx\right]^{1/q} \quad (2.1)$$

Proof. – By Poisson formula

$$\begin{aligned} |S(t) u_0(x)| &= (4\pi t)^{-1/2} \left| \int_{\mathbb{R}} u_0(s) \exp\left(-\frac{(x-s)^2}{4t}\right) ds \right| \\ &\leqq (4\pi t)^{-1/2} \left[\int_{\mathbb{R}} |u_0(s)|^q \exp\left(-\frac{s^2}{4}\right) ds\right]^{1/q} \\ &\quad \times \left[\int_{\mathbb{R}} \exp\left(-\left(\frac{(x-s)^2}{4t} - \frac{s^2}{4q}\right) q'\right) ds\right]^{1/q'} \end{aligned}$$

where $\dfrac{1}{q} + \dfrac{1}{q'} = 1$. Therefore

$$\int_{\mathbb{R}} |S(t) u_0(x)|^r \exp\left(-\frac{|x|^2}{4(1-t)}\right) dx \leqq I(r, q; t) \times \left[\int_{\mathbb{R}} |u_0(s)|^q \exp\left(-\frac{s^2}{4}\right) ds\right]^{r/q} \quad (2.2)$$

where

$$\begin{aligned} I(r, q; t) &\equiv (4\pi t)^{-r/2} \int_{\mathbb{R}} \exp\left(-\frac{|x|^2}{4(1-t)}\right) \times \left[\int_{\mathbb{R}} \exp\left(-q'\left(\frac{(x-s)^2}{4t} - \frac{s^2}{4q}\right)\right) ds\right]^{r/q'} dx \\ &\equiv (4\pi t)^{-r/2} \int_{\mathbb{R}} \exp\left(-\frac{x^2}{4(1-t)}\right) (J(q; x))^{r/q'} dx \end{aligned}$$

Since

$$q'\left(\frac{(x-s)^2}{4t}-\frac{s^2}{4q}\right)=\frac{q'}{4}\left(\frac{1}{t}-\frac{1}{q}\right)\left(s-\frac{qx}{q-t}\right)^2-\frac{q'}{4(q-t)}x^2$$

There holds

$$J(q;x)=\left[\int_{\mathbb{R}}\exp\left(-\frac{q'}{4}\left(\frac{1}{t}-\frac{1}{q}\right)s^2\right)ds\right]\exp\left(\frac{q'x^2}{4(q-t)}\right)$$

$$=2\left[\frac{\pi qt}{q'(q-t)}\right]^{1/2}\exp\left(\frac{q'x^2}{4(q-t)}\right)\leqq C_1 t^{1/2}\exp\left(\frac{q'x^2}{4(q-t)}\right)$$

for some constant $C_1=C_1(q)$. Therefore

$$I(r,q;t)\leqq C t^{-r/2q}\int_{\mathbb{R}}\exp\left(\left(\frac{r}{q-t}-\frac{1}{1-t}\right)\frac{x^2}{4}\right)dx$$

$$\leqq\frac{C(1-t)^{1/2}}{t^{r/2q}(t(r-1)-(r-q))^{1/2}}$$

where $C=C(r,q)>0$. Plugging this last inequality into (2.2), the result follows ■

Remark. – We should point out that, while Lemma 2.1 might look at first glance rather artificial, it is in fact sharp. To see this, consider the function

$$u_0(x)=\min\left\{1,\frac{1}{|x|^\alpha}\right\}\exp\left(\frac{x^2}{4q}\right)\equiv\tilde{u}_0(x)\exp\left(\frac{x^2}{4q}\right)\qquad\text{with}\quad q>1$$

If $\alpha q>1$, one readily sees that $\int_{\mathbb{R}}|u_0(x)|^q\exp\left(-\frac{x^2}{4}\right)dx<\infty$. To discuss the integrability of $S(t)u_0(x)$, we first write $S(t)u_0(x)=S(t)u_0(-x)$ in the form

$$S(t)u_0(x)=\pi^{-1/2}\exp\left(\frac{x^2}{4q}\right)$$

$$\times\int_{\mathbb{R}}\min\{1,|x+\sqrt{4t}\,\xi|^{-\alpha}\}\exp\left(-\left(1-\frac{t}{q}\right)\xi^2+\frac{x\sqrt{t}\,\xi}{q}\right)d\xi$$

Setting $\lambda=2q\left(1-\frac{t}{q}\right)\frac{\xi}{x\sqrt{t}}$, we then obtain that for x>0,

$$S(t)u_0(x)=\frac{\exp((x^2/4)(1+(t/q(1-t/q))\,x\sqrt{t}}{2\sqrt{\pi}\,q(1-(t/q))}$$

$$\times\int_{\mathbb{R}}\bar{u}_0(\xi(\lambda))\exp\left(-\frac{x^2t(\lambda-1)^2}{4q^2(1-(t/q))}\right)d\lambda$$

where $\bar{u}_0(\xi(\lambda))=\tilde{u}_0(x+\sqrt{4t}\,\xi(\lambda))$

Standard methods yield then

$$\int_{\mathbb{R}} \bar{u}_0(\xi(\lambda)) \exp\left(-\frac{x^2 t(\lambda-1)^2}{4q^2(1-(t/q))}\right) d\lambda \simeq \bar{u}_0(\xi(1)) \times \int_{\mathbb{R}} \exp\left(-\frac{x^2 t(\lambda-1)^2}{4^2(1-(t/q))}\right) d\lambda \simeq \frac{2q(1-(t/q))^{1/2}\pi^{1/2}}{|x|^{\alpha+1}\sqrt{t}\,|1+t(q(1-(t/q))^{-1}|^{\alpha}} \quad \text{as } |x| \to \infty$$

Therefore, convergence of $\int_{\mathbb{R}} |S(t)u_0(x)|^r \exp\left(-\frac{x^2}{4(1-t)}\right) dx$ requires, for general values of α, that

$$\frac{r}{q} + \frac{rt}{q^2(1-(t/q))} - \frac{1}{1-t} < 0$$

When $r \leqq q$, this inequality is always satisfied. However, if $r>q$ we need $t > \left(\frac{r-q}{r-1}\right)_+$. In this case we actually have to wait a bit to obtain the desired estimate ■

Let us change now variables as follows [*cf.* (1.8)]

$$S(t)u_0(x) = \zeta(y, \tau); \qquad y = x(1-t)^{-1/2}, \quad \tau = -\ln(1-t)$$

Then $\zeta(y, \tau)$ satisfies

$$\zeta_\tau = \zeta_{yy} - \frac{y}{2}\zeta_y; \qquad -\infty < y < +\infty, \quad \tau > 0, \tag{2.3a}$$

$$\zeta(y, 0) = \zeta_0(y) \equiv u_o(x) \quad \text{at } \tau = 0. \tag{2.3b}$$

Take now t^* such that $\left(\frac{r-q}{r-1}\right)_+ < t^* < 1$, and define τ^* as $\tau^* = -\ln(1-t^*)$.

When written in the new variables, Lemma 2.1 states that there exists $C = C(r, q, \tau^*)$ such that

$$\|\zeta(., \tau^*)\|_{r, w} \leqq C \|\zeta(., 0)\|_{q, w}$$

Moreover, since (2.3 *a*) is invariant under translations in this new time variable τ, we obtain

COROLLARY 2.2. – *Let $\zeta(y, \tau)$ be the solution of* (2.3), *where u_0 is as before. Then for any $r>1$, $q>1$ and $L>0$, there exist $\tau_0^* = \tau_0^*(q, r) > 0$ and $C = C(r, q, L) > 0$ such that*

$$\left.\begin{array}{l} \|\zeta(., \tau+\tau^*)\|_{r, w} \leqq C \|\zeta(., \tau)\|_{q, w} \quad \textit{for any } \tau \geqq 0 \\ \textit{and any } \tau^* \in [\tau_0^*, \tau_0^* + L]. \end{array}\right\} \tag{2.4}$$

We now go back to equation (1.1) with $F(u) = u^p$, $p > 1$. After performing the change (1.8), we obtain that $\varphi(y, \tau)$ solves (1.9 *a*), which can be viewed as a perturbed version of (2.3) above. Setting $\varphi(y, \tau)=(p-1)^{-(1/(p-1))}+\psi(y, \tau)$ then yields (1.11). We next extend our previous estimates as follows.

LEMMA 2.3. – *Assume that* $\psi(y, \tau)$ *solves* (1.11) *and* $|\psi| \leqq M < \infty$ *for some constant* M. *Then for any* $r>1$, $q>1$ *and* $L>0$, *there exist* $\tau_0^* = \tau_0^*(q, r)$ *and* $C = C(r, q, L) > 0$ *such that*

$$\|\psi(., \tau+\tau^*)\|_{r, w} \leqq C \|\psi(., \tau)\|_{q, w}$$

for any $\tau \geqq 0$ *and any* $\tau^* \in [\tau_0^*, \tau_0^* + L]$.

Proof. – We multiply both sides in (1.11 *a*) by (sgn ψ), and take advantage of Kato's inequality

$$\Delta\psi . \operatorname{sgn} \psi \leqq \Delta(|\psi|) \quad \text{in } D'(\mathbb{R}^N)$$

to obtain

$$(|\psi|)_\tau \leqq (|\psi|)_{yy} - \frac{y}{2}(|\psi|)_y + |\psi| + C(\psi|$$

for some $C>0$. Denoting by S_0 the semigroup corresponding to (2.3), we then have by comparison

$$|\psi(., \tau+\tau_0)| \leqq \exp((1+C)\tau_0) S_0(\tau_0) |\psi(., \tau)|$$

for any $\tau_0>0$, $\tau>0$. Take now $\tau_0=\tau^*$, where τ^* is as in Corrollary 2.2. We then have

$$\|\psi(., \tau+\tau^*)\|_{r, w} \leqq \exp((1+C)\tau^*) \|S_0(\tau^*)|\psi(., \tau)|\|_{r, w} \leqq C^* \|\psi(., \tau)\|_{q, w}$$

for some $C^* = C^*(r, q, L)$. This concludes the proof ■

3. HARNACK AND NONDEGENERACY ESTIMATES

Thoroughout this Section, we will assume that $\psi(y, \tau)$ is a bounded solution of (1.11), and shall write it in the form

$$\psi(y, \tau) = \sum_{k=0}^{\infty} a_k(\tau) H_k(y)$$

[*cf.* (1.17)]. Our first result is the following Harnack-type inequality which will be needed in Section 5 (*cf.* Lemma 5.6 and Proposition 5.7 there)

LEMMA 3.1. – *Let* $\psi(y, \tau)$ *be as above, and let* $\tau_1>0$, $A>0$ *be fixed. Assume that for some* $\beta>0$

$$\sum_{k \neq 2}^{\infty} a_k(\tau)^2 \leqq \beta a_2(\tau)^2 \quad \text{when } \tau \in [\tau_1, \tau_1 + A]$$

Then there exists $\theta=\theta(\beta, A)>0$ *such that*

$$\|\psi(., \tau_1)\|^2 \leqq \theta \|\psi(., \tau_1+A)\|^2 \tag{3.1}$$

Proof. — We substitute (1.17) in (1.11), make the scalar product of both sides with $H_2(y)$, and then multiply them by (sgn $a_2(\tau)$). Since ψ is bounded, $|f(\psi)| \leqq C|\psi|$ for some $C>0$, and we get

$$\frac{d}{d\tau}(|a_2(\tau)|) \geqq -C\|\psi(., \tau)\|$$

On the other hand, by assumption

$$\|\psi(., \tau)\|^2 \leqq (1+\beta) a_2(\tau)^2 \qquad \text{if} \quad \tau \in [\tau_1, \tau_1+A]$$

whence

$$\frac{d}{d\tau}(|a_2(\tau)|) \geqq -C(1+\beta)^{1/2}|a_2(\tau)| \qquad \text{if} \quad [\tau_1, \tau_1+A]$$

Integrating this inequality between τ_1 and τ_1+A, we obtain

$$\begin{aligned}\|\psi(., \tau_1+A)\| \geqq |a_2(\tau_1+A)| &\geqq \exp(-C(1+\beta)^{1/2}A)|a_2(\tau_1)| \\ &\geqq (1+\beta)^{-1/2}\exp(-C(1+\beta)^{1/2}A)\|\psi(., \tau_1)\|\end{aligned}$$

and the result follows ■

Our next goal consists in deriving a basic nondegeneracy result, namely Lemma 3.5 below, which roughly states that if $\|\psi(., \tau)\|$ decays in time faster than any exponential, then $\psi \equiv 0$. To show this, we shall introduce some notation. Let A be the modified Hermite operator in (1.15). Let λ be any real number, and for any $v \in L^2_w(\mathbb{R})$, let $P_\lambda v$ be the projection of v onto the subspace of $L^2_w(\mathbb{R})$ spanned by those eigenvectors of A whose eigenfunctions are not less than λ. Set also $R_\lambda v = v - P_\lambda v$. We then have

LEMMA 3.2. — *Assume that there exist* α *real,* $\tau_1 \geqq 0$ *and* $B>0$ *such that*

$$\|P_\alpha\psi(., \tau)\| \geqq \frac{1}{2}\|R_\alpha\psi(., \tau)\| \quad \textit{in } [\tau_1, \tau_1+B]$$

Then there exists $C_1>0$ *such that*

$$\|P_\alpha\psi(., \tau)\|^2 \leqq \exp((C_1-2\alpha)B)\|P_\alpha\psi(., \tau_1+B)\|_2, \ \textit{in } [\tau_1, \tau_1+B] \tag{3.2}$$

Proof. — By direct computation, we obtain

$$\begin{aligned}\frac{d}{d\tau}(\|P_\alpha\psi(., \tau)\|^2) &= 2\langle P_\alpha\psi(., \tau), P_\alpha A\psi(., \tau)\rangle \\ &\quad + 2\langle P_\alpha\psi(., \tau), P_\alpha f(\psi(., \tau))\rangle \\ &\geqq 2\alpha\|P_\alpha\psi(., \tau)\|^2 - 2\|P_\alpha\psi(., \tau)\| \ \|f(\psi)\|\end{aligned}$$

On the other hand, by assumption,

$$\|\psi(., \tau)\|^2 = \|P_\alpha\psi(., \tau)\|^2 + \|R_\alpha\psi(., \tau)\|^2 \leqq 5\|P_\alpha\psi(., \tau)\|^2,$$

and since $|f(\psi)| \leqq C|\psi|$ for some $C>0$, we arrive at

$$\frac{d}{d\tau}(\|P_\alpha \psi(.,\tau)\|^2) \geqq 2\alpha \|P_\alpha \psi(.,\tau)\|^2 - 2\sqrt{5}\,C\|P_\alpha \psi(.,\tau)\|^2$$

Integrating this inequality between τ_1 and τ_1+B, the conclusion follows. ■

We shall also need a technical Lemma which estimates the speed at which the L^2_w-norm of ψ moves towards the high frequencies in the spectrum of A. More precisely, by slightly adapting the arguments in Cohen and Lees ([CL], Lemmata 1 and 2), we obtain

LEMMA 3.3. — *a) Assume that, for some* $\bar{\tau} \geqq 0$ *and some real* λ

$$\|P_\lambda \psi(.,\bar{\tau})\| \geqq \|R_\lambda \psi(.,\bar{\tau})\| \tag{3.3}$$

then there exists $\delta>0$ *independent of* λ *and* $\bar{\tau}$ *such that*

$$\|P_\lambda \psi(.,\tau)\| \geqq \frac{1}{2}\|R_\lambda \psi(.,\tau)\| \quad \text{when } \tau \in [\bar{\tau}, \bar{\tau}+\delta]$$

b) Let $\bar{\tau}$, λ *be as in part a), and assume that* (3.3) *holds. Then there exists* $\sigma>0$ *such that*

$$\|P_\mu \psi(.,\bar{\tau}+\delta)\| \geqq \|R_\mu \psi(.,\bar{\tau}+\delta)\|$$

where $\mu=\lambda-\sigma$ ■

We shall also require the following nondegeneracy result which is due to P. D. Lax (*cf.* [L], Theorem 1).

LEMMA 3.4. — *Let* A *be the self-adjoint operator in* $L^2_w(\mathbb{R})$ *given in* (1.14). *Let* $u(\tau)$ *be a function of* τ *mapping* $0<\tau<\infty$ *into* $L^2_w(\mathbb{R})$, *whose values lie in the domain of* A. *Assume that* Au *is a continuous function of* τ, *and that u has a continuous strong derivative with respect to* τ. *Suppose that* $u(\tau)$ *satisfies the differential inequality*

$$\left\|\frac{du}{d\tau} - Au\right\| \leqq \theta \|u\|, \qquad 0<\tau<\infty.$$

The there exists a positive constant θ_0 *such that, if* $\theta \leqq \theta_0$, *the following result holds: If for any* $M>0$ *there is* $C=C(M)$ *such that*

$$\|u(\tau)\| \leqq C e^{-M\tau} \qquad \text{if} \quad \tau \geqq 0,$$

then $u(\tau) \equiv 0$ *for* $\tau \geqq 0$.

We are now in a position to show.

LEMMA 3.5. — *Assume that* $|\psi(y,\tau)|$ *is bounded, and suppose that for any* $M>0$ *there exists* $C=C(M)$ *such that*

$$\|\psi(.,\tau)\| \leqq C e^{-M\tau} \qquad \text{if} \quad \tau \geqq 0 \tag{3.4}$$

then $\psi(y,\tau) \equiv 0$.

Proof. – Let λ_0 be such that

$$\| P_{\lambda_0} \psi(., 0) \| \geqq \| R_{\lambda_0} \psi(., 0) \|$$

We now define the sequences

$$\tau_n = n\,\delta, \qquad \lambda_n = \lambda_0 - \sigma n$$

where δ, σ are as in Lemma 3.3. Using part *b*) in that result, we get

$$\| P_{\lambda_n} \psi(., \tau_n) \| \geqq \| R_{\lambda_n} \psi(., \tau_n) \|$$

whence, by part *a*) there

$$\| P_{\lambda_n} \psi(., \tau) \| \geqq \frac{1}{2} \| R_{\lambda_n} \psi(., \tau) \| \quad \text{when } \tau \in [\tau_n, \tau_{n+1}) \tag{3.5}$$

Define now $\Lambda(\tau) \equiv \lambda_n = \lambda - \sigma n$ for $\tau \in [\tau_n, \tau_{n+1})$. For such τ, we have $n \leqq \frac{\tau}{\delta}$, so that

$$\Lambda(\tau) \geqq \lambda_0 - \frac{\sigma \tau}{\delta}$$

and (3.5) can be recast as follows

$$\| P_{\Lambda(\tau)} \psi(., \tau) \| \geqq \frac{1}{2} \| R_{\Lambda(\tau)} \psi(., \tau) \| \qquad \text{for} \quad \tau \geqq 0. \tag{3.6}$$

Let us denote by C_1 a positive generic constant, depending only on the bound for $|\psi|$. We now consider (1.13), and notice that $|f(\psi)| \leqq C |\psi|^2$. Moreover, by standard theory the regularity assumptions on ψ, $A\psi$ required in Lemma 3.4 follow from the results recalled in an Appendix at the end of the paper. On the other hand, by Lemma 2.3 and (3.4), there exists $\tau^* > 0$ such that, for any $M > 0$.

$$\left\| \left(\frac{d\psi}{d\tau} - A\psi \right)(., \tau) \right\| \leqq C_1 \| \psi(., \tau) \|^2_{4, w} \leqq C_1 \| \psi(., \tau - \tau^*) \|^2$$
$$\leqq C e^{-M\tau} \| \psi(., \tau - \tau^*) \| \tag{3.7}$$

if τ is large enough, where C is now a generic constant depending on M and the bound for $|\psi|$. On the other hand, since $\Lambda(\tau)$ is nonincreasing in τ, and $\| P_\omega \psi(., \tau) \| \geqq \| P_\lambda \psi(., \tau) \|$ if $\omega \leqq \lambda$, we deduce from (3.6) that

$$\| P_{\Lambda(\tau)} \psi(., s) \| \geqq \frac{1}{2} \| R_{\Lambda(\tau)} \psi(., s) \| \qquad \text{for any} \quad s \in [\tau - \tau^*, \tau] \tag{3.8}$$

Then, by Lemma 3.2

$$\| P_{\Lambda(\tau)} \psi(., \tau - \tau^*) \|^2 \leqq \exp((C - 2\Lambda(\tau))\tau^*) \| P_{\Lambda(\tau)} \psi(., \tau) \|^2 \tag{3.9}$$

By (3.8) and (3.9) we have that

$$\| \psi(., \tau - \tau^*) \|^2 \leqq 5 \| P_{\Lambda(\tau)} \psi(., \tau - \tau^*) \|^2 \leqq 5 \exp((C - 2\Lambda(\tau))\tau^*) \| \psi(., \tau) \|^2$$

and substituting this into (3.7) yields

$$\left\|\left(\frac{d\psi}{d\tau}-\mathrm{A}\,\psi\right)(.,\tau)\right\|\leqq \mathrm{C}\exp\left(\frac{1}{2}(\mathrm{C}\,\tau^*-\Lambda(\tau)\,\tau^*-2\,\mathrm{M}\,\tau)\right)\|\psi(.,\tau)\|$$

we may select M such that

$$\left\|\left(\frac{d\psi}{d\tau}-\mathrm{A}\,\psi\right)(.,\tau)\right\|\leqq \mathrm{C}\,e^{-\varepsilon\tau}\|\psi(.,\tau)\|$$

for some $\varepsilon>0$ and any $\tau\geqq 0$ large enough. Taking now $\tau_0=\inf\{\tau:\mathrm{C}\,e^{-\varepsilon\tau}\leqq\theta:\ \theta$ as in Lemma 3.4$\}$, it follows that $\psi(y,\tau)\equiv 0$ for $\tau\geqq\tau_0$. Since the number of maxima cannot increase in time (*cf.* for instance [A], [AF]) this implies that $\psi(y,\tau)\equiv 0$ for any τ ■

4. ESTIMATING $|\mathbf{a}_0(\tau)|$ AND $|\mathbf{a}_1(\tau)|$ AS $\tau\to\infty$

Our aim in this Section consists in proving the following result

PROPOSITION 4.1. – *Let $\psi(y,\tau)$ be given in* (1.8), (1.10), *and assume that $|\psi(.,\tau)|$ is bounded and $\lim_{\tau\to\infty}\|\psi(.,\tau)\|=0$. Then, if*

$$\psi(y,\tau)=\sum_{k=0}^{\infty}a_k(\tau)\,\mathrm{H}_k(y),$$

we have that

$$\lim_{\tau\to\infty}\frac{|a_0(\tau)|+|a_1(\tau)|}{\|\psi(.,\tau)\|}=0. \tag{4.1}$$

To show (4.1), we shall argue by contradiction. Suppose that there exists a sequence $\{\tau_j\}$ with $\lim_{j\to\infty}\tau_j=\infty$, and a constant $\varepsilon>0$ such that

$$|a_0(\tau_j)|+|a_1(\tau_j)|\geqq\varepsilon\,\|\psi(.,\tau_j)\|\equiv\varepsilon\,\delta_j \qquad \text{where } \delta_j\to 0 \text{ as } j\to\infty \tag{4.2}$$

We then proceed in several steps. To begin with, we set

$$\psi(y,\tau)=\sigma_j(y,\tau)+\omega_j(y,\tau);\qquad j=1,2,\ldots \tag{4.3}$$

where for any such j, σ_j solves

$$(\sigma_j)_\tau=(\sigma_j)_{yy}-\frac{y}{2}(\sigma_j)_y+\sigma_j \qquad \text{if} \quad \tau>\tau_j, \tag{4.4a}$$

$$\sigma_j(y,\tau_j)=\psi(y,\tau_j) \tag{4.4b}$$

For fixed $j\geqq 1$, let us write $z=|\omega_j|$. We then have

LEMMA 4.2. – *For any* $\alpha \in (0, 1)$, *there exist constants* C_1, C_2 *depending on* α, *such that*

$$z_\tau \leqq z_{yy} - \frac{y}{2} z_y + C_1 z + g(|\sigma_j|) \qquad \textit{for} \quad \tau > \tau_j \tag{4.5 a}$$

where

$$g(s) \leqq \min\{C_2 s,\ C_2 (s^{1/(1-\alpha)} + s^{1+\alpha})\}. \tag{4.5 b}$$

Proof. – We substitute $\varphi = (p-1)^{-(1/(p-1))} + \sigma_j + \omega_j$ in equation (1.9 *a*). Dropping the subscript j for convenience, we readily check that ω satisfies

$$\begin{aligned}\omega_\tau = \omega_{yy} - \frac{y}{2}\omega_y - \frac{1}{p-1}\omega + \left[\frac{\varphi^p - (p-1)^{-(p/(p-1))}}{\varphi - (p-1)^{-(1/(p-1))}}\right]\omega \\ + \left[\frac{\varphi^p - (p-1)^{-(p/(p-1))}}{\varphi - (p-1)^{-(1/(p-1))}} - \frac{p}{p-1}\right]\sigma \\ \equiv \omega_{yy} - \frac{y}{2}\omega_y - \frac{1}{p-1}\omega + F_1(\varphi)\,\omega + F_2(\varphi)\,\sigma \end{aligned} \tag{4.6}$$

Since ψ is bounded by assumption, it follows that $|F_1(\varphi)| \leqq C_0$ for some $C_0 > 0$. As to F_2, we have that

$$|F_2(\varphi)| \leqq \min\{C,\ C|\sigma+\omega|\} \quad \text{for some } C>0.$$

Let us take $\alpha \in (0, 1)$ fixed, and denote henceforth by C a generic constant depending on α and the bound for $|\psi|$. Then $|F_2(\varphi)| \leqq C|\sigma+\omega|^\alpha$, whence

$$|F_2(\varphi)|\,|\sigma| \leqq C|\sigma+\omega|^\alpha |\sigma| \leqq C(|\sigma|^{1+\alpha} + |\sigma|^{1/(1-\alpha)} + |\omega|)$$

We now multiply both sides in (4.6) by $(\operatorname{sgn}\omega)$ and use there Kato's inequality as well as the previous bounds for F_1 and F_2 to obtain (4.5) ■

Since $\omega_j(y, \tau_j) = 0$, it follows from (4.5) that

$$|\omega_j(y, \tau)| \leqq e^{C_1(\tau-\tau_j)} \int_{\tau_j}^{\tau} S_0(\tau - s)\, g((|\sigma_j(.,s)|)\, ds$$

where S_0 is the semigroup associated to the operator A_0 given by $A_0 u = u'' - \frac{y}{2} u'$ with $D(A_0) = H_w^2(\mathbb{R})$. Fix now $R > 0$, and let $\chi_R(y) = 1$ if $|y| < R$, $\chi_R(y) = 0$ otherwise. We then write

$$\begin{aligned}|\omega_j(y, \tau)| \leqq e^{C_1(\tau-\tau_j)} \int_{\tau_j}^{\tau} S_0(\tau - s)\, \chi_R\, g((|\sigma_j(.,s)|)\, ds \\ + e^{C_1(\tau-\tau_j)} \int_{\tau_j}^{\tau} S_0(\tau - s)(1-\chi_R)\, g(|\sigma_j(.,s)|)\, ds \equiv S_1 + S_2.\end{aligned} \tag{4.7}$$

To proceed further, a suitable bound on $\|\omega_j(.,\tau)\|$ will be required. In view of (4.7) and the general results recalled in (A 3) in the Appendix, one has

$$\|\omega_j(.,\tau)\| \leqq C\left[\int_{\tau_j}^{\tau} \|\chi_R\, g(|\sigma_j(.,s)|)\|^2\, ds\right]^{1/2} + C\left[\int_{\tau_j}^{\tau} \|(1-\chi_R)\, g(|\sigma_j(.,s)|)\|^2\, ds\right]^{1/2} \quad (4.8)$$

where C_j depends on the time span $|\tau-\tau_j|$, remaining bounded whenever $|\tau-\tau_j|$ is bounded. We now have

LEMMA 4.3. – *For any fixed* $R>0$, $L>0$ *and* $\varepsilon\in(0, L)$, *there exist* $C>0$ *independent of* ε, R *and a function* $A(\varepsilon)$, *independent of* R, *such that* $\lim_{\varepsilon\downarrow 0} A(\varepsilon)=0$ *and*

$$\frac{1}{\|\psi(.,\tau_j)\|^2}\int_{\tau_j}^{\tau_j+L} \|(1-\chi_R)\, g(|\sigma_j(.,s)|)\|^2\, ds \leqq C\left[\varepsilon+\frac{1}{1-e^{-\varepsilon}}\int_{|z|\geqq A(\varepsilon)R} e^{-z^2}\, dz\right] \quad (4.9)$$

Proof. – By (4.4 *a*), it follows that

$$\|(1-\chi_R)|\sigma_j(.,s)|\|^2 \leqq e^{2(\tau-\tau_j)}\int_{|\xi|\geqq R} e^{-\xi^2/4} \times\left[\int_{\mathbb{R}} \frac{|\psi(\lambda,\tau_j)|}{(4\pi(1-e^{-(\tau-\tau_j)}))^{1/2}} \exp\left(-\frac{(\xi e^{-((\tau-\tau_j)/2)}-\lambda)^2}{4(1-e^{-(\tau-\tau_j)})}\right) d\lambda\right]^2 d\xi$$

$$= e^{2(\tau-\tau_j)}\int_{|\xi|\geqq R} e^{-\xi^2/4}\left[\int_{\mathbb{R}} |\psi(\lambda,\tau_j)|\, d\sigma(y)\right]^2 d\xi \equiv I$$

for some measure σ such that $\sigma(\mathbb{R})=1$. Then, by Jensen's inequality

$$I \leqq e^{2(\tau-\tau_j)}\int_{|\xi|\geqq R} e^{-\xi^2/4}\int_{\mathbb{R}} |\psi(\lambda,\tau_j)|^2\, d\sigma(\lambda)$$

$$= \frac{e^{2(\tau-\tau_j)}}{(4\pi(1-e^{-(\tau-\tau_j)}))^{1/2}}\int_{\mathbb{R}} |\psi(\lambda,\tau_j)|^2 e^{-\lambda^2/4} \times\left(\int_{|\xi|\geqq R} \exp\left(-\frac{\xi^2}{4}+\frac{\lambda^2}{4}-\frac{(\xi e^{-((\tau-\tau_j)/2)}-\lambda)^2}{4(1-e^{-(\tau-\tau_j)})}\right) d\xi\right) d\lambda \quad (4.10)$$

A quick computation reveals now that

$$-\frac{\xi^2}{4}+\frac{\lambda^2}{4}-\frac{(\xi e^{-((\tau-\tau_j)/2)}-\lambda)^2}{4(1-e^{-(\tau-\tau_j)})} = -\frac{(\xi-\lambda e^{-(\tau-\tau_j/2)})^2}{4(1-e^{-(\tau-\tau_j)})}$$

Therefore, recalling the definition of δ_j in (4.8), we get

For any $\varepsilon \in (0, 1)$,

$$\int_{\tau_j}^{\tau_j+\varepsilon} \|(1-\chi_R) g(|\sigma_j(.,s)|)\|^2 ds \leqq C\varepsilon\delta_j^2$$

where C is a positive constant independent of j and ε. (4.11)

On the other hand, by Hölder's inequality

$$\|(1-\chi_R)|\sigma(.,s)|\|^2 \leqq \frac{e^{2(\tau-\tau_j)}}{4\pi(1-e^{-(\tau-\tau_j)})}\delta_j^2 \times \left(\int_{|\xi|\geqq R} \exp\left(-\frac{\xi^2}{4}+\frac{\lambda^2}{4}-\frac{(\xi e^{-((\tau-\tau_j)/2)}-\lambda)^2}{2(1-e^{-(\tau-\tau_j)})}\right) d\xi\right) d\lambda$$

Since

$$-\frac{\xi^2}{4}+\frac{\lambda^2}{4}-\frac{(\xi e^{-((\tau-\tau_j)/2)}-\lambda)^2}{2(1-e^{-(\tau-\tau_j)})} = -\frac{(1+e^{-(\tau-\tau_j)})}{4(1-e^{-(\tau-\tau_j)})}\left[\lambda-\frac{2\xi e^{-(\tau-\tau_j)/2)}}{1+e^{-(\tau-\tau_j)}}\right]^2 - \frac{(1-e^{-(\tau-\tau_j)})\xi^2}{4(1+e^{-(\tau-\tau_j)})}$$

it follows that

$$\|(1-\chi_R)|\sigma(.,s)|\|^2 \leqq \frac{e^{2(\tau-\tau_j)}}{4\pi(1-e^{-(\tau-\tau_j)})}\delta_j^2 \int_{|\xi|\geqq R} \exp\left(-\frac{(1-e^{-(\tau-\tau_j)})\xi^2}{4(1+e^{-(\tau-\tau_j)})}\right) \left[\int_{\mathbb{R}} \exp\left(-\frac{(1+e^{-(\tau-\tau_j)})}{4(1-e^{-(\tau-\tau_j)})}\left[\lambda-\frac{2\xi e^{-((\tau-\tau_j)/2)}}{1+e^{-(\tau-\tau_j)}}\right]^2\right) d\lambda\right] d\xi$$

whence, for some $C>0$

$$\|(1-\chi_R)|\sigma(.,s)|\|^2 \leqq \frac{Ce^{2(\tau-\tau_j)}\delta_j^2}{(1-e^{-(\tau-\tau_j)})^{1/2}(1+e^{-(\tau-\tau_j)})^{1/2}} \times \int_{|\xi|\geqq R} \exp\left(-\frac{(1-e^{-(\tau-\tau_j)})\xi^2}{4(1+e^{-(\tau-\tau_j)})}\right) d\xi = \frac{Ce^{2(\tau-\tau_j)}\delta_j^2}{(1-e^{-(\tau-\tau_j)})}\int_{\Sigma} e^{-z^2} dz$$

where $\Sigma=\left\{z\in\mathbb{R}:|z|\geqq\left[\frac{1-e^{-(\tau-\tau_j)}}{2(1+e^{-(\tau-\tau_j)})}\right]^{1/2}\mathrm{R}\right\}$, and we finally obtain that

For any $\varepsilon>0$,

$$\int_{\tau_j+\varepsilon}^{\tau}\|(1-\chi_{\mathrm{R}})|\sigma(.,s)|\|^2\,ds\leqq\frac{Ce^{2(\tau-\tau_j)}}{(1-e^{-\varepsilon})}\delta_j^2\int_{|z|\geqq\mathrm{A}(\varepsilon)\mathrm{R}}e^{-z^2}\,dz \quad (4.12)$$

where $\mathrm{A}(\varepsilon)=\left[\frac{1-e^{-\varepsilon}}{2(1+e^{-\varepsilon})}\right]^{1/2}$.

We are now ready to derive (4.9). We just split the integral there into two, performed over intervals $(\tau_j, \tau_j+\varepsilon)$ and $(\tau_j+\varepsilon, \tau_j+\mathrm{L})$ respectively, and divide it by δ_j^2. Taking into account (4.11) and (4.12), the result follows ■

We next show

LEMMA 4.4. – *For any fixed* $\mathrm{L}>0$ *and* $\mathrm{R}>0$, *there exists* $\mathrm{C}>0$ *such that*

$$\frac{1}{\|\psi(.,\tau_j)\|}\left[\int_{\tau_j}^{\tau_j+\mathrm{L}}\|\chi_{\mathrm{R}}|\sigma_j(.,s)|^{5/4}\|^2\,ds\right]^{1/2}$$
$$\leqq\mathrm{C}\|\psi(.,\tau_j)\|^{1/4}(1+e^{c\mathrm{R}^2})^{5/4} \quad (4.13)$$

Proof. – Fix $\mathrm{R}>0$. By our choice of σ_j, we have

$$|\sigma_j(y,\tau)|\leqq\frac{e^{\tau-\tau_j}}{(4\pi(1-e^{-(\tau-\tau_j)}))^{1/2}}$$
$$\times\left\{\int_{|\lambda|\leqq 10\mathrm{R}}|\psi(\lambda,\tau_j)|\exp\left(\frac{(ye^{-((\tau-\tau_j)/2)}-\lambda)^2}{4(1-2^{-(\tau-\tau_j)})}\right)d\lambda\right.$$
$$\left.+\int_{|\lambda|\geqq 10\mathrm{R}}|\psi(\lambda,\tau_j)|\exp\left(-\frac{(ye^{-((\tau-\tau_j)/2)}-\lambda)^2}{4(1-e^{-(\tau-\tau_j)})}\right)d\lambda\right\}\equiv\mathrm{I}_1+\mathrm{I}_2$$

Using Hölder's inequality, we get that for some $\mathrm{C}>0$

$$\mathrm{I}_1\leqq\frac{e^{\tau-\tau_j}e^{c\mathrm{R}^2}}{(4\pi(1-e^{-(\tau-\tau_j)}))^{1/2}}\delta_j$$
$$\times\left[\int_{|\lambda|\leqq 10\mathrm{R}}\exp\left(-\frac{(ye^{-((\tau-\tau_j)/2)}-\lambda)^2}{2(1-e^{-(\tau-\tau_j)})}\right)d\lambda\right]^{1/2}$$
$$\leqq\frac{Ce^{\tau-\tau_l}e^{c\mathrm{R}^2}}{(4\pi(1-e^{-(\tau-\tau_k)}))^{1/4}}\delta_j$$

As to I_2, we notice that

$$\mathrm{I}_2\leqq\frac{e^{\tau-\tau_j}}{(4\pi(1-e^{-(\tau-\tau_j)}))^{1/2}}\delta_j\left[\int_{|\lambda|\geqq 10\mathrm{R}}\exp\left(\frac{\lambda^2}{4}-\frac{(ye^{-((\tau-\tau_j)/2)}-\lambda)^2}{2(1-e^{-(\tau-\tau_j)})}\right)d\lambda\right]^{1/2}$$

We now remark that

$$\frac{\lambda^2}{4}-\frac{(x-\lambda)^2}{2(1-e^{-(\tau-\tau_j)})}\leqq -\frac{\lambda^2}{16(1-e^{-(\tau-\tau_j)})}$$

whenever $|x|\leqq R$ and $|\lambda|\geqq 10R$

We then arrive at

$$I_2\leqq\frac{C\delta_j e^{(\tau-\tau_j)}}{(1-e^{-(\tau-\tau_j)})^{1/4}}\quad\text{for some } C>0$$

Putting together the bounds obtained for I_1 and I_2, it follows that

$$|\sigma_j(y,\tau)|^{5/4}\leqq\frac{C\delta_j^{5/4}e^{5/4(\tau-\tau_j)}}{(1-e^{-(\tau-\tau_j)})^{5/16}}(1+e^{cR^2})^{5/4}\qquad\text{if}\quad |y|\leqq R$$

whence

$$\left[\int_{\tau_j}^{\tau}\|\chi_R|\sigma_j(.,s)|^{5/4}\|^2 ds\right]^{1/2}$$
$$\leqq C\delta_j^{5/4}e^{5/4(\tau-\tau_j)}(1+e^{cR^2})^{5/4}\left[\int_{\tau_j}^{\tau}\frac{ds}{(1-e^{-(s-\tau_j)})^{5/16}}\right]^{1/2}$$

and (4.13) follows ■

Summing up our previous estimates, we obtain

LEMMA 4.5. – *Let* $\omega_j(y,\tau)$ *be given in* (4.3). *Then, for any fixed* $L>0$,

$$\lim_{j\to\infty}\frac{\|\omega_j(.,\tau)\|}{\|\psi(.,\tau_j)\|}=0\quad\textit{uniformly for}\ \tau\in[\tau_j,\tau_j+L]\tag{4.14}$$

Proof. – We set $\alpha=\frac{1}{4}$ in (4.5 *b*), and use then (4.8) with $g(s)=Cs$ when $|x|\geqq R$ [resp. $g(s)=Cs^{5/4}$ when $|x|\leqq R$]. Taking into account (4.9) and (4.13), the result follows by letting first $j\to\infty$, then $R\to\infty$ and finally $\varepsilon\to 0$ ■

Our next result is

LEMMA 4.6. – *Assume that* (4.2) *holds. Then for any fixed* $L>0$, *there exists* $K=K(L)$ *such that*

$$|a_0(\tau)|+|a_1(\tau)|\geqq\varepsilon K\|\psi(.,\tau)\|\quad\textit{when}\ \tau\in[\tau_j,\tau_j+L].\tag{4.15}$$

Proof. – Since $\psi(y,\tau)=\sigma_j(y,\tau)+\omega_j(y,\tau)$, and

$$\psi(y,\tau)=\sum_{k=0}^{\infty}a_k(\tau)H_k(y),$$

it follows from (4.4) that

$$a_k(\tau)=a_k(\tau_j)\,e^{(1-(k/2))(\tau-\tau_j)}+\langle \omega_j, H_k\rangle;\quad k=0, 1, 2, \ldots$$

In particular

$$|a_0(\tau)|\geqq|a_0(\tau_j)|\,e^{\tau-\tau_j}-\|\omega_j(.,\tau)\|;$$
$$|a_1(\tau)|\geqq|a_1(\tau_j)|\,e^{(\tau-\tau_j)/2}-\|\omega_j(.,\tau)\|$$

so that, using (4.2) and (4.14)

$$|a_0(\tau)|+|a_1(\tau)|\geqq\varepsilon\|\psi(.,\tau_j)\|-2\|\omega_j(.,\tau)\|\geqq\frac{\varepsilon}{2}\|\psi(.,\tau_j)\| \tag{4.16}$$

On the other hand, multiplying both sides in (1.11) by $\psi e^{-y^2/4}$ and integrating yields

$$\frac{1}{2}\frac{d}{d\tau}(\|\psi\|^2)\leqq(1+C)\|\psi\|^2$$

whence

$$\|\psi(.,\tau)\|\leqq\|\psi(.,\tau_j)\|\,e^{(1+C)(\tau-\tau_j)} \tag{4.17}$$

Putting together (4.16) and (4.17), the proof is concluded ■

We shall also need the following result.

LEMMA 4.7. – *Assume that there exist* $\sigma>0$ *and* $L>0$ *such that*

$$a_0(\tau)^2+a_1(\tau)^2\geqq\sigma\|\psi(.,\tau)\|^2 \quad \textit{for any } \tau\in[\tau_j, \tau_j+L], \tag{4.18}$$

then there exists $M=M(\sigma, L)$ *such that*

$$\|\psi(.,\tau_j)\|\leqq M\|\psi(.,\tau)\| \quad \textit{for any } \tau\in[\tau_j, \tau_j+L] \tag{4.19}$$

Proof. – By (1.18), we have

$$\dot{a}_k=\left(1-\frac{k}{2}\right)a_k+\langle f(\psi), H_k\rangle;\quad k=0, 1, 2, \ldots$$

Set now $S(\tau)=a_0(\tau)^2+a_1(\tau)^2$. Taking into account (4.18), we compute

$$\frac{dS}{d\tau}=2a_0\dot{a}_0+2a_1\dot{a}_1=2a_0^2+a_1^2+2a_0\langle f(\psi), H_0\rangle$$
$$+2a_1\langle f(\psi), H_1\rangle\geqq a_0^2+a_1^2-C_1a_0^2-C_1a_1^2=(1-C_1)S$$

where C_1 depends on σ. Integrating the inequality just obtained yields

$$\|\psi(.,\tau)\|^2\geqq S(\tau)\geqq e^{(1-C_1)(\tau-\tau_j)}(a_0(\tau_j)^2+a_1(\tau_j)^2)$$
$$\geqq\sigma\,e^{(1-C_1)(\tau-\tau_j)}\|\psi(.,\tau_j)\|^2$$

whence the result ■

We now extend (4.15) (or rather a variant of it) for arbitrarily large $\tau>0$.

LEMMA 4.8. – *Assume that* (4.2) *holds. Then there exists* $A>0$ *such that, for large enough* $\tau>0$

$$a_0(\tau)^2+a_1(\tau)^2 \geqq A \|\psi(.,\tau)\|^2 \tag{4.20}$$

Proof. – Consider the function

$$E(\tau)=E(\tau;\delta)=a_0(\tau)^2+a_1(\tau)^2-\delta\sum_{k=2}^{\infty} a_k(\tau)^2$$

where $\delta>0$ will be selected presently. A straightforward computation yields

$$\begin{aligned}\frac{dE}{d\tau} &\geqq a_0^2+a_1^2+2\delta\sum_{k=2}^{\infty}\left(\frac{k}{2}-1\right)a_k(\tau)^2+2a_0\langle f(\psi),H_0\rangle \\ &+2a_1\langle f(\psi),H_1\rangle-2\delta\sum_{k=2}^{\infty}a_k\langle f(\psi),H_k\rangle \geqq (a_0^2+a_1^2) \\ &+2\delta\sum_{k=2}^{\infty}\left(\frac{k}{2}-1\right)a_k(\tau)^2-2C(|a_0|+|a_1|)\|\psi\|^2 \\ &-2\delta\sum_{k=2}^{\infty}a_k\langle f(\psi),H_k\rangle \end{aligned}\tag{4.21}$$

Using Lemma 2.3, we now estimate the last term in the above inequality. Since $\sum_{k=2}^{\infty} a_k\langle f(\psi),H_k\rangle=\int_{\mathbb{R}}\left(\psi-\sum_{k=0}^{\infty}a_k H_k\right)f(\psi)e^{-y^2/4}dy$, we have that

$$\begin{aligned}\left|\sum_{k=2}^{\infty}a_k\langle f(\psi),H_k\rangle\right| &\leqq \int_{\mathbb{R}}|\psi||f(\psi)|e^{-y^2/4}dy \\ &+\sum_{k=0}^{1}|a_k|\int_{\mathbb{R}}|H_k|\psi^2 e^{-y^2/4}dy \leqq C\|\psi(.,\tau)\|_{3,w}^3 \\ &+C(|a_0|+|a_1|)\|\psi(.,\tau)\|_{4,w}^2 \leqq C\|\psi(.,\tau-\tau^*)\|^3 \\ &+C(|a_0|+|a_1|)\|\psi(.,\tau-\tau^*)\|^2 \end{aligned}\tag{4.22}$$

where C is a positive constant, which possibly changes from line to line, and $\tau^*>0$. We now set $L=\tau^*$, and select σ [which depends on ε, L; *cf.* (4.15)] such that (4.18) holds; finally, we set $\delta=\sigma$. One then has that $E\geqq 0$ for $\tau\in[\tau_j,\tau_j+L]$. On the other hand, using (4.19) and (4.18) we obtain at $\tau=\tau_j+L$,

$$\begin{aligned}\left|\sum_{k=2}^{\infty} a_k \langle f(\psi), H_k\rangle\right| & \\ &\leqq C_1 \|\psi(., \tau_j+L)\|^3 + C_1(|a_0|+|a_1|)\|\psi(., \tau_j+L)\|^2 \\ &\leqq \frac{C_1}{\sigma}\|\psi(., \tau_j+L)\|(a_0^2+a_1^2)+\frac{C_1}{\sigma}(a_0^2+a_1^2)(|a_0|+|a_1|) \\ &\leqq g(\tau_j+L)(a_0^2+a_1^2)\end{aligned}$$

for some $C_1>0$, where $g(s)\to 0$ as $s\to\infty$. The third term in the right of (4.21) can be estimated in a similar way, and we finally deduce that, ar $\tau=\tau_j+L$

$$\frac{dE}{d\tau}\geqq a_1^2+a_0^2+2\sigma\sum_{k=2}^{\infty}\left(\frac{k}{2}-1\right)a_k^2-h(\tau_j+L)(a_0^2+a_1^2)$$

where $h(s)\to 0$ as $s\to\infty$. Therefore, taking j large enough, we obtain that $\frac{dE}{d\tau}(\tau_j+L)\geqq 0$. The E stays nonnegative for any $\tau\geqq\tau_j+L$, and this yields (4.20) with $A=\sigma$ ■

End of the proof of Proposition 4.1. — We consider the function $S(\tau)=a_0(\tau)^2+a_1(\tau)^2$, already used in the proof of Lemma 4.7, and notice that, by our previous results

$$\begin{aligned}\dot S(\tau)=2a_0\dot a_0+2a_1\dot a_1 &= 2a_0^2+a_1^2+2a_0\langle f(\psi), H_0\rangle \\ &+2a_1\langle f(\psi), H_1\rangle \geqq a_0^2+a_1^2-C|a_0|\|\psi\|^2-C|a_1|\|\psi\|^2 \\ &\geqq S(\tau)(1-M(|a_0|+|a_1|))\end{aligned}$$

for some $M>0$. But then this inequality implies that $S(\tau)\geqq\alpha>0$ for some α and any τ large enough, and this contradicts the fact that $\|\psi(., \tau)\|\to 0$ as $\tau\to\infty$ ■

5. A BASIC ASYMPTOTIC ESTIMATE

This Section is devoted to conclude the Proof of Theorem 3. Namely, we show that if $\psi(y,\tau)\neq 0$ either,

$$\left\|\psi(., \tau)+\frac{c}{\tau}H_2(.)\right\|_{H_w^1}=o\left(\frac{1}{\tau}\right)\quad\text{as } \tau\to\infty \qquad (5.1\,a)$$

where $c=\dfrac{(4\pi)^{1/4}(p-1)^{-(1/(p-1))}}{\sqrt{2}\,p}$, or

There exist $m\geqq 3$ and $C\neq 0$ such that

$$\|\psi(.\,\tau)-C e^{(1-(m/2))\tau}H_m(.)\|_{H_w^1}=o(e^{(1-(m/2))\tau})\quad\text{as } \tau\to\infty \quad (5.1\,b)$$

(*cf.* Propositions 5.7 and 5.8 below). We then show in Proposition 5.9 that, under the assumptions of Theorem 1, (5.1 *a*) must necessarily hold. We also remark at the end of the Section that convergence in H^1_w implies convergence in $C^{k,\alpha}_{loc}$ for any $k \geqq 1$ and $\alpha \in (0, 1)$.

Let us describe briefly the way in which (5.1) is obtained. Consider the semilinear equation satisfied by ψ [*cf.* (1.11)]. By assumption, $\|\psi(.,\tau)\| \to 0$ as $\tau \to \infty$. If, moreover

$$\|\psi(.,\tau)\| = O(e^{-\varepsilon\tau}) \text{ as } \tau \to \infty \quad \text{for some } \varepsilon > 0,$$

we will use variation of constants formula in (1.11) to show that either (5.1 *b*) holds or $\|\psi(.,\tau)\| = O(e^{-\delta\tau})$ for any $\delta > 0$ as $\tau \to \infty$. In this last case, $\psi(.,\tau) \equiv 0$ by the nondegeneracy results in Section 3; *cf.* Proposition 5.8 below.

If we do not assume an exponential decay of $\|\psi(.,\tau)\|$, a more subtle approach is required. Basically, we then try to estimate the ratio

$$r(y,\tau) = \frac{\psi(y,\tau)}{\|\psi(.,\tau)\|}$$

as $\tau \to \infty$. Formally, $r(y,\tau)$ satisfies

$$r_\tau = r_{yy} - \frac{y}{2} r_y + r + \frac{f(r\|\psi\|)}{\|\psi\|} - r\frac{d}{d\tau}(\ln \|\psi\|)$$

Since all we know on $\|\psi\|$ is that $\lim_{\tau \to \infty} \|\psi(.,\tau)\| = 0$, we lack information on the coefficients of the above equation to describe the behaviour of $r(.,\tau)$ as $\tau \to \infty$. We then replace $\|\psi(.,\tau)\|$ in the definition of r by $\mu(\tau)$ [*cf.* (5.4) below]. where $\mu(\tau)$ is a function related to $\|\psi(.,\tau)\|$, but whose asymptotic properties as $\tau \to \infty$ can be accurately described. The fact that such a function exists follows from results due to Chen, Matano and Veron [CMV], and the relevant details are recalled in Lemma 5.1 below. We are then able to obtain asymptotic estimates on the new normalized variable (*cf.* Lemma 5.4). This is not enough to derive (5.1 *a*) yet; a suitable continuation argument on cones has to be implemented (*cf.* Lemma 5.5), and a careful asymptotic integration of the second Fourier coefficient in (1.18) needs to be done in proposition 5.7 to achieve our goal.

We now proceed to detail the arguments above. To start with, we recall the following result (*cf.* [CMV])

LEMMA 5.1. — *Let* $\rho(\tau)$ *be a nonnegative function such that* $p \in C([0,\infty)]$, $\lim_{\tau \to \infty} \rho(\tau) = 0$ *and* $\limsup_{\tau \to \infty} e^{\varepsilon\tau}\rho(\tau) = +\infty$ *for any* $\varepsilon > 0$ [*resp.* $\limsup_{\tau \to \infty} \tau\rho(\tau) = +\infty$]. *Then there exists a function* $\eta(\tau) \in C^\infty([0,\infty))$ *such that*

(i) $\eta>0$, $\eta'<0$, $\lim_{\tau\to\infty} \eta(\tau)=0$,

(ii) $0<\limsup_{\tau\to\infty} \frac{\rho(\tau)}{\eta(\tau)}<+\infty$,

(iii) $\lim_{\tau\to\infty} e^{\varepsilon\tau}\eta(\tau)=+\infty$ *for any* $\varepsilon>0$ [*resp.* $\lim_{\tau\to\infty} \tau\eta(\tau)=0$],

(iv) $\left(\frac{\eta'}{\eta}\right)'$ *and* $\left(\frac{\eta''}{\eta}\right)'$ *belong to* $L^1(0, \infty)$,

(v) $\lim_{\tau\to\infty} \frac{\eta'(\tau)}{\eta(\tau)}=\lim_{\tau\to\infty} \frac{\eta''(\tau)}{\eta(\tau)}=0$

We shall also need the following consequence of Lemma 5.1

LEMMA 5.2. – *Let* $\eta(\tau)$ *be the function obtained in the previous Lemma, and let* $a>0$ *be fixed. Then there exists* $C=C(a)$ *such that, for any* $\tau>0$

$$\eta(\tau)\leqq C\eta(\tau+a)$$

Proof. – It consists in a contradiction argument. Suppose that there exists a sequence $\{\tau_j\}$ such that $\lim_{j\to\infty} \tau_j=\infty$ and

$$\eta(\tau_j)\geqq j\eta(\tau_j+a)$$

Since $\eta\in C^1$, there exists $\xi_j\in(\tau_j, \tau_j+a)$ such that

$$\eta'(\xi_j)=\frac{1}{a}(\eta(\tau_j+a)-\eta(\tau_j)).$$

Further, since $\eta'<0$,

$$|\eta'(\xi_j)|\geqq\frac{1}{a}\left(1-\frac{1}{j}\right)\eta(\tau_j)\geqq\frac{1}{a}\left(1-\frac{1}{j}\right)\eta(\xi_j)$$

which contradicts (v) in Lemma 5.1 if $j\geqq 2$ ■

Define now

$$\rho(\tau)=\|\psi(.,\tau)\| \tag{5.2}$$

Then one of the following cases must necessarily hold

$$\limsup_{\tau\to\infty}(\tau\rho(\tau))=+\infty, \tag{5.3a}$$

There exist $\delta_0>0$ and $\delta_1>0$ such that

$$0<\delta_0\leqq\liminf_{\tau\to\infty}(\tau\rho(\tau))\leqq\limsup_{\tau\to\infty}(\tau\rho(\tau))\leqq\delta_1<+\infty, \tag{5.3b}$$

$$\left.\begin{array}{l}\liminf_{\tau\to\infty}(\tau\rho(\tau))=0\\ \text{and}\\ \limsup_{\tau\to\infty}(e^{\varepsilon\tau}\rho(\tau))=+\infty\quad\text{for any }\varepsilon>0\end{array}\right\} \tag{5.3c}$$

$$\rho(\tau) \leqq K e^{-\varepsilon\tau} \quad \text{for some } \varepsilon>0 \text{ and } K>0, \tag{5.3 d}$$

when $\tau>0$ is large enough.

We shall initially restrict our attention to cases *a*), *b*) and *c*) above. Let us define functions $\mu(\tau)$, $\chi(y, \tau)$ as follows

$$\mu(\tau)=\begin{cases} \eta(\tau) & \text{if (5.3 a) or (5.3 c) hold, where } \eta(\tau) \text{ is the corresponding function in Lemma 5.1} \\ \dfrac{1}{\tau} & \text{if (5.3 b) is satisfied} \end{cases} \tag{5.4 a}$$

$$\chi(y, \tau)=\frac{\psi(y, \tau)}{\mu(\tau)} \tag{5.4 b}$$

where, as before, ψ is given by (1.8), (1.14). We then have

LEMMA 5.3. – *For any* $\tau_0 \geqq 0$.

$$\int_{\tau_0}^{+\infty} \|\chi_\tau(., s)\|^2 ds<+\infty \tag{5.5}$$

Proof. – By Lemma 5.1, (ii), there exists $M>0$ such that

$$\|\chi(., \tau)\| \leqq M \tag{5.6}$$

On the other hand, by (1.13)-(1.14), χ satisfies

$$\chi_\tau=\chi_{yy}-\frac{Y}{2}\chi_y+\chi-\frac{\mu'}{\mu}\chi+\frac{f(\mu\chi)}{\mu} \tag{5.7 a}$$

where f is given in (1.11 *b*). By our choice of μ and Lemma 5.1, (v), $\left|\dfrac{\mu'}{\mu}\right|$ is bounded. Since $\mu^{-1}|f(\mu\chi)| \leqq C|\chi|$ and $\|\mu^{-1} f(\mu\chi)\| \leqq CM$, we then may consider (5.7) as a linear nonhomogeneous equation

$$\chi_\tau=A\chi+h(y, \tau); \quad y \in \mathbb{R}, \quad -\ln T<\tau<+\infty \tag{5.7 b}$$

where A is the linear operator given in (1.12), and

$$h(\gamma, \tau) \equiv h(\tau, y) \in L_{\text{loc}}^{\infty}((0, \infty); L_w^2(\mathbb{R})).$$

In particular, (A 3) in the Appendix holds in this case. Let us multiply both sides in (5.7) by $e^{-y^2/4}\chi_\tau$, and integrate first in space over $\mathbb{R}$ and then in time from τ_0 to τ_1, where $0<\tau_0<\tau_1<+\infty$, Using (5.6) and (A 3), we obtain

$$\int_{\tau_0}^{\tau_1}\int_{\mathbb{R}} \chi_\tau^2 e^{-y^2/4} dy\, ds \leqq N - \frac{1}{2}\int_{\tau_0}^{\tau_1} \frac{\mu'(\tau)}{\mu(\tau)}\left[\frac{d}{d\tau}\int_{\mathbb{R}} \chi(y, \tau)^2 e^{-y^2/4} dy\right] d\tau + \int_{\tau_0}^{\tau_1}\int_{\mathbb{R}} \frac{f(\mu\chi)}{\mu}\chi_\tau e^{-y^2/4} dy\, d\tau \equiv N+A_1+A_2 \tag{5.8}$$

for some finite constant N. Clearly

$$-\frac{1}{2}\int_{\tau_0}^{\tau_1}\frac{\mu'(\tau)}{\mu(\tau)}\left[\frac{d}{d\tau}\int_{\mathbb{R}}\chi^2 e^{-y^2/4}\,dy\right]d\tau$$
$$=\frac{1}{2}\frac{\mu'(\tau_0)}{\mu(\tau_0)}\int_{\mathbb{R}}\chi(y,\tau_0)^2 e^{-y^2/4}\,dy-\frac{1}{2}\frac{\mu'(\tau_1)}{\mu(\tau_1)}$$
$$\times\int_{\mathbb{R}}\chi(y,\tau_1)^2 e^{-y^2/4}\,dy+\frac{1}{2}\int_{\tau_0}^{\tau_1}\left(\frac{\mu'}{\mu}\right)'\left(\int_{\mathbb{R}}\chi^2 e^{-y^2/4}\,dy\right)d\tau$$

and the right-hand side above is bounded by (5.6) and Lemma 5.1, parts (iv) and (v). To estimate A_2 in (5.8), we proceed as follows. Consider the identity

$$\frac{dF}{d\tau}(y,\tau)\equiv\frac{d}{d\tau}\left[\frac{1}{\mu}\left[\frac{((p-1)^{-(1/(p-1))}+\mu\chi)^{1+p}}{(1+p)\mu}\right.\right.$$
$$\left.\left.-(p-1)^{-(p/(p-1))}\chi-\frac{p\mu\chi^2}{2(p-1)}-\frac{(p-1)^{-((p+1)/(p-1))}}{(1+p)\mu}\right]\right]$$
$$=\frac{f(\mu\chi)}{\mu}\chi_\tau+\frac{\mu'}{\mu^3}\left[-\frac{2}{1+p}((p-1)^{-(1/(p-1))}+\mu\chi)^{1+p}\right.$$
$$\left.+((p-1)^{-(1/(p-1))}+\mu\chi)^p\mu\chi+(p-1)^{-(p/(p-1))}\mu\chi+\frac{2(p-1)^{-((p+1)/(p-1))}}{1+p}\right]$$
$$\equiv\frac{f(\mu\chi)}{\mu}\chi_\tau+\Sigma(y,\tau)$$

Therefore

$$\int_{\tau_0}^{\tau_1}\int_{\mathbb{R}}\frac{f(\mu\chi)}{\mu}\chi_\tau e^{-y^2/4}\,dy\,d\tau=\int_{\mathbb{R}}F(y,\tau_1)e^{-y^2/4}$$
$$-\int_{\mathbb{R}}F(y,\tau_0)e^{-y^2/4}-\int_{\tau_0}^{\tau_1}\int_{\mathbb{R}}\Sigma(y,\tau)e^{-y^2/4}\,dy\,d\tau$$

Let us denote by C a generic constant, possibly changing from line to line, depending on τ_0, η and the bound for $|\psi|$. Recalling that $\mu\chi=\psi$, we see that $|F(y,\tau)|\leqq C|\psi|^2$, and the first two terms in the right hand side above are bounded. As to the third one, we notice that, by Taylor's Theorem $|\Sigma(y,\tau)|\leqq -C\frac{\mu'}{\mu^3}|\psi|^3$. Therefore, using Lemma 2.3 we get

$$\left|\int_{\tau_0}^{\tau_1}\int_{\mathbb{R}}\Sigma(y,\tau)e^{-y^2/4}\,dy\,d\tau\right|$$
$$\leqq -C\int_{\tau_0}^{\tau_1}\frac{\mu'(\tau)}{\mu(\tau)^3}\int_{\mathbb{R}}|\psi(y,\tau)|^3 e^{-y^2/4}\,dy\,d\tau$$

$$\leqq -C \int_{\tau_0}^{\tau_1} \frac{\mu'(\tau)}{\mu(\tau)^3} \left[\int_{\mathbb{R}} |\psi(y, \tau-a)|^2 e^{-y^2/4} dy \right]^{3/2} d\tau$$

$$\leqq -C \int_{\tau_0}^{\tau_1} \frac{\mu'(\tau)}{\mu(\tau)^3} \mu(\tau-a)^3 \left[\int_{\mathbb{R}} |\chi(y, \tau-a)|^2 e^{-y^2/4} dy \right]^{3/2} d\tau$$

so that, by (5.6) and Lemma 5.2

$$\left| \int_{\tau_0}^{\tau_1} \int_{\mathbb{R}} \Sigma(y, \tau) e^{-y^2/4} dy\, d\tau \right| \leqq C[\mu(\tau_0) - \mu(\tau_1)]$$

Taking into account that $\mu(\tau) \to 0$ as $\tau \to \infty$, we have obtained a bound for the term on the left in (5.8) which is independent of τ_j. This concludes the proof ■

Lemma 5.3 will be instrumental in proving our next result

LEMMA 5.4. – *There holds*

$$\lim_{\tau \to \infty} \| \chi(., \tau) - \langle \chi(. \tau), H_2(.) \rangle H_2(.) \|_{H_w^1} = 0 \qquad (5.9)$$

Proof. – Let us write $\chi(y, \tau) = \sum_{k=0}^{\infty} b_k(\tau) H_k(y)$, where $b_k(\tau) = \frac{a_k(\tau)}{\mu(\tau)}$, and the a_k's are the Fourier coefficients in (1.17). Fix now $R>0$, and consider equation (5.7) with initial value

$$\psi(y. \tau - R) = \sum_{k=0}^{\infty} b_k(\tau - R) H_k(y).$$

Denote by $S_A(t)$ the semigroup generated by operator A in (5.7). We then have

$$\chi(., \tau) = S_A(R)\, \chi(., \tau - R) + \int_{\tau-R}^{\tau} S_A(\tau - s)\, h(., s)\, ds$$

$$= \sum_{k=0}^{\infty} e^{(1-(k/2))R} b_k(\tau - R) H_k(y) + \int_{\tau-R}^{\tau} S_A(\tau - s)\, h(., s)\, ds$$

and therefore

$$\chi(y, \tau) - b_2(\tau) H_2(y) = \sum_{k=0}^{1} e^{(1-(k/2))R} b_k(\tau - R) H_k(y)$$
$$+ (b_2(\tau - R) - b_2(\tau)) H_2(y) + \sum_{k \geqq 3} e^{(1-(k/2))R} b_k(\tau - R) H_k(y)$$
$$+ \int_{\tau-R}^{\tau} S_A(\tau - s)\, h(., s)\, ds \equiv S_1 + S_2 + S_3 + S_4 \qquad (5.10)$$

Since $\| b_k H_k \| = \frac{|a_k|}{\|\psi\|} \cdot \frac{\rho}{\mu}$, it follows from Lemma 5.1 and Proposition 4.1 that

$$\| S_1 \| \to 0 \quad \text{as } \tau \to \infty$$

On the other hand, we certainly have

$$|\dot{b}_2(\tau) = |\langle \chi_\tau(.,\tau), H_2(.) \rangle| \leqq \left[\int_{\mathbb{R}} \chi_\tau(y,\tau)^2 e^{-y^2/4} dy \right]^{1/2}$$

so that

$$\begin{aligned} &|b_2(\tau) - b_2(\tau - R)| \\ &\quad \leqq \int_{\tau-R}^{\tau} |\dot{b}_2(s)| ds \leqq \int_{\tau-R}^{\tau} \left[\int_{\mathbb{R}} \chi_\tau(y,\tau)^2 e^{-y^2/4} dy \right]^{1/2} ds \\ &\qquad \leqq R^{1/2} \left[\int_{\tau-R}^{\tau} \left[\int_{\mathbb{R}} \chi_\tau(y,\tau)^2 e^{-y^2/4} dy \right] ds \right]^{1/2} \end{aligned} \tag{5.11}$$

whence, by Lemma 5.3

$$\| S_2 \|_{H_w^1} \to 0 \quad \text{as } \tau \to \infty$$

Consider now function $h(y, \tau)$ in S_4. By (5.7), it follows that

$$\| h(.,\tau) \| \leqq C_1 \frac{|\mu'(\tau)|}{\mu(\tau)} + C_2 \mu(\tau) \left[\int_{\mathbb{R}} |\chi(.,\tau)^4 e^{-y^2/4} dy \right]^{1/2}$$

for some constants C_1, C_2. Therefore, since $\chi = \frac{\psi}{\mu}$, taking into account Lemmata 2.3, 5.1 and 5.2, we deduce that $\lim_{\tau \to \infty} \| h(.,\tau) \| = 0$. We now set $H(y,\tau) = \int_{\tau-R}^{\tau} S_A(\tau - s) h(.,s) ds$, and notice that $H(y, \tau)$ solves (5.7*b*) with initial value $H(y, \tau - R) = 0$. We then may use (A 3) (with $\Phi_0 = 0$) to get

$$\| S_4 \|_{H_w^1} \to 0 \quad \text{as } \tau \to \infty$$

Only the term S_3 in (5.10) remains to be estimated. To do it, we notice that

$$\| \sum_{k \geqq 3} e^{(1-(k/2))R} b_k(\tau - R) H_k(.) \|_{H_w^1}^2 = \sum_{k \geqq 3} \left(\frac{k}{2} + 1 \right) e^{2(1-(k/2))R} b_k(\tau - R)^2$$

If we set $y=2\left(\frac{k}{2}-1\right)\mathrm{R}$, we have $y \geqq \frac{\mathrm{R}}{2}$ for $k=3, 4, \ldots,$ and $ye^{-y} \leqq \mathrm{C}\, e^{-y/2} \leqq \mathrm{C}\, e^{-\mathrm{R}/4}$ for some $\mathrm{C}>0$, so that

$$\|\mathrm{S}_3\|^2_{\mathrm{H}^1_w} \leqq \left(\frac{3\mathrm{C}}{2\mathrm{R}}\, e^{-\mathrm{R}/4}+e^{-\mathrm{R}}\right) \sum_{k \geqq 3} b_k(\tau-\mathrm{R})^2$$

$$\leqq \left(\frac{3\mathrm{C}}{2\mathrm{R}}\, e^{-\mathrm{R}/4}+e^{-\mathrm{R}}\right) \|\chi(.,\tau-\mathrm{R})\|^2$$

Putting together all these estimates and recalling (5.6), we obtain

$$\limsup_{\tau \to \infty} \|\chi(.,\tau)-b_2(\tau)\mathrm{H}_2(.)\|_{\mathrm{H}^1_w} \leqq \mathrm{M}\left(\frac{3\mathrm{C}}{2\mathrm{R}}\, e^{-\mathrm{R}/4}+e^{-\mathrm{R}}\right)$$

We now let $\mathrm{R} \to \infty$ to conclude ■

Lemma 5.4 is a first step towards obtaining (5.1). To proceed further, however, some technical results are required. The first one is

LEMMA 5.5. – *a) For any given* $\mathrm{L}>0$, *there exist* $\varepsilon_0>0$ *and a sequence* $\{\tau_j\}$ *with* $\lim_{j \to \infty} \tau_j=+\infty$, *such that*

$$|b_2(\tau)| \geqq \varepsilon_0 \text{ when } \tau \in [\tau_j, \tau_j+\mathrm{L}] \text{ with } j \text{ large enough} \tag{5.12}$$

b) For any $\varepsilon>0$ *and* $\mathrm{L}>0$ *there exists a sequence* $\{\tau_j\}$ *with* $\lim_{j \to \infty} \tau_j=+\infty$ *such that*

$$\sum_{k \neq 2} a_k(\tau)^2 \leqq \varepsilon\, a_2(\tau)^2 \quad \text{when } \tau \in [\tau_j, \tau_j+\mathrm{L}] \tag{5.13}$$

Proof. – *a)* Recalling our choice of $\rho(\tau)$ in (5.2), we see that by Lemma 5.1, (ii) there exist a sequence $\{\tau_j\}$ with $\lim_{j \to \infty} \tau_j=\infty$, and a constant $\mathrm{K}>0$ such that $\|\psi(.,\tau_j)\| \geqq \frac{1}{\mathrm{K}}\mu(\tau_j)$, or in another words $\|\chi(.,\tau_j)\| \geqq \frac{1}{\mathrm{K}}$. By (5.9), we must then have $|b_2(.,\tau_j)| \geqq \frac{1}{2\mathrm{K}}$ for large enough j. Using (5.11), we then deduce that, for $\tau \in [\tau_j, \tau_j+\mathrm{L}]$,

$$|b_2(\tau)-b_2(\tau_j)| \leqq \mathrm{L}^{1/2}\,\mathrm{A}(\tau_j;\mathrm{L})$$

where $\mathrm{A}(\tau_j;\mathrm{L}) \to 0$ as $j \to \infty$ for fixed L, whence

$$|b_2(\tau)| \geqq \frac{1}{4\mathrm{K}} \quad \text{for } \tau \in [\tau_j, \tau_j+\mathrm{L}] \text{ if } j \text{ is large enough.}$$

b) By lemma 5.4, $\sum_{k \neq 2} b_k(\tau)^2 \to 0$ as $\tau \to \infty$, and by part *a*) above (5.12) holds. Therefore, for any $\delta > 0$

$$\sum_{k \neq 2} a_k(\tau)^2 = \mu(\tau)^2 \sum_{k \neq 2} b_k(\tau)^2 \leqq \frac{\delta}{\varepsilon_0^2} \mu(\tau)^2 b_2(\tau)^2 = \frac{\delta}{\varepsilon_0^2} a_2(\tau)^2$$

if $\tau \in [\tau_j, \tau_j + L]$ [$\{\tau_j\}$ as in part *a*)], and j is large enough. ■

We next improve (5.13) as follows

LEMMA 5.6. – *We have*

$$\lim_{\tau \to \infty} \frac{\sum_{k \neq 2} a_k(\tau)^2}{a_2(\tau)^2} = 0 \tag{5.14}$$

Proof. – Assume that the result is false. Then, by (4.1) there exist $\sigma > 0$ and a sequence $\{s_j\}$ such that $\lim_{j \to \infty} s_j = \infty$ and

$$\sum_{k=3}^{\infty} a_k(s_j)^2 \geqq \sigma a_2(s_j)^2 \tag{5.15}$$

We now take up an approach already used in Lemma 4.8. We fix $\varepsilon > 0$ with $\varepsilon < \frac{\sigma}{2}$, and consider the function

$$E_\varepsilon(\tau) = \varepsilon a_2(\tau)^2 - \sum_{k=3}^{\infty} a_k(\tau)^2 \tag{5.16}$$

By part *b*) in Lemma 5.5 $\left(\text{with } \varepsilon \text{ replaced by } \frac{\varepsilon}{2}\right)$, we have that for any fixed $L > 0$ and some sequence $\{\tau_j\}$ such that $\lim_{j \to \infty} \tau_j = \infty$.

$$E_\varepsilon(\tau) \geqq \frac{\varepsilon}{2} a_2(\tau)^2 \quad \text{when } \tau \in [\tau_j, \tau_j + L] \tag{5.17}$$

On the other hand, a routine computation yields

$$\begin{aligned} \frac{dE_\varepsilon}{d\tau}(\tau) &= 2\varepsilon a_2 \langle f(\psi), H_2 \rangle - 2 \sum_{k=3}^{\infty} a_k^2 \left(1 - \frac{k}{2}\right) \\ &\quad - 2 \sum_{k=3}^{\infty} a_k \langle f(\psi), H_k \rangle \geqq 2\varepsilon a_2 \langle f(\psi), H_2 \rangle \\ &\quad + \sum_{k=3}^{\infty} a_k^2 - 2 \int_{\mathbb{R}} \left(\psi - \sum_{k=0}^{2} a_k H_k\right) f(\psi)\, e^{-y^2/4}\, dy \end{aligned} \tag{5.18}$$

By (5.15), $E_\varepsilon(s_j) < 0$, whereas by (5.17), $E_\varepsilon(\tau_j) \geqq 0$ for $j = 1, 2, \ldots$

Let us denote by $\{\theta_j\}$ a sequence of exit times from the cones where $E_\varepsilon \geqq 0$, *i.e.* let us define

$$\theta_j = \inf\{\tau \geqq \tau_j : E_\varepsilon(\tau) < 0\}; \qquad j = 1, 2, \ldots$$

Obviously, $\theta_j \geqq \tau_j + L$, and, if $\varepsilon < \frac{1}{2}$,

$$[\theta_j - L, \theta_j] \subset \{\tau : E_\varepsilon(\tau) \geqq 0\} \subset \{\tau : E_{1/2}(\tau) \geqq 0\} \tag{5.19}$$

We next set out to estimate the right-hand side of (5.18) at $\tau = \theta_j$. By continuity, $E_\varepsilon(\theta_j) = 0$. Taking into account (4.1), we obtain

$$\left(1 + \frac{1}{\varepsilon}\right) \sum_{k=3}^{\infty} a_k(\theta_j)^2 \geqq \frac{1}{2} \sum_{k=0}^{\infty} a_k(\theta_j)^2 \text{ for large enough } j \tag{5.20}$$

The third term on the right in (5.18) is dealt with as follows

$$|S| \equiv \left| \int_{\mathbb{R}} \left(\psi - \sum_{k=0}^{2} a_k H_k\right) f(\psi) e^{-y^2/4} dy \right|$$
$$\leqq C\left[\int_{\mathbb{R}} |\psi|^3 e^{-y^2/4} dy + \sum_{k=0}^{2} |a_k| \left(\int_{\mathbb{R}} |\psi|^4 e^{-y^2/4} dy\right)^{1/2}\right]$$

We now use Lemma 2.3 and set $L = \tau^*$ to obtain

$$|S(\theta_j)| \leqq C\left(\|\psi(., \theta_j - L)\|^3 + \|\psi(., \theta_j - L)\|^2 \sum_{k=0}^{2} |a_k|\right)$$

Recalling (5.19), it follows from (4.1) that

$$[\theta_j - L, \theta_j] \subset \{\tau : a_2(\tau)^2 \geqq \sum_{k \neq 2} a_k(\tau)^2\}$$

for j large enough. We then take advantage of Lemma 3.1 to get

$$|S(\theta_j)| \leqq C\left(\|\psi(., \theta_j)\|^3 + \|\psi(., \theta_j)\|^2 \sum_{k=0}^{2} |a_k|\right) \tag{5.21}$$

In a similar way, by the usual quadratic bound for $f(\psi)$,

$$|2\varepsilon a_2(\theta_j) \langle f(\psi), H_2 \rangle| \leqq 2C\varepsilon |a_2(\theta_j)| \|\psi(., \theta_j)\|^2 \tag{5.22}$$

where here and henceforth C denotes a generic positive constant.

Substituting (5.20)-(5.22) into (5.18), we obtain

$$\frac{dE_\varepsilon}{d\tau}(\theta_j) \geqq \frac{\varepsilon}{2(1+\varepsilon)} \|\psi(., \theta_j)\|^2 - C\varepsilon |a_2(\theta_j)| \|\psi(., \theta_j)\|^2$$
$$- C\left(\|\psi(., \theta_j)\|^3 + \|\psi(., \theta_j)\|^2 \sum_{k=0}^{2} |a_k|\right)$$

As $\lim_{j \to \infty} \| \psi(., \theta_j) \| = 0$ [whence $\lim_{j \to \infty} | a_2(\theta_j) | = 0$], we arrive at

$$\frac{dE_\varepsilon}{d\tau}(\theta_j) \geq \frac{\varepsilon}{4(1+\varepsilon)} \| \psi(., \theta_j) \|^2 \quad \text{for large } j$$

Since we may assume without loss of generality that $\| \psi(., \theta_j) \| > 0$ for large j, we would then have $E_\varepsilon(\tau) > 0$ for $\tau \in [\theta_j, \theta_j + \delta]$ and some $\delta > 0$, which contradicts the definition of θ_j. Taking again into account (4.1), the result follows ■

We next show

PROPOSITION 5.7. – *The following alternative holds. Either*

$$\| \psi(., \tau) + \frac{c}{\tau} H_2(.) \|_{H^1_w} = o\left(\frac{1}{\tau}\right) \quad as \ \tau \to \infty \tag{5.23}$$

where $c = \dfrac{(4\pi)^{1/4}(p-1)^{-(1/(p-1))}}{\sqrt{2}p}$, *or*

$$\| \psi(., \tau) \| = O(e^{-\varepsilon\tau}) \quad as \ \tau \to \infty \quad for\ some\ \varepsilon > 0. \tag{5.24}$$

Proof. – Statement (5.24) follows if (5.3 *d*) holds. If otherwise one of the assumptions (5.3 *a*)-(5.3 *c*) is fulfilled, our previous results in this Section apply. We then write

$$\psi(y, \tau) = a_2(\tau) H_2(y) + \theta(y, \tau)$$

Write now

$$f(\psi) = \frac{p(p-1)^{(1/(p-1))}}{2} \psi^2 + g(\psi) \equiv \nu \psi^2 + g(\psi),$$

where $| g(\psi) | \leq C | \psi |^3$. Then $a_2(\tau)$ satisfies

$$a_2 = \nu a_2^2 \langle H_2^2, H_2 \rangle + 2\nu a_2 \langle \theta, H_2^2 \rangle + \nu \langle \theta^2, H_2 \rangle + \langle g(\psi), H_2 \rangle \tag{5.25}$$

We now examine the first term on the right in (5.25). Set

$$A_{n, m, l} = \int_{\mathbb{R}} H_n(y) H_m(y) H_l(y) e^{-y^2/4} dy$$

It has been shown in [GHV] that

$$A_{n, m, l} \neq 0 \text{ if and only if } (n+m+l) \text{ is even and } n \leq m+l,\ m \leq n+l, \quad l \leq m+n. \tag{5.26}$$

In this case, we have

$$A_{n, m, l} = \left((4\pi)^{-1/4}(n!)(m!)(l!)\right)^{1/2} \left[\left[\frac{m+n-l}{2}\right]! \left[\frac{n+l-m}{2}\right]! \left[\frac{m+l-n}{2}\right]!\right]^{-1}$$

A derivation of (5.26) is recalled in Appendix B at the end of this paper for the reader's convenience.

It then follows that

$$\nu a_2^2 \langle H_2^2, H_2 \rangle = A_{2,2,2}\, \nu a_2^2 = \frac{\sqrt{2}\, p\,(p-1)^{(1/(p-1))}}{(4\pi)^{1/4}}\, a_2^2$$

We want to show now that the last three terms on the right in (5.25) are negligible with respect to the first one there. To begin with, we have that

$$|\langle g(\psi), H_2 \rangle| \leq C \| \psi(., \tau) \|_{6, w}^3 \\ \leq C \| \psi(., \tau-\tau^*) \|^3 \leq C \| \psi(., \tau) \|^3 \leq C |a_2(\tau)|^3$$

where use has been made of Lemmata 2.3 and 3.1 (this last can be applied since Lemma 5.6 holds). Furthermore, we readily check that

$$|a_2 \langle \theta, H_2^2 \rangle| \leq |a_2(\tau)| \, \| \theta(., \tau) \|_{4, w} \| H_2^2 \|_{4/3, w},$$
$$|\langle \theta^2, H_2 \rangle| \leq \| \theta(., \tau) \|_{4, w}^2$$

We now claim that

$$\lim_{\tau \to \infty} \frac{\| \theta(., \tau) \|_{4, w}}{|a_2(\tau)|} = 0 \tag{5.27}$$

To keep the flow of the main arguments here, we assume (5.27) for the moment and continue. Substituting our previous estimates in (5.25) yields

$$\dot{a}_2(\tau) = c a_2(\tau)^2 (1 + \varepsilon(\tau))$$

where c is given in (5.23) and $\varepsilon(\tau) \to 0$ as $\tau \to \infty$. We now integrate this last equation to get

$$-\frac{1}{a_2(\tau)} + \frac{1}{a_2(\tau_0)} = c\left((\tau - \tau_0) + \int_{\tau_0}^{\tau} \varepsilon(s)\, ds\right)$$

whence

$$\lim_{\tau \to \infty} \left[\frac{1}{\tau a_2(\tau)} \right] = -c$$

If either (5.3 *a*) or (5.3 *c*) holds, this would give a contradiction. Hence (5.3 *b*) must be satisfied, and then the result follows from Lemma 5.4.

To conclude the proof, it remains to show (5.27). To this end, we notice that θ solves

$$\theta_\tau = \theta_{yy} - \frac{y}{2}\theta_y + \theta + [f(\psi) - \langle f(\psi), H_2 \rangle H_2] \equiv A\theta + D(y, \tau)$$

Fix now $R > 0$. Using once more variation of constants formula, and keeping to the notations in Lemma 5.4, we may represent $\theta(y, \tau)$ in the form

$$\theta(y,\tau)=S_A(R)\theta(.,\tau-R)+\int_{\tau-R}^{\tau} S_A(\tau-s)\,D(.,s)\,ds \qquad (5.28)$$

Denoting by C a generic constant independent of ψ, we have by Lemmata 5.6 and 3.1 that

$$\|\theta(.,\tau-R)\|\ll|a_2(\tau-R)|\leqq\|\psi(.,\tau-R)\|\leqq C\|\psi(.,\tau)\|\leqq C|a_2(\tau)|$$

On the other hand, setting $1-e^{-\tau}=t$, a straightforward adaptation of the arguments leading to Lemma 2.1 and Corollary 2.2 yields

$$\|S_A(\tau)\Phi_0\|_{p,w}\leqq\frac{C\,e^{\tau}}{(1-e^{-\tau})^{1/2q}}\|\Phi_0\|_{q,w} \qquad (5.29)$$

for any $\Phi_0\in L^q_w(\mathbb{R}^N)$, $1\leqq p\leqq q<\infty$. Taking $R>0$ sufficiently large, we have that

$$\begin{aligned}\|S_A(R)\theta(.,\tau-R)\|_{4,w}&\leqq C\|\theta(.,\tau-R)\|\\ &\ll|a_2(.,\tau-R)|\leqq\|\psi(.,\tau-R)\|\leqq C\|\psi(.,\tau)\|\leqq C|a_2(\tau)|\end{aligned}$$

Therefore

$$\lim_{\tau\to\infty}\frac{\|S_A(R)\theta(.,\tau-R)\|_{4,w}}{|a_2(\tau)|}=0 \qquad (5.30)$$

We next examine the second term on the right of (5.28). Since

$$\begin{aligned}\|D(.,s)\|_{5,w}\leqq C\|\psi(.,s)\|^2_{10,w}&\leqq C\|\psi(.,\tau-\tau^*)\|^2\\ &\leqq C\|\psi(.,\tau)\|^2\leqq C\,a_2(\tau)^2\end{aligned}$$

if follows from (5.29) that

$$\begin{aligned}&\int_{\tau-R}^{\tau}\|S_A(\tau-s)D(.,s)\|_{4,w}\,ds\\ &\quad\leqq C\,e^{R}\int_{\tau-R}^{\tau}\frac{\|D(.,s)\|_{5,w}}{(1-e^{-(\tau-s)})^{1/10}}\,ds\leqq C\,a_2(\tau)^2\int_{\tau-R}^{\tau}\frac{ds}{(1-e^{-(\tau-s)})^{1/10}}\\ &\quad\leqq C\,a_2(\tau)^2\ll|a_2(\tau)|\quad\text{as } \tau\to\infty \qquad (5.31)\end{aligned}$$

Putting together (5.30) and (5.31), (5.27) follows ■

We next examine the case where (5.24) holds

PROPOSITION 5.8. – *Assume that*

$$\|\psi(.,\tau)\|=O(e^{-\varepsilon\tau})\quad\text{as } \tau\to\infty \quad \textit{for some } \varepsilon>0$$

Then the following alternative holds. Either there exists $m\geqq 3$ *and* $C\neq 0$ *such that*

$$\|\psi(.,\tau)-C\,e^{(1-(m/2))\tau}H_m(.)\|_{H^1_w}=o(e^{(1-(m/2))\tau})\quad \textit{as } \tau\to\infty, \qquad (5.32)$$

or

$$\psi\equiv 0.$$

Proof. – As we have repeatidly done so far, we represent $\psi(y, \tau)$ in the form

$$\psi(y, \tau)=S_A(\tau)\psi_0+\int_0^\tau S_A(\tau-s) f(\psi(.,s))\,ds$$

Suppose now that $\|\psi(.,\tau)\|\leqq M e^{-\varepsilon\tau}$ for some $M>0$ and $\varepsilon>0$, where $2\varepsilon\neq\frac{l}{2}-1$ for any integer l (this last condition can always be satisfied by replacing, if necessary ε by some $\varepsilon'<\varepsilon$). Take now an integer $k_0\geqq 2$ such that

$$\frac{k_0}{2}-1<2\varepsilon<\frac{k_0+1}{2}-1$$

Since

$$S_A(\tau)\psi=\sum_{k=0}^{\infty}\alpha_k e^{(1-(k/2))\tau}H_k(y) \qquad \text{with} \quad \sum_{k=0}^{\infty}\alpha_k^2<+\infty,$$

and

$$f(\psi)=\sum_{k=0}^{\infty}\langle f(\psi), H_k\rangle H_k,$$

we have

$$\begin{aligned}\psi(y,\tau)=&\sum_{k=0}^{k_0}\alpha_k e^{(1-(k/2))\tau}H_k(y)+\sum_{k_0+1}^{\infty}\alpha_k e^{(1-(k/2))\tau}H_k(y)\\&+\sum_{k=0}^{k_1}H_k(y)\int_0^\tau e^{(1-(k/2))(\tau-s)}\langle f(\psi), H_k\rangle ds\\&+\sum_{k_0+1}^{\infty}H_k(y)\int_0^\tau e^{(1-(k/2))(\tau-s)}\langle f(\psi), H_k\rangle ds\\&\equiv T_1+T_2+T_3+T_4 \quad (5.33)\end{aligned}$$

We now proceed to estimate the H_w^1-norm of the various terms in the right hand side in (5.33). Note first that

$$\begin{aligned}\left\|\sum_{k_0+1}^{\infty}\alpha_k e^{(1-(k/2))\tau}H_k(.)\right\|_{H_w^1}&\leqq\sum_{k_0+1}^{\infty}\alpha_k(1+(k/2))^{1/2}e^{(1-(k/2))\tau}\\&\leqq\left[\sum_{k_0+1}^{\infty}\alpha_k^2\right]^{1/2}\left[\sum_{k_0+1}^{\infty}(1+(k/2)e^{2(1-(k/2))\tau}\right]^{1/2}\end{aligned}$$

Clearly

$$\sum_{l}^{\infty} (1+(k/2))\, e^{2(1-(k/2))\tau} = e^{(2-l)\tau} \sum_{j=0}^{\infty} (1+((j+l)/2))\, e^{-j\tau}$$

On the other hand, $\left(1+\dfrac{j+l}{2}\right) e^{-j\tau} = \dfrac{(1+((j+l)/2))}{j\tau+1}\,(j\tau+1)\, e^{-j\tau}$. If $\tau \geqq 1$, $\dfrac{1+((j+l)/2)}{j\tau+1} \leqq C$ for some $C=C(l)$, whereas $(y+1)\,e^{-y} \leqq C\,e^{-y/2}$ for some constant C. Setting $l=k_0+1$, we then obtain

$$\left\| \sum_{k_0+1}^{l} \alpha_k\, e^{(1-(k/2))\tau} H_k(\,.\,) \right\|_{H_w^1} \leqq C \left[\sum_{k=0}^{\infty} \alpha_k^2 \right]^{1/2} e^{(1-((k_0+1)/2))\tau}$$

For some $C=C(l)$ and $\tau \geqq 1$. By our choice of k_0, this gives

$$\left\| \sum_{k_0+1}^{\infty} \alpha_k\, e^{(1-(k/2))\tau} H_k(\,.\,) \right\|_{H_w^1} \leqq C\, e^{-2\varepsilon\tau} \tag{5.34}$$

for some constant $C>0$ and $\tau \geqq 1$.

As a next step, we estimate the L_w^2-norm of T_4 in (5.33) as follows

$$\| T_4 \| \leqq \sum_{k_0+1}^{\infty} \int_0^{\tau} e^{(1-(k/2))(\tau-s)} \left| \langle f(\psi),\, H_k \rangle \right| ds \leqq \int_0^{\tau} \left[\sum_{k_0+1}^{\infty} e^{2(1-(k/2))(\tau-s)} \right]^{1/2} \| f(\psi) \|\, ds$$

Notice that

$$\sum_{k=l}^{\infty} e^{2(1-(k/2))\tau} = e^{(2-l)r} (1-e^{-r})^{-1}$$

Since $|f(s)| \leqq C s^2$ for some $C>0$, we then have $\| f(\psi) \| \leqq C\, e^{-2\epsilon\tau}$ and

$$\| T_4 \| \leqq C\, e^{-2\epsilon\tau} \quad \text{for some } C>0 \tag{5.35}$$

We then observe that $T_4 = T_4(y, \tau)$ solves

$$z_\tau = A z + f(\psi) - \sum_{k=0}^{k_0} \langle f(\psi),\, H_k \rangle H_k \equiv A z + \sigma(y, \tau)$$

where $\|\sigma(.,\tau)\| \leqq \|f(\psi)\| \leqq C e^{-2\epsilon\tau}$ for some $C>0$. By variation of constants formula, for any $R \in (0, \tau)$,

$$T_4(.,\tau) = S_A T_4(.,\tau - R) + \int_{\tau-R}^{\tau} S_A(.,\tau - s)\,\sigma(.,s)\,ds$$

Taking into account (5.35) and (A3), we then obtain

$$\|T_4(.,\tau)\|_{H^1_w} \leqq C e^{-2\epsilon\tau} \quad \text{for some } C>0 \tag{5.36}$$

Finally, for $0 \leqq k \leqq k_0$

$$\begin{aligned}\int_0^{\tau} e^{-(1-(k/2))s} \langle f(\psi), H_k \rangle\, ds &= \int_0^{\infty} e^{-(1-(k/2))s} \langle f(\psi), H_k \rangle\, ds - \int_{\tau}^{\infty} e^{-(1-(k/2))s} \langle f(\psi), H_k \rangle\, ds \\ &\equiv \beta_k - \int_{\tau}^{\infty} e^{-(1-(k/2))s} \langle f(\psi), H_k \rangle\, ds \end{aligned} \tag{5.37a}$$

where

$$\left| \int_{\tau}^{\infty} e^{-(1-(k/2))s} \langle f(\psi), H_k \rangle\, ds \right| \leqq C e^{-2\epsilon\tau} \quad \text{for some } C>0. \tag{5.37b}$$

Assume now that $k_0 \geqq 3$. Substituting (5.34)-(5.37) into (5.33), we obtain

$$\begin{aligned}\psi(y,\tau) = \sum_{k=0}^{k_0} (\alpha_k + \beta_k)\, e^{(1-(k/2))\tau} H_k(y) + T_2(y,\tau) + T_4(y,\tau) \\ - \sum_{k=0}^{k_0} H_k(y) \int_{\tau}^{\infty} e^{(1-(k/2))(\tau - s)} \langle f(\psi), H_k \rangle\, ds \end{aligned} \tag{5.38}$$

By our previous bounds for T_2 and T_4, the fact that $\|f(\psi)\| \leqq C e^{-2\epsilon\tau}$, and our choice of k_0, it follow that

$$\psi(y,\tau) = \sum_{k=0}^{k_0} (\alpha_k + \beta_k)\, e^{(1-(k/2))\tau} H_k(y) + R(y,\tau)$$

where $\|R(y,\tau)\|_{H^1_w} = O(e^{-2\epsilon\tau})$ for $\tau \geqq 1$. Since we are assuming that $\|\psi(.,\tau)\| = O(e^{-\epsilon\tau})$, we necessarily have

$$\alpha_k + \beta_k = 0, \qquad \text{for} \quad 0, 1, 2$$

Then two possibilities arise. There may be an integer $m \in [3, k_0]$ such that $\alpha_m + \beta_m \neq 0$ but $\alpha_k + \beta_k = 0$ for $k < m$.

In this case, we would have

$$\psi(.,\tau) = (\alpha_m + \beta_m)\, e^{(1-(m/2))\tau} + Q(y,\tau)$$

where $\|Q(y,\tau)\|_{H^1_w} = o(e^{(1-(m/2))\tau}$ as $\tau \to \infty$, and (5.32) holds.

On the other hand, if $\alpha_m+\beta_m=0$ for any $m\in[3, k_0]$, then we would obtain

$$\|\psi(.,\tau)\|=O(e^{-2\varepsilon\tau}) \quad \text{as } \tau\to\infty \tag{5.39}$$

Iterating our previous argument, we would arrive at (5.32) in a finite number of steps. Otherwise (3.4) would hold, and therefore $\psi\equiv 0$. Finally, if $k_0=2$, we get (5.39), and we may start our procedure again, this time with $k_0\geqq 3$. This concludes the proof ■

We now summarize, by Proposition 5.7 and 5.8, if $\psi\not\equiv 0$, either

$$\psi(y,\tau)=-\frac{(4\pi)^{1/4}(p-1)^{-(1/(p-1))}}{\sqrt{2}p}\cdot\frac{H_2(y)}{\tau}+Q(y,\tau),$$

where

$$\|Q(.,\tau)\|_{H^1_w}=o\left(\frac{1}{\tau}\right) \quad \text{as } \tau\to\infty, \tag{5.40}$$

or

$$\left.\begin{array}{c}\psi(y,\tau)=C\,e^{(1-(m/2))\tau}H_m(y)+R(y,\tau),\\ \text{for some } C\neq 0 \text{ and } m\geqq 3, \text{ where}\\ \|R(.,\tau)\|_{H^1_w}=o(e^{(1-(m/2))\tau}) \text{ as } \tau\to\infty.\end{array}\right\} \tag{5.41}$$

In particular, convergence takes place in $H^1(B_R)$ for any ball with radius $R>0$, whence also in $C^{0,\alpha}(B_R)$ for some $\alpha\in(0, 1)$. Convergence in $C^{k,\alpha}_{loc}$ with $k\geqq 1$ and $\alpha\in(0, 1)$ follows then by considering the equation satisfied by $Q(y,\tau)$ [resp. $R(y,\tau)$] and using standard Schauder estimates in bounded regions. This concludes the proof of Theorem 3.

Our last result in this Section is

PROPOSITION 5.9. – *Under the assumptions of Theorem* 1, (5.40) *holds. Furthermore, for any* $k\geqq 1$ *and* $\alpha\in(0.1)$, *convergences takes also place in* $C^{k,\alpha}_{loc}$ *there.*

Proof. – It suffices to show that (5.41) is to be discarded in this case, and this follows at once by contradiction. Indeed, if (5.41) holds, by symmetry m must be an even number, and then H_m has $\left(\frac{m}{2}\right)$ maxima. Therefore, taking $R>0$ large enough, our solution have at least $\left(\frac{m}{2}\right)$ maxima for large enough τ, and this is impossible, since by the results in [A], [AF], the number of maxima cannot increase in time ■

6. EXTENDING CONVERGENCE TO LARGE SETS

In this Section we shall show that (5.40) actually holds not only on bounded sets $|y|\leqq C<+\infty$, but also in the larger region described in the statement of Theorem 1. To begin with, we obtain

LEMMA 6.1. — *Under the assumptions of Theorem* 1, *we have*

$$(1-t)^{1/(p-1)}u(\xi((1-t)|\log(1-t)|)^{1/2}, t)$$
$$\geqq\left[(p-1)(1+\left(\frac{p-1}{4p}\right)\xi^2+o(1-t))\right]^{-(1/(p-1))} \quad \textit{as } t\uparrow 1, \quad (6.1)$$

uniformly on sets $|\xi|\leqq R$ *with* $R>0$.

Proof. — Let s be a real parameter, $0<s<1$, and consider the auxillary functions

$$v_s(x, t)=(1-s)^{1/(p-1)}.u(x\sqrt{1-s}, s+t(1-s)) \quad (6.2)$$

We readily see that, for any fixed s, v_s solves (1.1).

Moroever, by our choice of ψ and (5.40), we have that if c is given in (5.23)

$$v_s(x, 0)=(p-1)^{-(1/(p-1))}-\frac{c\,H_2(x)}{|\ln(1-s)|}+o\left(\frac{1}{|\ln(1-s)|}\right) \quad \text{as } s\to 1$$

in H_w^1 [as well as in $C_{loc}^{k,\alpha}$ for some $\alpha\in(0, 1)$ and any $k\geqq 1$].

Consider now the function

$$z_s(x, t)=((S(t)v_s(x, 0))^{-(p-1)}-(p-1)t)^{-(1/(p-1))}$$

Obviously, $z_s(x, 0)=v_s(x, 0)$, and a routine computation reveals that

$$z_{s,t}-z_{s,xx}\leqq z_s^P$$

We then deduce that, for fixed s

$$v_s(x, t)\geqq z_s(x, t) \quad \text{whenever } x\in\mathbb{R}, \quad 0<t<1 \quad (6.3)$$

Set now $\tau_s=-\ln(1-s)$. By the results in Section 5 and well known representation formulas for caloric functions (*cf.* for instance [Wi], Chapter 10) we have,

$$S(t)v_s(x, 0)=S(t)((p-1)^{-(1/(p-1))}+\psi(x, \tau_s))$$
$$=(p-1)^{-(1/(p-1))}+\sum_{k=0}^{\infty}a_k(\tau_s)(1-t)^{k/2}H_k\left(\frac{x}{\sqrt{1-t}}\right)=(p-1)^{-(1/(p-1))}$$
$$-\left[\frac{c}{|\ln(1-s)|}+o\left(\frac{1}{|\ln(1-s)|}\right)\right](1-t)H_2\left(\frac{x}{\sqrt{1-t}}\right)$$
$$+\sum_{k=3}^{\infty}a_k(\tau_s)(1-t)^{k/2}H_k\left(\frac{x}{\sqrt{1-t}}\right)+o\left(\frac{1}{|\ln(1-s)|}\right) \quad \text{as } s\to 1 \quad (6.4)$$

We now intend to get rid of the third term on the right in (6.4). To this end, we proceed as follows. First, (6.2) can be written in the form

$$v_s(x, t) = (1-s)^{1/(p-1)} u(r, \tilde{t})$$

where $r = x\sqrt{1-s}$, $\tilde{t} = s + t(1-s)$. To get (6.1), we need a lower bound on $u(r, \tilde{t})$ along sets where $r = \xi(1-\tilde{t})^{1/2} |\ln(1-\tilde{t})|^{1/2}$. In terms of x and t, this means $x = \xi(1-t)^{1/2} |\ln(1-t)(1-s)|^{1/2}$. We then have

$$\begin{aligned} S \equiv \sum_{k=3}^{\infty} a_k(\tau_s)(1-t)^{k/2} H_k\left(\frac{x}{\sqrt{1-t}}\right) &= \sum_{k=3}^{\infty} a_k(\tau_s)(1-t)^{k/2} H_k(\xi |\ln((1-s)(1-t))|^{1/2}) \\ &= \sum_{k=3}^{\infty} a_k(\tau_s)(1-t)^{k/2} |\ln((1-s)(1-t))|^{k/2} \\ &\qquad \cdot \frac{H_k(\xi |\ln((1-s)(1-t))|^{1/2})}{|\ln((1-s)(1-t))|^{k/2}} \end{aligned}$$

We now select $t = t(s)$ as follows

$$1 = (1-t) |\ln((1-s)(1-t))| \tag{6.5}$$

so that $(1-t) \simeq |\ln(1-s)|^{-1}$ as $s \uparrow 1$, and set

$$1 - \lambda = \frac{1}{|\ln((1-s)(1-t)|}$$

Then S can be recast in the form

$$S(\xi, \lambda) = \sum_{k=3}^{\infty} a_k(\tau_s)(1-\lambda)^{k/2} H_k\left(\frac{\xi}{\sqrt{1-\lambda}}\right),$$

whence S solves the heat equation with datum

$$\|S(\xi, 0)\| = o\left(\frac{1}{|\ln(1-s)|}\right)$$

as $s \uparrow 1$. Using Poisson formula for S, we readily see that

$$|S_\xi(\xi, \lambda)| + |S_\lambda(\xi, \lambda)| \leqq C \|S(., 0)\| \quad \text{if} \quad |\xi| \leqq R, \quad 0 < \delta < \lambda \leqq 1$$

for any $R > 0$. On the other hand, since $S(0, 1) = 0$ application of Taylor's expansion yields $S(\xi, \lambda) = S_\xi(0, 1)\xi + S_\lambda(0, 1)\lambda + \ldots$ We thus obtain that, if $|\xi| \leqq R$ and λ is close enough to one,

$$|S(\xi, \lambda)| = o\left(\frac{1}{|\ln(1-s)|}\right) \quad \text{as } s \uparrow 1$$

Recalling (6.4) and (6.5), we get

$$S(t)\,v_s(x,0)=(p-1)^{-(1/(p-1))}-\left[\frac{c}{|\ln(1-s)|}+o\left(\frac{1}{|\ln(1-s)|}\right)\right]$$
$$\times(1-t)\,H_2(\xi\,|\ln(1-s)(1-t))|^{1/2})+o\left(\frac{1}{|\ln(1-s)|}\right)$$
$$=(p-1)^{-(1/(p-1))}-(c(1-t)+o(1-t))(1-t)\,H_2\left(\frac{\xi}{\sqrt{1-t}}\right)$$
$$+o(1-t)\quad\text{as } t\uparrow 1.$$

Since $H_2\left(\dfrac{\xi}{\sqrt{1-t}}\right)=\dfrac{c_2\xi^2}{1-t}-2c_2$, where c_2 is given in (1.15), we finally obtain

$$S(t)\,v_s(x,0)=(p-1)^{-(1/(p-1))}-cc_2(1-t)\xi^2+o(1-t)\quad\text{as } t\uparrow 1$$

uniformly on sets $|\xi|\leqq R$, $R>0$. Using (6.3), we obtain

$$v_s(x,t)\geqq[((p-1)^{-(1/(p-1))}-cc_2(1-t)\xi^2$$
$$+o(1-t))^{-(p-1)}-(p-1)\,t]^{-(1/(p-1))}$$
$$=[(p-1)(1+(p-1)^{p/(p-1)}cc_2(1-t)\xi^2+o(1-t))-(p-1)\,t]^{-(1/(p-1))}$$
$$=[(p-1)(1+(p-1)^{p/(p-1)}cc_2\xi^2+o(1))]^{-(1/(p-1))}(1-t)^{-(1/(p-1))}$$

Since $1-\tilde{t}=(1-t)(1-s)$, (6.1) follows (with t replaced by $\tilde{t}$). This concludes the proof ■

We now turn to the task of obtaining an upper bound complementing (6.1). To this end, several technical steps are to be taken. Let φ be the auxiliary function introduced in (1.8). We begin by showing

LEMMA 6.2. – *Assume that u_0 satisfies the hypotheses in Theorem* 1. *Then for any* $R>0$, *there exists* $C=C(p,R)$ *such that*

$$|\varphi_y(\xi\sqrt{\tau},\tau)|\leqq\frac{C}{\sqrt{\tau}},\tag{6.6}$$

uniformly for $|\xi|\leqq R$ *and large enough* $\tau>0$.

Proof. – We differentiate with respect to y in the equation satisfied by φ [*see* (1.8)], multiply both sides by $(\operatorname{sgn}\varphi_y)$ and use Kato's inequality to get the following inequality for $z=|\varphi_y|$

$$z_\tau\leqq z_{yy}-\frac{y}{2}z_y+\left(\frac{1}{2}-\frac{p}{p-1}\right)z+p\,\varphi^{p-1}z\tag{6.7}$$

By assumption, the solution of (1.1), (1.2) is symmetric, and the same happens for φ. By Proposition 5.9, we then have

$$\varphi(y,\tau)^{p-1}\leqq\varphi(0,\tau)^{p-1}\leqq\frac{1}{p-1}+\frac{C}{\tau},$$

for some $C>0$ (the same symbol will denote a generic constant henceforth), and for large enough $\tau>0$. We now use variation of constants formula in (6.7) starting from $\tau=\tau_0$, to get

$$z(y, \tau) \leqq \frac{e^{((1/2)+(C/\tau_0))(\tau-\tau_0))}}{(4\pi(1-e^{-(\tau-\tau_0)}))^{1/2}} \times \int_{\mathbb{R}} \exp\left(-\frac{(ye^{-((\tau-\tau_0)/(2))}-\lambda)^2}{4(1-e^{-(\tau-\tau_0)})}\right) z(\lambda, \tau_0)\, d\lambda$$

$$\leqq \frac{e^{((1/2)+(C/\tau_0))(\tau-\tau_0))}}{(4\pi(1-e^{-(\tau-\tau_0)}))^{1/2}} \| z(., \tau_0) \| \times \left[\int_{\mathbb{R}} \exp\left(-\frac{(ye^{-((\tau-\tau_0)/2)}-\lambda)^2}{2(1-e^{-(\tau-\tau_0)})}+\frac{\lambda^2}{4}\right) d\lambda\right]^{1/2} \tag{6.8}$$

For any given $\tau_0>0$, we take τ such that

$$e^{\tau-\tau_0}=\tau \tag{6.9}$$

Notice that then $\tau=\tau_0+\ln\tau=\tau_0+\ln\tau_0+\ldots$ as $\tau_0\to\infty$, and there exist constants C_1, C_2 such that $C_1\tau_0\leqq\tau\leqq C_2\tau_0$. We now proceed to estimate z along lines $y=\xi\sqrt{\tau}$ with ξ bounded. Using (6.8) and the fact that $\|z\|=o\left(\frac{1}{\tau}\right)$ as $\tau\to\infty$ [*cf.* (5.40)], we obtain

$$z(\xi\sqrt{\tau}, \tau) \leqq \frac{C}{\tau} e^{((1/2)+(C/\tau_0))(\tau-\tau_0)} \left[\int_{\mathbb{R}} \exp\left(\frac{\lambda^2}{4}-\frac{(\xi-\lambda)^2}{2(1-(1/\tau))}\right) d\lambda\right]^{1/2} \leqq \frac{C\tau^{1/2+(C/\tau_0)}}{\tau_0} \leqq C\tau_0^{-1/2}.\tau_0^{C/\tau_0}$$

and since $\lim_{s\to\infty} s^{C/s}=1$, the result follows. ■

To proceed further, we set

$$W=\varphi^{-(p-1)} \tag{6.10}$$

so that W solves

$$W_\tau=W_{yy}-\frac{y}{2}W_y+W-(p-1)-\frac{p}{p-1}\cdot\frac{W_y^2}{W} \tag{6.11}$$

We next show

LEMMA 6.3. — *There exists* $C>0$ *such that*

$$\| W(., \tau)-(p-1) \| \leqq \frac{C}{\tau} \quad as\ \tau\to\infty. \tag{6.12}$$

Proof. – Let δ be a positive number to be selected presently. Then

$$\|W(.\tau)-(p-1)\|^2=\int_{|y|\leq\delta\sqrt{\tau}}|\varphi^{-(p-1)}-(p-1)|^2 e^{-y^2/4}\,dy + \int_{|y|\geq\delta\sqrt{\tau}}|\varphi^{-(p-1)}-(p-1)|^2 e^{-y^2/4}\,dy$$

On the other hand, estimate (6.1) can be written as follows

$$\varphi(y,\tau)\geqq[(p-1)(1+k\delta^2+o(e^{-\tau}))]^{-(1/(p-1))}\quad \text{as } \tau\to\infty,$$

for some constant $k=k(p)>0$, provided that $|y|\leqq\delta\sqrt{\tau}$. In particular, $\varphi(y,\tau)\geqq\alpha>0$ for large τ and y as before, and there exists $C=C(\alpha)>0$ such that $|\varphi^{-(p-1)}-(p-1)|\leqq C|\varphi-(p-1)^{-(1/(p-1))}|$, whence, recalling (5.40)

$$\int_{|y|\leqq\delta\sqrt{\tau}}|\varphi^{-(p-1)}-(p-1)|^2 e^{-y^2/4}\,dy\leqq C\|\varphi-(p-1)^{-(1/(p-1))}\|^2 \leqq\frac{C^2}{\tau^2}\quad \text{as } \tau\to\infty \qquad (6.13\,a)$$

Estimating this integral when $|y|>\delta\sqrt{\tau}$ requires of a different approach. To begin with, we notice that the solution $u(x,t)$ of (1.1) is a supercaloric function. Therefore, for $t\in(t_0,1)$ and large enough x, there exist constants C and θ such that

$$u(x,t)\geqq C e^{-\theta x^2}$$

(*cf.* for instance [Wa]); In terms of the variable φ, this reads $\varphi(x,t)\geqq C(1-t)^{1/(p-1)}e^{-\theta x^2}$. Recalling that $y=x(1-t)^{-1/2}$, $\tau=-\log(1-t)$, we obtain

$$\varphi(y,\tau)^{-(p-1)}\leqq C e^{\tau}e^{\theta(p-1)y^2e^{-\tau}}$$

In particular, if τ is large enough

$$\int_{|y|\geqq\delta\sqrt{\tau}}\varphi^{-2(p-1)}e^{-y^2/4}\,dy\leqq C e^{2\tau}\int_{|y|\geqq\delta\sqrt{\tau}}e^{-y^2/8}\,dy \simeq\frac{C e^{(2-(\delta^2/8))\tau}}{\delta\sqrt{\tau}}\,\text{as } \tau\to\infty \qquad (6.13\,b)$$

Taking now $\delta>4$, the result follows from (6.13) ■

Let us write now

$$G=W-(p-1)$$

Taking into account (6.11), we now obtain

$$G_\tau=G_{yy}-\frac{y}{2}G_y+G-\frac{p}{p-1}\cdot\frac{(G_y)^2}{(p-1)+G} \qquad (6.14)$$

Set

$$L(y, \tau) = \frac{p}{p-1} \cdot \frac{(G_y)^2}{(p-1)+G} \tag{6.15}$$

Then for $\tau \geqq_0 > 0$, $G(y, \tau)$ can be written in the form

$$\begin{aligned} G(y, \tau) = & \frac{e^{\tau-\tau_0}}{(4\pi(1-e^{-(\tau-\tau_0)}))^{1/2}} \\ & \times \int_{\mathbb{R}} \exp\left(-\frac{(ye^{-((\tau-\tau_0)/2)}-\lambda)^2}{4(1-e^{-(\tau-\tau_0)})}\right) G(\lambda, \tau_0)\, d\lambda \\ & - \int_{\tau_0}^{\tau} \frac{e^{(\tau-s)}\, ds}{(4\pi(1-e^{-(\tau-s)}))^{1/2}} \int_{\mathbb{R}} \exp\left(-\frac{(ye^{-((\tau-s)/2)}-\lambda)^2}{4(1-e^{-(\tau-s)})}\right) L(\lambda, s)\, d\lambda \\ & \equiv J_1(y, \tau) + J_2(y, \tau) \end{aligned} \tag{6.16}$$

We next extimate J_1, J_2 as $\tau \to \infty$

LEMMA 6.4. – *There holds*

$$\lim_{\tau \to \infty} J_1(\xi\sqrt{\tau}, \tau) = \frac{(p-1)^2}{4p} \xi^2 \tag{6.17}$$

uniformly on sets $|\xi| \leqq C$ *with* $C > 0$.

Proof. – We fix $R > 0$ and split the integral in J_1 into two, $J_{1,1}^R$ and $J_{1,2}^R$, to be performed in the regions $|\lambda| \leqq R$ and $|\lambda| > R$ respectively. Clearly

$$\begin{aligned} |J_{1,2}^R(y, \tau)| \leqq & \frac{e^{\tau-\tau_0} \| G(., \tau_0) \|}{(4\pi(1-e^{-(\tau-\tau_0)}))^{1/2}} \\ & \times \left[\int_{|\lambda| \geqq R} \exp\left(-\frac{ye^{-((\tau-\tau_0)/2)}-\lambda)^2}{2(1-e^{-(\tau-\tau_0)})} + \frac{\lambda^2}{4}\right) d\lambda \right]^{1/2} \end{aligned}$$

Settint $y = \xi\sqrt{\tau}$, relating τ and τ_0 as in (6.9), and using (6.12) we get

$$|J_{1,2}^R(y, \tau)| \leqq \frac{C\tau}{(4\pi(1-(1/\tau)))^{1/2}\tau_0} \left[\int_{|\lambda| \geqq R} \exp\left(\frac{\lambda^2}{4} - \frac{(\xi-\lambda)^2}{2(1-(1/\tau))}\right) d\lambda \right]^{1/2}$$

and the integral above converges to zero as $R \to \infty$, uniformly on sets $|\xi| \leqq K$. Recalling that $\tau \simeq \tau_0$ as $\tau_0 \to \infty$, we have thus shown that

$$|J_{1,2}^R(y, \tau)| \leqq B(R), \text{ uniformly on } \tau, \quad \text{where } \lim_{R \to \infty} B(R) = 0 \tag{6.18}$$

Consider now $J_{1,1}^R$. By Proposition 5.9, we have that, for any $R > 0$

$$G(y, \tau) = b.\frac{H_2(y)}{\tau} + g_R(y; \tau) \quad \text{for } \tau > 0 \text{ large enough}$$

where

$$b=\frac{(4\pi)^{1/4}(p-1)^2}{\sqrt{2}p}, \qquad \text{and} \qquad g_R(y;\tau)=o\left(\frac{1}{\tau}\right) \text{ as } \tau\to\infty, \tag{6.19}$$

uniformly on sets $|y|\leqq R$. Setting $y=\xi\sqrt{\tau}$ and relating τ and τ_0 as in (6.9), we then have

$$J^R_{1,1}(y,\tau)=\frac{b\tau}{(4\pi(1-(1/\tau)))^{1/2}\tau_0}\int_{|\lambda|\leqq R} H_2(\lambda)\exp\left(-\frac{(\xi-\lambda)^2}{4(1-(1/\tau))}\right)d\lambda$$
$$+\frac{\tau}{(4\pi(1-(1/\tau)))^{1/2}}\int_{|\lambda|\leqq R} g_R(\lambda,\tau)\exp\left(-\frac{(\xi-\lambda)^2}{4(1-(1/\tau))}\right)d\lambda\equiv Z_1+Z_2$$

and we readily see that

$$\text{For fixed } R>0,\ \lim_{\tau\to\infty} Z_2=0 \tag{6.20}$$

On the other hand, $\lim_{\tau\to\infty} Z_1=\frac{b}{(4\pi)^{1/2}}\int_{|\lambda|\leqq R} H_2(\lambda)\exp\left(-\frac{(\xi-\lambda)^2}{4}\right)d\lambda.$

We let first $\tau\to\infty$ and then $R\to\infty$. Since

$$\exp\left(\frac{2\xi\lambda-\xi^2}{4}\right)=\sum_{n=0}^{\infty}\frac{H_n(\lambda)\xi^n}{2^n n!\,c_n};$$
$$c_n=[2^{n/2}(4\pi)^{1/4}(n!)^{1/2}]^{-1},$$

we finally obtain (6.17) ■

Our last result in this Section is

LEMMA 6.5. – *Let* $J_2(y,\tau)$ *be given in* (6.16). *Then*

$$\lim_{\tau\to\infty} J_2(\xi\sqrt{\tau},\tau)=0 \tag{6.21}$$

uniformly on sets $|\xi|\leqq C$ *with* $C>0$.

Proof. – To get (6.21), we shall split J_2 into several terms. Namely, we take $A>0$ such that $\tau_0+A<\tau$, $R>0$ arbitrary and $\delta>0$ to be selected later. We then write

$$J_2(y,\tau)=\int_{\tau_0}^{\tau_0+A}\frac{e^{(\tau-s)}}{(4\pi(1-e^{-(\tau-s)}))^{1/2}}$$
$$\times\left[\int_{R<|\lambda|<\delta\sqrt{\tau_0}}\exp\left(-\frac{(ye^{-((\tau-s)/2)}-\lambda)^2}{4(1-e^{-(\tau-s)})}\right)L(\lambda,s)\,d\lambda\right]ds$$
$$+\int_{\tau_0}^{\tau_0+A}\frac{e^{(\tau-s)}}{(4\pi(1-e^{-(\tau-s)}))^{1/2}}\left[\int_{|\lambda|\leqq R}\exp\left(-\frac{(ye^{-((\tau-s)/2)}-\lambda)^2}{4(1-e^{-(\tau-s)})}\right)L(\lambda,s)\,d\lambda\right]ds$$
$$+\int_{\tau_0+A}^{\tau}\frac{e^{(\tau-s)}}{(4\pi(1-e^{-(\tau-s)}))^{1/2}}\left[\int_{|\lambda|\leqq\delta\sqrt{\tau_0}}\exp\left(-\frac{(ye^{-((\tau-s)/2)}-\lambda)^2}{4(1-e^{-(\tau-s)})}\right)L(\lambda,s)\,d\lambda\right]ds$$

$$+\int_{\tau_0}^{\tau}\frac{e^{(\tau-s)}}{(4\pi(1-e^{-(\tau-s)}))^{1/2}}\left[\int_{|\lambda|\geqq\delta\sqrt{\tau_0}}\exp\left(-\frac{(ye^{-((\tau-s)/2)}-\lambda)^2}{4(1-e^{-(\tau-s)})}\right)L(\lambda,s)\,d\lambda\right]ds$$
$$\equiv J_{2,1}+J_{2,2}+J_{2,3}+J_{2,4}. \quad (6.22)$$

We now estimate $J_{2,1},\ldots,J_{2,4}$ as follows. By (6.6) and the lower bound in (6.1)

$$|J_{2,3}|\leqq C\int_{\tau_0+A}^{\tau}\frac{e^{(\tau-s)}\,ds}{(4\pi(1-e^{-(\tau-s)}))^{1/2}s}\left[\int_{|\lambda|<\delta\sqrt{\tau_0}}\exp\left(-\frac{(ye^{-((\tau-s)/2)}-\lambda)^2}{4(1-e^{-(\tau-s)})}\right)d\lambda\right]$$
$$\leqq\frac{C}{\tau_0}\int_{\tau_0+A}^{\tau}e^{\tau-s}\,ds\leqq Ce^{-A}. \quad (6.23)$$

As to $J_{2,1}$, we notice that $ye^{-((\tau-s)/2)}=\xi e^{(s-\tau_0)/2}$, so that if $s\leqq\tau_0+A$ and τ_0 is large enough, we have $|ye^{-((\tau-s)/2)}|\leqq|\xi|e^{A/2}$. Since $\tau-s\geqq b>0$ for some b, whenever $s\leqq\tau_0+A$, we then have

$$|J_{2,1}|\leqq\frac{Ce^{\tau-\tau_0}}{\tau_0}\int_{\tau_0}^{\tau_0+A}\left[\int_{|\lambda|\geqq R}\exp\left(-\frac{(ye^{-((\tau-s)/2)}-\lambda)^2}{k}\right)d\lambda\right]ds,$$

where $k=k(b)>0$. Using the inequality $-\frac{1}{k}(\theta-\lambda)^2\leqq\frac{\theta^2}{k}-\frac{\lambda^2}{2k}$, we obtain

$$|J_{2,1}|\leqq CA\,e^{\delta^2A/k}\int_{|\lambda|\geqq R}\exp\left(-\frac{\lambda^2}{2k}\right)d\lambda. \quad (6.24)$$

Consider now $J_{2,2}$. In the region $|y|\leqq R$, we can use. Proposition 5.9 to get $|L(y,\tau)|\leqq\frac{C}{\tau^2}$ as $\tau\to\infty$. Therefore

$$\|J_{2,2}\|\leqq\frac{C}{\tau_0^2}\int_{\tau_0}^{\tau_0+A}e^{(\tau-s)}\,ds\leqq\frac{C}{\tau_0} \quad (6.25)$$

for some constant C depending on A. It only remains to estimate $J_{2,4}$. The main problem now is to derive a suitable bound for $L(y,\tau)$ given in (6.15) in the region where $|y|\geqq\delta\sqrt{\tau_0}$. This is done as follows. By definition

$$L(y,\tau)=\frac{p}{p-1}(u(x,t))^{-(1+p)}(u_x(x,t))^2$$

where, as usual, $y=\frac{x}{\sqrt{1-t}}$, $\tau=-\ln(1-t)$. As remarked in the proof of Lemma 6.3, $u(x,t)$ is supercaloric and therefore for any $\delta\in(0,1)$,

$$u(x,t)\geqq Ce^{-\theta x^2}\quad\text{whenever } x\in\mathbb{R}\text{ and } t\in(\delta,1), \quad (6.26)$$

for some constants C, θ depending on δ. On the other hand, since $u(x,t)\leqq C(1-t)^{-(1/(p-1))}$ for any $x\in\mathbb{R}$, $t<1$ and some $C>0$ (*cf.*

Theorem 4.2 in [GP]), we may use Proposition 1 in [GK1] to get

$$|u_x(x, t)| \leqq C(1-t)^{-(1/(p-1))-(1/2)} \quad \text{for any } x \in \mathbb{R} \text{ and } t<1 \quad (6.27)$$

Putting together (6.26) and (6.27), we obtain

$$h(y, \tau) \leqq C \exp(\theta y^2 e^{-\tau}) \exp\left(\left(1+\frac{2}{p-1}\right)\tau\right) \quad (6.28)$$

We shall use this estimate to bound $J_{2,4}$. Notice that, when $s \in (\tau_0, \tau)$ with τ_0, τ related to (6.9) and τ_0 is large enough, it follows that $s \leqq 2\tau_0$, so that $\exp\left(\left(1+\frac{2}{p-1}\right)s\right) \leqq \exp\left(\left(2+\frac{4}{p-1}\right)\tau_0\right)$, and $e^{\tau-s} \leqq e^{\tau-\tau_0} = \tau \leqq 2\tau_0$. We then have

$$|J_{2,4}(y, \tau)| \leqq C \tau_0 e^{(2+(4/(p-1)))\tau_0} \cdot \int_{\tau_0}^{\tau} \frac{ds}{(4\pi(1-e^{-(\tau-s)}))^{1/2}}$$
$$\times \left[\int_{|\lambda| \geqq \delta\sqrt{\tau_0}} \exp\left(-\frac{(ye^{-((\tau-s)/2)}-\lambda)^2}{4(1-e^{-(\tau-s)})} + \theta\lambda^2 e^{-\tau_0}\right) d\lambda\right] \quad (6.29)$$

Assume now that $|\xi| \leqq K$, $K>0$, and take $\delta \geqq 2K$. We then claim that

$$\theta\lambda_e^{2-\tau_0} - \frac{(\lambda - ye^{-((\tau-s)/2)})^2}{4(1-e^{-(\tau-s)})} \leqq -\frac{(\lambda - ye^{-((\tau-s)/2)})^2}{8(1-e^{-(\tau-s)})} \quad \text{if} \quad |\lambda| \geqq \delta\sqrt{\tau_0} \quad (6.30)$$

Let us assume for the moment that (6.30) holds and continue. We use this inequality to bound the argument of the exponential in the second integral in (6.29), and then make the change of variables $z=(1-e^{-(\tau-s)})^{-1/2}(\lambda - e^{-((\tau-s)/2)}y)$, to get

$$|J_{2,4}(y, \tau)| \leqq C \tau_0 e^{(2+(4/(p-1))\tau_0} \int_{\tau_0}^{\tau} \left(\int_{\Sigma} e^{-(z^2/8)} dz\right) ds$$

where

$$\Sigma = \left\{ z \in \mathbb{R} : |z + e^{-((\tau-s)/2)}(1-e^{-(\tau-s)})^{-1/2} y| \geqq \delta(1-e^{-(\tau-s)})^{-1/2}\sqrt{\tau_0} \right\}.$$

Since $|ye^{-((\tau-s)/2)}| \leqq |\xi|\sqrt{\tau} \leqq \left(1+\frac{1}{10}\right)|\xi|\sqrt{\tau_0}$ for large τ, we readily see that $\Sigma \subset \left\{ z \in \mathbb{R} : |z| \geqq \frac{\delta}{3}\sqrt{\tau_0} \right\}$ provided that δ is large enough. We then use the fact that $\int_x^{\infty} e^{-s^2/8} ds \simeq 4xe^{-x^2/8}$ as $x \to \infty$, to arrive at

$$|J_{2,4}(y, \tau)| \leqq C \tau_0 e^{(2+(4/(p-1)))\tau_0} \cdot e^{-\alpha\delta^2\tau_0}$$

for some constant $\alpha>0$. Redefining δ if necessary, we may take it large enough so that

$$|J_{2,4}(y,\tau)|\to 0 \quad \text{as } \tau_0\to\infty. \tag{6.31}$$

uniformly for $|\xi|\leqq K$. Taking into account (6.22)-(6.25) and (6.31), the conclusion follows by taking $\tau_0\to\infty$, $R\to\infty$ and $A\to\infty$, in this order.

We conclude by showing (6.30). As $|ye^{-((\tau-s)/2)}|\leqq|\xi|\sqrt{\tau_0}\leqq K\sqrt{\tau_0}$ and $|\lambda|\geqq 2K\sqrt{\tau_0}$, it suffices to obtain that

$$|\lambda|-|y|e^{-((\tau-s)/2)}\geqq\sqrt{8\theta}\,e^{-(\tau_0/2)}|\lambda|$$

i.e.

$$|\lambda|(1-\sqrt{8\theta}\,e^{-(\tau_0/2)})\geqq k\sqrt{\tau_0}$$

which is indeed satisfied if τ_0 is large enough ■

7. THE EXPONENTIAL CASE

In this Section we shall sketch those variants of the previous arguments which are required to prove Theorems 2 and 4. We normalize again the blow-up time by setting $T=1$, and define ζ as follows

$$\left.\begin{array}{l} u(x,t)=-\log(1-t)+\zeta(y,\tau) \qquad \text{where } y=\dfrac{x}{\sqrt{1-t}}, \\ \tau=-\log(1-t) \end{array}\right\} \tag{7.1}$$

so that ζ satisfies

$$\zeta_\tau=\zeta_{yy}-\frac{y}{2}\zeta_y+e^\zeta-1\equiv A\zeta+f(\zeta), \tag{7.2}$$

where A is the operator defined in (1.14), and

$$f(\zeta)=e^\zeta-\zeta-1=\frac{\zeta^2}{2}+\ldots, \quad \text{as } \zeta\to 0 \tag{7.3}$$

By (1.5), ζ is bounded above. Then, by (7.3), there exists $C>0$ such that $|f(\zeta)|\leqq C\min\{|\zeta|^2, |\zeta|\}$. Since $\zeta(y,\tau)\to 0$ as $\tau\to\infty$, uniformly on compact sets of y [*cf.* (1.4 *b*)], and $|\zeta_y(.,\tau)|$ is bounded [by (1.5 *b*)], we get $|\zeta(y)|\leqq C(1+|y|)$ for some $C>0$. Therefore $\|\zeta(.,\tau)\|\to 0$ as $\tau\to\infty$. We now repeat our previous approach with ψ replaced by ζ, $\psi(y,\tau)$ being the function give in (1.10). In particular, we set

$$\zeta(y,\tau)=\sum_{k=0}^{\infty}c_k(\tau)H_k(y)$$

and note that results in Section 2 and 3 remain true when we substitute there ψ by ζ. A minor modification of the arguments in Section 5 [in particular, a change in the constant in the first term in the right in (5.25)] eventually yields that, if $\zeta \not\equiv 0$.

$$\zeta(y, \tau) = -\frac{(4\pi)^{1/4}}{\sqrt{2}} \cdot \frac{H_2(y)}{\tau} + o\left(\frac{1}{\tau}\right) \quad \text{as } \tau \to \infty, \qquad (7.4a)$$

or

$$\zeta(y, \tau) = C e^{(1-(m/2))\tau} H_m(y) + o(e^{(1-(m/2))\tau}) \quad \text{as } \tau \to \infty \qquad (7.4b)$$

in $H^1_w(\mathbb{R})$ and also in $C^{k,\alpha}_{loc}(\mathbb{R})$ for some $\alpha \in (0, 1)$ and any $k \geqq 1$ and Theorem 4 follows. A few details are then to be changed in Section 6, more precisely in Lemma 6.1 there. To begin with, we replace the family of auxiliary functions given in (6.2) by

$$v_s(x, t) = \log(1-s) + u(x\sqrt{1-s}, s + t(1-s)); \qquad 0 < s < 1$$

Clearly, for any such fixed s, v_s solves (1.1) with $F(u) = e^u$, and moreover,

$$v_s(x, 0) = \varphi(x, -\log(1-s)) = -\frac{(4\pi)^{1/4}}{\sqrt{2}} \cdot \frac{H_2(x)}{|\log(1-s)|} + o\left(\frac{1}{|\log(1-s)|}\right) \quad \text{as } s \uparrow 1.$$

Instead of $z_s(x, t)$ in (6.3), we now take

$$z_s(x, t) = -\log(\exp(-S(t) v_s(x, 0)) - t)$$

and it is readily seen that $z_{s,t} - z_{s,xx} - e^{z_s} \leqq 0$.

Repeating the steps in Lemma 6.1, we obtain, for $\xi = \frac{y}{\sqrt{\tau}}$,

$$S(t) v_s(x, 0) = -\frac{\xi^2}{4} + o(1-t) \quad \text{as } t \uparrow 1$$

which eventually leads to

$$v_s(x, t) \geqq z_s(x, t) \geqq -\log(1-t) - \log\left(1 + \frac{\xi^2}{4}\right) + \ldots \text{as } t \uparrow 1$$

whence the lower bound

$$\log(1-t) + u(\xi((1-t)|\log(1-t)|)^{1/2}, t) \geqq -\log\left(1 + \frac{\xi^2}{4}\right) \qquad (7.5)$$

uniformly for $|\xi| \leqq R$ with $R > 0$. Introducing now a new variable

$$\zeta = -\ln W \qquad (7.6)$$

leads to the equation

$$W_\tau = AW - \frac{W_y^2}{W} - 1$$

Note that now $W \to 1$ as $\zeta \to 0$. Using (7.5) we obtain, instead of (6.12),

$$\| W(., \tau) - 1 \| \leqq \frac{C}{\tau} \quad \text{as } \tau \to \infty$$

We finally set

$$G = W - 1, \tag{7.7}$$

replace (6.14) by

$$G_\tau = G_{yy} - \frac{y}{2} G_y + G - \frac{G_y^2}{1+G} \equiv AG + L(y, \tau) \equiv J_1'(y, \tau) + J_2'(y, \tau),$$

and repeat the arguments in Lemmata 6.4 and 6.5 to get

$$\lim_{\tau \to \infty} G(\xi \sqrt{\tau}, \tau) = \frac{\xi^2}{4} \tag{7.8}$$

uniformly on sets $|\xi| \leqq C$ with $C>0$. In view of (7.6)-(7.8), we finally obtain equality of both sides in (7.5), and the proof of Theorem 2 is concluded ■

APPENDIX A

During the proof of Theorem 1, we have used some *a priori* estimates on solutions of linear evolution problems associated to operator A given in (1.14). Such results are rather classical, but we shall state them here for the readers convenience. Let m be a fixed nonnegative integer, and consider the linear operator in $L_w^2(\mathbb{R})$ given by

$$A_m \varphi = \varphi'' - \frac{y}{2} \varphi' + m\varphi; \qquad D(A_m) = H_w^2(\mathbb{R})$$

Then A_m is self-adjoint in $L_w^2(\mathbb{R})$, and has eigenvalues $\lambda_n = m - \frac{n}{2}$; $m=0$, 1, 2, ... with corresponding eigenfunctions $H_n(y)$ given in (1.15).

Consider now the initial-value problem

$$\Phi_\tau = A_m \Phi + g(., \tau); \qquad y \in \mathbb{R}, \quad \tau > 0 \tag{A 1}$$

$$\Phi(y, 0) = \Phi_0(y); \qquad y \in \mathbb{R} \tag{A 2}$$

where $\Phi_0(y) \in L^2_w(\mathbb{R})$ and $g(y, \tau) \equiv g(\tau, y) \in L^2_{loc}((0, \infty); L^2_w(\mathbb{R}))$: Existence of solutions of (A 1), (A 2) follows from standard semigroup theory. Moreover, we have the following estimates

PROPOSITION A 1. – *There exists a positive constant* C, *such that for any* $T > 0$, *there holds*

$$\|\Phi(.,\tau)\|^2 + t\|\Phi_y(.,\tau)\|^2 + \int_0^T t^2 \|\Phi_{yy}(.,\tau)\|^2 d\tau + \int_0^T t^2 \|\Phi_\tau(.,\tau)\|^2 d\tau$$
$$\leq C(T + T^2 + e^{2mT})\left(\|\Phi_0\|^2 + \int_0^T \|g(.,\tau)\|^2 d\tau\right) \quad (A\,3)$$

Proof. – Set $\Phi(y, \tau) = \sum_{k=0}^{\infty} a_k(\tau) H_k(y)$, $g(y, \tau) = \sum_{k=0}^{\infty} g_k(\tau) H_k(y)$.

Then

$$A_m \Phi(y, \tau) = \sum_{k=0}^{\infty} \left(m - \frac{k}{2}\right) a_k(\tau) H_k(y)$$

and

$$S_{A_m}(\tau)\Phi_0 = \sum_{k=0}^{\infty} a_k(0) e^{(m-(k/2))\tau} H_k(y).$$

We then have

$$\Phi(.,\tau) = S_{A_m}(\tau)\Phi_0 + \int_0^\tau S_{A_m}(\tau - s)\, g(.,s)\, ds = S_{A_m}(\tau)\Phi_0$$
$$+ \sum_{k=0}^{\infty}\left[\int_0^\tau e^{(m-(k/2))(\tau-s)} g_k(s)\, ds\right] H_k(y) \equiv Y_1(.,\tau) + Y_2(.,\tau)$$

From now on, we shall denote by C a generic constant depending only on m which may change from line to line. Clearly

$$\|Y_1(.,\tau)\|^2 = \sum_{k=0}^{\infty} a_k(0)^2 e^{2(m-(k/2))\tau} \leq e^{2m\tau}\|\Phi_0\|^2,$$

$$\|Y_2(.,\tau)\|^2 = \sum_{k=0}^{\infty}\left[\int_0^\tau e^{(m-(k/2))(\tau-s)} g_k(s)\, ds\right]^2$$
$$\leq \sum_{k=0}^{\infty}\left[\int_0^\tau e^{2(m-(k/2))(\tau-s)} ds\right].$$

$$\sum_{k=0}^{\infty}\left[\int_0^\tau e^{2(m-(k/2))(\tau-s)} ds\right]\left[\int_0^\tau g_k(s)^2 ds\right] \leq \sum_{k \neq 2m}\left[\frac{1 - e^{(2m-k)\tau}}{k - m}\right]\int_0^\tau g_k(s)^2 ds$$
$$+ \tau \int_0^\tau g_{2m}(s)^2 ds \leq (C e^{2m\tau} + \tau)\int_0^\tau \|g(.,s)\|^2 ds$$

Noting that $ye^{-y} \leqq e^{-1}$, we also have

$$\left\| \frac{\partial Y_1}{\partial y}(.,\tau) \right\|^2 = \frac{1}{2} \sum_{k=0}^{\infty} \frac{a_k(0)^2}{\tau} k \tau e^{(2m-k)\tau}$$

$$\leqq \frac{m}{\tau} \left[1 + e^{2m\tau} \sum_{k \neq 2m}^{N} \frac{k\tau^{-1}}{k-2m} a_k(0)^2 \right]$$

$$\leqq \frac{C}{\tau} (1 + e^{2m\tau} \| \Phi_0 \|^2) \left\| \frac{\partial Y_2}{\partial y}(.,\tau) \right\|^2$$

$$\leqq \sum_{k=0}^{\infty} \frac{k}{2} \left[\int_0^\tau e^{(2m-\tau)(k-s)} ds \right] \left[\int_0^\tau g_k(s)^2 ds \right]$$

$$\leqq \frac{C}{\tau} (e^{2m\tau} + \tau) \| \Phi_0 \|^2$$

In particular, we obtain

$$\| \Phi_y(.,\tau) \|^2 \leqq C \sum_{k=1}^{\infty} k a_k(\tau)^2 \leqq \frac{C}{\tau} (\tau + e^{2m\tau}) \| \Phi_0 \|^2 \tag{A 4}$$

We have thus shown the corresponding bounds for $\Phi(.,\tau)$ and $\Phi_y(.,\tau)$. To estimate the remaining derivatives, it is more convenient to start from the ODE's satisfied for the Fourier coefficients a_k

$$\dot{a}_k = \left(m - \frac{k}{2} \right) a_k + g_k \tag{A 5}$$

We multiply in (A 5) by $t^2 \dot{a}_k$, add all the terms from $k=0$ to ∞ and integrate over (O, T) to get

$$\int_0^T \tau^2 \sum_{k=0}^{\infty} \dot{a}_k^2 d\tau \leqq \frac{T^2}{2} \sum_{k=0}^{\infty} \left(m - \frac{k}{2} \right) a_k(T)^2$$

$$- \int_0^T \tau \sum_{k=0}^{\infty} \left(m - \frac{k}{2} \right) a_k(\tau)^2 d\tau + \int_0^T \tau^2 \sum_{k=0}^{\infty} \dot{a}_k(\tau) g_k(\tau) d\tau$$

Using Cauchy-Schwartz inequality in the last term above, we obtain

$$\int_0^T \tau^2 \sum_{k=0}^{\infty} \dot{a}_k^2 d\tau \leqq T^2 m \| \Phi(T) \|^2$$

$$+ C \int_0^T \tau \sum_{k=0}^{\infty} k a_k(\tau)^2 d\tau + CT^2 \int_0^T \| g(.,s) \|^2 ds$$

Taking into account (A 4), the bound for $\Phi_\tau(.,\tau)$ is obtained, and the corresponding estimate for $\Phi_{yy}(.,\tau)$ follows now at once from (A 1) ■

We finally show

PROPOSITION A 2. — *Assume that* Φ *solves* (1.11) *and* $|\Phi|$ *is bounded. Then, for some* $\alpha \in (0, 1)$

$$A\Phi, \qquad \frac{d\Phi}{d\tau} \in C^{\alpha}_{\text{loc}}((0, \infty); L^2_w(\mathbb{R}))$$

Proof. — By hypothesis

$$\frac{d\Phi}{d\tau} = A\Phi + f(\Phi) \text{ with } f(\Phi) \text{ given in } (1.11\,b) \tag{A 6}$$

Using the boundedness of $\Phi(., \tau)$ we have

$$\| f(\Phi(., \tau)) - f(\Phi(., s)) \| \leqq C \| \Phi(., \tau) - \Phi(., s) \| = C \left\| \int_s^T \frac{d\Phi}{d\tau}(., \lambda)\, d\lambda \right\| \leqq \tilde{C}(\tau - s)^{1/2}$$

where we have used (A 3). Then

$$\Phi(., \tau) = S_A(\tau)\Phi(., \tau_0) + \int_{\tau_0}^{\tau} S_A(\tau - s) f(\Phi(., s))\, ds \equiv G_1(., \tau) + G_2(., \tau).$$

Arguing as in Proposition A 1 we easily get $\frac{dG_1}{d\tau} \in C^{\alpha}_{\text{loc}}((\tau_0, \infty); L^2_w(\mathbb{R}))$ and since $f(\Phi) \in C^{1/2}_{\text{loc}}([\tau_0, \infty); L^2_w(\mathbb{R}))$, an argument similar to the used in [H], Lemma 3.5.1 yields $\frac{dG_2}{d\tau} \in C^{\alpha}_{\text{loc}}((\tau_0, \infty); L^2_w(\mathbb{R}))$ for some $\alpha > 0$. This gives the result for $\frac{d\Phi}{d\tau}$, and the assertion for $A\Phi$ follows from (A 6).

APPENDIX B

We compute here the terms $A_{n, m, l}$ defined in (5.26). To this end we first consider the integrals

$$I_{n, m, l} = \int_{-\infty}^{\infty} H_n(x) H_m(x) H_l(x) e^{-x^2}\, dx,$$

where, changing slightly our previous notation, $H_n(x)$ represents now the n-th Hermite polynomial. By a well known generation formula

$$\exp(2tx - t^2 + 2sx - s^2 + 2rx - r^2) = \sum_{n, m, l = 0}^{\infty} \frac{H_n(x) H_m(x) H_l(x)}{n!\, m!\, l!} t^n s^m r^l. \tag{B 1}$$

We now recall Cauchy integral formula in polydiscs for functions $f(z)$ analytical in $\mathbb{C}^N$

$$f(z)=\frac{1}{(2\pi i)^N}\times\int_{\partial D_N}\int_{\partial D_{N-1}}\dots\int_{\partial D_1}\frac{f(\xi_1,\xi_2,\dots,\xi_N)}{(\xi_1-z_1)(\xi_2-z_2)\dots(\xi_N-z_N)}\,d\xi_1\dots d\xi_N$$
$$=\frac{1}{(2\pi i)^N}\sum_{j_1,\dots,j_N=0}^{\infty} z_1^{j_1}\dots z_N^{j_N}\times\int\dots\int_{|\xi_N-w_N|=r_N,\,|\xi_1-w_1|=r_1}\frac{f(\xi_1,\xi_2,\dots,\xi_N)}{\xi_1^{j_1+1}\dots\xi_N^{j_N+1}}\,d\xi_1\dots d\xi_N,\qquad \text{(B 2)}$$

where

$$z=(z_1,z_2,\dots,z_N)\in D_1\times D_2\times\dots\times D_N=D(w_1,r_1)\times\dots\times D(w_N,r_N),$$

$D(w_k, r_k)$ being a disc in $\mathbb{C}$ centered at w_k with radius r_k. Set now

$$\Phi(t,s,r)=\exp(-(t^2+s^2+r^2))\int_{-\infty}^{+\infty}\exp(-x^2+2(t+s+r)x)\,dx.$$

We now multiply in (B 1) by $\exp(-x^2)$ and integrate over $\mathbb{R}$.

$$\Phi(t,s,r)=\sum_{n,m,l=0}^{\infty}\frac{I_{n,m,l}\,t^n s^m r^l}{n!\,m!\,l!}.\qquad \text{(B 3)}$$

From (B 2) and (B 3) we deduce that for some $\rho>0$

$$\frac{I_{n,m,l}}{n!\,m!\,l!}=\frac{1}{(2\pi i)^3}\int_{|t|=\rho}\int_{|s|=\rho}\int_{|r|=\rho}\frac{\Phi(t,s,r)}{t^{1+n}s^{1+m}r^{1+l}}\,dt\,ds\,dr.$$

Notice that, since

$$\int_{-\infty}^{+\infty}\exp(-x^2+2\alpha x)\,dx=\exp(\alpha^2)\int_{-\infty}^{+\infty}\exp(-(x-\alpha)^2)\,dx=\sqrt{\pi}\exp(\alpha^2),$$

we have that

$$\Phi(t,s,r)=\sqrt{\pi}\exp(2(ts+tr+sr)).$$

whence

$$I_{n,m,l}=\sqrt{\pi}\,\frac{n!\,m!\,l!}{(2\pi i)^3}\int_{|t|=\rho}\int_{|s|=\rho}\int_{|r|=\rho}\frac{\exp(2(ts+tr+sr))}{t^{1+n}s^{1+m}r^{1+l}}\,dt\,ds\,dr.$$

Moreover, setting $a=2(s+r)$, we obtain

$$\frac{n!}{2\pi i}\int_{|t|=\rho}\frac{\exp(2(ts+tr))}{t^{1+n}}\,dt = \frac{n!}{2\pi i}\int_{|t|=\rho}\frac{\exp(at)}{t^{1+n}}\,dt=\frac{d^n}{dt^n}(e^{ta})\big|_{t=0}=a^n. \quad \text{(B 4)}$$

Therefore

$$\mathrm{I}_{n,m,l}=\sqrt{\pi}\,\frac{m!\,l!}{(2\pi i)^2}\int_{|s|=\rho}\int_{|r|=\rho}\frac{\exp(2sr)(2(s+r))^n}{s^{1+m}r^{1+l}}\,ds\,dr$$
$$=2^n\sqrt{\pi}\,\frac{m!\,l!}{(2\pi i)^2}\sum_{k=0}^{n}\binom{n}{k}\int_{|s|=\rho}\int_{|r|=\rho}\frac{\exp(2sr)}{s^{1+m+k}r^{1+l-n+k}}\,ds\,dr.$$

Assume for instance that $m\geqq n$. Then arguing as in (B 4)

$$\int_{|s|=\rho}\frac{\exp(2sr)}{s^{1+m-k}}\,ds=(2r)^{m-k}\,\frac{2\pi i}{(m-k)!},$$

so that

$$\mathrm{I}_{n,m,l}=\frac{2^n\sqrt{\pi}\,m!\,l!}{2\pi i}\sum_{k=0}^{n}\frac{\binom{n}{k}2^{m-k}}{(m-k)!}\int_{|r|=\rho}r^{-(1+l-n-m+2k)}\,dr,$$

and since

$$\int_{|r|=\rho}r^{-(1+l-n-m+2k)}\,dr=(2\pi i)\,\delta_{2k,\,m+n-l},$$

we arrive at

$$\mathrm{I}_{n,m,l}=2^n\sqrt{\pi}\,m!\,l!\sum_{k=0}^{n}\frac{\binom{n}{k}2^{m-k}}{(m-k)!}\,\delta_{2k,\,m+n-l}. \quad \text{(B 5)}$$

Therefore $\mathrm{I}_{n,m,l}\neq 0$ if $m+n-l$ is an even integern $0\leqq m+n-l\leqq 2n$, whence $m+n+l$ has to be even and $l\leqq m+n$, $m\leqq l+n$. Then $k=\frac{m+n-l}{2}$, and (B 5) gives

$$\mathrm{I}_{n,m,l}=\frac{2^n\sqrt{\pi}\,m!\,l!\,2^{m-((m+n-l)/2)}}{(m-m+n-l/2)!}\binom{n}{((m+n-l)/2)}$$
$$=2^{n+m+l/2}\sqrt{\pi}\,\frac{n!\,m!\,l!}{((m+l-n)/2)!\,((m+n-l)/2)!\,((n+l-m)/2)!}.$$

having obtained (B 6), we turn our attention to the integral

$$A_{n,m,l}=\int_{-\infty}^{+\infty}\tilde{H}_n(x)\tilde{H}_m(x)\tilde{H}_l(x)\exp\left(-\frac{x^2}{4}\right)dx.$$

where $\tilde{H}_n(x)=c_n H_n\left(\frac{x}{2}\right)$, $c_n=(2^{n/2}(4\pi)^{1/4}(n!)^{1/2})^{-1}$, and we just notice that

$$A_{n,m,l}=2\,c_n\,c_m\,c_l\,I_{n,m,l},$$

where the result.

REFERENCES

[A] S. ANGENENT, The Zero Set of a Solution of a Parabolic Equation, *J. reine angew Math.*, Vol. **390**, 1988, pp. 79-96.

[AF] S. B. ANGENENT and B. FIEDLER, The Dynamics of Rotating Waves in Scalar Reaction-Diffusion Equations, *Trans. Amer. Math. Soc.*, Vol. **307**, 1988, pp. 545-568.

[AW] D. G. ARONSON and H. F. WEINBERGER, Multidimensional Nonlinear Diffusion arising in Population Genetics, *Advances in Math.*, Vol. **30**, 1978, pp. 33-76.

[BBE] J. BEBERNES, A. BRESSAN and D. EBERLY, A Description of Blow-up for the Solid Fuel Ignition Model, *Indiana Univ. Math. J.*, Vol. **36**, 1987, pp. 131-136.

[B] A. BRESSAN, On the Asymptotic Shape of Blow-up, *Indiana Univ. Math. J.*, Vol. **39**, 1990, pp. 947-960.

[CMV] X. Y. CHEN, H. MATANO and L. VERON, Anisotropic Singularities of Solutions of Nonlinear Elliptic Equations in $\mathbb{R}^2$, *J. Funct. Anal.*, Vol. **83**, 1989, pp. 50-93.

[CL] P. J. COHEN and M. LEES, Asymptotic decay of Differential Inequalities, *Pacific J. Math*, Vol. **11,** 1961, pp. 1235-1249.

[D] J. DOLD, Analysis of the Early Stage of Thermal Runaway, *Quart. J. Mech. Appl. Math.*, Vol. **38**, 1985, pp. 361-387.

[FH] A. FRIEDMAN and J. B. MCLEOD, Blow-up of positive Solutions of Semilinear Heat Equations, *Indiana Univ. Math. J.*, Vol. **34**, 1985, pp. 425-447.

[Fu] H. FUJITA, On the Blowing-up of Solutions of the Cauchy Problem for $u_t=\Delta u+u^{1+\alpha}$, *J. Fac. Sci. Univ. of Tokio*, Section I, Vol. **13**, 1966, pp. 109-124.

[GHV] V. A. GALAKTIONOV, M. A. HERRERO and J. J. L. VELÁZQUEZ, The Space Structure near a Blow-up Point for Semilinear Heat Equations: a formal Approach, *Soviet J. Comput. Math. and Math. Physics*, Vol. **31**, 1991, pp. 399-411.

[GP] V. A. GALAKATIONOV and S. A. POSASHKOV, Application of new Comparison Theorems in the Investigation of Unbounded Solutions of nonlinear Parabolic Equations, *Diff. Urav.*, Vol. **22**, 7, 1986, pp. 1165-1173.

[GK1] Y. GIGA and R. V. KOHN, Asymptotically Self-Similar Blow-up of Semilinear Heat Equations, *Comm. Pure Appl. Math.*, Vol. **38**, 1985, pp. 297-319.

[GK2] Y. GIGA and R. V. KOHN, Characterizing Blow-up using Similarity Variables, *Indiana Univ. Math., J.*, Vol. **36**, 1987, pp. 1-40.

[GK3] Y. GIGA and R. V. KOHN, Nondegeneracy of Blow-up for Semilinear Heat Equations, *Comm. Pure Appl. Math.*, Vol. **42**, 1989, pp. 845-884.

[H] D. HENRY, *Geometric Theory of Semilinear Parabolic Equations*, Springer Lecture Notes in Mathematics No. 840, 1981.

[HV1] M. A. HERRERO and J. J. L. VELÁZQUEZ, Flat Blow-up in One-Dimensional Semilinear Heat Equations, *Differential and Integral Equations*, Vol. **5**, 1992, pp. 973-997.

[L] A. A. LACEY, The Form of Blow-up for Nonlinear Parabolic Equations, *Proc. Royal Soc. Edinburgh*, Vol. **98 A**, 1984, pp. 183-202.

[La] P. D. LAX, A Stability Theorem for Solutions of Abstract Differential Equations, and its Application to the Study of the Local behaviour of Solutions of Elliptic Equations, *Comm. Pure Appl. Math*, Vol. **9**, 1956, pp. 747-766.

[Li] W. LIU, The Blow-up Rate of Solutions of Semilinear Heat Equations, *J. Diff. Equations* Vol. **77**, 1989, pp. 104-122.

[MW] C. E. MÜLLER and F. B. WEISSLER, Single Point Blow-up for a General Semilinear Heat Equation, *Indiana Univ. Math., J.*, Vol. **34**, 1983, pp. 881-913.

[W] F. B. WEISSLER, Single Point Blow-up of Semilinear Initial Value Problems, *J. Diff. Equations*, Vol. **55**, 1984, pp. 204-224.

[Wa] N. A. WATSON, *Parabolic Equations on an Infinite Strip*, Monographs and Textbooks in Pure and Applied Mathematics, Vol. **127**, Marcel Dekker, 1988.

[Wi] D. V. WIDDER, *The Heat Equation*, Academic Press, New York, 1975.

(*Manuscript received February* 25, 1991;
accepted March 5, 1991.)