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The Dirichlet-problem for harmonic maps from the disk into a lorentzian warped product

by

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ABSTRACT. – In this paper we prove the existence and the regularity of a harmonic map from the disk of $\mathbf{R}^2$ into the Lorentz manifold $S^2 \times_\beta \mathbf{R}$, with a given boundary condition. Since the energy functional is not bounded from below, we search for its critical points which are not minima.

Key words : Harmonic maps, Dirichlet problem, Lorentz manifold, critical point theory.

RÉSUMÉ. – Dans cet article, on démontre l'existence et la régularité d'une fonction harmonique du disque de $\mathbf{R}^2$ dans une variété de Lorentz $S^2 \times_\beta \mathbf{R}$, dont la valeur est prescrite sur la frontière du disque. Puisque la fonctionnelle de l'énergie n'est pas bornée inférieurement, on cherche des points critiques de cette fonctionnelle qui ne sont pas des minima.

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1. INTRODUCTION AND STATEMENT OF THE RESULT

Let (M, g) be a m-dimensional riemannian manifold with boundary ∂M, and let (N, h) be a n-dimensional semiriemannian manifold. We are interested in the existence of harmonic maps $w : \mathrm{M} \to \mathrm{N}$ which satisfies a boundary condition $w_{|\partial \mathrm{M}} = \gamma$, where $\gamma : \partial \mathrm{M} \to \mathrm{N}$ is a given smooth function. Let $\mathrm{H}_{\gamma}^{k,p}(\mathrm{M}, \mathrm{N})$ be the Sobolev space of functions $w : \mathrm{M} \to \mathrm{N}$ whose the k-th derivatives belong to L^p, and such that $w_{|\partial \mathrm{M}} = \gamma$. A map $w \in \mathrm{H}_{\gamma}^{1,2}(\mathrm{M}, \mathrm{N})$ is harmonic if it is a critical point of the energy functional $\mathrm{E} : \mathrm{H}_{\gamma}^{1,2}(\mathrm{M}, \mathrm{N}) \to \mathbf{R}$:

$$\mathrm{E}(w) = \int_{\mathrm{M}} \sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{m} h_{ij}(w(x)) \frac{\partial w^i(x)}{\partial x_\alpha} \frac{\partial w^j(x)}{\partial x_\beta} g_{\alpha\beta}(x)\, d\mathrm{V},$$

where w^i, $i = 1, 2, \ldots, n$ are the local coordinates of $w(x)$ in N.

In this paper we set $\mathrm{M} = \Omega = \{(x_1, x_2) \in \mathbf{R}^2 \,|\, x_1^2 + x_2^2 < 1\}$, and suppose that $(\mathrm{N}, h) = \mathrm{S}^2 \times_{\beta} \mathbf{R}$ is the Lorentzian warped product of $\mathrm{S}^2 = \{(x_1, x_2, x_3) \in \mathbf{R}^3 \,|\, x_1^2 + x_2^2 + x_3^2 = 1\}$ times $\mathbf{R}$ (*see* [12]). In other words, we consider S^2 with the canonical metric tensor $\tilde{h}$ induced from $\mathbf{R}^3$, and $\mathrm{N} = \mathrm{S}^2 \times \mathbf{R}$ with the tensor h whose components at the point $w = (u, t) \in \mathrm{S}^2 \times \mathbf{R}$ are $h_{ij}(w) = \tilde{h}_{ij}(u)$ if $i, j = 1, 2$; $h_{i3}(w) = h_{3i}(w) = 0$ if $i = 1, 2$, and $h_{33}(w) = -\beta(u)$, where $\beta : \mathrm{S}^2 \to]0, +\infty[$ is given C^1 function. The main result of this paper is the following theorem.

THEOREM 1.1. – *Let* $\gamma = (\nu, \tau) \in \mathrm{C}^{2,\delta}(\partial\Omega, \mathrm{S}^2 \times \mathbf{R})$; *then, if* ν *is not constant, there exists a harmonic map* $w \in \mathrm{C}^{\infty}(\Omega, \mathrm{N}) \cap \mathrm{C}^{2,\delta}(\bar{\Omega}, \mathrm{N})$ *such that* $w_{|\partial\Omega} = \gamma$.

The existence of harmonic maps between riemannian manifolds has been extensively studied by many authors (*see* [1], [3], [4], [5], [13], [14] and its references). More recently has been considered the case in which the starting manifold (M, g) or the target manifold (N, h) is a Lorentzian manifold (*see* [8], [15] or, respectively, [10]). In the latter case, which is our case also, the energy functional is not bounded from below, so its critical point are not minima.

In [10], because of suitable symmetry assumptions, the problem is reduced to the existence of geodesics in a Lorentzian manifold, and the methods of [2] are used.

In order to prove Theorem 1.1, we consider, as in [13], a perturbed functional $\mathrm{E}_\alpha(w) = \mathrm{E}_\alpha(u, t)$ from $\mathrm{H}_{\nu}^{1,2\alpha} \times \mathrm{H}_{\tau}^{1,2}$ to $\mathbf{R}$, such that E_1 is the energy functional. Because the fact that the target manifold is a Lorentzian warped product, we have that $-\mathrm{E}_\alpha(u, .)$ is convex, and it possesses a minimum point t_u. Moreover the functional $u \mapsto \mathrm{E}_\alpha(u, t_u)$ is bounded from below, and there exists a minimum point u_α, so $w_\alpha = (u_\alpha, t_{u_\alpha})$ is a critical point of the functional E_α, for every $\alpha \geqq 1$. (In particular, w_1 is a critical point of the energy functional in $H^{1,2}(\Omega, N)$). Finally, we show that w_α converges to a smooth harmonic map $w \in C^\infty(\Omega, N) \cap C^{2,\delta}(\bar{\Omega}, N)$.

2. PROOF OF THE RESULT

The energy of a function $w=(u, t)$ from Ω to the warped product $S^2 \times_\beta \mathbf{R}$ is given by

$$E(u, t)=\int_\Omega \sum_{i, j, k=1}^{2} \tilde{h}_{ij}(u) \frac{\partial u^i}{\partial x_k} \frac{\partial u^j}{\partial x_k} dx - \int_\Omega \beta(u) \sum_{k=1}^{2} \left(\frac{\partial t}{\partial x_k}\right)^2 dx,$$

where $\tilde{h}$ is the metric tensor on S^2 induced from $\mathbf{R}^3$, and $u^i(x)$ ($i=1, 2$) are the local coordinates of the point $u(x)$ in S^2.

Since S^2 is isometrically imbedded on $\mathbf{R}^3$, we have

$$E(u, t)=\int_\Omega |\nabla \bar{u}|^2 dx - \int_\Omega \beta(\bar{u}) |\nabla t|^2 dx,$$

where

$$|\nabla \bar{u}|^2 = |\bar{u}_{x_1}|^2 + |\bar{u}_{x_2}|^2 = \sum_{i=1}^{3} \left|\frac{\partial \bar{u}^i}{\partial x_1}\right|^2 + \sum_{i=1}^{3} \left|\frac{\partial \bar{u}^i}{\partial x_2}\right|^2, \qquad |\nabla t|^2 = \left|\frac{\partial t}{\partial x_1}\right|^2 + \left|\frac{\partial t}{\partial x_2}\right|^2,$$

and $\bar{u}^i(x)$ ($i=1, 2, 3$) are the coordinates of the point $u(x) \in S^2$ in $\mathbf{R}^3$.

Let $\gamma=(\nu, \tau) \in C^{2,\delta}(\partial\Omega, S^2 \times \mathbf{R})$, and set, for $p \geqq 2$:

$$H_\nu^{k,p} = H_\nu^{k,p}(\Omega, S^2), \qquad H_\tau^{k,p} = H_\tau^{k,p} = H_\tau^{k,p}(\Omega, \mathbf{R}).$$

For every $\alpha \geqq 1$, let $E_\alpha : H_\nu^{1,2\alpha} \times H_\tau^{1,2} \to \mathbf{R}$ be the functional (in the following we shall write u instead of $\bar{u}$):

$$E_\alpha(u, t)=\int_\Omega (1+|\nabla u|^2)^\alpha dx - \int_\Omega \beta(u) |\nabla t|^2 dx.$$

Clearly, the critical points of E_1 are harmonic maps.

Remark 2.1. – Let $\tilde{\beta} \in C^1(\mathbf{R}^3,]0, +\infty[)$ be such that $\tilde{\beta}_{|S^2}=\beta$ and $\tilde{\beta}(u)=1$ for $|u|>2$. It is easy to see that the critical points of E_α, $\alpha \geqq 1$, are weakly solutions of the following nonlinear elliptic system:

$$\left.\begin{aligned} &-\sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left[(1+|\nabla u|^2)^{\alpha-1} \frac{\partial u^p}{\partial x_i}\right] \\ &= (1+|\nabla u|^2)^{\alpha-1} |\nabla u|^2 u^p + \frac{1}{2\alpha} Z^p(u) |\nabla t|^2 \\ &\sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left[\tilde{\beta}(u) \frac{\partial t}{\partial x_i}\right] = 0 \end{aligned}\right\} \tag{2.1}$$

where $p=1, 2, 3$ and $Z(u)=\tilde{\beta}'(u)-(\tilde{\beta}'(u)\,|\,u)\,u$.

In fact, let $w=(u, t)\in H_v^{1,2\alpha}\times H_\tau^{1,2}$ be a critical point of E_α; for every $\varphi\in C_0^\infty(\Omega, \mathbf{R}^3)$, $\psi\in C_0^\infty(\Omega, \mathbf{R})$, we set $\Gamma(\varepsilon)=((u+\varepsilon\varphi)\,|\,u+\varepsilon\varphi\,|^{-1}, t+\varepsilon\psi)$; then

$$\frac{dE_\alpha(\Gamma(\varepsilon))}{d\varepsilon}_{|\varepsilon=0}=2\alpha\int_\Omega(1+|\nabla u|^2)^{\alpha-1}[(\nabla u\,|\,\nabla\varphi)-|\nabla u|^2(u\,|\,\varphi)]\,dx$$
$$-\int_\Omega[(\tilde{\beta}'(u)\,|\,\varphi)-(\tilde{\beta}'(u)\,|\,u)(u\,|\,\varphi)]\,|\nabla t|^2\,dx-2\int_\Omega\tilde{\beta}(u)(\nabla t\,|\,\nabla\psi)\,dx=0$$

for every φ, ψ, so we get (2.1).

LEMMA 2.2. – *Fix* $\alpha\geqq 1$ *and* $u\in H_v^{1,2\alpha}$; *then, the functional* $E_\alpha(u, .): H_\tau^{1,2}\to\mathbf{R}$ *has a unique maximum point* t_u.

Proof. – Easy. ■

Remark 2.3. – Clearly, there exists $c>0$ such that $\|t_u\|_{H_\tau^{1,2}}\leqq c$ for every $\alpha\geqq 1$ and $u\in H_v^{1,2\alpha}$. In fact, if we set $\beta_0=\min_{S^2}\beta$, $\beta_\infty=\max_{S^2}\beta$, and fix $t_0\in H_\tau^{1,2}$, we have $E_\alpha(u, t_u)\geqq E_\alpha(u, t_0)$, so

$$\int_\Omega|\nabla t_u|^2\,dx\leqq(\beta_\infty/\beta_0)\int_\Omega|\nabla t_0|^2\,dx.$$

For every $\alpha\geqq 1$, we consider the functional $F_\alpha: H_v^{1,2\alpha}\to\mathbf{R}$ given by: $F_\alpha(u)=E_\alpha(u, t_u)$. We have the following lemma that we shall prove in section 3.

LEMMA 2.4. – *For every* $\alpha\geqq 1$, $F_\alpha\in C^1$, *and*

$$\langle F'_\alpha(u), v\rangle=\left\langle\frac{\partial E_\alpha}{\partial u}(u, t_u), v\right\rangle.$$

So, $u\in H_v^{1,2\alpha}$ *is a critical point of* F_α *if and only if* $w=(u, t_u)$ *is a critical point of* E_α.

LEMMA 2.5. – *Let* $\alpha\geqq 1$. *Then, the functional* $F_\alpha: H_v^{1,2\alpha}\to\mathbf{R}$ *is coercive and weakly lower semicontinuous, so there exists* $u_\alpha\in H_v^{1,2\alpha}$ *such that* $F_\alpha(u_\alpha)=\min F_\alpha(u)$.

Proof. – The coerciveness of F_α follows from Remark 2.3. We fix now $t\in H_\tau^{1,2}$. The functional $u\mapsto\int_\Omega(1+|\nabla u|^2)^\alpha\,dx$ is clearly weakly lower semicontinuous; moreover, since $H^{1,2\alpha}(\Omega)$ is compactly imbedded in $L^q(\Omega)$ ($q\geqq 1$) for $\alpha=1$ and in $C(\bar{\Omega})$ for $\alpha>1$, and since $u\mapsto\int_\Omega\beta(u)\,|\nabla t|^2\,dx$ is continuous from $L^q(\Omega)$ or $C(\bar{\Omega})$ to $\mathbf{R}$, we get that $u\mapsto E_\alpha(u, t)$ is weakly lower semicontinuous.

Let (u_n) be a minimizing sequence such that $u_n \to u$ weakly in $H_v^{1,2\alpha}$. Then $E_\alpha(u_n, t_u) \leqq E_\alpha(u_n, t_{u_n}) = F_\alpha(u_n)$, so

$$F_\alpha(u) = E_\alpha(u, t_u) \leqq \liminf_{n \to \infty} E_\alpha(u_n, t_u) \leqq \liminf_{n \to \infty} F_\alpha(u_n),$$

and the lemma is proved. ■

Remark 2.6. – Because Lemma 2.4 and Lemma 2.5, there exists a critical point $w=(u, t) \in H_v^{1,2} \times H_\tau^{1,2}$ of the functional E_1; however, in order to show that $w \in C^\infty(\Omega) \cap C^{2,\delta}(\bar{\Omega})$, we use an approximation procedure developed later.

LEMMA 2.7. – *Let* $\gamma \in C^{2,\delta}(\partial\Omega)$; *then, there exists* $\alpha_0 > 1$ *such that, if* $1 < \alpha \leqq \alpha_0$ *and* $E'_\alpha(w) = 0$, *then* $w \in C^\infty(\Omega) \cap C^{2,\delta}(\bar{\Omega})$.

Proof. – Let $w=(u, t) \in H_v^{1,2\alpha} \times H_\tau^{1,2}$ be such that $E'_\alpha(w)=0$, so that w is a weakly solution of the nonlinear elliptic system (2.1). Let $p=(p_a^i) \in \mathbf{R}^8$, $a=1, 2$, $i=1, \ldots, 4$, and let $z=(z_i)_{i=1,\ldots,4} \in \mathbf{R}^4$; we set $\bar{p}=(p_a^i)$ with $a=1, 2$ and $i=1, 2, 3$, and $\bar{z}=(z_i)_{i=1,2,3}$. Then, we can define the following functions $A_i^a, B_i : \mathbf{R}^4 \times \mathbf{R}^8 \to \mathbf{R}$:

$$A_i^a(z, p) = \begin{cases} (1+|\bar{p}|^2)^{\alpha-1} p_a^i & \text{for} \quad i=1, 2, 3 \quad \text{and} \quad a=1, 2, \\ \tilde{\beta}(\bar{z}) p_a^4 & \text{for} \quad i=4 \quad \text{and} \quad a=1, 2; \end{cases}$$

$$B_i(z, p) = \begin{cases} (1+|\bar{p}|^2)^{\alpha-1} |\bar{p}|^2 z_i + \dfrac{1}{2\alpha} Z^i(\bar{z})((p_1^4)^2 + (p_2^4)^2) & \text{for} \quad i=1, 2, 3, \\ 0 \quad \text{for} \quad i=4. \end{cases}$$

The system (2.1) became:

$$-\sum_{a=1}^{2} \frac{\partial}{\partial x_a} A_i^a(w, \nabla w) = B_i(w, \nabla w), \qquad i=1, \ldots, 4.$$

It is easy to check that the assumptions (1.10.8) of [11] are satisfied, so, as in [13], Prop. 2.3, we get $w \in H_{\text{loc}}^{2,2\alpha}(\Omega)$, and then $w \in C^\infty(\Omega)$. Since $u \in H^{1,2\alpha}(\Omega)$, we have $\partial\tilde{\beta}(u)/\partial x_i \in L^{2\alpha}(\Omega)$, so, because of [9], Theorem 15.1, p. 187, applied to the fourth equation of the system (2.1), we have $t \in H^{2,2\alpha}(\Omega)$, which implies $t \in C^1(\bar{\Omega})$. Now, in order to get the regularity of u up to the boundary, we prove first that $\nabla u \in L^\infty(\Omega)$; the proof is similar to the proof of Theorem 3.1 in [1].

Suppose $\|\nabla u\|_{L^\infty(\Omega)} = +\infty$, and let $(r_k) \subset]0, 1[$ be such that $r_k \to 1$ as $k \to \infty$. Let $\theta_k = \max_{\bar{\Lambda}_k} |\nabla u| = |\nabla u(a_k)|$, where $\Lambda_k = B(0, r_k)$. Clearly $\theta_k \to +\infty$ and $d(a_k, \partial\Lambda_k) \to 0$ as $k \to \infty$. We can assume that $a_k \to a \in \partial\Omega$

as $k \to \infty$. From the system (2.1) we have, in Ω:

$$-\Delta u - \frac{2(\alpha-1)}{1+|\nabla u|^2} \sum_{\substack{i,j=1,2\\ q=1,2,3}} \frac{\partial u}{\partial x_i}\,\frac{\partial u^q}{\partial x_j}\,\frac{\partial^2 u^q}{\partial x_i \partial x_j}$$

$$= |\nabla u|^2 u + \frac{Z(u)\,|\nabla t|^2}{2\alpha(1+|\nabla u|^2)^{\alpha-1}}. \qquad (2.2)$$

Then

$$-\Delta u + (\alpha-1) \sum_{\substack{i,j=1,2\\ q=1,2,3}} A^q_{ij}(x) \frac{\partial u^q}{\partial x_i \partial x_j} = |\nabla u|^2 u + B(x) \qquad (2.3)$$

in Ω, where A^q_{ij} and B are continuous and bounded from Ω to $\mathbf{R}^3$. Now, we distinguish two cases.

1) Case: $\theta_k d(a_k, \partial\Lambda_k) \to +\infty$ as $k \to \infty$;

Let $\rho_k = d(a_k, \partial\Lambda_k)$; clearly we can assume $\rho_k > 0$ for every $k \in \mathbf{N}$. Set $\Omega_k = \{x \in \mathbf{R}^2 \,|\, \theta_k^{-1} x + a_k \in B(a_k, \rho_k)\}$, and let $u_k : \bar{\Omega}_k \to \mathbf{R}^3$ be such that $u_k(x) = u(\theta_k^{-1} x + a_k)$. Notice that, for every $R > 0$, there exists $k_0 \in \mathbf{N}$ such that $B(0, R) \subset \Omega_k$ for $k \geq k_0$. Moreover, since $\theta_k^{-1} x + a_k \to a$ for every $x \in \mathbf{R}^2$, $u_k \to v \equiv u(a)$ in $C^0_{\mathrm{loc}}(\mathbf{R}^2)$. Since

$$\Delta u_k(x) = \theta_k^{-2} \Delta u(\theta_k^{-1} x + a_k), \qquad \nabla u_k(x) = \theta_k^{-1} \nabla u(\theta_k^{-1} x + a_k),$$

from the equation (2.3) we get, in Ω_k:

$$-\Delta u_k + (\alpha-1) \sum_{\substack{i,j=1,2\\ q=1,2,3}} A^q_{ij}(\theta_k^{-1} x + a_k) \frac{\partial^2 u_k^q}{\partial x_i \partial x_j} = |\nabla u_k|^2 u_k + \theta_k^{-2} B(\theta_k^{-1} x + a_k).$$

Since $|\nabla u_k| \leq 1$ in Ω_k, from standard estimates in PDE (*see* e.g. [6], Sec. 11.2), we have that, for every $R > 0$, there exists $\gamma = \gamma(R) \in]0, 1[$, such that (u_k) is bounded in $C^{1,\gamma}(\bar{B}(0, R))$, so $u_k \to v \equiv u(a)$ in $C^1_{\mathrm{loc}}(\mathbf{R}^2)$. On the other hand, $|\nabla u_k(0)| = \theta_k^{-1} |\nabla u(a_k)| = 1$, so we get a contradiction.

2) Case: $\theta_k d(a_k, \partial\Lambda_k) \to \rho < +\infty$ as $k \to \infty$; then, we can assume $a = (-1, 0)$ and $a_k \neq (r_k, 0)$ for every $k \in \mathbf{N}$. Let $U =]1/2, +\infty[\times \mathbf{R}$ ($\subset \mathbf{R}^2$), and let $T : \bar{\Omega} \setminus \{(1, 0)\} \to \bar{U}$ be such that

$$(\bar{x}_1, \bar{x}_2) = T(x_1, x_2) = \left(\frac{1 - x_1}{(1-x_1)^2 + x_2^2}, \frac{x_2}{(1-x_1)^2 + x_2^2} \right).$$

Let $\bar{u} : \bar{U} \to \mathbf{R}^3$ be such that $\bar{u}(\bar{x}) = u(r_k T^{-1}(\bar{x}))$, so $u(x) = \bar{u}(T(r_k^{-1} x))$ for $x \in \Lambda_k$. Since

$$\Delta u(x) = r_k^{-2} |T(r_k^{-1} x)|^4 \Delta \bar{u}(T(r_k^{-1} x)),$$
$$|\nabla u(x)|^2 = r_k^{-2} |T(r_k^{-1} x)|^4 \, |\nabla \bar{u}(T(r_k^{-1} x))|^2,$$

from (2.3) we get, in U:

$$-r_k^{-2}|\bar{x}|^4\Delta\bar{u}(\bar{x})+(\alpha-1)\sum_{\substack{i,j,h,l=1,2\\ q=1,2,3}} r_k^{-1}A^q_{ij}(r_k\mathrm{T}^{-1}(\bar{x}))\times\frac{\partial \mathrm{T}_h(r_k^{-1}x)}{\partial x_i}\,\frac{\partial \mathrm{T}_l(r_k^{-1}x)}{\partial x_j}\,\frac{\partial^2\bar{u}(\bar{x})}{\partial x_h\,\partial x_l}$$
$$+(\alpha-1)\sum_{\substack{i,j,h=1,2\\ q=1,2,3}} r_k^{-1}A^q_{ij}(r_k\mathrm{T}^{-1}(\bar{x}))\frac{\partial^2\mathrm{T}_h(r_k^{-1}x)}{\partial x_i\,\partial x_j}\,\frac{\partial\bar{u}(\bar{x})}{\partial x_h}$$
$$=r_k^{-2}|\bar{x}|^4\,|\nabla\bar{u}(\bar{x})|^2\,\bar{u}(\bar{x})+\mathrm{B}(r_k\mathrm{T}^{-1}(\bar{x})),\quad(2.4)$$

where $\bar{x}=\mathrm{T}(r_k^{-1}x)$, and T_h are the components of T. It is easy to check that

$$\left|\frac{\partial\mathrm{T}_h(r_k^{-1}x)}{\partial x_i}\right|\leqq|\mathrm{T}(r_k^{-1}x)|^2=r_k^{-2}|x|^2,$$
$$\left|\frac{\partial^2\mathrm{T}_h(r_k^{-1}x)}{\partial x_i\,\partial x_j}\right|\leqq|\mathrm{T}(r_k^{-1}x)|^4=r_k^{-4}|x|^4.$$

Then, from (2.4), we get

$$-\Delta\bar{u}+\sum_{\substack{i,j,h,l=1,2\\ q=1,2,3}}\mathrm{C}^q_{ijhl}\frac{\partial^2\bar{u}}{\partial x_h\,\partial x_l}=\sum_{\substack{i,j,h=1,2\\ q=1,2,3}}\mathrm{D}^q_{ijh}\frac{\partial\bar{u}}{\partial x_h}+|\nabla\bar{u}|^2\,\bar{u}+\mathrm{F},\quad(2.5)$$

where C^q_{ijhl}, D^q_{ijh} and F are continuous and bounded (independently of k) in U. Let $a_k=(x_k, y_k)$, so $x_k\to -1$ and $y_k\to 0$ as $k\to\infty$, and let $\tilde{u}_k:\bar{\mathrm{U}}\to\mathbf{R}^3$ be such that $\tilde{u}_k(\tilde{x},\tilde{y})=\bar{u}\left(\frac{1}{2}+\theta_k^{-1}\left(\tilde{x}-\frac{1}{2}\right),\ \theta_k^{-1}\tilde{y}+y_k\right)$. Clearly, since $\theta_k\to+\infty$, $\tilde{u}_k\to v\equiv\bar{u}\left(\frac{1}{2},0\right)=u(a)$ in $\mathrm{C}^0_{\mathrm{loc}}(\bar{\mathrm{U}})$. Using (2.5) we show that $\tilde{u}_k\to v$ in $\mathrm{C}^1_{\mathrm{loc}}(\bar{\mathrm{U}})$. In fact, from (2.5) we have

$$-\Delta\tilde{u}_k+\sum_{\substack{i,j,h,l=1,2\\ q=1,2,3}}\tilde{\mathrm{C}}^q_{ijhl}\frac{\partial^2\tilde{u}_k}{\partial x_h\,\partial x_l}=\frac{1}{\theta_k}\sum_{\substack{i,j,h=1,2\\ q=1,2,3}}\tilde{\mathrm{D}}^q_{ijh}\frac{\partial\tilde{u}_k}{\partial x_h}+|\nabla\tilde{u}_k|^2\,\tilde{u}_k+\tilde{\mathrm{F}},\quad(2.6)$$

where $\tilde{\mathrm{C}}^q_{ijhl}$, $\tilde{\mathrm{D}}^q_{ijh}$ and $\tilde{\mathrm{F}}$ are continuous and bounded (independently of k) in U. Notice that $\nabla\tilde{u}_k(\tilde{x},\tilde{y})=\theta_k^{-1}\nabla\bar{u}\left(\frac{1}{2}+\theta_k^{-1}\left(\tilde{x}-\frac{1}{2}\right),\ \theta_k^{-1}\tilde{y}+y_k\right)$; since $|\nabla\bar{u}(\bar{x})|=r_k|\bar{x}|^{-2}|\nabla u(x)|$, where $\bar{x}\in\mathrm{U}$ and $x=r_k\mathrm{T}^{-1}(\bar{x})\in\Lambda_k$, we have $|\nabla\tilde{u}_k|\leqq 4$ in U. Then, for every $\mathrm{R}>0$, from (2.6) and [6], Sec. 11.2, there exists $\gamma=\gamma(\mathrm{R})\in\,]0,1[$, such that $(\tilde{u}_k)$ is bounded in $\mathrm{C}^{1,\gamma}(\mathrm{U}_\mathrm{R})$, where $\mathrm{U}_\mathrm{R}=\bar{\mathrm{U}}\cap\mathrm{B}(0,\mathrm{R})$. So, modulo subsequences, $\tilde{u}_k\to v\equiv u(a)$ in $\mathrm{C}^1_{\mathrm{loc}}(\bar{\mathrm{U}})$.

On the other hand, let $\tilde{x}_k = \frac{1}{2} + \theta_k \left(\bar{x}_k - \frac{1}{2} \right)$ [we recall that $(\bar{x}_k, \bar{y}_k)$ are the coordinates of $\bar{a}_k = \mathrm{T}(r_k^{-1} a_k)$], and consider the sequence $\tilde{a}_k = (\tilde{x}_k, 0)$. Since

$$\tilde{x}_k - \frac{1}{2} = \theta_k \left(\frac{1 - r_k^{-1} x_k}{(1 - r_k^{-1} x_k)^2 + (r_k^{-1} x_k)^2} - \frac{1}{2} \right) = \frac{\theta_k\, d(a_k, \partial \Lambda_k)(r_k + |a_k|)}{2[(r_k - x_k)^2 + y_k^2]},$$

from our assumptions we have that $(\tilde{a}_k)$ is bounded.

Moreover $|\nabla \tilde{u}_k(\tilde{a}_k)| = \theta_k^{-1} |\nabla \bar{u}(\bar{x}_k, y_k)| \approx \theta_k^{-1} |\nabla \bar{u}(\bar{a}_k)| \approx 4$, and this is impossible since $\nabla \tilde{u}_k \to \nabla v = 0$ in $C^0_{\mathrm{loc}}(\bar{U})$.

Then, we have proved that $\nabla u \in L^\infty(\Omega)$. In particular, $\nabla u \in L^4(\Omega)$. As in [13], Proposition 2.3, we consider the linear operator

$$-\Delta_\alpha u = -\Delta u + (\alpha - 1) \sum_{\substack{i, j = 1, 2 \\ q = 1, 2, 3}} A^q_{ij}(x) \frac{\partial^2 u^q}{\partial x_i \partial x_j}.$$

Since for $\alpha - 1$ small enough Δ_α is close to $\Delta : H^{2,4}(\Omega) \to L^4(\Omega)$, from (2.3) we get $u \in H^{2,4}(\Omega)$ and then, by Sobolev, $u \in C^1(\bar{\Omega})$, Finally, the fact that $w = (u, t) \in C^{2,\delta}(\bar{\Omega})$ follows, for instance, from [11], Theorem 1.11.3. ■

Let (w_α), $\alpha > 1$, be such that $E'_\alpha(w_\alpha) = 0$. For $\alpha \leqq \alpha_0$, we have $w_\alpha \in C^\infty(\Omega) \cap C^{2,\delta}(\bar{\Omega})$; let $w_\alpha = (u_\alpha, t_\alpha)$ and $\theta_\alpha = \max_{\bar{\Omega}} |\nabla u_\alpha|$ for $1 < \alpha \leqq \alpha_0$. Then, we have the following lemma.

LEMMA 2.8. – *Suppose* $(\|\nabla u_\alpha\|_{L^{2\alpha}})$ *and* $(\|t_\alpha\|_{L^2})$ *bounded independently to* α. *Then, for every* $\varepsilon \in]0, 1[$ *there exist* $\alpha_1 \in]1, \alpha_0]$, $a, b > 0$ *such that* $\max_{\bar{\Omega}} |\nabla t_\alpha| \leqq a \theta_\alpha^\varepsilon + b$ *for every* $\alpha \in]1, \alpha_1[$.

Proof. – We shall write $w = (u, t)$, θ instead of $w_\alpha = (u_\alpha, t_\alpha)$, θ_α. The fourth equation of the system (2.1) became:

$$\tilde{\beta}(u) \Delta t + \sum_{i=1}^{2} \left(\tilde{\beta}'(u) \left| \frac{\partial u}{\partial x_i} \right. \right) \frac{\partial t}{\partial x_i} = 0.$$

From our assumptions, $(\|(\tilde{\beta}'(u) | \partial u/\partial x_i)\|_{L^{2\alpha}})$ is bounded, so, from [9], Theorem 17.1, p. 207 and p. 209, we get $(\|t\|_{H^{2,2}})$ bounded independently to α. Now, we fix $\varepsilon \in]0, 1[$; let $\sigma > 0$ be such that $\varepsilon = 1 - 2/(2 + \sigma)$, and $q = (2 + \sigma)(4 + \sigma)/\sigma$. Since $(\|\nabla t\|_{H^{1,2}})$ is bounded, we have, by Sobolev, that $(\|\nabla t\|_{L^q})$ is bounded. Let $t_0 = t - \tau$, so $t_0 \in H_0^{2,2}(\Omega)$ and satisfies the equation:

$$\Delta t_0 = f \equiv -\Delta \tau - \sum_{i=1}^{2} \tilde{\beta}(u)^{-1} \left(\tilde{\beta}'(u) \left| \frac{\partial u}{\partial x_i} \right. \right) \frac{\partial t}{\partial x_i}. \tag{2.7}$$

Let $p = 2 + \sigma/2$, $\alpha_1 = 1 + \sigma/2$; we claim that there exist $a_1, b_1 > 0$ such that $\|f\|_{L^p} \leqq a_1 \theta^\varepsilon + b_1$ for every $\alpha \in]1, \alpha_1[$. Then, from (2.7) and standard estimates (*see* e. g. [9], Ch. III, Sec. 11), we get $\|t\|_{H^{2,p}} \leqq c(\|f\|_{L^p} + \|t\|_{L^2})$, where c does not depend on α. Since $H^{2,p}(\Omega) \subsetneq C^1(\bar{\Omega})$, the Lemma is proved. In order to prove the claim, we must show that $\left\| \frac{\partial u^j}{\partial x_i} \frac{\partial t}{\partial x_i} \right\|_{L^p} \leqq a_1 \theta^{\varepsilon}$ for every $\alpha \in]1, \alpha_1[$ ($i=1, 2, j=1, 2, 3$). In fact,

$$\left| \frac{\partial u^j}{\partial x_i} \frac{\partial t}{\partial x_i} \right|^p \leqq |\nabla u|^p \, |\nabla t|^p \leqq \theta^{(1-k)p} |\nabla u|^{kp} \, |\nabla t|^p,$$

where $k=2\alpha/(2+\sigma)$ (Notice that $k<1$ if $\alpha<\alpha_1=1+\sigma/2$). By Young inequality (*see* e.g. [9], p. 37), we have

$$|\nabla u|^{kp} \, |\nabla t|^p \leqq \frac{p}{2+\sigma} (|\nabla u|^{kp})^{(2+\sigma)/p} + \frac{1}{r} |\nabla t|^{pr} = \frac{p}{2+\sigma} |\nabla u|^{2\alpha} + \frac{\sigma}{2(2+\sigma)} |\nabla t|^q$$

where $r=((2+\sigma)/p)'$, so that $pr=q$. Then

$$\left\| \frac{\partial u^j}{\partial x_i} \frac{\partial t}{\partial x_i} \right\|_{L^p}^p \leqq \theta^{(1-k)p} \left(\frac{p}{2+\sigma} \|\nabla u\|_{L^{2\alpha}} + \frac{\sigma}{2(2+\sigma)} \|\nabla t\|_{L^q} \right) \leqq a_1^p \, \theta^{(1-k)p} \leqq a_1^p \, \theta^{\varepsilon},$$

and the claim is proved. ■

We close this section with the proof of Theorem 1.1. For $\alpha \geqq 1$, let $u_\alpha \in H^{1,2\alpha}_v$ be such that $F_\alpha(u_\alpha) = \min \{ F_\alpha(u) \mid u \in H^{1,2\alpha}_v \}$ (*see* Lemma 2.5). If we set $w_\alpha = (u_\alpha, t_{u_\alpha})$, we have $E'_\alpha(w_\alpha)=0$ because of Lemma 2.4. From Remark 2.3, we have $\|t_{u_\alpha}\|_{H^{1,2}} \leqq c$; moreover it is easy to check that $(\|u_\alpha\|_{H^{1,2\alpha}})$ is bounded (if α is bounded). Then, if we fix $\varepsilon \in]0, 1[$ and set $\alpha_2 = \min \{\alpha_0, \alpha_1\}$ (*see* Lemmas 2.7 and 2.8), we have

$$w_\alpha \in C^\infty(\Omega) \cap C^{2,\delta}(\bar{\Omega})$$

and for every $\alpha \in]1, \alpha_2[$:

$$\max_{\bar{\Omega}} |\nabla t_{u_\alpha}| \leqq a \theta_\alpha^{\varepsilon} + b, \tag{2.8}$$

where $\theta_\alpha = \max_{\bar{\Omega}} |\nabla u_\alpha|$, and a, b does not depend on α.

Let $(\alpha_k) \subset]1, \alpha_2]$ be such that $\alpha_k \to 1$. We shall write w_k, u_k, t_k, θ_k instead of w_{α_k}, u_{α_k}, $t_{u_{\alpha_k}}$, θ_{α_k}. Since (u_k), (t_k) are bounded in $H^{1,2}$, we can assume $u_k \to u$, $t_k \to t$ weakly in $H^{1,2}$.

Proof of Theorem 1.1. – Since $w_k \in C^\infty(\Omega) \cap C^{2,\alpha}(\bar{\Omega})$, u_k satisfies the equation (2.2), with α and t replaced by α_k and t_k. If the sequence (θ_k) is bounded, we have $(\|\nabla t_k\|_{C^0(\bar{\Omega})})$ is bounded because of (2.8). Then, as in [1], we get $u_k \to u$ and $t_k \to t$ in $C^1(\bar{\Omega})$, and $w=(u, t)$ satisfies the system (2.1) with $\alpha=1$. The fact that $w \in C^{2,\delta}(\bar{\Omega})$ follows from [11], Theorem 1.11.3, so Theorem 1.1 is proved in this case.

We prove now that the case (θ_k) unbounded does not occur. In fact, arguing by contradiction, suppose $\theta_k \to +\infty$, and let $(a_k) \subset \bar{\Omega}$ be such that $|\nabla u_k(a_k)| = \theta_k$. We can assume $a_k \to a \in \bar{\Omega}$ as $k \to \infty$.

1) Case: $\lim_{k\to\infty} \theta_k d(a_k, \partial\Omega) = +\infty$.

Set $\Omega_k = \{x \in \mathbf{R}^2 \mid \theta_k^{-1} x + a_k \in \Omega\}$, and

$$v_k(x) = u_k\left(\frac{x}{\theta_k} + a_k\right), \qquad r_k(x) = t_k\left(\frac{x}{\theta_k} + a_k\right)$$

for $x \in \bar{\Omega}_k$. Then (v_k, r_k) satisfies, on Ω_k, the equations:

$$-\Delta v_k - \frac{2(\alpha_k - 1)}{\theta_k^{-2} + |\nabla v_k|^2} \sum_{\substack{i,j=1,2\\ q=1,2,3}} \frac{\partial v_k}{\partial x_i} \frac{\partial v_k^q}{\partial x_j} \frac{\partial^2 v_k^q}{\partial x_i \partial x_j} = |\nabla v_k|^2 v_k + \frac{Z(v_k)|\nabla r_k|^2}{2\alpha_k(1 + \theta_k^2 |\nabla v_k|^2)^{\alpha_k - 1}}.$$

Because of (2.8), $|\nabla r_k(x)| = \theta_k^{-1} |\nabla t_k(\theta_k^{-1} x + a_k)| \leq \theta_k^{-1}(a\theta_k^{\varepsilon} + b) \to 0$. Then, as in [1], we have $v_k \to v$ in $C^1_{\text{loc}}(\mathbf{R}^2)$. Set $\rho_k = d(a_k, \partial\Omega)$; since $\theta_k \rho_k \to +\infty$, we can assume $\rho_k > 0$ for every $k \in \mathbf{N}$. Set $\Lambda_k = B(a_k, \rho_k)$.

Fix $\varepsilon > 0$; then there exists $r > 0$ such that $\int_D |\nabla u|^2 dx \leq \varepsilon$, where $D = \{x \in \Omega \mid |x - a| < r\}$. Clearly $\Lambda_k \subset D$ for k large enough. Let

$$\Gamma_k = \{x \in \mathbf{R}^2 \mid \theta_k^{-1} x + a_k \in \Lambda_k\}.$$

It is easy to check that, for every $R > 0$, there exists k_0 such that $B(0, R) \subset \Gamma_k$ for $k \geq k_0$. Then

$$\int_{B(0,R)} |\nabla v_k|^2 dx \leq \int_{\Gamma_k} |\nabla v_k|^2 dx = \int_{\Lambda_k} |\nabla u_k|^2 dx,$$

so, for $k \to \infty$ we get

$$\int_{B(0,R)} |\nabla v|^2 dx \leq \liminf_{k\to\infty} \int_{\Lambda_k} |\nabla u_k|^2 dx,$$

and then for $R \to +\infty$, $\int_{\mathbf{R}^2} |\nabla v|^2 dx \leq \liminf_{k\to\infty} \int_{\Lambda_k} |\nabla u_k|^2 dx$.

On the other hand,

$$\int_\Omega |\nabla u_k|^2 dx = \int_{\Omega\setminus\Lambda_k} |\nabla u_k|^2 dx + \int_{\Lambda_k} |\nabla u_k|^2 dx$$
$$\geq \int_{\Omega\setminus D} |\nabla u_k|^2 dx + \int_{\Lambda_k} |\nabla u_k|^2 dx,$$

so

$$\liminf_{k\to\infty}\int_\Omega |\nabla u_k|^2\,dx \geqq \int_{\Omega\setminus \mathrm{D}} |\nabla u|^2\,dx+\int_{\mathbf{R}^2}|\nabla v|^2\,dx$$
$$\geqq \int_\Omega |\nabla u|^2\,dx+\int_{\mathbf{R}^2}|\nabla v|^2\,dx-\varepsilon.$$

For $\varepsilon\to 0$, we get

$$\liminf_{k\to\infty}\int_\Omega |\nabla u_k|^2\,dx \geqq \int_\Omega |\nabla u|^2\,dx+\int_{\mathbf{R}^2}|\nabla v|^2\,dx.$$

We recall now that $\alpha_k\to 1$ and $\int_\Omega \beta(u_k)\,|\nabla t_{u_k}|^2\,dx \leqq \int_\Omega \beta(u_k)\,|\nabla t_u|^2\,dx$, so it is easy to check that

$$\liminf_{k\to\infty} \mathrm{F}_{\alpha_k}(u_k)\geqq \mathrm{F}_1(u)+\int_{\mathbf{R}^2}|\nabla v|^2\,dx.$$

Since $\mathrm{F}_{\alpha_k}(u_k)=\min \mathrm{F}_{\alpha_k}$, we have $\mathrm{F}_{\alpha_k}(u)\geqq \mathrm{F}_{\alpha_k}(u_k)$, and then, for $k\to\infty$, $\mathrm{F}_1(u)\geqq \mathrm{F}_1(u)+\int_{\mathbf{R}^2}|\nabla v|^2\,dx$, so that $\nabla v\equiv 0$.

On the other hand, $|\nabla v(0)|=\lim\limits_{k\to\infty}|\nabla v_k(0)|=\lim\limits_{k\to\infty}\theta_k^{-1}\,|\nabla u_k(a_k)|=1$, and we have a contradiction.

2) Case: $\lim\limits_{k\to\infty}\theta_k\, d(a_k,\partial\Omega)=\rho<+\infty$.

Then $a_k\to a\in\partial\Omega$, and we can assume $a=(-1, 0)$. Let U and T as in the proof of Lemma 2.7, and let

$$\bar{u}_k=u_k\circ \mathrm{T}^{-1},\qquad \bar{t}_k=t_k\circ \mathrm{T}^{-1}.$$

Then $\bar{u}_k$ and $\bar{t}_k$ are well-defined on $\bar{\mathrm{U}}$. From (2.1) we have

$$-\Delta u_k+(\alpha_k-1)\sum_{\substack{i,j=1,2\\ q=1,2,3}} \mathrm{A}^q_{ij}(x)\frac{\partial^2 u^q_k}{\partial x_i\,\partial x_j} =|\nabla u_k|^2\,u_k+\frac{\mathrm{Z}(u_k)\,|\nabla t_k|^2}{2\,\alpha_k\,(1+|\nabla u_k|^2)^{\alpha_k-1}},\qquad (2.9)$$

where $\mathrm{A}^q_{ijk}(x)$ are bounded and continuous functions from $\bar{\Omega}$ to $\mathbf{R}^3$. If we set $\mathrm{T}(x)=\bar{x}$, from (2.9) we get

$$-\Delta\bar{u}_k+(\alpha_k-1)\sum_{\substack{i,j=1,2\\ q=1,2,3}} \mathrm{B}^q_{ijk}(\bar{x})\frac{\partial^2 \bar{u}^q_k}{\partial x_i\,\partial x_j} =(\alpha_k-1)\sum_{\substack{i=1,2\\ q=1,2,3}} \mathrm{C}^q_{ik}(\bar{x})\frac{\partial \bar{u}^q_k}{\partial x_i}+|\nabla \bar{u}_k|^2\,\bar{u}_k+\mathrm{D}_k(\bar{x})\,|\nabla \bar{t}_k|^2,\qquad (2.10)$$

where B^q_{ijk}, C^q_{ik} and D_k are bounded and continuous.

Let $a_k=(x_k, y_k)$; then we define

$$\tilde{u}_k(\tilde{x}, \tilde{y})=\bar{u}_k\left(\frac{1}{2}+\frac{1}{\theta_k}\left(\tilde{x}-\frac{1}{2}\right), \frac{1}{\theta_k}\tilde{y}+y_k\right),$$
$$\tilde{t}_k(\tilde{x}, \tilde{y})=\bar{t}_k\left(\frac{1}{2}+\frac{1}{\theta_k}\left(\tilde{x}-\frac{1}{2}\right), \frac{1}{\theta_k}\tilde{y}+y_k\right);$$

from (2.10) we get

$$-\Delta\tilde{u}_k+(\alpha_k-1)\sum_{\substack{i, j=1, 2\\ q=1, 2, 3}}\tilde{B}^q_{ijk}\frac{\partial^2\tilde{u}^q_k}{\partial x_i\,\partial x_j}$$
$$=\frac{1}{\theta_k}(\alpha_k-1)\sum_{\substack{i=1, 2\\ q=1, 2, 3}}\tilde{C}^q_{ik}\frac{\partial\tilde{u}^q_k}{\partial x_i}+|\nabla\tilde{u}_k|^2\tilde{u}_k+\tilde{D}_k\frac{1}{\theta^2_k}|\nabla\tilde{t}_k|^2.$$

As in [1], we have that $\tilde{u}_k$ converges to some $\tilde{u}\in C^1_{\text{loc}}(\tilde{U})$, and moreover, by (2.8), $\theta_k^{-2}|\nabla\tilde{t}_k|^2\to 0$, so $\tilde{u}$ satisfies the equation $-\Delta\tilde{u}=\tilde{u}|\nabla\tilde{u}|^2$ in U. At this point, we get a contradiction arguing as in [1]. ■

3. PROOF OF LEMMA 2.4

Let $\tilde{\beta}\in C^1(\mathbf{R}^3,]0, +\infty[)$ as in Remark 2.1, and let $\tilde{E}_\alpha: H^{1, 2\alpha}_v(\Omega, \mathbf{R}^3)\to\mathbf{R}$ be the functional

$$\tilde{E}_\alpha(u, t)=\int_\Omega|\nabla u|^2\,dx-\int_\Omega\tilde{\beta}(u)|\nabla t|^2\,dx,$$

so $\tilde{E}_\alpha(u, t)=E_\alpha(u, t)$for every u such that $|u|=1$. Then we set

$$\tilde{F}_\alpha(u)=\tilde{E}_\alpha(u, t_u), \text{so } \tilde{F}_\alpha(u)=F_\alpha(u)\text{ if }|u|=1.$$

For every $\varphi\in H^{1, 2\alpha}_0(\Omega, \mathbf{R}^3)$ we set $\Gamma(\varepsilon)=(u+\varepsilon\varphi)|u+\varepsilon\varphi|^{-1}$; since

$$\langle F'_\alpha(u), \varphi\rangle=\frac{d}{d\varepsilon}\tilde{F}_\alpha(\Gamma(\varepsilon))_{|\varepsilon=0}=\langle\tilde{F}'_\alpha(u), \varphi-u(u|\varphi)\rangle,$$

and

$$\left\langle\frac{\partial E_\alpha}{\partial u}(u, t_u), \varphi\right\rangle=\left\langle\frac{\partial\tilde{E}'_\alpha}{\partial u}(u, t_u), \varphi-u(u|\varphi)\right\rangle,$$

in order to prove Lemma 2.4, it is enough to show that

$$\langle F'_\alpha(u), v\rangle=\left\langle\frac{\partial\tilde{E}_\alpha}{\partial u}(u, t_u), v\right\rangle\tag{3.1}$$

for every $u \in H^{1,2\alpha}_{\mathrm{V}}(\Omega, \mathbf{R}^3)$ and every $v \in H^{1,2\alpha}_0(\Omega, \mathbf{R}^3)$. The proof of (3.1) is similar to the proof of Lemma 2.2 in [7]. We sketch it for the reader convenience.

Step 1. – $\tilde{F}_\alpha$ is continuous. In fact,

$$\tilde{E}_\alpha(u, t_v) - \tilde{E}_\alpha(v, t_v) \leqq \tilde{E}_\alpha(u, t_u) - \tilde{E}_\alpha(v, t_v)$$
$$= \tilde{F}_\alpha(u) - \tilde{F}_\alpha(v) \leqq \tilde{E}_\alpha(u, t_u) - \tilde{E}_\alpha(v, t_u), \text{ so } \tilde{F}_\alpha(u) - \tilde{F}_\alpha(v) \to 0 \text{ as } v \to u.$$

Step 2. – The map $u \mapsto t_u$ is continuous from $H^{1,2\alpha}_{\mathrm{V}}(\Omega, \mathbf{R}^3)$ to $H^{1,2}_\tau(\Omega, \mathbf{R}^3)$. For if not, there exist $u \in H^{1,2\alpha}_{\mathrm{V}}(\Omega, \mathbf{R}^3)$, $(u_n) \subset H^{1,2\alpha}_{\mathrm{V}}(\Omega, \mathbf{R}^3)$ and $\varepsilon > 0$ such that $u_n \to u$ and $\| t_u - t_{u_n} \| \geqq \varepsilon$. Since $t \mapsto \int_\Omega \beta(u) |\nabla t|^2 \, dx$ verifies the Palais-Smale condition, there exists $\delta > 0$ such that

$$\sup \{ \tilde{E}_\alpha(u, t) \mid t \in H^{1,2}_\tau, \ \| t - t_u \| = \varepsilon/2 \} \leqq \tilde{E}_\alpha(u, t_u) - \delta. \tag{3.2}$$

From $\tilde{F}_\alpha(u_n) \to \tilde{F}_\alpha(u)$ and $\tilde{E}_\alpha(u_n, t_u) \to \tilde{F}_\alpha(u)$, we get, for n large enough, $\tilde{E}_\alpha(u, t_u) - \delta/2 \leqq \tilde{E}_\alpha(u_n, t_{u_n})$, $\tilde{E}_\alpha(u, t_u) - \delta/2 \leqq \tilde{E}_\alpha(u_n, t_u)$. Since $\tilde{E}_\alpha(u_n, .)$ is concave, we have

$$\min \{ \tilde{E}_\alpha(u_n, t) \mid t = t_u + \lambda (t_n - t_u), \quad \lambda \in [0, 1] \} \geqq \tilde{E}_\alpha(u, t_u) - \delta/2. \tag{3.3}$$

Let $r_n \in \partial B(t_u, \varepsilon/2) \cap \{ t_u + \lambda(t_n - t_u) \mid \lambda \in [0, 1] \}$. Then, by (3.2), (3.3):

$$\tilde{E}_\alpha(u, t_u) - \delta \geqq \tilde{E}_\alpha(u, r_n) = \tilde{E}_\alpha(u, r_n) - \tilde{E}_\alpha(u_n, r_n)$$
$$+ \tilde{E}_\alpha(u_n, r_n) \geqq \tilde{E}_\alpha(u, r_n) - \tilde{E}_\alpha(u_n, r_n) + \tilde{E}_\alpha(u, t_u) - \delta/2,$$

so we get $\delta/2 \leqq \tilde{E}_\alpha(u_n, r_n) - \tilde{E}_\alpha(u, r_n)$. It is easy to check that the right-hand side of the last inequality tends to zero as $n \to \infty$, so we have a contradiction, and the claim follows.

Step 3. – Fix $u \in H^{1,2\alpha}_{\mathrm{V}}(\Omega, \mathbf{R}^3)$, and let $v \in H^{1,2\alpha}_0(\Omega, \mathbf{R}^3)$, $\sigma > 0$. Since $\tilde{E}_\alpha(u, t_{u+\sigma v}) \leqq \tilde{E}_\alpha(u, t_u)$, we have:

$$\frac{\tilde{F}_\alpha(u + \sigma v) - \tilde{F}_\alpha(u)}{\sigma} \leqq \frac{\tilde{E}_\alpha(u + \sigma v, t_{u+\sigma v}) - \tilde{E}_\alpha(u, t_{u+\sigma v})}{\sigma}.$$

From step 2 we get:

$$\limsup_{\sigma \to 0^+} \frac{\tilde{F}_\alpha(u + \sigma v) - \tilde{F}_\alpha(u)}{\sigma} \leqq \left\langle \frac{\partial \tilde{E}_\alpha}{\partial u}(u, t_u), v \right\rangle.$$

Moreover $\tilde{F}_\alpha(u + \sigma v) - \tilde{F}_\alpha(u) \geqq \tilde{E}_\alpha(u + \sigma v, t_u) - \tilde{E}_\alpha(u, t_u)$, so

$$\left\langle \frac{\partial \tilde{E}_\alpha}{\partial u}(u, t_u), v \right\rangle \leqq \liminf_{\sigma \to 0^+} \frac{\tilde{F}_\alpha(u + \sigma v) - \tilde{F}_\alpha(u)}{\sigma},$$

and we get (3.1). ■

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