Ann. Inst. Henri Poincaré,

Vol. 11, n° 2, 1994, p. 135-158. *Analyse non linéaire*

On parabolic initial-boundary value problems with critical growth for the gradient

by

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ABSTRACT. – We prove the existence of weak solutions for the initial-boundary value problem of the quasilinear parabolic equation

$$\frac{\partial u}{\partial t} + \mathrm{A}(u) + g(x, t, u, \mathrm{D}u) = f.$$

Here A is a Leray-Lions type operator from $\mathscr{V} = \mathrm{L}^p(0, \mathrm{T}; \mathrm{W}_0^{1, p}(\Omega))$ to its dual space $\mathscr{V}^*$, g is a nonlinear term with critical growth with respect to $\mathrm{D}u$ satisfying a sign condition and no growth condition with respect to u; f is a given element in $\mathscr{V}^*$.

Key words : Critical growth, lack of compactness.

Classification A.M.S. : 35 K 60, 47 H 05.

(*) The work of the first author was partly supported by SFB 256, Bonn, FRG, SFB 123 Heidelberg, FRG and the University of Oklahoma Research Council, Norman, OK, U.S.A.

RÉSUMÉ. — Nous démontrons l'existence des solutions faibles du problème parabolique quasilinéaire

$$\frac{\partial u}{\partial t}+\mathrm{A}(u)+g(x, t, u, \mathrm{D}u)=f.$$

avec conditions aux limites et initiales. Ici A est un opérateur de type Leray-Lions de l'espace $\mathscr{V}=\mathrm{L}^p(0, \mathrm{T}; \mathrm{W}_0^{1,p}(\Omega))$ à valeurs dans $\mathscr{V}^*$, g est un terme non linéaire à croissance critique en $\mathrm{D}u$, qui satisfait une condition de signe et dont la croissance en u n'est pas limitée; f est un élément donné de $\mathscr{V}^*$.

1. INTRODUCTION

On a cylinder $\mathrm{Q_T}=\Omega\times]0, \mathrm{T}[$, over the bounded smooth domain $\Omega\subset\mathbb{R}^\mathrm{N}$ we consider the parabolic initial-boundary value problem

$$(\mathrm{P})\qquad\begin{cases}\dfrac{\partial u}{\partial t}+\mathrm{A}(u)+g(x, t, u, \mathrm{D}u)=f \quad \text{in } \mathrm{Q_T},\\ u(x, t)=0 \quad \text{on } \partial\Omega\times]0, \mathrm{T}[,\\ u(x, 0)=0 \quad \text{in } \Omega,\end{cases}$$

where $\mathrm{A}(u)=-\sum_{i=1}^{\mathrm{N}}\mathrm{D}_i\mathrm{A}_i\ (x, t, u, \mathrm{D}u)$ is a classical divergence operator of Leray-Lions type with respect to the Sobolev space $\mathscr{V}=\mathrm{L}^p(0, \mathrm{T}; \mathrm{W}_0^{1,p}(\Omega))$ for some $p\in]1, \infty[$ and the perturbation g satisfies the growth condition

$$|g(x, t, u, \mathrm{D}u)|\leqq h(|u|)(|\mathrm{D}u|^p+\lambda_1(x, t))$$

for some continuous function $h:\mathbb{R}^+\to\mathbb{R}^+$. The growth is called critical since it is restricted only by the integration exponent of the underlying Sobolev space $\mathscr{V}$. In this situation we are lacking the compactness arguments of bounded sequences used to show the existence of weak solutions of (P) for general elements f in the dual space $\mathscr{V}^*$, *cf.*: [Li].

Recently, in the case of the corresponding elliptic equations the existence of weak solution was shown independently by Del Vecchio in [D] and by the first author in [La3]; see also [BBM]. For the parabolic problem only some partial results are obtained in the paper by Boccardo and Murat so far. Note also that the classical theory using $\mathrm{C}^\alpha-$*a priori* estimates needs more regularity properties of the data and inhomogenuity than those given by our hypotheses.

We are facing the situation that it is rather easy to construct weakly convergent sequences of approximating solutions with various methods, yet, in general it seems to be impossible to verify that their weak limits are indeed weak solutions. Only for one particular approximating sequence u_n we are going to show the pointwise convergence and the weak compactness of $|Du_n|^p$ in $L^1(Q_T)$, implying the strong convergence in $\mathscr{V}$ and hence existence of weak solutions.

The estimates to verify these two properties rest on the monotonicity of the leading differential operator. But the operator $\partial/\partial t$ is known to be monotone only if the domain of definition is small enough, see for instance, [Z, pp. 845]. In [LM] we dealt with unbounded operators of lower order and we had been able to establish enough "monotonicity properties" in a situation where partial integration still can be verified for test function, which have not compact support in [0, T]. This requires some *a priori* regularity of the solution which can be shown if the perturbation is relatively weakly compact in $L^1(Q_T)$. Boccardo and Murat pointed out that the difficulties with partial integration can be avoided, if it is possible to use cut-off functions with respect to time. Yet, their approach also needs the relatively weak compactness in $L^1(Q_T)$ which cannot be provided by the approximation schemes, if the growth of the perturbation is critical in the gradient *cf.*: [BM].

With the help of certain mollification with respect to time and cut-off functions as in [BM] we will show that there is enough "monotonicity" to justify the estimates providing the pointwise convergence of this particular sequence. However, we only get the equi-integrability locally in time. That leads to a solution in a very weak sense, in particular, it seems to be impossible to establish the energy equality. Therefore we consider an "extended" problem with the same weak assumptions on [0, T] as before, but with stronger condition on $[T, \mathscr{T}]$, say. This allows us to show the existence in the weak sense described above first, and verify the desired regularity properties of the solution, *a posteriori*.

In Section 2 we give the precise setting of the problem and state the main result. We gather properties of the associated truncated problem in Section 3 and we generalize a Lemma by Frehse to the parabolic case in Section 4 providing the pointwise convergence of the gradient. In Section 5 we show the strong convergence for the gradient. The final step of the existence proof and the regularity properties are presented in Section 6. In Section 7 we prove that the distribution u' has enough regularity to justify the estimates needed in Section 5.

2. ASSUMPTIONS AND THE MAIN RESULT

For the coefficients A_i, ($i=1, 2, \ldots, N$), of the operator A we introduce the following hypotheses.

(A_1) The functions $A_i(x, t, \eta, \xi)$ from $Q_T \times \mathbb{R} \times \mathbb{R}^N$ to $\mathbb{R}$ are measurable in $(x, t) \in Q_T$ and continuous in $(\eta, \xi) \in \mathbb{R} \times \mathbb{R}^N$.

(A_2) For all $(x, t) \in Q_T$ and $(\eta, \xi) \in \mathbb{R} \times \mathbb{R}^N$

$$|A_i(x, t, \eta, \xi)| \leqq c_1 (|\eta|^{p-1} + |\xi|^{q-1} + k_1(x, t))$$

with $1<p<\infty$, $q \leqq \dfrac{pN}{N-p}$, $c_1>0$ and $k_1 \in L^{p'}(Q_T)$, $p' = \dfrac{p}{p-1}$.

(A_3) For all $(x, t) \in Q_T$, $\eta \in \mathbb{R}$ and $\xi \neq \xi^*$ in $\mathbb{R}^N$

$$\sum_{i=1}^{N} \{A_i(x, t, \eta, \xi) - A_i(x, t, \eta, \xi^*)\}(\xi_i - \xi_i^*) > 0.$$

(A_4) For all $(x, t) \in Q_T$ and $(\eta, \xi) \in \mathbb{R} \times \mathbb{R}^N$

$$\sum_{i=1}^{N} A_i(x, t, \eta, \xi)\xi_i \geqq c_2 |\xi|^p - k_2(x, t)$$

with $c_2>0$ and $k_2 \in L^1(Q_T)$.

Conditions (A_1)-(A_3) are the parabolic versions of the so called Leray-Lions conditions providing the pseudomonotonicity of the quasilinear operator in the elliptic case. We need the strict ellipticity (A_4), the coercivness of the combined differential operator seems to be not sufficient.

The assumptions on the perturbation g as a function from $Q_T \times \mathbb{R} \times \mathbb{R}^N$ read as follows:

(G_1) $g(x, t, \eta, \xi)$ is measurable in $(x, t) \in Q_T$ and continuous in $(\eta, \xi) \in \mathbb{R} \times \mathbb{R}^N$.

(G_2) For all $(x, t) \in Q_T$ and $(\eta, \xi) \in \mathbb{R} \times \mathbb{R}^N$

$$|g(x, t, \eta, \xi)| \leqq h(|\eta|)(|\xi|^p + \lambda_1(x, t))$$

with $\lambda_1 \in L^1(Q_T)$ and some continuous non-decreasing function $h: \mathbb{R}_+ \to \mathbb{R}_+$.

(G_3) For all $(x, t) \in Q_T$ and $(\eta, \xi) \in \mathbb{R} \times \mathbb{R}^N$

$$g(x, t, \eta, \xi)\eta \geqq -\lambda_2(x, t)$$

with some nonnegative $\lambda_2 \in L^1(Q_T)$.

Even though (A_4) and (G_3) can be weakened by some obvious applications of the embedding theorems, the sign condition (G_3) is needed to obtain the existence of solutions for all elements f in the dual space. It should be noted that there is no growth restriction on the "lower order nonlinearity" of g as a function in u. Hence our work includes earlier results by Brezis and Browder and the authors, *cf.*: [BB, B, LM]. We introduce now the notion of weak solution of the problem (P) used here.

DEFINITION. — *Let* $\mathscr{V}=L^p(0, T; W_0^{1,p}(\Omega))$. *A function* u *in* $\mathscr{V} \cap L^\infty(0, T; L^2(\Omega))$ *with* $g(.,.,u, Du) \in L^1(Q_T)$ *is called a weak solution of* (P) *if*

$$-\int_{Q_T} u \frac{\partial \phi}{\partial t} dx\, dt + \int_{Q_T} \sum_{i=1}^{N} A_i(x, t, u, Du) D_i \phi\, dx\, dt + \int_{Q_T} g(x, t, u, Du) \phi\, dx\, dt = \langle f, \phi \rangle \quad (2.1)$$

for all $\phi \in \mathscr{V} \cap L^\infty(Q_T) \cap C^1([0, T]; L^2(\Omega))$ *with* $\phi(t)=0$ *in a neighborhood of* T. *The inhomogenuity* f *is a prescribed element in* $\mathscr{V}^*$, *the dual space of* $\mathscr{V}$.

We introduced a rather weak notion of solution and account for the regularity properties of the solution in the statements of the theorem, since this properties are additional informations furnished by the particular approximation scheme.

THEOREM. — *Suppose that the conditions* (A_1)-(A_4) *and* (G_1)-(G_3) *are satisfied. Then the problem* (P) *admits a weak solution for any given* $f \in \mathscr{V}^*$. *Furthermore, the weak solution obtained by the approximating scheme below has the properties*:

(i) $u \in C([0, T]; L^2(\Omega))$,

(ii) $$\int_\Omega u(\tau)\phi(\tau)\, dx - \int_{Q_\tau} u \phi'\, dx\, dt + \int_{Q_\tau} \sum_{i=1}^{N} A_i(x, t, u, Du) D_i \phi\, dx\, dt + \int_{Q_\tau} g(x, t, u, Du)\phi\, dx\, dt = \langle f, \phi \rangle_{Q_\tau}$$

for all $\tau \in (0, T]$ *and for all* $\phi \in \mathscr{V} \cap L^\infty(Q_T) \cap C^1([0, T]; L^2(\Omega))$,

(iii) (*Energy equality*) *For all* $\tau \in (0, T]$ *we have*

$$\frac{1}{2}\int_\Omega u^2(\tau)\, dx + \int_{Q_\tau} \sum_{i=1}^{N} A_i(x, t, u, Du) D_i u\, dx\, dt + \int_{Q_\tau} g(x, t, u, Du) u\, dx\, dt = \langle f, u \rangle_{Q_\tau}.$$

First we investigate the solutions of the related problem where the perturbation is truncated at the levels $\pm n$, say. One important advantage of this approach is that for every approximation u_n it is possible to use all of $\mathscr{V}$ as testspace and not only certain, in case of a Galerkin scheme, even finite dimensional subspaces.

3. THE TRUNCATED PROBLEM

For each $n \in \mathbb{N}$ we define the truncated perturbation g_n as a function

$$g_n(x, t, \eta, \xi) = \begin{cases} n \dfrac{g(x, t, \eta, \xi)}{|g(x, t, \eta, \xi)|}, & \text{if } |g(x, t, \eta, \xi)| > n, \\ g(x, t, \eta, \xi), & \text{if } |g(x, t, \eta, \xi)| \leqq n. \end{cases}$$

Then we consider the initial boundary value problem

$$\text{(TP)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} + \mathrm{A}(u) + g_n(x, t, u, \mathrm{D}u) = f \quad \text{in } \mathrm{Q_T}, \\ u(x, t) = 0 \quad \text{on } \partial\Omega \times]0, \mathrm{T}[, \\ u(x, 0) = 0 \quad \text{in } \Omega. \end{cases}$$

For $u, v \in \mathscr{V}$ we define

$$\langle \mathrm{F}(u), v \rangle = \int_{\mathrm{Q_T}} \sum_{i=1}^{\mathrm{N}} \mathrm{A}_i(x, t, u, \mathrm{D}u) \mathrm{D}_i v \, dx \, dt$$

and

$$\langle \mathrm{G}_n(u), v \rangle = \int_{\mathrm{Q_T}} g_n(x, t, u, \mathrm{D}u) v \, dx \, dt.$$

The classical theory of pseudomonotone mappings can be used to show the existence of weak solutions u_n if $p \geqq 2$ (*see* [Li], for example). In case $p < 2$ the existence can be shown, for instance, by time dependent Galerkin approximations as in [LM]. Hence for all $p > 1$ there is a sequence $\{u_n\}$ in $\mathrm{C}([0,\mathrm{T}], \mathrm{L}^2(\Omega)) \cap \mathscr{V}$ such that

$$\langle u_n', v \rangle + \langle \mathrm{F}(u_n), v \rangle + \langle \mathrm{G}_n(u_n), v \rangle = \langle f, v \rangle \tag{3.1}$$

for all $n \in \mathbb{N}$ and $v \in \mathscr{V}$, with $u(0) = 0$ in $\mathrm{L}^2(\Omega)$. Note that $\langle u_n', v \rangle$ is defined in the sense of distributions. Since $\langle \mathrm{F}(u_n), . \rangle$, $\langle \mathrm{G}_n(u_n), . \rangle$ and $\langle f, . \rangle$ are in $\mathscr{V}^*$ we can extend $\langle u_n', v \rangle$ to all $v \in \mathscr{V}$. For $\phi \in \mathrm{C}^1([0, \mathrm{T}], \mathrm{L}^2(\Omega)) \cap \mathscr{V}$ with $u(0) = 0$ we have

$$\langle u_n', \phi \rangle = \int_\Omega u_n(\mathrm{T}) \phi(\mathrm{T}) \, dx - \int_{\mathrm{Q_T}} u_n \phi' \, dx \, dt. \tag{3.2}$$

Furthermore, using similar arguments as in [LM] (*cf.* Section 7, too) we can show for all n that there is at least one solution u_n satisfying

$$\int_\Omega u_n(\tau)\phi(\tau)\,dx-\int_{Q_\tau} u_n\phi'\,dx\,dt + \int_{Q_\tau}\sum_{i=1}^{N} A_i(x,t,u_n,Du_n)\,D_i\phi\,dx\,dt + \int_{Q_\tau} g_n(x,t,u_n,Du_n)\,\phi\,dx\,dt=\langle f,\phi\rangle_{Q_\tau} \quad (3.3)$$

for all $\tau\in(0,T]$ and $\phi\in C^1([0,\tau];L^2(\Omega))\cap L^p(0,\tau;W_0^{1,p}(\Omega))$, and

$$\frac{1}{2}\int_\Omega u_n^2(\tau)\,dx+\int_{Q_\tau}\sum_{i=1}^{N} A_i(x,t,u_n,Du_n)\,D_i u_n\,dx\,dt + \int_{Q_\tau} g_n(x,t,u_n,Du_n)\,u_n\,dx\,dt=\langle f,u_n\rangle_{Q_\tau}. \quad (3.4)$$

For $\tau=T$ we get

$$\langle F(u_n),u_n\rangle\leqq\|f\|_{\mathscr{V}^*}\|u_n\|_{\mathscr{V}}+\|\lambda_2\|_{L^1(Q_T)}$$

which yields by (A_4) that $\{u_n\}$ is bounded in $\mathscr{V}$. Hence also $\{\langle F(u_n),u_n\rangle\}$ and $\{\langle G_n(u_n),u_n\rangle\}$ are bounded sequences. With

$$Q_T^{m,n}=\{(x,t)\in Q_T \mid |u_n(x,t)|\leqq m\}$$

we get by (G_2) and (G_3)

$$\int_{Q_T}|g_n(x,t,u_n,Du_n)|\,dx\,dt\leqq\int_{Q_T^{m,n}}|g_n(x,t,u_n,Du_n)|\,dx\,dt + \int_{Q_T\setminus Q_T^{m,n}}|g(x,t,u_n,Du_n)|\,|u_n|\,dx\,dt$$
$$\leqq h(m)(\|u_n\|_{\mathscr{V}}^p+\|\lambda_1\|_{L^1(Q_T)})+\langle G_n(u_n),u_n\rangle+2\|\lambda_2\|_{L^1(Q_T)}$$

implying that $\{|g_n(.,.,u_n,Du_n)|\}$ remains bounded in $L^1(Q_T)$.

Denoting $u_n'=(f-F(u_n))+(-G(u_n))$ we observe that $z_n=f-F(u_n)$ is bounded in $\mathscr{V}^*$ and $w_n=-G(u_n)$ is bounded in $L^1(Q_T)$. Thus we can invoke a result of [B, p. 162] to conclude that $\{u_n\}$ is strongly relatively compact in $L^p(Q_T)$. Collecting the results we get

PROPOSITION 1. — *Let* $\{u_n\}\subset\mathscr{V}$ *be the sequence of solutions of the truncated problem* (TP). *Then there are constants* K_1, K_2 *and* K_3 *such that* $\|u_n\|_{\mathscr{V}}\leqq K_1$, $|\langle G_n(u_n),u_n\rangle|\leqq K_2$ *and* $\|G_n(u_n)\|_{L^1(Q_T)}\leqq K_3$ *for all* $n\in\mathbb{N}$. *Moreover, for a subsequence we have* $u_n\rightharpoonup u$ *in* $\mathscr{V}$, $u_n\to u$ *in* $L^p(Q_T)$, $u_n(x,t)\to u(x,t)$ a.e. *in* Q_T, $D_i u_n\rightharpoonup D_i u$ *in* $L^p(Q_T)$ *for each* $i=1,2,\ldots,N$ *and* $F(u_n)\rightharpoonup\Lambda$ *in* $\mathscr{V}^*$.

4. A PARABOLIC VERSION OF A LEMMA BY FREHSE

Testing the equation (3.1) with $\psi \in \mathcal{V} \cap L^\infty(Q_T)$ we have

$$\langle u_n', \psi \rangle + \langle F(u_n), \psi \rangle + \langle G_n(u_n), \psi \rangle = \langle f, \psi \rangle. \tag{4.1}$$

Hence

$$|\langle u_n' + F(u_n), \psi \rangle| \leqq K_3 \|\psi\|_{L^\infty(Q_T)} + |\langle f, \psi \rangle| \tag{4.2}$$

holds for all $n \in \mathbb{N}$. The following parabolic generalization of a lemma by Frehse [F] provides the a.e. convergence of the gradients (*cf.* also [La3]).

LEMMA 1. — *Assume that* $\{u_n\}$ *is a bounded sequence in* $\mathcal{V}$, $u_n \to u$ *in* $L^p(Q_T)$ *and* $u_n \rightharpoonup u$ *in* $\mathcal{V}$. *Assume further that the inequality* (4.2) *holds for all* $n \in \mathbb{N}$ *and* $\psi \in \mathcal{V} \cap L^\infty(Q_T)$. *Then there exists a subsequence of* $\{u_n\}$ *such that* $Du_n(x, t) \to Du(x, t)$ a.e. *in* Q_T.

Before proving the Lemma we introduce some notations. For each $v \in \mathcal{V}$ we define

$$v_\nu(x, t) = \nu \int_{-\infty}^{t} \bar{v}(x, s)\, e^{\nu(s-t)}\, ds,$$

where $\bar{v}$ is the zero extension of v and $\nu > 0$. Throughout the paper the index ν always indicates this mollification with respect to time. We have $v_\nu \to v$ in $\mathcal{V}$ as $\nu \to \infty$ and $\dfrac{\partial}{\partial t}(v_\nu) = \nu\,(v - v_\nu)$ if $v(0) = 0$ (*see* [La2]). We also denote

$$T_\theta(s) = \begin{cases} s, & \text{if } |s| \leqq \theta, \\ \theta \operatorname{sgn} s, & \text{if } |s| > \theta, \end{cases}$$

and

$$S_\theta(r) = \int_0^r T_\theta(s)\, ds.$$

for each $\theta > 0$. Let ϕ be a function in $C_0^\infty(Q_T)$ such that $\|\phi\|_{L^\infty(Q_T)} = 1$ and $\phi \geqq 0$. In the sequel we shall use $T_\varepsilon[u_n - (u_k)_\nu]\phi$ as a test function ψ in (4.2) with $\varepsilon > 0$, $\nu > 0$ and $n, k \in \mathbb{N}$. We begin with the following

PROPOSITION 2.

$$|\langle u_n' - ((u_k)_\nu)', T_\varepsilon[u_n - (u_k)_\nu]\phi \rangle| \leqq o\left(\frac{1}{n}\right) + o\left(\frac{1}{\nu}\right) + o^\nu\left(\frac{1}{k}\right),$$

where the Landau symbols $o(\rho)$ *and* $o^\mu(\rho)$ *are real numbers such that, if* $\rho \to 0$ *also* $o(\rho) \to 0$, *respectively* $o^\mu(\rho) \to 0$ *for any fixed* μ.

Proof. — As in [BM] we obtain

$$\langle u_n' - ((u_k)_\nu)', T_\varepsilon[u_n - (u_k)_\nu]\phi \rangle = -\int_{Q_T} S_\varepsilon[u_n - (u_k)_\nu]\, \phi'\, dx\, dt.$$

Since S_ε is a convex function we have

$$\left|\int_{Q_T}\phi' S_\varepsilon\left\{3\frac{u_n-u}{3}+3\frac{u-u_\nu}{3}+3\frac{u_\nu-(u_k)_\nu}{3}\right\}dx\,dt\right|$$
$$\leqq \|\phi'\|_{L^\infty(Q_T)}\frac{1}{3}\int_{Q_T}\{S_\varepsilon[3(u_n-u)]+S_\varepsilon[3(u-u_\nu)]+S_\varepsilon[3(u_\nu-(u_k)_\nu]\}\,dx\,dt$$

and hence the assertion follows from the continuity of $S_\varepsilon(\tau)$ and its linear growth as $\tau\geqq\varepsilon$.

PROPOSITION 3.

$$\langle((u_k)_\nu)',\ T_\varepsilon[u_n-(u_k)_\nu]\phi\rangle\geqq o^{\nu,n}\left(\frac{1}{k}\right)+o^\nu\left(\frac{1}{n}\right).$$

Proof. — By the definition of $(u_k)_\nu$ we have

$$\langle((u_k)_\nu)',\ T_\varepsilon[u_n-(u_k)_\nu]\phi\rangle=\nu\int_{Q_T}(u_k-(u_k)_\nu)\,T_\varepsilon[u_n-(u_k)_\nu]\,\phi\,dx\,dt$$
$$=\nu\int_{Q_T}(u-u_\nu)\,T_\varepsilon[u-u_\nu]\,\phi\,dx\,dt$$
$$+\nu\int_{Q_T}(u-u_\nu)\{T_\varepsilon[u_n-u_\nu]-T_\varepsilon[u-u_\nu]\}\,\phi\,dt\,dt$$
$$+\nu\int_{Q_T}(u-u_\nu)\{T_\varepsilon[u_n-(u_k)_\nu]-T_\varepsilon[u_n-u_\nu]\}\,\phi\,dx\,dt$$
$$+\nu\int_{Q_T}\{(u_k-(u_k)_\nu)-(u-u_\nu)\}\,T_\varepsilon[u_n-(u_k)_\nu]\,\phi\,dx\,dt.$$

The first integral is non-negative, the second integral is of the form $o^\nu\left(\frac{1}{n}\right)$ and the third integral of the form $o^{\nu,n}\left(\frac{1}{k}\right)$. Also the last integral is of the form $o^\nu\left(\frac{1}{k}\right)$, since $u_k\to u$ in $L^p(Q_T)$ as $k\to\infty$ and $\|T_\varepsilon[u_n-(u_k)_\nu]\phi\|_{L^\infty(Q_T)}\leqq C_\varepsilon$ where C_ε is a constant independent on n, k and ν. Hence the assertion follows.

Proof of Lemma 1. — Testing the inequality (4.2) with $\psi=T_\varepsilon[u_n-(u_k)_\nu]\phi$ and using the Propositions 2 and 3 we obtain for each

$\varepsilon>0$ the estimate

$$\int_{Q_T}\sum_{i=1}^{N} A_i(x, t, u_n, Du_n)\, D_i\{T_\varepsilon[u_n-(u_k)_\nu]\,\phi\}\,dx\,dt$$
$$\leqq K_3\,\|T_\varepsilon[u_n-(u_k)_\nu]\,\phi\|_{L^\infty(Q_T)}+|\langle f, T_\varepsilon[u_n-(u_k)_\nu]\,\phi\rangle|$$
$$+o^\nu\left(\frac{1}{n}\right)+o\left(\frac{1}{n}\right)+o\left(\frac{1}{\nu}\right)+o^\nu\left(\frac{1}{k}\right)+o^{\nu, n}\left(\frac{1}{k}\right).$$

Since $(u_k)_\nu \to u_\nu$ in $\mathscr{V}$ we can keep n and ν fixed and let $k\to\infty$ to get

$$\int_{Q_T}\sum_{i=1}^{N} A_i(x, t, u_n, Du_n)\, D_i\{T_\varepsilon[u_n-u_\nu]\,\phi\}\,dx\,dt$$
$$\leqq o\left(\frac{1}{n}\right)+o\left(\frac{1}{\nu}\right)+o^\nu\left(\frac{1}{n}\right). \quad (4.3)$$

We have $u_\nu \to u$ in $\mathscr{V}$ and $u_n\to u$ in $L^p(Q_T)$. Hence we are able to choose a sequence $\{Q_m\}_{m=1}^\infty$ of subsets of Q_T and a subsequence $\{u_{n_\nu}\}_{\nu=1}^\infty$ of $\{u_n\}$ with the following properties:

(i) $Q_m \subset Q_{m+1}$, $m\in\mathbb{N}$, and $\left|Q_T\setminus \bigcup_{m=1}^\infty Q_m\right|=0$,

(ii) $\|u\|_{L^\infty(Q_m)}\leqq m$, $\|Du\|_{L^\infty(Q_m)}\leqq m$,

(iii) $|u_{n_\nu}(x, t)-u_\nu(x, t)|\leqq \dfrac{1}{m}$, in Q_m for all $\nu \geqq \nu_0(m)$,

(iv) $I_\nu := \displaystyle\int_{Q_T}\left\{\sum_{i=1}^{n} A_i(x, t, u_{n_\nu}, Du_{n_\nu})\, D_i T_\varepsilon[u_{n_\nu}-u_\nu]\right\}\phi\,dx\,dt \leqq o\left(\frac{1}{\nu}\right).$

Note that (iv) follows from (4.3) since $T_\varepsilon[u_{n_\nu}-u_\nu]\,D_i\phi \to 0$ in $L^p(Q_T)$. For $\varepsilon=\dfrac{1}{m}$ and $\nu\geqq\nu_0(m)$ we can write

$$\int_{Q_m}\left\{\sum_{i=1}^{N} A_i(x, t, u_{n_\nu}, Du_{n_\nu})\, D_i[u_{n_\nu}-u_\nu]\right\}\phi\,dx\,dt$$
$$=\int_{Q_m}\left\{\sum_{i=1}^{N} A_i(x, t, u_{n_\nu}, Du_{n_\nu})\, D_i T_\varepsilon[u_{n_\nu}-u_\nu]\right\}\phi\,dx\,dt$$
$$=I_\nu-\int_{Q_T\setminus Q_m}\left\{\sum_{i=1}^{N} A_i(x, t, u_{n_\nu}, Du_{n_\nu})\, D_i T_\varepsilon[u_{n_\nu}-u_\nu]\right\}\phi\,dx\,dt$$
$$\leqq o^m\left(\frac{1}{\nu}\right)-I.$$

Denoting

$$Q_m^\nu = \left\{ (x,\, t) \in Q_T \,\middle|\, |u_{n_\nu}(x,\, t) - u_\nu(x,\, t)| \leqq \frac{1}{m} \right\}$$

we have by (A_4) and by the facts that $D_i u_\nu \to D_i u$ in $L^p(Q_T)$ and $D_i T_\varepsilon [u_{n_\nu} - u_\nu] = 0$ on $Q_T \setminus Q_m^\nu$ the following estimate

$$\begin{aligned} -I &= -\int_{(Q_T \setminus Q_m) \cap Q_m^\nu} \left\{ \sum_{i=1}^N A_i(x,\, t,\, u_{n_\nu},\, Du_{n_\nu}) D_i(u_{n_\nu} - u_\nu) \right\} \phi \, dx \, dt \\ &\leqq \int_{(Q_T \setminus Q_m) \cap Q_m^\nu} k_2(x,\, t) \, dx \, dt \\ &\quad + \int_{(Q_T \setminus Q_m) \cap Q_m^\nu} \sum_{i=1}^N A_i(x,\, t,\, u_{n_\nu},\, Du_{n_\nu}) D_i(u_\nu \phi) \, dx \, dt \\ &\leqq o\left(\frac{1}{m}\right). \end{aligned}$$

Since $A_i(x,\, t,\, u_{n_\nu},\, Du_\nu)$ converges strongly in $L^{p'}(Q_T)$ we have

$$\begin{aligned} 0 &\leqq \int_{Q_m} \sum_{i=1}^N \{ A_i(x,\, t,\, u_{n_\nu},\, Du_{n_\nu}) - A_i(x,\, t,\, u_{n_\nu},\, Du_\nu) \} (D_i u_{n_\nu} - D_i u_\nu) \phi \, dx \, dt \\ &\leqq o\left(\frac{1}{m}\right) + o^m\left(\frac{1}{\nu}\right). \end{aligned}$$

Now we are in the position to employ Lemma 6 of [La1] yielding $D_i u_{n_\nu}(x,\, t) \to D_i u(x,\, t)$ a.e. in $\operatorname{supp}(\phi)$ for each $i = 1, 2, \ldots, N$ and for all suitable $\phi \in C_0^\infty(\Omega)$; hence the proof is complete.

5. STRONG CONVERGENCE OF THE GRADIENTS IN $L^p(Q_T)$

In order to establish the strong convergence it is enough to show that the sequence $\{ |Du_n|^p \}$ is equi-integrable since we already have the convergence a.e. from the previous section. First of all we remark, however, that the sequence of solutions $\{u_n\}$ of (TP) can be obtained on the extended cylinder $Q_{\mathscr{T}}$ with $\mathscr{T} = T + 1$, say, where we assume in addition to (A_1)-(A_4) and (G_1)-(G_3) on $Q_{\mathscr{T}}$ that g has a more restrictive growth in $Q_{\mathscr{T}} \setminus Q_T$. For instance the assumption

(G_4) $\quad |g(x,\, t,\, \eta,\, \xi)| \leqq c_3 (|\xi|^{p-1} + |\eta|^{p-1} + \lambda_3(x,\, t))$ $\quad$ for all $(x,\, t) \in Q_{\mathscr{T}} \setminus Q_T$
with some constant $c_3 > 0$ and $\lambda_3 \in L^1(Q_{\mathscr{T}} \setminus Q_T)$.
is sufficient for our purpose. Obviously all the facts of the previous section are true for problem (P) on $Q_{\mathscr{T}}$, also. We therefore have a sequence of

solutions $\{u_n\} \subset \mathscr{V} = L^p(0, \mathscr{T}; W_0^{1,p}(\Omega))$ of the truncated problem satisfying

$$\langle u_n', v \rangle + \langle F(u_n), v \rangle + \langle G_n(u_n), v \rangle = \langle f, v \rangle \tag{5.1}$$

for all $v \in \mathscr{V}$. The following results will be proved in section 7.

LEMMA 2. — *Let* $\phi \in C^1(\Omega \times [0, \mathscr{T}])$ *with* $\phi \equiv 1$ *in* $\Omega \times [0, T]$ *and* $\phi(x, \mathscr{T}) = 0$ *in* Ω. *Then*

$$\langle u_n', \phi [T_\theta(\phi u_n) - (T_\theta(\phi u_k))_\nu] \rangle \geqq o\left(\frac{1}{\nu}\right) + o^{n, \nu}\left(\frac{1}{k}\right) + o^\nu\left(\frac{1}{n}\right)$$

for all $\theta > 0$. (*As always in this note, the index* ν *indicates the smoothing with respect to time as defined above.*)

LEMMA 3. — *Let* ϕ *be as in Lemma* 2. *Then for all* $\sigma > 0$ *we have*

$$\langle u_n', \phi [\phi u_n - (T_\sigma(\phi u_k))_\nu] \rangle \geqq o\left(\frac{1}{\nu}\right) + o^{n, \nu}\left(\frac{1}{k}\right) + o\left(\frac{1}{n}\right).$$

The next propositions provide the equi-integrability of $\{|Du_n|^p\}$. Since our problem is defined in $Q_{\mathscr{T}}$ now we always can assume that $\phi \equiv 1$ in Q_T. For each $\theta > 0$ and $n \in \mathbb{N}$ we define

$$P_n^\theta = \{(x, t) \in Q_T \mid |u_n(x, t)| > \theta\},$$
$$B_n^\sigma = \{(x, t) \in Q_T \mid |Du_n(x, t)| > \sigma\}.$$

With this notation we have

PROPOSITION 4. — *If* $\theta > 0$ *satisfies the condition* $\theta h(\theta) < \frac{1}{4} c_2$, *then the sequence* $|Du_n|^p \chi_{\complement P_n^\theta} \chi_{[0, T]}$ *is equi-integrable.*

Proof. — It will be clearly sufficient to show that

$$\int_{Q_T} |Du_n|^p \chi_{\complement P_n^\theta} \chi_{B_n^\sigma} \, dx \, dt \leqq \omega$$

with $\omega = \omega(\sigma, \nu, n) = o\left(\frac{1}{\sigma}\right) + o^\sigma\left(\frac{1}{\nu}\right) + o^{\sigma, \nu}\left(\frac{1}{n}\right)$ for some parameter ν. By (A_4) we have

$$c_2 \int_{Q_T \cap \complement P_n^\theta \cap B_n^\sigma} |Du_n|^p \, dx \, dt \leqq o\left(\frac{1}{\sigma}\right) + \int_{Q_T \cap \complement P_n^\theta \cap B_n^\sigma} \sum_{i=1}^N A_i(x, t, u_n, Du_n) D_i u_n \, dx \, dt$$
$$= o\left(\frac{1}{\sigma}\right) + \int_{Q_T \cap \complement P_n^\theta \cap B_n^\sigma} \sum_{i=1}^N A_i(x, t, u_n, Du_n) D_i [T_\theta(u_n) - T_\theta(u)] \, dx \, dt$$

Choosing $\phi \in C^1(\Omega \times [0, \mathscr{T}])$ as in Lemma 2 we have

$$\int_{Q_{\mathscr{T}} \setminus (\complement P_n^\theta \cap B_n^\sigma)} \sum_{i=1}^{N} A_i(x, t, u_n, Du_n) D_i \{ \phi [T_\theta(\phi u_n) - T_\theta(\phi u)] \} \, dx\, dt \geqq o^\sigma\left(\frac{1}{n}\right).$$

Hence

$$c_2 \int_{Q_T \cap \complement P_n^\theta \cap B_n^\sigma} |Du_n|^p \, dx\, dt \leqq o\left(\frac{1}{\sigma}\right) + o^\sigma\left(\frac{1}{n}\right) + \langle F(u_n), \phi [T_\theta(\phi u_n) - T_\theta(\phi u)] \rangle$$

$$= o\left(\frac{1}{\sigma}\right) + o^\sigma\left(\frac{1}{n}\right) + \langle F(u_n), \phi [T_\theta(\phi u_n) - (T_\theta(\phi u_k))_\nu] \rangle$$

$$+ \langle F(u_n), \phi [(T_\theta(\phi u_k))_\nu - (T_\theta(\phi u))_\nu] \rangle$$

$$+ \langle F(u_n), \phi [(T_\theta(\phi u))_\nu - T_\theta(\phi u)] \rangle$$

$$= \omega + o^{n,\nu}\left(\frac{1}{k}\right) + \langle F(u_n), \phi [T_\theta(\phi u_n) - (T_\theta(\phi u_k))_\nu] \rangle.$$

Using Lemma 2 and then the equation (5.1) we obtain

$$c_2 \int_{Q_T \cap \complement P_n^\theta \cap B_n^\sigma} |Du_n|^p \, dx\, dt \leqq \omega + o^{n,\nu}\left(\frac{1}{k}\right) + \langle F(u_n), \phi [T_\theta(\phi u_n) - (T_\theta(\phi u_k))_\nu] \rangle$$

$$+ \langle u_n', \phi [T_\theta(\phi u_n) - (T_\theta(\phi u_k))_\nu] \rangle$$

$$= \omega + o^{n,\nu}\left(\frac{1}{k}\right) - \langle G_n(u_n), \phi [T_\theta(\phi u_n) - (T_\theta(\phi u_k))_\nu] \rangle$$

$$+ \langle f, \phi [T_\theta(\phi u_n) - (T_\theta(\phi u_k))_\nu] \rangle$$

$$= \omega + o^{n,\nu}\left(\frac{1}{k}\right) - \int_{Q_T} g_n(x, t, u_n, Du_n) \phi [T_\theta(\phi u_n) - (T_\theta(\phi u_k))_\nu] \, dx\, dt$$

$$- \int_{Q_{\mathscr{T}} \setminus Q_T} g_n(x, t, u_n, Du_n) \phi [T_\theta(\phi u_n) - (T_\theta(\phi u_k))_\nu] \, dx\, dt.$$

In the latter integral we can use the polynomial growth restriction (G_4) and hence include it to the above remainder terms. Thus we are left only with the former integral. For each $(x, t) \in P_n^\theta \cap Q_T$ we have $\operatorname{sign}(u_n) T_\theta(u_n) \geqq \operatorname{sign}(u_n) (T_\theta(u_k))_\nu$. Hence by (G_3)

$$g_n(x, t, u_n, Du_n) \{ T_\theta(u_n) - (T_\theta(u_k))_\nu \} \geqq -2\lambda_2(x, t).$$

On $\complement P_n^\theta \setminus B_n^\sigma$ the sequence $g_n(x, t, u_n, Du_n)$ is equi-integrable yielding

$$\int_{\complement P_n^\theta \setminus B_n^\sigma} g_n(x, t, u_n, Du_n) \{ T_\theta(u_n) - (T_\theta(u_k))_\nu \} \, dx\, dt$$

$$\leqq o\left(\frac{1}{n}\right) + o^\nu\left(\frac{1}{k}\right) + o\left(\frac{1}{\nu}\right).$$

Finally by (G_2) we get

$$-\int_{Q_T \cap \complement P_n^\theta \cap B_n^\sigma} g_n(x, t, u_n, Du_n)\{T_\theta(u_n)-(T_\theta(u_k))_\nu\}\,dx\,dt \leqq \int_{Q_T \cap \complement P_n^\theta \cap B_n^\sigma} 2\theta h(\theta)(|Du_n|^p+\lambda_1(x, t))\,dx\,dt.$$

Therefore we conclude

$$c_2 \int_{Q_T \cap \complement P_n^\theta \cap B_n^\sigma} |Du_n|^p\,dx\,dt \leqq \omega + o^{n,\nu}\left(\frac{1}{k}\right) + 2\theta h(\theta) \times \int_{Q_T \cap \complement P_n^\theta \cap B_n^\sigma} |Du_n|^p\,dx\,dt.$$

Choosing now $\theta>0$ small enough to make $\theta h(\theta)<\frac{1}{4}c_2$ the assertion follows if we let $k\to\infty$.

PROPOSITION 5. — $|Du_n|^p \chi_{\complement P_n^\rho} \chi_{[0, T]}$ *is equi-integrable for any given* $\rho>0$.

Proof. — Let $\rho>0$ be given. We can choose $\theta>0$ and $K\in\mathbb{N}$ such that $K\theta=\rho$ and $\theta<\frac{c_2}{4h(\rho)}$. Hence θ meets the condition of Proposition 4. We argue by induction to show that

$$|Du_n|^p \chi_{\complement P_n^{\varkappa\theta}} \chi_{[0, T]}$$

is equi-integrable for each $\varkappa=1, 2, \ldots, K$. For $\varkappa=1$ the claim is true by Proposition 4. Assume now that $|Du_n|^p \chi_{\complement P_n^{(\varkappa-1)\theta}} \chi_{[0, T]}$ is equi-integrable. Using similar estimates as in the proof of Proposition 4 and the assumption of induction we get

$$c_2 \int_{Q_T \cap \complement P_n^{\varkappa\theta} \cap B_n^\sigma} |Du_n|^p\,dx\,dt \leqq \omega + o^{n,\nu}\left(\frac{1}{k}\right)$$

$$-\int_{Q_T \cap \complement P_n^{\varkappa\theta} \cap B_n^\sigma} g_n(x, t, u_n, Du_n)\{T_{\varkappa\theta}(u_n)-(T_{\varkappa\theta}(u_k))_\nu\}\,dx\,dt$$

$$=\omega + o^{n,\nu}\left(\frac{1}{k}\right)$$

$$-\int_{Q_T \cap B_n^\sigma \cap (\complement P_n^{\varkappa\theta} \setminus \complement P_n^{(\varkappa-1)\theta})} g_n(x, t, u_n, Du_n)\left\{\varkappa\theta\frac{u_n}{|u_n|}-(T_{\varkappa\theta}(u_k))_\nu\right\}dx\,dt$$

$$-\int_{Q_T \cap B_n^\sigma \cap (\complement P_n^{\varkappa\theta} \setminus \complement P_n^{(\varkappa-1)\theta})} g_n(x, t, u_n, Du_n)\left\{T_{\varkappa\theta}(u_n)-\varkappa\theta\frac{u_n}{|u_n|}\right\}dx\,dt$$

In $\pi = Q_T \cap B_n^\sigma \cap (\complement P_n^{\varkappa\theta} \setminus \complement P_n^{(\varkappa-1)\theta})$ we have $|(T_{\varkappa\theta}(u_k))_\nu| \leqq k\theta$, further $\left|T_{\varkappa\theta}(u_n) - \kappa\theta \frac{u_n}{|u_n|}\right| \leqq \theta$, and sign $(u_n)\left[\kappa\theta \frac{u_n}{|u_n|} - (T_{\varkappa\theta}(u_k))_\nu\right] \geqq 0$; therefore

$$\int_\pi |g_n(x, t, u_n, Du_n)| \theta \, dx \, dt \leqq \theta \, h(\rho) \int_\pi (|Du_n|^p + \lambda_1(x, t)) \, dx \, dt.$$

by (G_2). Taking the assumption $\theta \, h(\rho) < \frac{c_2}{4}$ into account we can conclude

$$\frac{c_2}{2} \int_{Q_T \cap B_n^\sigma \cap \complement P_n^{\varkappa\theta}} |Du_n|^p \, dx \, dt \leqq \omega.$$

Hence the step of induction is established and the proof is complete.

PROPOSITION 6. – *$\{|Du_n|^p\}$ is equi-integrable.*

Proof. – In view of Proposition 5 it will be sufficient to show

$$\int_{P_n^\rho} |Du_n|^p \, dx \, dt < \omega$$

with $\omega = \omega(\rho, \nu, n) = o\left(\frac{1}{\rho}\right) + o^\rho\left(\frac{1}{\nu}\right) + o^{\rho, \nu}\left(\frac{1}{n}\right)$ for some parameter ν. Since $u_n \to u$, $Du_n \to Du$, $u_\nu \to u$ and $Du_\nu \to Du$ a.e. in $Q_{\mathscr{T}}$ we can choose a sequence of subsets $\{\tilde{Q}_m\}$ of $Q_{\mathscr{T}}$ such that $\tilde{Q}_m \subset \tilde{Q}_{m+1}$, $\left|Q_{\mathscr{T}} \setminus \bigcup_{m=1}^{\infty} \tilde{Q}_m\right| = 0$ and $\{u_n\}$, $\{Du_n\}$, $\{u_\nu\}$ and $\{Du_\nu\}$ are uniformly convergent on each $\tilde{Q}_m$. Therefore

$$\int_{\complement P_n^\rho} \sum_{i=1}^{N} A_i(x, t, u_n, Du_n) D_i \{\phi[\phi u_n - (T_\rho(\phi u_k))_\nu]\} \, dx \, dt = \omega + o^{n, \nu}\left(\frac{1}{k}\right)$$

and

$$\int_{\complement P_n^\rho} g_n(x, t, u_n, Du_n) \phi[\phi u_n - (T_\rho(\phi u_k))_\nu] \, dx \, dt = \omega + o^{n, \nu}\left(\frac{1}{k}\right).$$

Using (A_4), the fact that $|P_n^\rho| = o\left(\frac{1}{\rho}\right)$ and similar arguments as in the proof of Proposition 4 we estimate

$$c_2 \int_{P_n^\rho} |Du_n|^p \, dx \, dt \leqq \int_{P_n^\rho} \sum_{i=1}^{N} A_i(x, t, u_n, Du_n) D_i u_n \, dx \, dt + o\left(\frac{1}{\rho}\right)$$

$$\leqq \int_{P_n^\rho} \sum_{i=1}^{N} A_i(x, t, u_n, Du_n) D_i \{u_n - (T_\rho(u_k))_\nu\} \, dx \, dt + \omega + o^{\rho, \nu, n}\left(\frac{1}{k}\right)$$

$$\leqq \int_{Q_T} \sum_{i=1}^{N} A_i(x, t, u_n, Du_n) D_i \{ u_n - (T_\rho(u_k))_\nu \} \, dx \, dt + \omega + o^{\rho, \nu, n}\left(\frac{1}{k}\right)$$

$$\leqq \langle F(u_n), \phi [\phi u_n - (T_\rho(\phi u_k))_\nu] \rangle + \omega + o^{\rho, \nu, n}\left(\frac{1}{k}\right)$$

Now Lemma 3 yields

$$c_2 \int_{P_n^\rho} | Du_n |^p \, dx \, dt \leqq - \langle u'_n, \phi [\phi u_n - (T_\rho(\phi u_k))_\nu] \rangle$$

$$- \langle G_n(u_n), \phi [\phi u_n - (T_\rho(\phi u_k))_\nu] \rangle$$

$$+ \langle f, \phi [\phi u_n - (T_\rho(\phi u_k))_\nu] \rangle + \omega + o^{\rho, n, \nu}\left(\frac{1}{k}\right)$$

$$\leqq - \int_{Q_{\mathcal{T}} \cap P_n^\rho} g_n(x, t, u_n, Du_n)(u_n - (T_\sigma(u_k))_\nu) \, dx \, dt + \omega + o^{\rho, n, \nu}\left(\frac{1}{k}\right)$$

$$\leqq \omega + o^{\rho, n, \nu}\left(\frac{1}{k}\right).$$

providing the desired estimate as $k \to \infty$.

COROLLARY. — *There exists a subsequence of* $\{ u_n \}$ *such that* $D_i u_n \to D_i u$ *in* $L^p(Q_T)$ *for each* $i = 1, 2, \ldots, N$.

6. PROOF OF THE THEOREM

In order to prove our main theorem of Section 2 we have first

PROPOSITION 7. — $g_n(x, t, u_n, Du_n)$ *is equi-integrable in* Q_T.

Proof. — Let $\varepsilon > 0$ be given. In view of Proposition 1 we can choose $\gamma = \gamma(\varepsilon)$ such that

$$0 \leqq \frac{1}{\gamma} \int_{Q_T} \{ g_n(x, t, u_n, Du_n) u_n + \lambda_2(x, t) \} \, dx \, dt < \frac{\varepsilon}{2}.$$

Since $| Du_n |^p$ is equi-integrable in Q_T by Proposition 6, there exists $\delta > 0$ such that

$$\int_{\tilde{Q}} \{ | Du_n |^p + \lambda_1(x, t) \} \, dx \, dt < \frac{\varepsilon}{2(h(\gamma) + 1)}$$

whenever $\tilde{Q} \subset Q_T$ and $|\tilde{Q}| < \delta$. By (G_2) we have

$$\begin{aligned}
\int_{\tilde{Q}} |g_n(x, t, u_n, Du_n)|\, dx\, dt &\leqq h(\gamma) \int_{\tilde{Q} \cap \complement P_n^\gamma} \{ |Du_n|^p + \lambda_1(x, t) \}\, dx\, dt \\
&+ \int_{\tilde{Q} \cap P_n^\gamma} |g_n(x, t, u_n, Du_n)|\, dx\, dt \\
&\leqq \frac{\varepsilon}{2} + \frac{1}{\gamma} \int_{\tilde{Q} \cap P_n^\gamma} |g_n(x, t, u_n, Du_n)|\, |u_n|\, dx\, dt \\
&\leqq \frac{\varepsilon}{2} + \frac{1}{\gamma} \int_{Q_T} \{ g_n(x, t, u_n, Du_n) + \lambda_2(x, t) \}\, dx\, dt \\
&< \varepsilon
\end{aligned}$$

and hence the assertion follows.

By the previous results we now conclude that

$$\begin{gathered}
g_n(.,., u_n, Du_n) \to g(.,., u, Du) \quad \text{in } L^1(Q_T), \\
\langle G_n(u_n), \psi \rangle \to \langle G(u), \psi \rangle \quad \text{for any } \psi \in L^\infty(Q_T), \\
F(u_n) \rightharpoonup F(u) \quad \text{in } \mathscr{V}^*.
\end{gathered}$$

From (5.1) with smooth testfunction ψ supported in $\bar{Q}_T$ and letting $n \to \infty$ we get the equation

$$-\langle u, \psi' \rangle + \langle F(u), \psi \rangle + \langle G(u), \psi \rangle = \langle f, \psi \rangle.$$

Therefore (2.1) follows by approximation.

To verify the continuity properties we remark that $g_n(x, t, u_n, Du_n)$ converges in $L^1(Q_T)$. With this additional information we consider (3.3) again. Testing with $\phi(t, x) = \phi(x) \in C_0^\infty(\Omega)$ for indices n and k we get for all $\tau \in]0, T[$:

$$\begin{aligned}
\int_\Omega & \{ u_n(\tau) - u_k(\tau) \} \phi\, dx \\
&= \int_{Q_\tau} \sum_{i=1}^{N} \{ A_i(x, t, u_k, Du_k) - A_i(x, t, u_n, Du_n) \} D_i \phi\, dx\, dt \\
&+ \int_{Q_\tau} \{ g_k(x, t, u_k, Du_k) - g_n(x, t, u_n, Du_n) \} \phi\, dx\, dt \\
&= o\left(\frac{1}{n}\right) + o\left(\frac{1}{k}\right).
\end{aligned}$$

Also $\| u_n(t) \|_{L^2(\Omega)} \leqq C$ by (3.4). Hence $u_n(t)$ is weakly convergent for all t. Further

$$\lim_{n \to \infty} \int_\Omega \{ u_n(t_1) - u_n(t_2) \} \phi \, dx$$
$$= \lim_{n \to \infty} \int_{t_1}^{t_2} \int_\Omega \sum_{i=1}^N A_i(x, t, u_n, Du_n) D_i \phi \, dx \, dt + \int_{t_1}^{t_2} \int_\Omega g_n(x, t, u_n, Du_n) \phi \, dx \, dt$$
$$= o(t_2 - t_1)$$

providing the weak continuity of $u(t)$ in $L^2(\Omega)$.

To prove the continuity with respect to the strong topology of $L^2(\Omega)$ we observe

$$\lim_{\theta \to \infty} \lim_{n \to \infty} \int_{Q_\tau} g_n(x, t, u_n, Du_n) T_\theta(u) \, dx \, dt$$
$$= \lim_{\theta \to \infty} \int_{Q_\tau} g(x, t, u, Du) T_\theta(u) \, dx \, dt$$
$$= \int_{Q_\tau} g(x, t, u, Du) u \, dx \, dt \qquad (6.1)$$

for all $\tau \in]0, T[$. Hence for every θ we may choose ν_θ large enough such that

$$\lim_{\theta \to \infty} \lim_{n \to \infty} \int_{Q_\tau} g_n(x, t, u_n, Du_n) (T_\theta(u))_{\nu_\theta} \, dx \, dt$$
$$= \lim_{\theta \to \infty} \int_{Q_\tau} g(x, t, u, Du) (T_\theta(u))_{\nu_\theta} \, dx \, dt$$
$$= \int_{Q_\tau} g(x, t, u, Du) u \, dx \, dt.$$

On the other hand we have

$$\lim_{n \to \infty} \int_{Q_\tau} (T_\theta(u))_{\nu_\theta} (u_n - (T_\theta(u))_{\nu_\theta}) \, dx \, dt$$
$$= \nu_\theta \int_{Q_\tau} (T_\theta(u) - (T_\theta(u))_{\nu_\theta}) (u - (T_\theta(u))_{\nu_\theta}) \, dx \, dt \geqq 0, \qquad (6.2)$$

providing

$$
\begin{aligned}
0 &\leqq \frac{1}{2}\| u(\tau)-(\mathrm{T}_\theta(u))_{\nu_\theta}(\tau)\|^2_{\mathrm{L}^2(\Omega)} \\
&= \lim_{n\to\infty} \langle u_n' - (\mathrm{T}_\theta(u))'_{\nu_\theta}, u_n-(\mathrm{T}_\theta(u))_{\nu_\theta}\rangle \\
&\leqq -\lim_{n\to\infty}\Big[\int_{\mathrm{Q}_\tau}\sum_{i=1}^{\mathrm{N}} \mathrm{A}_i(x, t, u_n, \mathrm{D}u_n)\,\mathrm{D}_i(u_n-(\mathrm{T}_\theta(u))_{\nu_\theta})\,dx\,dt \\
&\quad + \int_{\mathrm{Q}_\tau} g_n(x, t, u_n, \mathrm{D}u_n)(u_n-(\mathrm{T}_\theta(u))_{\nu_\theta})\,dx\,dt - \langle f, u_n-(\mathrm{T}_\theta(u))_{\nu_\theta}\rangle_{\mathrm{Q}_\tau} \\
&\leqq o\left(\frac{1}{\nu_\theta}\right)
\end{aligned}
$$

where $o\left(\frac{1}{\nu_\theta}\right)$ does not depend on τ.

Now we consider the sequence $(\mathrm{T}_\theta(u))_{\nu_\theta}\in \mathrm{C}([0, \mathrm{T}], \mathrm{L}^2(\Omega))$. For $\tau\in(0, \mathrm{T}]$ we have

$$
\begin{aligned}
&\frac{1}{2}\|(\mathrm{T}_\theta(u))_{\nu_\theta}(\tau)-(\mathrm{T}_\sigma(u))_{\nu_\sigma}(\tau)\|^2_{\mathrm{L}^2(\Omega)} \\
&\quad\leqq \lim_{n\to\infty}\|(\mathrm{T}_\theta(u))_{\nu_\theta}(\tau)-u_n(\tau)\|^2_{\mathrm{L}^2(\Omega)} + \lim_{n\to\infty}\|(\mathrm{T}_\sigma(u))_{\nu_\sigma}(\tau)-u_n(\tau)\|^2_{\mathrm{L}^2(\Omega)} \\
&\quad\leqq o\left(\frac{1}{\theta}\right)+o\left(\frac{1}{\sigma}\right).
\end{aligned}
$$

Hence $(\mathrm{T}_\theta(u))_{\nu_\theta}$ is a Cauchy sequence in $\mathrm{C}([0, \mathrm{T}]; \mathrm{L}^2(\Omega))$ converging to u and consequently $u\in \mathrm{C}([0, \mathrm{T}], \mathrm{L}^2(\Omega))$.

The energy equality now follows easily. Indeed, as above we get

$$
\begin{aligned}
o(\theta) &\geqq \lim_{n\to\infty}\langle u_n'-(\mathrm{T}_\theta(u))'_{\nu_\theta}, u_n-(\mathrm{T}_\theta(u))_{\nu_\theta}\rangle \\
&= \frac{1}{2}\lim_{n\to\infty}\int_\Omega (u_n-(\mathrm{T}_\theta(u))_{\nu_\theta})^2(\tau)\,dx \geqq 0,
\end{aligned}
$$

since $(\mathrm{T}_\theta(u))_{\nu_\theta}(\tau)$ converges to $u(\tau)$ in the strong topology so does $u_n(\tau)$, and hence by (6.2)

$$\lim_{n\to\infty}\langle u_n', u_n-(\mathrm{T}_\theta(u))_{\nu_\theta}\rangle \geqq 0.$$

Consequently by Fatou's Lemma and (6.1)

$$0 \leqq \lim_{n \to \infty} \int_{Q_T} g_n(x, t, u_n, Du_n)(u_n - u)\, dx\, dt$$

$$= \lim_{n \to \infty} \int_{Q_T} g_n(x, t, u_n, Du_n)(u_n - T_\theta(u)_{v_\theta})\, dx\, dt \leqq o(\theta)$$

which proves that we can go to the limit in each of the terms in (3.4).

7. PROOFS OF LEMMATA 2 AND 3

In order to prove Lemma 2 and Lemma 3 of Section 5 we need some properties of derivatives of distributions which are not valid in general but can be verified in our situation.

PROPOSITION 8. – *Let* $v \in C(0, \mathscr{T}; L^2(\Omega))$ *and* $g \in \mathscr{V}^*$ *such that* $v(0) = 0$ *and* $v' = g$ *in the sense of distributions on* $Q_{\mathscr{T}}$, *and let* $\phi \in C^1(0, \mathscr{T}; \mathbb{R})$ *such that* $\phi(\mathscr{T}) = 0$. *Then*

$$\langle (\phi v)', T_\theta(\phi v) \rangle = \langle g, \phi T_\theta(\phi v) \rangle + \langle \phi', v T_\theta(\phi v) \rangle = 0.$$

Proof. – Let $\varepsilon > 0$ be given and let $\psi \in C^1(0, \mathscr{T}; \mathbb{R})$ be such that $\psi = 1$ on $[2\varepsilon, \mathscr{T} - 2\varepsilon]$ and $\psi = 0$ on $[0, \varepsilon] \cup [\mathscr{T} - \varepsilon, \mathscr{T}]$. Let σ and ρ indicate Friedrich's mollification with respect to t with mollifiers supported in $\left[-\frac{\varepsilon}{16}, \frac{\varepsilon}{16}\right]$, say. Then using trivial extensions and the properties of mollification we get

$$\lim_{\sigma \to 0} \int_{\mathbb{R} \times \Omega} \psi \frac{d}{dt}(S_\theta((\phi v)_\sigma))\, dx\, dt$$

$$= - \lim_{\sigma \to 0} \lim_{\rho \to 0} \int_{\mathbb{R} \times \Omega} [(\psi (T_\theta(\phi v))_\sigma)_\rho]' (\phi v)_\sigma\, dx\, dt$$

$$= - \lim_{\sigma \to 0} \lim_{\rho \to 0} \int_{\mathbb{R} \times \Omega} \frac{d}{dt}[(\psi (T_\theta(\phi v)_\sigma)_\rho)_\sigma \phi]\, v\, dx\, dt$$

$$+ \lim_{\sigma \to 0} \lim_{\rho \to 0} \int_{\mathbb{R} \times \Omega} \phi' [\psi (T_\theta(\phi v)_\sigma)_\rho)_\sigma]\, v\, dx\, dt$$

$$= \langle g, \psi \phi T_\theta(\phi v) \rangle + \int_{\mathbb{R} \times \Omega} \psi \phi' v T_\theta(\phi v)\, dx\, dt$$

$$:= I_\psi.$$

Choosing $\psi=\psi_\varepsilon$ such that $\psi_\varepsilon=1$ on $[\varepsilon, \mathcal{T}-\varepsilon]$ and letting $\varepsilon\to 0$ we get

$$\lim_{\varepsilon\to 0} \mathrm{I}_{\psi_\varepsilon}=\langle g, \phi\,\mathrm{T}_\theta(\phi v)\rangle+\int_{\mathbb{R}\times\Omega}\phi' v\,\mathrm{T}_\theta(\phi v)\,dx\,dt.$$

On the other hand,

$$\begin{aligned}|\mathrm{I}_{\psi_\varepsilon}| &= \lim_{\sigma\to 0}\left|\int_{\mathbb{R}\times\Omega}\psi_\varepsilon\frac{d}{dt}(\mathrm{S}_\theta(\phi v)_\sigma)\,dx\,dt\right|\\ &=\left|\int_{\mathbb{R}\times\Omega}\psi'_\varepsilon\,\mathrm{S}_\theta(\phi v)\,dx\,dt\right|\\ &\leqq 2\varepsilon\frac{1}{\varepsilon}\Big\{\max_{t\in[0,\varepsilon]}\|(\phi v)(t)\|^2_{\mathrm{L}^2(\Omega)}+\max_{t\in[\mathcal{T}-\varepsilon,\mathcal{T}]}\|(\phi v)(t)\|^2_{\mathrm{L}^2(\Omega)}\Big\}\end{aligned}$$

which converges to zero as $\varepsilon\to\infty$ because of our assumptions that $v\in\mathrm{C}([0, \mathcal{T}]; \mathrm{L}^2(\Omega))$, $v(0)=0$, $\phi(\mathcal{T})=0$.

PROPOSITION 9. – *In addition to the assumptions of Proposition* 8 *we assume that* $\phi\geqq 0$ *and* $\phi'\leqq 0$. *Then*

$$\langle g, \phi v\rangle=\langle v', \phi v\rangle\geqq 0.$$

Proof. – As in the previous proof we obtain for smooth $\tilde{\phi}$, that

$$\mathrm{I}_\varepsilon := \lim_{\sigma\to 0}\int_{\mathbb{R}\times\Omega}\psi_\varepsilon\frac{d}{dt}\left(\left(v\tilde{\phi}\right)_\sigma\right)^2 dx\,dt$$

converges to zero as $\varepsilon\to 0$. On the other hand,

$$\begin{aligned}&\lim_{\sigma\to 0}\int_{\mathbb{R}\times\Omega}\psi\frac{d}{dt}((v\tilde{\phi})_\sigma)^2\,dx\,dt\\ &\quad=\lim_{\sigma\to 0}\lim_{\rho\to 0} 2\int_{\mathbb{R}\times\Omega}\psi(v\tilde{\phi})_\sigma((v_\rho\tilde{\phi})_\sigma)'\,dx\,dt\\ &\quad=\lim_{\sigma\to 0}\lim_{\rho\to 0} 2\int_{\mathbb{R}\times\Omega}(\psi(\tilde{\phi} v)_\sigma)_\sigma(v_\rho)'\tilde{\phi}\,dx\,dt\\ &\qquad+\lim_{\sigma\to 0}\lim_{\rho\to 0} 2\int_{\mathbb{R}\times\Omega}(\psi(\tilde{\phi} v)_\sigma)_\sigma v_\rho\tilde{\phi}'\,dx\,dt\\ &\quad=2\langle g, \psi\tilde{\phi}^2 v\rangle+2\int_{\mathbb{R}\times\Omega}\psi v^2\tilde{\phi}\tilde{\phi}'\,dx\,dt\end{aligned}$$

Using the same arguments as in the proof of Proposition 8 we obtain

$$I_\varepsilon \to 2\langle g, \tilde{\phi}^2 v\rangle + 2\int_{\mathbb{R}\times\Omega} v^2 \tilde{\phi}\tilde{\phi}'\, dx\, dt.$$

With $\tilde{\phi}=(\sqrt{\phi})_\delta$, the Friedrichs' mollification of $\sqrt{\varphi}$, the assertion follows as $\delta\to 0$, since then $\tilde{\phi}\geqq 0$ and $\tilde{\phi}'\leqq 0$.

Proof of Lemma 2. – We have to show that

$$I := \langle u_n', \phi[T_\theta(\phi u_n) - (T_\theta(\phi u_k))_\nu]\rangle \geqq o\left(\frac{1}{\nu}\right) + o^{n,\nu}\left(\frac{1}{k}\right) + o^{\nu}\left(\frac{1}{n}\right).$$

Indeed, by Proposition 8 we have

$$\langle u_n', \phi T_\theta(\phi u_n)\rangle = -\int_{Q_T} u_n \phi' T_\theta(\phi u_n)\, dx\, dt.$$

On the other hand

$$-\langle u_n', \phi(T_\theta(\phi u_k))_\nu\rangle = \int_{Q_{\mathcal{T}}} u_n (T_\theta((\phi u_k)_\nu)' \phi\, dx\, dt + \int_{Q_{\mathcal{T}}} u_n \phi' (T_\theta(\phi u_k))_\nu\, dx\, dt.$$

Hence

$$I = \int_{Q_{\mathcal{T}}} u_n ((T_\theta(\phi u_k))_\nu)' \phi\, dx\, dt + o\left(\frac{1}{n}\right) + o\left(\frac{1}{\nu}\right) + o^{n,\nu}\left(\frac{1}{k}\right).$$

But as in the proof of Proposition 3 we get

$$\int_{Q_{\mathcal{T}}} u_n (T_\theta(\phi u_k))_\nu)' \phi\, dx\, dt = \nu \int_{Q_{\mathcal{T}}} \phi u_n [T_\theta(\phi u_k) - (T_\theta(\phi u_k))_\nu]\, dx\, dt$$

$$= \nu \int_{Q_{\mathcal{T}}} u\phi [T_\theta(\phi u) - (T_\theta(\phi u))_\nu]\, dx\, dt + o^{n,\nu}\left(\frac{1}{k}\right) + o^{\nu}\left(\frac{1}{n}\right)$$

and the result follows, because the last integral is nonnegative as we shall show in the appendix.

Proof of Lemma 3. – We have to show that

$$J := \langle u_n', \phi[\phi u_n - (T_\sigma(\phi u_k)_\nu]\rangle \geqq o\left(\frac{1}{\nu}\right) + o^{n,\nu}\left(\frac{1}{k}\right) + o^{\nu}\left(\frac{1}{n}\right).$$

By the same arguments as above we can write

$$J = \langle u_n', \phi^2 u_n\rangle + \int_{Q_{\mathcal{T}}} u_n [(T_\sigma(\phi u_k))_\nu]' \phi\, dx\, dt + \int_{Q_{\mathcal{T}}} u_n (T_\sigma(\phi u_k))_\nu \phi'\, dx\, dt$$

$$\geqq \langle u_n', \phi^2 u_n\rangle + \int_{Q_{\mathcal{T}}} u_n (T_\sigma(\phi u_k))_\nu \phi'\, dx\, dt + o^{n,\nu}\left(\frac{1}{k}\right) + o^{\nu}\left(\frac{1}{n}\right).$$

Since $\langle u_n', \phi^2 u_n \rangle = -\int_{Q_{\mathcal{T}}} u_n^2 \phi\phi' \, dx\, dt$, and since

$$-\int_{Q_{\mathcal{T}}} u_n^2 \phi\phi' \, dx\, dt + \int_{Q_{\mathcal{T}}} u_n (T_\sigma(\phi u_k))_\nu \phi' \, dx\, dt$$
$$= \int_{Q_{\mathcal{T}}} \phi' u_n [(T_\sigma(\phi u_k))_\nu - \phi u_n] \, dx\, dt$$
$$\geqq o\left(\frac{1}{\nu}\right) + o^\nu\left(\frac{1}{n}\right) + o^\nu\left(\frac{1}{k}\right) + \int_{Q_{\mathcal{T}}} \phi' [u(T_\sigma(\phi u) - \phi u)] \, dx\, dt$$

the assertion follows from the fact that the last integral is nonnegative.

APPENDIX

We owe the short proof of the following fact to the referee. Our original proof was much more complicated.

LEMMA. — *Let $v \in L^2(D)$ for some $D \subset \mathbb{R}^k$ and $\theta > 0$. Then*

$$\int_D v\, T_\theta(v)\, dx \geqq \int_D vw\, dx$$

for all $w \in S := \{ u \in L^2(D) \mid \| u \|_{L^2(D)} \leqq \| T_\theta(v) \|_{L^2(D)}$ and $\| u \|_\infty \leqq \theta \}$.

Proof. — For $w \in S$ *we have*

$$\int_D vw\, dx$$
$$\leqq \int_D T_\theta(|v|)\, |w|\, dx + \int_D (|v| - \theta)^+ \, |w|\, dx$$
$$\leqq \| T_\theta(v) \|_2 \, \| w \|_2 + \int_D (|v| - \theta)^+ \, \theta\, dx$$
$$\leqq \| T_\theta(v) \|_2^2 + \int_D (|v| - \theta)^+ \, \theta\, dx$$
$$= \int_D v\, T_\theta(v)\, dx.$$

ACKNOWLEDGEMENTS

The authors are grateful for the valuable comments of the referee. In particular the proof of the Lemma in the Appendix is his.

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(*Manuscript received November* 30, 1992;
revised August 12, 1993.)