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Existence of geodesics for the Lorentz metric of a stationary gravitational field (*)

by

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ABSTRACT. — Let $g=g(z)$ $(z=(z_0, \ldots, z_3) \in \mathbb{R}^4)$ be a Lorentz metric (with signature $+, -, -, -$) on the space-time manifold $\mathbb{R}^4$. Suppose that g is stationary, *i.e.* g does not depend on z_0. Then we prove, under some other mild assumptions on g, that for any two points $a, b \in \mathbb{R}^4$ there exists a geodesic, with respect to g, joining a and b.

Key words : Lorentz metric, geodesic, critical point.

RÉSUMÉ. — Soit $g=g(z)$ $(z=(z_0, \ldots, z_3) \in \mathbb{R}^4)$ une métrique de Lorentz (avec signature $+, -, -, -$) sur l'espace-temps $\mathbb{R}^4$. On suppose que g soit stationnaire, c'est-à-dire indépendante de z_0. Nous démontrons, sous des autres convenable hypothèses sur g, l'existence d'arcs de géodésique joignant deux points a, b arbitrairement donné dans $\mathbb{R}^4$.

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0. INTRODUCTION AND STATEMENT OF THE RESULTS

In General Relativity a gravitational field is described by a symmetric, second order tensor

$$g \equiv g(z)[.,.], \qquad z=(z_0, \ldots, z_3) \in \mathbb{R}^4$$

on the space-time manifold $\mathbb{R}^4$. The tensor g is assumed to have the signature $+, -, -, -$; namely for all $z \in \mathbb{R}^4$ the bilinear form $g(z)[.,.]$ possesses one positive and three negative eigenvalues. The "pseudometric" induced by g is called Lorentz-metric.

In this paper we study the existence of geodesics, with respect to g, connecting two points a, $b \in \mathbb{R}^4$.

To this end we consider the "action" functional related to g, *i.e.*

$$f(z)=\frac{1}{2}\int_0^1 g(z(s))[\dot{z}(s), \dot{z}(s)]\, ds = \frac{1}{2}\int_0^1 \sum_{i,j=0}^{3} g_{ij}(z(s))\, \dot{z}_i(s)\, \dot{z}_j(s)\, ds \qquad (0.1)$$

where g_{ij} $(i,j=0, \ldots, 3)$ denote the components of g and $z=z(s)$ belongs to the Sobolev space

$$H^1 \equiv H^1((0,1), \mathbb{R}^4)$$

of the curves $z:(0,1) \to \mathbb{R}^4$ which are square integrable with their first derivative $\dot{z}=\dfrac{dz}{ds}$. If g is smooth, f defined in (0.1) is Fréchet differentiable in H^1. Let a, $b \in \mathbb{R}^4$, then a geodesic joining a and b is a critical point of f on the manifold

$$M=\{ z \in H^1 \mid z(0)=a, z(1)=b \}. \qquad (0.2)$$

Due to the indefinitess of the metric g it is easy to see that the functional (0.1) is unbounded both from below and from above even modulo submanifolds of finite dimension or codimension. Then the Morse index of a geodesic is $+\infty$, in contrast with the situation for positive definite Riemannian spaces. This fact causes difficulties in the research of a geodesic connecting a and b and actually such a geodesic, in general, does not exist (*cf.* [3], § 5.2 or [5], remark 1.14).

However the above difficulties can be overcome if the events a, b are causally related, namely if a, b can be joined by a smooth curve $z=z(s)$ such that

$$g(z(s))[\dot{z}(s), \dot{z}(s)] \geqq 0 \quad \text{for all } s \in (0,1). \qquad (0.3)$$

Such a curve is called causal.

In this case, under mild assumptions on g, the existence of a causal geodesic joining a, b can be achieved just maximizing the functional

$$f^*(z)=\int_0^1 \sqrt{g(z)(s))[\dot{z}(s),\dot{z}(s)]}\,ds$$

over all the causal curves in M (*cf.* [1], [8] or [3], chapt. 6).

Here we are interested to find sufficient conditions on the metric tensor g which guarantee the existence of geodesics connecting *any* two given points $a, b \in \mathbb{R}^4$.

We shall prove the following result.

THEOREM 0.1. – *Let* $g_{ij}(i,j=0,\ldots,3)$ *denote the components of the metric tensor* g. *We assume that:*

(g_1) $g_{ij} \in C^1(\mathbb{R}^4, \mathbb{R})$ $(i,j=0,\ldots,3)$.

(g_2) $g_{00}(z) \geqq \nu > 0$ *for all* $z \in \mathbb{R}^4$.

(g_3) *There exists* $\mu > 0$ *s. t.*

$$-\sum_{i,j=1}^{3} g_{ij}(z)\,\xi_i\,\xi_j \geqq \mu\,|\,\xi\,|^2 \quad \textit{for all } z \in \mathbb{R}^4$$

and all

$$\xi=(\xi_1,\xi_2,\xi_3)\in\mathbb{R}^3.$$

(g_4) *The functions* $g_{0i}(i=0,\ldots,3)$ *are bounded.*

(g_5) $\dfrac{\partial g_{ij}}{\partial z_0}(z)=0$ *for all* $z \in \mathbb{R}^4$.

Then for any two points $a, b \in \mathbb{R}^4$ *there exists a geodesic, with respect to the metric* g, *joining* a *and* b.

The assumptions $(g_1), \ldots, (g_4)$ are reasonably mild.

The most restrictive assumption is (g_5) which means that the gravitational field is stationary (*cf.* [4], § 88). The proof of theorem 0.1 is attained by using some minimax arguments which have been recently developed in the study of nonlinear differential equations having a variational structure (*cf.* e. g. [7] for a review on these topics).

1. PROOF OF THEOREM 0.1

The manifold M in H^1 defined in (0.2) can be written as follows

$$M=\bar{z}+H_0^1$$

where

$$\bar{z}=a+(b-a)s, \qquad s\in(0,1)$$

and

$$H_0^1=\{z\in H^1\,|\,z(0)=z(1)=0\}.$$

In order to prove theorem 0.1 we shall first carry out a finite dimensional approximation.

Let $n\in\mathbb{N}$ and set

$$M_n=\bar{z}+H_n \tag{1.1}$$

where

$$H_n=\text{span}\{\varphi_j\sin\pi ls: j=0,\ldots,3;\, l=1,\ldots,n\}$$

$\varphi_j\,(j=0,\ldots,3)$ being the canonical base in $\mathbb{R}^4$.

Moreover we set

$$\begin{aligned} V_n&=\text{span}\{\varphi_0\sin\pi ls; l=1,\ldots,n\}\\ W_n&=\text{span}\{\varphi_j\sin\pi ls: j=1,2,3;\, l=1,\ldots,n\}\\ S_n&=\bar{z}+V_n\\ Q_n(R)&=\bar{z}+W_n\cap B_R \end{aligned} \tag{1.2}$$

where

$$B_R=\{z\in H_0^1\,|\,\|z\|\leqq R\},\qquad R>0$$

and $\|.\|$ denotes the standard norm in the Sobolev space H^1. Finally we set

$$f_n=f_{|M_n} \tag{1.3}$$

where f denotes the functional defined in (0.1). First we prove the existence of a critical point of f_n, that is to say of a point $z_n\in M_n$ such that

$$\langle f'(z_n),\zeta\rangle=0\quad\text{for all}\,\zeta\in H_n$$

where f' is the Fréchet-differential of f and $\langle .,.\rangle$ denotes the pairing between H^1 and its dual. More precisely the following theorem holds.

THEOREM 1.1. — *Suppose that g satisfies the assumptions of theorem 0.1. Then there exists a critical point $z_n\in M_n$ of f_n such that*

$$c'\leqq f(z_n)\leqq c'' \tag{1.4}$$

where c' and c'' are two constants independent on n.

The proof of theorem 1.1 is based on a variant of the "saddle point theorem" of P. H. Rabinowitz [*cf.* [6] or propositions 2.1 and 2.2 in [2]). We need some lemmas.

LEMMA 1.2. — *Fix $n\in\mathbb{N}$ and $R>0$. Then S_n and the boundary $\partial Q_n(R)$ of $Q_n(R)$ link, namely for any continuous map $h: M_n\to M_n$ s. t. $h(z)=z$ for all $z\in\partial Q_n(R)$, we have*

$$h(Q_n(R))\cap S_n\neq\varnothing$$

Proof. — Let $h: \mathrm{M}_n \to \mathrm{M}_n$ s.t. $h(z)=z$ for all $z \in \partial \mathrm{Q}_n(\mathrm{R})$ and define

$$\tilde{h}: \mathrm{H}_n \to \mathrm{H}_n \qquad \text{s.t.} \quad \forall \gamma \in \mathrm{H}_n: \ \tilde{h}(\gamma) = h(\gamma + \bar{z}) - \bar{z}$$

It is easy to see that

$$\tilde{h}(\gamma) = \gamma, \qquad \forall \gamma \in \partial(\mathrm{B}_\mathrm{R} \cap \mathrm{W}_n)$$

Then by using the Brower degree (*cf.* [2], prop. 2.1 or [6]) it can be shown that there exists $\gamma \in \tilde{h}(\mathrm{W}_n \cap \mathrm{B}_\mathrm{R}) \cap \mathrm{V}_n$ and therefore $\bar{z} + \gamma \in h(\mathrm{Q}_n(\mathrm{R})) \cap \mathrm{S}_n$. □

We denote by $f'_{|\mathrm{M}_n}$ the Fréchet differential of f on the manifold M_n and by $\|.\|$ the standard norm in H^1. Moreover we set

$$t = z_0 \qquad \text{and} \qquad x = (z_1, z_2, z_3)$$

Now we prove that $f_{|\mathrm{M}_n}$ satisfies the Palais-Smale condition. More precisely the following lemma holds.

LEMMA 1.3. — *Let g satisfy the assumptions of Theorem 0.1. Let $\{z_k\}$ be a sequence in M_n such that*

$$f'_{|\mathrm{M}_n}(z_k) \to 0 \quad \text{as } k \to \infty \tag{1.5}$$

and

$$\{f(z_k)\} \text{ is bounded} \tag{1.6}$$

Then $\{z_k\}$ is bounded in the H^1 norm and consequently it is precompact.

Proof. — Since $z_k \in \mathrm{M}_n$, we can set

$$z_k = (t_k, x_k) = \bar{z} + (\tau_k, \xi_k)$$

with $\tau_k \in \mathrm{V}_n$ and $\xi_k \in \mathrm{W}_n$ [*cf.* (1.1), (1.2)].

By (1.5) we deduce that

$$\langle f'(z_k), \zeta \rangle = \varepsilon_k \|\zeta\| \quad \text{for all } \zeta \in \mathrm{H}_n \tag{1.7}$$

where $\varepsilon_k \to 0$ as $k \to \infty$.

Then for all $\zeta = (\tau, \xi)$, with $\tau \in \mathrm{V}_n$ and $\xi = (\xi_1, \xi_2, \xi_3) \in \mathrm{W}_n$, we have

$$\int_0^1 g(x_k)[\dot{z}_k, \dot{\zeta}]\, ds + \frac{1}{2} \int_0^1 \sum_{i,j=0}^{3} \sum_{l=1}^{3} \frac{\partial g_{ij}}{\partial x_l}(x_k) \,.\, \xi_l (\dot{z}_k)_i \,.\, (\dot{z}_k)_j \, ds = \varepsilon_k \|\zeta\|. \tag{1.8}$$

And, if we take $\zeta = (\tau_k, 0) = \tau_k$, we get

$$\int_0^1 g(x_k)[\dot{z}_k, \dot{\tau}_k]\, ds = \varepsilon_k \|\tau_k\| \tag{1.9}$$

Now set

$$T=\begin{pmatrix} +1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}. \tag{1.10}$$

Then

$$\tau_k=\frac{1}{2}[z_k-\bar{z}+T(z_k-\bar{z})]$$

and from (1.9) we get

$$\frac{1}{2}\int_0^1 g(x_k)[\dot{z}_k, \dot{z}_k-\dot{\bar{z}}]\,ds-\varepsilon_k\|\tau_k\|=-\frac{1}{2}\int_0^1 g(x_k)[\dot{z}_k, T(\dot{z}_k-\dot{\bar{z}})]\,ds. \tag{1.11}$$

By (1.6) there exists $c_1>0$ such that for all $k\in\mathbb{N}$

$$|f(z_k)|=\frac{1}{2}\left|\int_0^1 g(x_k)[\dot{z}_k, \dot{z}_k]\,ds\right|\leqq c_1.$$

From (1.11) we get

$$\begin{aligned}\frac{1}{2}\int_0^1 & g(x_k)[\dot{z}_k, T\dot{z}_k]\,ds \\ &\leqq c_1+\frac{1}{2}\int_0^1 g(x_k)[\dot{z}_k, \dot{\bar{z}}]\,ds+\frac{1}{2}\int_0^1 g(x_k)[\dot{z}_k, T\dot{\bar{z}}]\,ds+\varepsilon_k\|\tau_k\| \\ &=c_1+\int_0^1 \Big(g_{00}(x_k)\dot{t}_k+\sum_{i=1}^{3} g_{0i}(x_k)(\dot{x}_k)_i\Big)\dot{\bar{t}}.\,ds+\varepsilon_k\|\tau_k\| \end{aligned} \tag{1.12}$$

where $(\bar{t}, \bar{x})=\bar{z}$.

Since g_{0i} $(i=0, 1, 2, 3)$ are bounded, from (1.12) we easily get

$$\int_0^1 g(x_k)[\dot{z}_k, T\dot{z}_k]\,ds\leqq 2c_1+c_2\|z_k\|+2\varepsilon_k\|\tau_k\| \tag{1.13}$$

where c_2 is a positive constant depending on $\bar{t}$ and g_{0i} $(i=0, \ldots, 3)$.

Now it can be easily verified that

$$g(x_k)[\dot{z}_k, T\dot{z}_k]=g_{00}(x_k)\dot{t}_k^2-\sum_{i,j=1}^{3} g_{ij}(x_k)(\dot{x}_k)_i.(\dot{x}_k)_j. \tag{1.14}$$

From (1.13) and (1.14) and by using (g_2), (g_3) we get

$$c_3\|z_k\|^2\leqq 2c_1+c_2\|z_k\|+2\varepsilon_k\|\tau_k\| \tag{1.15}$$

where c_3 is a positive constant.

From (1.15) we deduce that

$$\{z_k\} \text{ is bounded in } H^1. \quad \square$$

Proof of Theorem 1.1. – Set

$$W=\overline{\sum_{n\in\mathbb{N}} W_n}, \qquad V=\overline{\sum_{n\in\mathbb{N}} V_n}$$

(the closures are taken in the H_0^1-norm)

$$S=\bar{z}+V, \qquad Q=Q(R)=\bar{z}+W\cap B_R.$$

It is easy to see that

$$f(z)\to-\infty \quad \text{as } \|z\|\to\infty, \qquad z\in\bar{z}+W$$

and

$$\inf f(S)>-\infty.$$

Then if R is large enough we get

$$\sup f(\partial Q(R))<\inf f(S).$$

Let $n\in\mathbb{N}$ and set

$$c_n=\inf_{h\in\mathscr{H}_n}\sup f(h(Q_n)) \tag{1.16}$$

where

$$\mathscr{H}_n=\{h: M_n\to M_n, h \text{ continuous and s.t. } h(u)=u, \forall u\in\partial Q_n\}$$

and Q_n is defined in (1.2).

By Lemma 1.2 c_n is well defined and

$$c'=\inf f(S)\leqq c_n\leqq \sup f(Q)=c''.$$

Moreover by lemma 1.3 $f_{|M_n}$ satisfies the Palais-Smale condition; then, by the saddle point theorem (*cf.* [6] or Theorem 2.3 in [2]), c_n defined by (1.16) is a critical value of $f_{|M_n}$. □

We are now ready to prove Theorem 0.1.

Proof of Theorem 0.1. – Consider the sequence $\{z_n\}$ of the critical points of $f_{|M_n}$ found in Theorem 1.1.

The same arguments used in proving lemma 1.3 show that $\{z_n\}$ is bounded in H^1, then there exists a subsequence, which we continue to call $\{z_n\}$ such that

$$z_n\to z^* \quad \text{weakly in } H^1. \tag{1.17}$$

We shall prove that

$$z_n\to z^* \quad \text{strongly in } H^1. \tag{1.18}$$

We set

$$z_n=\bar{z}+\zeta_n, \qquad \zeta_n=(\tau_n,\xi_n)$$

$$z^* = \bar{z} + \zeta^*, \qquad \zeta^* = (\tau^*, \xi^*)$$

and $(\tau_n^*, \xi_n^*) = \zeta_n^* = P_n \zeta^*$

P_n being the projection on H_n.

Since z_n are critical points of $f_{|M_n}$ we have

$$\langle f'(z_n), T(\zeta_n - \zeta_n^*) \rangle = \int_0^1 g(x_n)[\dot{z}_n, T(\dot{\zeta}_n - \dot{\zeta}_n^*)]\, ds - \frac{1}{2}\int_0^1 \sum_{i,j=0}^{3} \sum_{l=1}^{3} \frac{\partial g_{ij}}{\partial x_l}(x_n).(\xi_n - \xi_n^*)_l.(\dot{z}_n)_i.(\dot{z}_n)_j\, ds = 0 \quad (1.19)$$

where T is defined in (1.10) and $(t_n, x_n) = z_n$.

H^1 is compactly embedded into L^∞, then by (1.17), $\xi_n \to \xi^*$ in L^∞ and $\{z_n\}$ is bounded in L^∞. Therefore

$$\frac{\partial g_{ij}}{\partial x_l}(x_n)(\xi_n - \xi_n^*)_l \to 0 \quad \text{in } L^\infty \qquad (i,j = 0, \ldots, 3 \text{ and } l = 1, 2, 3) \quad (1.20)$$

Then from (1.19), (1.20), (1.17) we deduce that

$$\int_0^1 g(x_n)[\dot{z}_n, T(\dot{\zeta}_n - \dot{\zeta}_n^*)]\, ds = O(1). \quad (1.21)$$

In (1.21) and in the sequel $O(1)$ denotes a sequence converging to zero.

Since $z_n = \bar{z} + \zeta_n$ we have

$$\int_0^1 g(x_n)[\dot{\bar{z}}, T(\dot{\zeta}_n - \dot{\zeta}_n^*)]\, ds + \int_0^1 g(x_n)[\dot{\zeta}_n, T(\dot{\zeta}_n - \dot{\zeta}_n^*)]\, ds = O(1)$$

Then, since

$$T(\dot{\zeta}_n - \dot{\zeta}_n^*) \rightharpoonup 0 \quad \text{weakly in } L^2 \quad (1.22)$$

and $g_{ij}(x_n)(\dot{\bar{z}})_i$ converges (strongly) in L^∞, we get

$$\int_0^1 g(x_n)[\dot{\zeta}_n, T(\dot{\zeta}_n - \dot{\zeta}_n^*)]\, ds = O(1) \quad (1.23)$$

which can also be written as

$$\int_0^1 g(x_n)[(\dot{\zeta}_n - \dot{\zeta}_n^*), T(\dot{\zeta}_n - \dot{\zeta}_n^*)]\, ds + \int_0^1 g(x_n)[\dot{\zeta}_n^*, T(\dot{\zeta}_n - \dot{\zeta}_n^*)]\, ds = O(1)$$

by (1.22) and since $g_{ij}(x_n)(\dot{\zeta}_n^*)_i$ converges in L^2, we get

$$\int_0^1 g(x_n)[(\dot{\zeta}_n - \dot{\zeta}_n^*), T(\dot{\zeta}_n - \dot{\zeta}_n^*)]\, ds = O(1). \quad (1.24)$$

On the other hand,

$$\int_0^1 g(x_n)[(\dot{\zeta}_n - \dot{\zeta}_n^*), T(\dot{\zeta}_n - \dot{\zeta}_n^*)]\, ds \geq \text{const.} \|\dot{\zeta}_n - \dot{\zeta}_n^*\|_{L^2}^2. \qquad (1.25)$$

From (1.24) and (1.125) and since $\zeta_n \to \zeta^*$ in H_0^1 we get

$$z_n \to z^* \quad \text{in}\, H^1. \qquad (1.26)$$

Let us finally show that z^* is a critical point of $f_{|M}$. By (1.26) we have

$$\forall \zeta \in H_0^1, \quad \langle f'(z_n), \zeta \rangle \to \langle f'(z^*), \zeta \rangle \quad \text{as } n \to \infty. \qquad (1.27)$$

On the other hand

$$\langle f'(z_n), \zeta \rangle = \langle f'(z_n), \zeta_n \rangle + \langle f'(z_n), \zeta - \zeta_n \rangle \qquad (1.28)$$

where $\zeta_n = P_n \zeta$.

Since z_n is a critical point of $f_{|M_n}$ and $\zeta - \zeta_n \to 0$ as $n \to \infty$, from (1.28) we deduce that

$$\langle f'(z_n), \zeta \rangle = O(1). \qquad (1.29)$$

Finally from (1.27) and (1.29) we deduce that

$$\forall \zeta \in H_0^1, \quad \langle f'(z^*), \zeta \rangle = 0$$

and therefore z^* is a critical point of $f_{|M}$. □

REFERENCES

[1] A. AVEZ, Essais de géométrie riemannienne hyperbolique globale. Application à la Relativité Générale, *Ann. Inst. Fourier*, Vol. **132**, 1963, pp. 105-190.

[2] P. BARTOLO, V. BENCI and D. FORTUNATO, Abstract Critical Point Theorems and Applications to Some Nonlinear Problems with "Strong Resonance" at Infinity, *Journal of nonlinear Anal. T.M.A.*, Vol. **7**, 1983, pp. 981-1012.

[3] S. W. HAWKING and G. F. R. ELLIS, *The Large scale Structure of Space-Time*, Cambridge University Press, 1973.

[4] L. LANDAU and E. LIFCHITZ, *Théorie des champs*, Mir, 1970.

[5] R. PENROSE, Techniques of Differential Topology in Relativity, *Conference board of Math. Sc.*, Vol. **7**, S.I.A.M., 1972.

[6] P. H. RABINOWITZ, *Some Mini-Max Theorems and Applications to Nonlinear Partial Differential Equations, Nonlinear Analysis*, CESARI, KANNAN, WEINBERGER Ed., Academic Press, 1978, pp. 161-177.

[7] P. H. RABINOWITZ, Mini-Max Methods in Critical Point Theory with Applications to Differential Equations, *Conf. board Math. Sc. A.M.S.*, Vol. **65**, 1986.

[8] H. J. SEIFERT, Global Connectivity by Time Like Geodesic, *Zs. f. Naturfor.*, Vol. **22 a**, 1967, pp. 1256-1360.

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