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Continuous dependence in L^2 for discontinuous solutions of the viscous p-system

by

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ABSTRACT. — We prove that discontinuous solutions of the Navier-Stokes equations for isentropic or isothermal flow depend continuously on their initial data in L^2. This improves earlier results in which continuous dependence was known only in a much stronger norm, a norm inappropriately strong for the physical model. We also apply our continuous dependence theory to obtain improved rates of convergence for certain finite difference approximations.

Key words : Navier-Stokes equations, continuous dependence.

RÉSUMÉ. — Nous prouvons que les solutions discontinues des équations de Navier-Stokes, pour des flots isentropiques ou isothermaux, dépendent continuement des conditions initiales dans L^2. Ceci améliore les résultats précédents dans lesquels la continuité de la dépendance n'était connue

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que pour une norme beaucoup plus forte, inappropriée pour le modèle physique. Nous appliquons aussi cette théorie de la dépendance continue pour obtenir une amélioration des taux de convergence pour certaines approximations aux différences finies.

1. INTRODUCTION

We prove the continuous dependence on initial data of discontinuous solutions of the Navier-Stokes equations for compressible, isentropic or isothermal flow:

$$v_t - u_x = F_1 \tag{1.1}$$

$$u_t + p(v)_x = \left(\frac{\varepsilon u_x}{v}\right)_x + F_2, \qquad t>0, \qquad -1<x<1, \tag{1.2}$$

with initial data

$$v(x, 0) = v_0(x) \qquad \text{and} \qquad u(x, 0) = u_0(x) \tag{1.3}$$

and boundary conditions

$$u(-1, t) = u_l(t) \qquad \text{and} \qquad u(1, t) = u_r(t). \tag{1.4}$$

Here v, u, and p represent the specific volume, velocity, and pressure in a fluid, t is time, and x is the Lagrangean coordinate, so that the lines $x=\text{Const.}$ correspond to particle trajectories. ε is a positive viscosity constant, and the source terms F_1 and F_2 depend upon x, t, y, v, and u, where y is the Eulerian coordinate

$$y(v(x, t)) = \int_{-1}^{x} v(s, t)\, ds. \tag{1.5}$$

We also apply our continuous dependence theory to obtain improved rates of convergence for certain finite difference approximations to solutions of (1.1)-(1.4).

Previous results concerning continuous dependence on initial data for discontinuous solutions are obtained in Zarnowski and Hoff [8], Theorem 5.2, and its extension to the nonisentropic case, Hoff [5], Theorem 1.4. These results measure the difference between two solutions in an exceptionally strong norm, one which dominates the local variation in perturbations of the discontinuous variable v. Since $1/v$ is the fluid density, which itself is a gradient, these results must therefore be regarded as unsatisfactory from the physical point of view. In addition, the question of continuous dependence always plays a crucial role in the derivation of error bounds for approximate solutions. In particular, error bounds for certain finite difference approximations to solutions of (1.1)-(1.4) were derived in [8]-[9]; these rates of convergence appear to be unrealistically low, however, precisely because they are formulated in a norm which is inappropriately strong.

The goal of the present paper is therefore to show that, under assumptions consistent with the known existence theory, discontinuous solutions depend continuously on their initial data in L^2, which clearly is a more suitable norm for the physical problem. This result is stated precisely in Theorem 1.4 below, and is proved in section 2. (A nearly identical result can be formulated for the corresponding Cauchy problem.) The key idea is to replace a direct estimate for the difference between two solutions with an adjoint-equation argument; the adjoint functions are estimated in fractional Sobolev norms, and L^2 information is extracted by interpolation. A somewhat more detailed sketch of the main issues is given below following the statement of Theorem 1.4. In section 3 we apply our result in a reexamination of the difference approximations studied in [8]-[9]. We show that, for fairly general discontinuous initial data, the error bound can be improved from $O(h^{1/4-\delta})$ to $O(h^{1/2})$; and for H^1 initial data, from $O(h^{1/2})$ to $O(h)$. (Of course, the more favorable convergence rates are measured in a weaker norm.) We also point out that the $O(h^{1/2})$ estimate for these discontinuous solutions is reminiscent of error bounds in average-norm for approximations to discontinuous solutions of other compressible flow models; *see* for example Kuznetsov [7] and Hoff and Smoller [6], Theorem 5.1.

The general source terms F_1 and F_2 in (1.1)-(1.2), and the general boundary conditions (1.4) are included mostly for the sake of completeness; they play little role in the analysis because they essentially "subtract out" for the equations satisfied by the difference between two solutions. Physically, the most important case of (1.1)-(1.2) is that in which $F_1=0$ and the force F_2 depends only on t and on the Eulerian coordinate y. The boundary conditions (1.4) require that the leftmost fluid particle moves with velocity u_l, and the rightmost with velocity u_r. In physical space this means that the fluid is confined to the region between two pistons, moving with respective velocities u_l and u_r. The problem of a fluid moving in a fixed domain therefore corresponds to the case that $u_l=u_r=0$.

We now give a precise formulation of our results. First we fix a positive time T, and we assume that the functions appearing in (1.1)-(1.4) satisfy the following conditions:

A1. $p\in C^2([\underline{v}, \bar{v}])$, where $[\underline{v}, \bar{v}]$ is a fixed interval in $(0, \infty)$, and $p'(v)<0$ on $[\underline{v}, \bar{v}]$;

A2. $u_l, u_r\in L^\infty([0, T])$;

A3. F_1 and F_2 are sufficiently regular that, whenever $v, u \in C([0, T]; L^2[-1, 1])$ with $v \in [\underline{v}, \bar{v}]$ a.e., then

$$F_j(x, t, y(v(x, t)), v(x, t), u(x, t)) \in L^1([-1, 1] \times [0, T]), \qquad j=1, 2;$$

A4. there is a constant C_F such that

$$|F_j(x, t, y_2, v_2, u_2) - F_j(x, t, y_1, v_1, u_1)| \leq C_F(|y_2 - y_1| + |v_2 - v_1| + |u_2 - u_1|), \qquad j=1, 2,$$

for all $(x, t) \in [-1, 1] \times [0, T]$ and $(y_i, v_i, u_i) \in [0, 2\bar{v}] \times [\underline{v}, \bar{v}] \times \mathbb{R}$, $i=1, 2$.

We define weak solutions of the system (1.1)-(1.4) as follows:

DEFINITION 1.1. — *We say that the pair* (v, u) *is a weak solution of the system* (1.1)-(1.4) *provided*

$$v, u \in C([0, T]; L^2([-1, 1])); \tag{1.6}$$

$$v(., 0) = v_0 \quad \textit{and} \quad u(., 0) = u_0; \tag{1.7}$$

$$v(x, t) \in [\underline{v}, \bar{v}] \text{ a.e.}; \tag{1.8}$$

$$u - \bar{u} \in L^\infty_{loc}((0, T); H^1([-1, 1])) \cap L^2((0, T); H^1_0([-1, 1])), \tag{1.9}$$

where $\bar{u}$ *is the function*

$$\bar{u}(x, t) = \frac{1}{2}[(1-x)u_l(t) + (1+x)u_r(t)];$$

(v, u) *satisfies the equations* (1.1)-(1.2) *in the sense that, for all* $t_1 < t_2$ *in* $[0, T]$ *and all* $\varphi \in C^1([-1, 1] \times [t_1, t_2])$,

$$\int_{-1}^{1} (v\varphi)(x, .)\,dx\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_{-1}^{1} (-v\varphi_t + u\varphi_x)\,dx\,dt = \int_{t_1}^{t_2}\int_{-1}^{1} F_1(x, t, y(v(x, t)), v(x, t), u(x, t))\,\varphi(x, t)\,dx\,dt; \tag{1.10}$$

and for all $t_1 < t_2$ *in* $[0, T]$ *and all* $\psi \in C^1([-1, 1] \times [t_1, t_2])$ *satisfying* $\psi(\pm 1, 0) = 0$,

$$\int_{-1}^{1} (u\psi)(x, .)\,dx\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2}\int_{-1}^{1} [u\psi_t + p(v)\psi_x]\,dx\,dt + \int_{t_1}^{t_2}\int_{-1}^{1} \frac{\varepsilon u_x \psi_x}{v}\,dx\,dt = \int_{t_1}^{t_2}\int_{-1}^{1} F_2(x, t, y(v(x, t)), v(x, t), u(x, t))\,\psi(x, t)\,dx\,dt. \tag{1.11}$$

The existence of weak solutions of the Cauchy problem for (1.1)-(1.2), but without source terms F_1 and F_2, is proved in Hoff [2]-[3]. An extension of these results to the initial-boundary value problem (1.1)-(1.4) is included in [8]-[9]. In both cases it is shown that, when $u_0, v_0 \in BV$ with v_0 positive, global weak solutions exist and satisfy the following regularity and smoothness conditions:

$$u_x, v_t \in C((0, T]; L^2([-1, 1])); \tag{1.12}$$

there are positive constants $r<1$ and C_0 such that

$$\sup_{0\leq t\leq T}[\|v(.,t)\|+\|u(.,t)\|+t^r(\|v_t(.,t)\|_\infty+\|u_x(.,t)\|_\infty)] + \left(\int_0^T\int_{-1}^1 u_x^2\,dx\,dt\right)^{1/2}\leqq C_0; \quad (1.13)$$

and there is a positive constant C_1 such that

$$\|u_x(.,t)\|\leqq C_1 t^{-1/4}, \qquad 0<t\leqq T. \quad (1.14)$$

(Here and throughout this paper $\|.\|_p$ denotes the usual L^p norm on $[-1, 1]$, with the subscript omitted when $p=2$.) Actually, a great deal more technical information is obtained in [2] and [3] concerning the regularity and qualitative properties of solutions. We mention here only that discontinuities in v are shown to persist for all time, convecting along particle paths and decaying exponentially in time, at a rate inversely proportional to ε. Analogous statements can be proved for the more complicated system corresponding to nonisentropic flow; see Hoff ([4]-[5]). These facts are of great importance in the general theory; they will play no role in the present analysis, however. Here we deal with solutions in the sense of Definition 1.1, and assume only that they satisfy the regularity conditions (1.12)-(1.14).

Next we introduce the fractional-order Sobolev spaces, and we recall several of their basic properties:

DEFINITION 1.2. — *Let* $\{\varphi_k\}_{k=1}^\infty$ *be an orthonormal basis for* $L^2([-1, 1])$ *consisting of eigenfunctions in* H_0^1 *of* $-d^2/dx^2$. *Thus*

$$\varphi_k''+\lambda_k\varphi_k=0, \qquad -1<x<1,$$
$$\varphi_k(\pm 1)=0.$$

For $w\in H_0^1([-1, 1])$ *and* $\alpha\in[-1, 1]$, *we then define*

$$|w|_\alpha=\left(\sum_1^\infty \lambda_k^\alpha |\langle w, \varphi_k\rangle|^2\right)^{1/2},$$

where $\langle .,.\rangle$ *is the usual* L^2 *inner product on* $[-1, 1]$.

The following elementary facts are easily derived: for $w, z\in H_0^1$,

$$\alpha_1\leqq\alpha_2\Rightarrow |w|_{\alpha_1}\leqq|w|_{\alpha_2}; \quad (1.15)$$

$$|w|_0=\|w\| \quad \text{and} \quad |w|_1=\|w_x\|; \quad (1.16)$$

$$|\langle w, z\rangle|\leqq|w|_\alpha|z|_{-\alpha}; \quad (1.17)$$

$$|w|_\alpha=\sup_{z\in H_0^1-\{0\}}\frac{|\langle w, z\rangle|}{|z|_{-\alpha}}; \quad (1.18)$$

$$|w|_\alpha\leqq|w|_{\alpha_1}^{(\alpha_2-\alpha)/(\alpha_2-\alpha_1)}|w|_{\alpha_2}^{(\alpha-\alpha_1)/(\alpha_2-\alpha_1)}, \qquad \alpha_1\leqq\alpha\leqq\alpha_2. \quad (1.19)$$

Next we introduce the *weak truncation error* for approximate solutions of the system (1.1)-(1.4). The weak truncation error measures the extent to which an approximate solution fails to be an exact weak solution, in the sense of Definition 1.2.

DEFINITION 1.3. — *Let* (v^h, u^h) *be an approximate solution of* (1.1)-(1.4) *for which* $v^h \in [\underline{v}, \bar{v}]$ a.e. *Given* $t_1 < t_2$ *in* $[0, T]$ *we then define the linear functionals* $\mathscr{L}_1(t_1, t_2, .)$ *and* $\mathscr{L}_2(t_1, t_2, .)$ *as follows: for* $\varphi, \psi \in C^1([-1, 1] \times [t_1, t_2])$ *with* $\psi(\pm 1, t) = 0$,

$$\mathscr{L}_1(t_1, t_2, \varphi) = \int_{-1}^{1} (v^h \varphi)(x, .)\, dx \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_{-1}^{1} (-v^h \varphi_t + u^h \varphi_x)\, dx\, dt$$
$$- \int_{t_1}^{t_2} \int_{-1}^{1} F_1(x, t, y(v^h(x, t)), v^h(x, t), u^h(x, t))\, \varphi(x, t)\, dx\, dt, \quad (1.20)$$

and

$$\mathscr{L}_2(t_1, t_2, \psi) = \int_{-1}^{1} (u^h \psi)(x, .)\, dx \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \int_{-1}^{1} [u^h \psi_t + p(v^h) \psi_x]\, dx\, dt$$
$$+ \int_{t_1}^{t_2} \int_{-1}^{1} \frac{\varepsilon u_x^h \psi_x}{v^h}\, dx\, dt$$
$$- \int_{t_1}^{t_2} \int_{-1}^{1} F_2(x, t, y(v^h(x, t)), v^h(x, t), u^h(x, t))\, \psi(x, t)\, dx\, dt. \quad (1.21)$$

Then given $\alpha \in [0, 1]$, *we define the weak truncation error* Q_α *associated to* (v^h, u^h) *by*

$$Q_\alpha(t_1, t_2) = \sup_{s_1, s_2, \varphi, \psi} \frac{|\mathscr{L}_1(s_1, s_2, \varphi)| + |\mathscr{L}_2(s_1, s_2, \psi)|}{|||(\varphi, \psi)|||_{\alpha, [s_1, s_2]}}, \quad (1.22)$$

where the sup *is taken over* $s_1 < s_2$ *in* $[t_1, t_2]$, *and over* φ, ψ *as described above, and where*

$$|||(\varphi, \psi)|||_{\alpha, [s_1, s_2]} = \sup_{s_1 \leqq t \leqq s_2} [\|\varphi(., t)\| + |\psi(., t)|_\alpha + (s_2 - t)^{(1-\alpha)/2} \|\psi_x(., t)\|]$$
$$+ \left(\int_{s_1}^{s_2} \int_{-1}^{1} [\psi_x^2 + (s_2 - t)^{1-\alpha} \psi_t^2]\, dx\, dt \right)^{1/2}. \quad (1.23)$$

In the following theorem we give bounds for the error

$$E_{-\alpha}(t) = \|(v - v^h)(., t)\| + |(u - u^h)(., t)|_{-\alpha} \quad (1.24)$$

in terms of the weak truncation error Q_α and the initial error:

THEOREM 1.4. — *Assume that the functions and parameters in* (1.1)-(1.4) *satisfy the hypotheses* A1-A4, *let* T *be a fixed positive time, and let constants* C_0, C_1, *and* r *be given, as in* (1.13) *and* (1.14), *with* $0 < r < 1$.

(*a*) *If* F_1 *and* F_2 *are independent of* u, *then given* $\alpha \in (0, 1]$, *there is a constant* $C = C(\alpha)$, *depending only on* α, T, ε, $p|_{[\underline{v}, \bar{v}]}$, C_F, C_0, C_1, *and* r, *such that, if* (v, u) *is a solution of* (1.1)-(1.4) *satisfying* (1.6)-(1.14), *and* *if* (v^h, u^h) *is an approximate solution, satisfying* (1.6), (1.8), (1.9), *and* (1.12)-(1.14), *then for any* $\tau \in [0, T]$,

$$\sup_{\tau \leqq t \leqq T} E_{-\alpha}(t) \leqq C(\alpha)[E_{-\alpha}(\tau) + Q_\alpha(\tau, T)]. \tag{1.25}$$

(In general, the constant $C(\alpha)$ *may become unbounded as* $\alpha \downarrow 0$.) *In addition, for* $t > \tau \geqq 0$ *and* $\alpha \in (0, 1]$,

$$E_0(t) \leqq C(\alpha)[\mathscr{E} + t^{-\alpha/4(1+\alpha)} \mathscr{E}^{1/(1+\alpha)}], \tag{1.26}$$

where $\mathscr{E}$ *is the term in brackets in* (1.25).

(*b*) *In the general case that the* F_j *do depend on* u, *there are positive constants* C *and* $\bar{C}_1$, *depending only on* T, ε, $p|_{[\underline{v}, \bar{v}]}$, C_F, C_0, *and* r, *such that, if* (v, u) *and* (v^h, u^h) *are as in* (*a*), *and if* $C_1 \leqq \bar{C}_1$, *then for any* $\tau \in [0, T]$,

$$\sup_{\tau \leqq t \leqq T} E_0(t) \leqq C[E_0(\tau) + Q_0(\tau, T)]. \tag{1.27}$$

Thus in the case that (v^h, u^h) is an *exact* weak solution, $Q_\alpha = 0$, and the theorem asserts continuous dependence on initial data in L^2. In the general case (*b*), this initial data must be assumed to be small. When the F_j are independent of u, however, this smallness assumption can be eliminated, but at the expense of further weakening the topology of continuous dependence, as in (1.25), or of introducing an initial layer, as in (1.26).

We now give a brief sketch of the key ideas in the proof of Theorem 1.4; complete details are presented in section 2. Thus let (v_j, u_j), $j = 1, 2$, be *exact* solutions of (1.1)-(1.4) as described above, and let $\Delta v = v_2 - v_1$ and $\Delta u = u_2 - u_1$. Then formally, from (1.1)-(1.2),

$$\Delta v_t - \Delta u_x = 0$$
$$\Delta u_t + \Delta p_x = \varepsilon \left(\frac{\Delta u_x}{v_2} - \frac{\Delta v}{v_1 v_2} u_{1\,x} \right)_x.$$

The direct approach of ([8]-[9]) consists in simply multiplying these two equations by Δv and Δu, respectively, and integrating over $[-1, 1] \times [0, t]$. Applying the Cauchy-Schwartz inequality in an elementary way, one thus obtains that

$$\int_{-1}^{1} (\Delta v^2 + \Delta u^2)(x, t)\, dx + \int_0^t \int_{-1}^{1} \Delta u_x^2 (x, s)\, dx\, ds$$
$$\leqq C \int_{-1}^{1} (\Delta v_0^2 + \Delta u_0^2)\, dx + C \int_0^t \int_{-1}^{1} [\Delta v^2 + |\Delta u_x u_{1\,x} \Delta v|]\, dx\, ds. \tag{1.28}$$

Now, a simple Gronwall argument takes care of the term $\iint \Delta v^2$ on the right-hand side here. But if we are attempting to obtain an estimate for E_0, then we are constrained to bound the other term in this double integral by

$$\frac{\delta}{2}\iint \Delta u_x^2 + \frac{1}{2\delta}\sup_{0\leqq s\leqq t}\|\Delta v(.,s)\|^2\int_0^t \|u_{1\,x}(.,s)\|_\infty^2\,ds. \tag{1.29}$$

The first term here can be absorbed into the left side of (1.28) when δ is small. The problem, however, is that the integral in the other term is in general divergent. Indeed, solutions w of the heat equation with data in $L^2 \cap BV$ achieve the rate of smoothing $\|w_x(.,t)\|_\infty \leqq C t^{-1/2}$, but no better. The approach taken in ([8]-[9]) was therefore to replace the estimate (1.29) with the bound

$$\frac{\delta}{2}\iint \Delta u_x^2 + \frac{1}{2\delta}\sup_{0\leqq s\leqq t}\|\Delta v(.,t)\|_\infty^2 \iint u_{1\,x}^2\,dx\,ds.$$

This succeeds because the second integral here is small for small time, by (1.14). This success occurs, however, at the expense of having to measure the perturbation $(\Delta v, \Delta u)$ in a norm which dominates $\|\Delta v\|_\infty$; and this we regard as unsatisfactory.

In the present paper we apply a completely different approach to circumvent this difficulty. Specifically, we subtract the weak forms (1.10)-(1.11) satisfied by the two solutions, and choose the test functions φ and ψ to satisfy appropriate adjoint equations, solved backward in time, with "initial" data given at a positive time t. Omitting all but the key terms, we obtain

$$\left|\int (\Delta v\,\varphi + \Delta u\,\psi)\big|_0^t\,dx\right| \leqq \iint |\Delta v\,\psi_x u_{1\,x}|\,dx\,ds + \ldots \tag{1.30}$$

The term on the right here clearly plays the same role in this approach as the troublesome term on the right side of (1.28). Now, in order to obtain information from (1.30) about $\|\Delta v(.,t)\|$ and $\|\Delta u(.,t)\|$ by duality, we should take $\varphi(.,t)$, $\psi(.,t)\in L^2$. We therefore need to show that, for such data, the solution (φ, ψ) of our adjoint system satisfies

$$\|\psi_x(.,s)\|_\infty \leqq \|\psi(.,t)\|(t-s)^{-3/4}, \tag{1.31}$$

which again is sharp even for solutions of the heat equation with L^2 data. Assuming that (1.31) has been proved, we may then apply (1.14) to bound the term on the right side of (1.30) by

$$\sup_{0\leqq s\leqq t}\|\Delta v(.,s)\|\int_0^t \|\psi_x(.,s)\|_\infty\,\|u_{1\,x}(.,s)\|\,ds$$
$$\leqq \sup_{0\leqq s\leqq t}\|\Delta v(.,s)\|\,C_1\,\|\psi(.,t)\|\int_0^t (t-s)^{-3/4}s^{-1/4}\,ds.$$

Since the integral on the right here is a constant, this term may be absorbed provided that C_1 is small. This gives part (*b*) of Theorem 1.4.

Observe that we succeeded by this adjoint-equation device in splitting the nonintegrable singularity $1/t$ occurring in (1.29) into the integrable product $(t-s)^{-3/4}s^{-1/4}$ of two singularities of the same total strength. (Actually, we have oversimplified here a bit. Since the discontinuous functions v_j determine the coefficients of the adjoint system, ψ will satisfy a parabolic equation whose coefficients are not smooth. It is therefore necessary to replace ψ by a lower-order perturbation of ψ in the above argument.)

In the case that C_1 is not small, we choose the initial data $\psi(.,t)$ to be somewhat smoother $-\psi(.,t)\in H^\alpha$ for $\alpha>0$. This results in a slightly more favorable smoothing rate in (1.31), so that the integral in (1.29) is replaced by one which is small when t is small. We can then obtain from (1.30) a bound only for $\|\Delta v(.,t)\|+|\Delta u(.,t)|_{-\alpha}$, which is the estimate in (1.25). The L^2 bound (1.26) is then recovered from this result and the bound (1.14) by a simple interpolation argument.

Most of the work in the proof of Theorem 1.4 consists in obtaining the required rates of smoothing for the adjoint solution (φ, ψ), particularly in the H^α norms. These results are achieved by applying standard energy estimates, together with certain interpolation-theory arguments based on the proof of the Riesz-Thorin Theorem.

2. PROOF OF THEOREM 1.4

We begin by deriving an estimate for $E_{-\alpha}(t_2)$ in terms of $E_{-\alpha}(t_1)$ when $0<t_1<t_2<T$ and $\alpha\in[0,1]$. Thus the two cases of the theorem will be combined for the present, and will be distinguished only at the end of the argument. For the time being, C will denote a generic positive constant depending only on

$$T,\ \varepsilon,\ p|_{[\underline{v},\bar{v}]},\ C_F,\ C_0 \text{ and } r. \tag{2.1}$$

Thus fix times $t_1<t_2$ in $(0,T)$ and $\alpha\in[0,1]$. We subtract the definitions (1.20) and (1.21) of the functionals $\mathscr{L}_1$ and $\mathscr{L}_2$ from the corresponding weak forms (1.10) and (1.11). Adding the resulting two equations, we obtain

$$\begin{aligned}\int_{-1}^{1}&(\Delta v\,\varphi+\Delta u\,\psi)(x,.)\,dx\Big|_{t_1}^{t_2}\\&+\int_{t_1}^{t_2}\int_{-1}^{1}\Big[-\Delta v\Big(\varphi_t+\frac{\Delta p}{\Delta v}\psi_x\Big)-\Delta u(\psi_t-\varphi_x)\\&\qquad+\frac{\varepsilon\Delta u_x}{v}\psi_x+\varepsilon\Big(\frac{1}{v}-\frac{1}{v^h}\Big)u_x^h\psi_x\Big]\,dx\,dt\\&=\int_{t_1}^{t_2}\int_{-1}^{1}(\Delta F_1\,\varphi+\Delta F_2\,\psi)\,dx\,dt-\mathscr{L}_1(t_1,t_2,\varphi)-\mathscr{L}_2(t_1,t_2,\psi).\end{aligned}$$

Now, for $\delta>0$ we can choose smooth approximations v^δ and $v^{h\delta}$ to v and v^h which satisfy

$$v^\delta(x,t),\ v^{h\delta}(x,t)\in[\underline{v},\bar{v}] \tag{2.3}$$

and

$$\|v_t^\delta(.,t)\|_\infty,\qquad \|v_t^{h\delta}(.,t)\|_\infty\leq C_0 t^{-r}, \tag{2.4}$$

as in (1.8) and (1.13), and such that

$$v^\delta\to v\quad\text{ and }\quad v^{h\delta}\to v^h\text{ a.e. in } [-1,1]\times[t_1,t_2]\quad\text{as } \delta\to 0. \tag{2.5}$$

We then let a^δ denote the divided difference

$$a^\delta(x,t)=p[v^\delta(x,t),v^{h\delta}(x,t)]=\begin{cases}\dfrac{p(v^\delta)-p(v^{h\delta})}{v^\delta-v^{h\delta}}, & v^\delta\neq v^{h\delta},\\ p'(v^\delta), & v^\delta=v^{h\delta},\end{cases} \tag{2.6}$$

which is therefore a smooth function for $(x,t)\in[-1,1]\times[t_1,t_2]$.

We now fix functions f and g in $C_0^\infty(-1,1)$, and we take $\varphi=\varphi^\delta$ and $\psi=\psi^\delta$ in (2.2) to be the solutions of the following adjoint system:

$$\left.\begin{aligned}&\varphi_t+a^\delta\psi_x=0\\ &\psi_t-\varphi_x+\left(\frac{\varepsilon\psi_x}{v^\delta}\right)_x=0,\qquad -1<x<1,\ t<t_2,\\ &\varphi(.,t_2)=f,\qquad \psi(.,t_2)=g,\\ &\psi(\pm1,t)=0.\end{aligned}\right\} \tag{2.7}$$

It will suffice to consider "initial" data (f,g) satisfying

$$\|f\|=|g|_\alpha=1. \tag{2.8}$$

Substituting the first two equations in (2.7) into (2.2), we then obtain

$$\begin{aligned}\int_{-1}^{1}[\Delta v(.,t_2)f+\Delta u(.,t_2)g]\,dx&=\int_{-1}^{1}(\Delta v\,\varphi^\delta+\Delta u\,\psi^\delta)(x,t_1)\,dx\\ &+\int_{t_1}^{t_2}\int_{-1}^{1}\Big[\Delta v\,(p[v,v^h]-a^\delta)\,\psi_x^\delta+\varepsilon\Delta u_x\left(\frac{1}{v^\delta}-\frac{1}{v}\right)\psi_x^\delta\\ &+\varepsilon\left(\frac{1}{v^h}-\frac{1}{v}\right)u_x^h\psi_x^\delta+(\Delta F_1\,\varphi^\delta+\Delta F_2\,\psi^\delta)\Big]\,dx\,dt\\ &-\mathscr{L}_1(t_1,t_2,\varphi^\delta)-\mathscr{L}_2(t_1,t_2,\psi^\delta).\end{aligned}$$

Before estimating the various terms appearing in (2.9) above, we first collect together various properties of the adjoint functions φ^δ and ψ^δ:

LEMMA 2.1. — *There is a constant* C, *depending only on the quantities in* (2.1), *but not on* α, t_1, t_2, δ, f, *or* g, *such that the solution* $(\varphi^\delta,\psi^\delta)$ *of* *the system* (2.7)-(2.8) *satisfies*

$$|||(\varphi^\delta, \psi^\delta)|||_{\alpha, [t_1, t_2]} \equiv \sup_{t_1 \leqq t \leqq t_2} [\| \varphi^\delta(., t) \| + | \psi^\delta(., t) |_\alpha + (t_2 - t)^{(1-\alpha)/2} \| \psi_x^\delta(., t) \|] + \left(\int_{t_1}^{t_2} \int_{-1}^{1} [(\psi_x^\delta)^2 + (t_2 - t)^{1-\alpha} (\psi_t^\delta)^2] \, dx \, dt \right)^{1/2} \leqq C, \quad (2.10)$$

and

$$\left\| \left(\frac{\varepsilon \psi_x^\delta}{v^\delta} - \varphi^\delta \right)(., t) \right\|_\infty \leqq C (t_2 - t)^{(\alpha-1)/2} \left[1 + \left((t_2 - t)^{1-\alpha} \int_{-1}^{1} (\psi_t^\delta)^2 \, dx \right)^{1/4} \right]. \quad (2.11)$$

The proof of Lemma 2.1 is somewhat lengthy, and will therefore be deferred to the end of this section.

We now show that the first two terms in the double integral on the right-hand side of (2.9) approach zero as $\delta \to 0$. We shall present the argument only for the second of these, which is the more difficult. Triangulating, we may bound this term by

$$C \iint | \Delta u_x (v - v^\delta) | \left| \frac{\varepsilon \psi_x^\delta}{v^\delta} - \varphi^\delta \right| + C \iint | \Delta u_x (v - v^\delta) | \, | \varphi^\delta |. \quad (2.12)$$

The second term in (2.12) is bounded by

$$C \left(\iint \Delta u_x^2 (v - v^\delta)^2 \right)^{1/2}$$

by (2.10). The integrand here approaches zero a.e., and is bounded by $C \Delta u_x^2 \in L^1([-1, 1] \times [t_1, t_2])$, by (2.3) and (1.8). This integral therefore approaches zero as $\delta \to 0$.

We apply (2.11) to bound the first term in (2.12) by

$$C \int_{t_1}^{t_2} \left((t_2 - t)^{-1/2} \left[1 + \left(\int_{-1}^{1} (t_2 - t) (\psi_t^\delta)^2 \, dx \right)^{1/4} \right] \int_{-1}^{1} | \Delta u_x (v - v^\delta) | \, dx \right) dt.$$

We apply Hölder's inequality with exponents 4 and 4/3, and then appeal to (2.10) to obtain the bound

$$C \left[\int_{t_1}^{t_2} (t_2 - t)^{-2/3} \left(\int_{-1}^{1} | \Delta u_x (v - v^\delta) | \, dx \right)^{4/3} dt \right]^{3/4}. \quad (2.13)$$

Now, $\Delta u_x (v - v^\delta) \to 0$ a.e. in $[-1, 1] \times [t_1, t_2]$, as $\delta \to 0$, so that, by Fubini's theorem, there is a set A of measure zero for which $\Delta u_x (x, t) (v - v^\delta)(x, t) \to 0$ for almost all x when $t \notin A$. Since

$$| \Delta u_x (., t) (v - v^\delta)(., t) | \leqq C | \Delta u_x (., t) | \in L^1([-1, 1])$$

by (1.12), we thus obtain that

$$\int_{-1}^{1} |\Delta u(x, t)(v-v^{\delta})(x, t)|\, dx \to 0 \quad \text{as } \delta \to 0$$

when $t \notin \mathrm{A}$. This shows that the integrand of the time integral in (2.13) approaches zero as $\delta \to 0$ for almost all t. This integrand is bounded by

$$\begin{aligned}(t_2-t)^{-2/3}\left(\int_{-1}^{1} |\Delta u_x (v-v^{\delta})|\, dx\right)^{4/3} &\leqq (t_2-t)^{-2/3}\left(\int \Delta u_x^2\, dx\right)^{2/3}\left(\int (v-v^{\delta})^2\, dx\right)^{2/3}\\ &\leqq \mathrm{C}(t_2-t)^{-2/3}\, t^{-1/3} \in \mathrm{L}^1([t_1, t_2]),\end{aligned}$$

by (1.14). (The precise exponents are not really crucial here, since $t_1 > 0$.) This proves that the expression in (2.13) approaches zero as $\delta \to 0$, and completes the proof that the first two terms in the double integral in (2.9) vanish in the limit as $\delta \to 0$.

The third term in this double integral is bounded by

$$\mathrm{C}\iint |u_x^h \Delta v| \left|\frac{\varepsilon \psi_x^{\delta}}{v^{\delta}} - \varphi^{\delta}\right| + \mathrm{C}\iint |u_x^h \Delta v|\, |\varphi^{\delta}|. \tag{2.14}$$

We apply (2.10) and (1.13) to bound the second of these terms by

$$\begin{aligned}\int_{t_1}^{t_2} \| u_x^h(\cdot, t)\|_{\infty} \left(\int \Delta v^2\, dx\right)^{1/2} \left(\int (\varphi^{\delta})^2\, dx\right)^{1/2} dt &\leqq \mathrm{C} \sup_{t_1 \leqq t \leqq t_2} \left(\int \Delta v^2\, dx\right)^{1/2} \int_{t_1}^{t_2} t^{-r}\, dt\\ &\leqq \mathrm{C}(t_2^{1-r} - t_1^{1-r}) \sup_{t_1 \leqq t \leqq t_2} \mathrm{E}_{-\alpha}(t).\end{aligned}$$

Applying (2.11), (1.14), and (2.10), we can bound the first term in (2.14) by

$$\begin{aligned}\mathrm{C}\int_{t_1}^{t_2} (t_2-t)^{(\alpha-1)/2}&\left[1+\left(\int_{-1}^{1} (t_2-t)^{1-\alpha}(\psi_t^{\delta})^2\, dx\right)^{1/4}\right]\\ &\left(\int_{-1}^{1} \Delta v^2\, dx\right)^{1/2}\left(\int_{-1}^{1} (u_x^h)^2\, dx\right)^{1/2} dt\\ &\leqq \mathrm{C}\mathrm{C}_1 \sup_{t_1 \leqq t \leqq t_2} \mathrm{E}_{-\alpha}(t) \int_{t_1}^{t_2} (t_2-t)^{(\alpha-1)/2}\, t^{-1/4}\\ &\left[1+\left(\int_{-1}^{1} (t_2-t)^{1-\alpha}(\psi_t^{\delta})^2\, dx\right)^{1/4}\right] dt\end{aligned}$$

$$\leqq CC_1 \sup E_{-\alpha}(t)\left[\int_{t_1}^{t_2}(t_2-t)^{2(\alpha-1)/3}\, t^{-1/3}\, dt\right]^{3/4}$$
$$\leqq CC_1 (t_2-t_1)^{\alpha/2} \sup_{t_1 \leqq t \leqq t_2} E_{-\alpha}(t),$$

because

$$\int_0^{t_2}(t_2-t)^{-2/3}\, t^{-1/3}\, dt = \int_0^1 (1-s)^{-2/3}\, s^{-1/3}\, ds$$

is a constant independent of t_2. (This is the point in the argument where the specific exponent in (1.14) is crucial.) Combining these two estimates, we then have that the third term in the double integral in (2.9) is bounded by

$$C[(t_2^{1-r}-t_1^{1-r})+C_1(t_2-t_1)^{\alpha/2}] \sup_{t_1 \leqq t \leqq t_2} E_{-\alpha}(t).$$

To bound the ΔF_j terms in (2.9) we use the fact that, by the definition (1.5),

$$|\Delta y(x,t)| \leqq \int_{-1}^{x} |\Delta v(s,t)|\, ds \leqq C \|\Delta v(.,t)\|.$$

Therefore by A4 and (2.10), for case (*b*) of Theorem 1.4,

$$\left|\iint \Delta F_2\, \psi^\delta\, dx\, dt\right| \leqq C \int_{t_1}^{t_2} (\|\Delta v(.,t)\| + \|\Delta u(.,t)\|)\, \|\psi^\delta(.,t)\|\, dt$$
$$\leqq C(t_2-t_1) \sup_{t_1 \leqq t \leqq t_2} E_0(t) = C(t_2-t_1) \sup_{t_1 \leqq t \leqq t_2} E_{-\alpha}(t),$$

and similarly for the F_1 term in (2.9). The same estimate holds in case (*a*) that the F_j are independent of u, because $\|\psi^\delta\| \leqq |\psi^\delta|_\alpha$ for $\alpha \geqq 0$.

Finally, the single integral on the right-hand side of (2.9) is bounded by $CE_{-\alpha}(t_1)$; and (2.10) and the definition (1.22) show that

$$|\mathscr{L}_1(t_1,t_2,\varphi^\delta)| + |\mathscr{L}_2(t_1,t_2,\psi^\delta)| \leqq Q_\alpha(t_1,t_2)\, |||(\varphi^\delta,\psi^\delta)|||_{\alpha,[t_1,t_2]}$$
$$\leqq CQ_\alpha(t_1,t_2).$$

Combining all these estimates, we therefore obtain from (2.9) that

$$\int_{-1}^{1} [\Delta v(x,t_2) f(x) + \Delta u(x,t_2) g(x)]\, dx$$
$$\leqq C\big(E_{-\alpha}(t_1) + Q_\alpha(t_1,t_2)$$
$$+ [C_1(t_2-t_1)^{\alpha/2} + (t_2^{1-r}-t_1^{1-r}) + (t_2-t_1)] \sup_{t_1 \leqq t \leqq t_2} E_{-\alpha}(t)\big)$$

for all f and g satisfying (2.8). Taking the sup over such f and g, we may then replace the left-hand side in (2.15) by $E_{-\alpha}(t_2)$. Recall that the constant C in (2.15) depends only on the quantities in (2.1). In case (*a*),

α is positive, and we can therefore assert that there is a positive number Δt, depending only on the quantities in (2.1) and on α and C_1, such that

$$\sup_{t_1 \leqq t \leqq t_2} E_{-\alpha}(t) \leqq C[E_{-\alpha}(t_1) + Q_\alpha(t_1, t_2)] \tag{2.16}$$

when $0 < t_2 - t_1 \leqq \Delta t$. In case (*b*), α is zero, and (2.15) shows instead that there are positive constants Δt and $\bar{C}_1$, depending only on the quantities in (2.1), such that

$$\sup_{t_1 \leqq t \leqq t_2} E_0(t) \leqq C[E_0(t_1) + Q_0(t_1, t_2)] \tag{2.17}$$

provided $C_1 \leqq \bar{C}_1$ and $0 < t_2 - t_1 \leqq \Delta t$. A standard Gronwall argument then shows that (2.16) and (2.17) hold for all $t_1 < t_2$ in (0, T), and therefore for all $t_1 < t_2$ in [0, T], by (1.6). This proves (1.25) and (1.27). To prove (1.26) we observe that, by (1.14),

$$|\Delta u(., t)|_1 \leqq C t^{-1/4}$$

for all t. (1.26) then follows from this and (1.25) via the interpolation result (1.19). This completes the proof of Theorem 1.4. □

Proof of Lemma 2.1. – We let C denote a generic positive constant depending only on the quantities listed in (2.1), and we surpress the dependence on δ of the solution (φ, ψ) of the adjoint system (2.7)-(2.8).

To derive an L^2 bound for (φ, ψ), we multiply the first equation in (2.7) by $-\varphi/a^\delta$ and the second by ψ and integrate. Applying the boundary condition, we obtain that, for $t_1 \leqq s_1 < s_2 \leqq t_2$,

$$\int_{-1}^{1} \left(-\frac{\varphi^2}{2a^\delta} + \psi^2 \right)(x, .)\, dx \Big|_{s_1}^{s_2} + \int_{s_1}^{s_2} \int_{-1}^{1} \left[-\frac{\varphi^2}{2(a^\delta)^2} a_t^\delta - \frac{\varepsilon \psi_x^2}{v^\delta} \right] dx\, dt = 0.$$

Rearranging and using the fact that $C^{-1} < -a^\delta \leqq C$ (*see* A1 and (2.6)), we get that

$$\begin{aligned} \int_{-1}^{1} (\varphi^2 + \psi^2)(x, s_1)\, dx + \int_{s_1}^{s_2} \int_{-1}^{1} \psi_x^2\, dx\, dt \\ \leqq C \Bigg[\int_{-1}^{1} (\varphi^2 + \psi^2)(x, s_2)\, dx \\ + \left(\sup_{s_1 \leqq t \leqq s_2} \int \varphi(x, t)^2\, dx \right) \int_{s_1}^{s_2} \| a_t^\delta(., t) \|_\infty\, dt \Bigg]. \end{aligned} \tag{2.18}$$

However, (2.4) shows that

$$\int_{s_1}^{s_2} \| a_t^\delta(., t) \|_\infty\, dt \leqq C_0 (s_2^{1-r} - s_1^{1-r}).$$

Applying a standard Gronwall argument in (2.18), we may then conclude that

$$\sup_{t_1 \leqq t \leqq t_2} \left[\int_{-1}^{1} (\varphi^2 + \psi^2)(x, t)\, dx \right] + \int_{t_1}^{t_2} \int_{-1}^{1} \psi_x^2\, dx\, dt \leqq C \int_{-1}^{1} (f^2 + g^2)\, dx. \quad (2.19)$$

Next, we derive simultaneously two different H^1 bounds for ψ, corresponding to different norms of the initial function g. To do this, we let σ denote either of the functions $\sigma(t) = t_2 - t$, or $\sigma(t) \equiv 1$. We multiply the second equation in (2.7) by $\sigma(t)\psi_t(x, t)$ and integrate and apply the boundary condition $\psi_t(\pm 1, t) = 0$ to obtain that, for $t_1 \leqq s_1 < s_2 \leqq t_2$,

$$\int_{s_1}^{s_2} \int_{-1}^{1} \sigma \psi_t^2\, dx\, dt - \varepsilon \int_{s_1}^{s_2} \int_{-1}^{1} \frac{\sigma}{v^\delta} \left(\frac{\psi_x^2}{2} \right)_t dx\, dt = \int_{s_1}^{s_2} \int_{-1}^{1} \sigma \psi_t \varphi_x\, dx\, dt$$
$$= -\sigma(.) \int_{-1}^{1} (\psi_x \varphi)(x, .)\, dx \Big|_{s_1}^{s_2} + \int_{s_1}^{s_2} \int_{-1}^{1} (\sigma_t \varphi + \sigma \varphi_t) \psi_x\, dx\, dt.$$

Rearranging and applying the Cauchy-Schwartz inequality to the single integral on the right, and applying (2.19) in an elementary way, we then obtain that

$$\int_{s_1}^{s_2} \int_{-1}^{1} \sigma \psi_t^2\, dx\, dt + \sigma(s_1) \int_{-1}^{1} \psi_x(x, s_1)^2\, dx$$
$$\leqq C \left[\int_{-1}^{1} (f^2 + g^2)\, dx + \sigma(s_2) \int_{-1}^{1} \psi_x(x, s_2)^2\, dx + \int_{s_1}^{s_2} \int_{-1}^{1} \sigma (\psi_x^2 |v_t^\delta| + |\psi_x \varphi_t|)\, dx\, dt \right].$$

We have that

$$\int_{s_1}^{s_2} \int_{-1}^{1} \sigma \psi_x^2 |v_t^\delta|\, dx\, dt \leqq C (s_2^{1-r} - s_1^{1-r}) \sup_{s_1 \leqq t \leqq s_2} \left[\sigma(t) \int \psi_x(x, t)^2\, dx \right]$$

by (2.4), and

$$\iint |\psi_x \varphi_t| \leqq C \iint \psi_x^2 \leqq C \int_{-1}^{1} (f^2 + g^2)\, dx$$

by (2.7) and (2.19). Thus

$$\int_{s_1}^{s_2}\int_{-1}^{1}\sigma\psi_t^2\,dx\,dt+\sigma(s_1)\int_{-1}^{1}\psi_x(x,s_1)^2\,dx$$
$$\leqq C\left[\int_{-1}^{1}(f^2+g^2)\,dx+\sigma(s_2)\int_{-1}^{1}\psi_x(x,s_2)^2\,dx\right.$$
$$\left.+(s_2^{1-r}-s_1^{1-r})\sup_{s_1\leqq t\leqq s_2}\left(\sigma(t)\int_{-1}^{1}\psi_x(x,t)^2\,dx\right)\right].$$

A standard Gronwall argument then shows that

$$\int_{t_1}^{t_2}\int_{-1}^{1}\sigma\psi_t^2\,dx\,dt+\sup_{t_1\leqq t\leqq t_2}\sigma(t)\int\psi_x(x,t)^2\,dx$$
$$\leqq C\left[\int_{-1}^{1}(f^2+g^2)\,dx+\sigma(t_2)\int_{-1}^{1}g'(x)^2\,dx\right],\qquad(2.20)$$

where either $\sigma(t)=t_2-t$, or $\sigma(t)\equiv 1$.

Observe that (2.19) and (2.20) prove (2.10) in the special case that $\alpha=0$. For other values of α we shall have to appeal to some standard results of interpolation theory; these are stated in the following lemma.

LEMMA 2.2. – *Let* $H=H_0^1([-1,1])$.

(*a*) *Suppose that* $B: H\times H\to\mathbb{C}$ *is bilinear, and that there are indices* $\alpha_j,\beta_j\in[-1,1]$ *and constants* M_1 *and* M_2 *such that*

$$|B(g,g^*)|\leqq M_j|g|_{\alpha_j}|g^*|_{\beta_j},\qquad j=1,2,\qquad(2.21)$$

for all $g, g^*\in H$. *Then for any* $\theta\in[0,1]$, *and for all* $g, g^*\in H$,

$$|B(g,g^*)|\leqq M_1^{1-\theta}M_2^{\theta}|g|_\alpha|g^*|_\beta,\qquad(2.22)$$

where

$$\alpha=\alpha_1+\theta(\alpha_2-\alpha_1)\qquad\text{and}\qquad\beta=\beta_1+\theta(\beta_2-\beta_1).$$

(*b*) *Let* $S: H\to L^2([-1,1]\times[t_1,t_2])$ *be a linear operator, and suppose that there are constants* M_0 *and* M_1 *such that*

$$\left(\int_{t_1}^{t_2}\int_{-1}^{1}(t_2-t)|Sg|^2\,dx\,dt\right)^{1/2}\leqq M_0|g|_0$$

and

$$\left(\int_{t_1}^{t_2}\int_{-1}^{1}|Sg|^2\,dx\,dt\right)^{1/2}\leqq M_1|g|_1.$$

Then for any $\alpha\in[0,1]$ *and for all* $g\in H_0^1$,

$$\left(\int_{t_1}^{t_2}\int_{-1}^{1}(t_2-t)^{1-\alpha}|Sg|^2\,dx\,dt\right)^{1/2}\leqq M_0^{1-\alpha}M_1^{\alpha}|g|_\alpha.$$

(*a*) and (*b*) are special cases of standard results in the general theory of interpolation of Banach spaces. For the reader's benefit, we sketch simple, self-contained proofs at the end of this section. (These proofs are little more than appropriate adaptations of the proof of the Riesz-Thorin theorem.)

We now apply part (*a*) of Lemma 2.2 to derive a bound for the term

$$\sup_{t_1 \leqq t \leqq t_2} (t_2 - t)^{1-\alpha} \int_{-1}^{1} \psi_x(x, t)^2 \, dx$$

appearing in (2.10). Fixing $t \in [t_1, t_2)$, we define a linear mapping $S : L^2 \times H_0^1 \to L^2$ by $S(f, g) = \psi_x(., t)$, where (φ, ψ) is the solution of the system (2.7) with data (f, g). We then define a bilinear form B on $H_0^1 \times H_0^1$ by

$$B(g, g^*) = \langle S(0, g), g^* \rangle.$$

(Recall that $\langle ., . \rangle$ is the usual inner product in $L^2([-1, 1])$.) The two cases of (2.20) then show that

$$\begin{aligned} |B(g, g^*)| &\leqq |S(0, g)|_0 \, |g^*|_0 = \| \psi_x(., t) \| \, |g^*|_0 \\ &\leqq C (t_2 - t)^{-1/2} |g|_0 \, |g^*|_0, \end{aligned}$$

and

$$\begin{aligned} |B(g, g^*)| &\leqq C (\|g\| + \|g'\|) |g^*|_0 \\ &\leqq C |g|_1 |g^*|_0, \end{aligned}$$

by the Poincaré inequality. Lemma 2.2 therefore implies that

$$|\langle S(0, g), g^* \rangle| = |B(g, g^*)| \leqq C (t_2 - t)^{(\alpha - 1)/2} |g|_\alpha |g^*|_0$$

for all g^*, so that

$$\| S(0, g) \| \leqq C (t_2 - t)^{(\alpha - 1)/2} |g|_\alpha.$$

We also have from the $\sigma \equiv 1$ case of (2.20) that

$$\| S(f, 0) \| \leqq C \| f \|,$$

so that, by the linearity of S,

$$\| \psi_x(., t) \| = \| S(f, g) \| \leqq C (t_2 - t)^{(\alpha - 1)/2} (\| f \| + |g|_\alpha). \tag{2.23}$$

A similar argument shows that

$$|\psi(., t)|_\alpha \leqq C (\| f \| + |g|_\alpha). \tag{2.24}$$

To bound the last term on the right-hand side of (2.10), we now define $S(f, g) = \psi_t$, where (φ, ψ) is the solution of (2.7) with data (f, g). The mapping $g \mapsto S(0, g)$ from H_0^1 into $L^2([-1, 1] \times [t_1, t_2])$ is then linear, and the two cases of (2.20) show that

$$\left(\int_{t_1}^{t_2} \int_{-1}^{1} (t_2 - t) |S(0, g)|^2 \, dx \, dt \right)^{1/2} \leqq C |g|_0$$

and

$$\left(\int_{t_1}^{t_2}\int_{-1}^{1} |S(0, g)|^2\, dx\, dt\right)^{1/2} \leqq C(\|g\| + \|g'\|)$$
$$\leqq C|g|_1.$$

Part (*b*) of Lemma 2.2 therefore implies that

$$\left(\int_{t_1}^{t_2}\int_{-1}^{1} (t_2-t)^{1-\alpha} |S(0, g)|^2\, dx\, dt\right)^{1/2} \leqq C|g|_\alpha.$$

We also have from the $\sigma \equiv 1$ case of (2.20) that

$$\left(\iint (t_2-t)^{1-\alpha} |S(f, 0)|^2\right)^{1/2} \leqq C\left(\iint |S(f, 0)|^2\right)^{1/2}$$
$$\leqq C\|f\|.$$

Therefore by the linearity of S,

$$\left(\int_{t_1}^{t_2}\int_{-1}^{1} (t_2-t)^{1-\alpha} \psi_t^2\, dx\, dt\right)^{1/2} \leqq C(\|f\| + |g|_\alpha). \tag{2.25}$$

Combining (2.19), (2.20), and (2.23)-(2.25), we therefore obtain that

$$|||(\varphi, \psi)|||_{\alpha, [t_1, t_2]} \leqq C(\|f\| + \|g\| + |g|_\alpha)$$
$$\leqq C(\|f\| + |g|_\alpha),$$

which proves (2.10).

Finally, to prove (2.11), we fix $t < t_2$ and let

$$z = \left(\frac{\varepsilon \psi_x}{v^\delta} - \varphi\right)(., t).$$

Assuming that the normalization (2.8) is in effect, we then obtain from (2.10) and the second equation in (2.7) that

$$\begin{aligned} \|z\|_\infty^2 &\leqq C[\|z\|^2 + \|z\|\,\|z_x\|] \\ &\leqq C\left((t_2-t)^{\alpha-1} + 1 + [(t_2-t)^{(\alpha-1)/2} + 1]\left[\int_{-1}^{1} \psi_t^2\, dx\right]^{1/2}\right) \\ &\leqq C(t_2-t)^{\alpha-1}\left(1 + \left[(t_2-t)^{1-\alpha}\int_{-1}^{1} \psi_t^2\, dx\right]^{1/2}\right), \end{aligned}$$

as required. □

Proof of Lemma 2.2. — To prove (*a*) it suffices to establish the estimate (2.21) when g and g^* are the finite sums

$$g = \sum a_k \varphi_k \quad \text{and} \quad g^* = \sum b_j \varphi_j, \tag{2.26}$$

where the $\{\varphi_k\}$ are as in Definition 1.2. Fixing such g and g^*, we then define for $z \in \mathbb{C}$ the function

$$w(z) = B\big(\sum_k \lambda_k^{(\alpha-\alpha_1)/2+(\alpha_1-\alpha_2)/2z} a_k \varphi_k, \sum_j \lambda_j^{(\beta-\beta_1)/2+(\beta_1-\beta_2)/2z} b_j \varphi_j\big),$$

where α_j, β_j, α, and β are as in the statement of the lemma. It follows easily from the bilinearity of B and the positivity of the λ_k that w is an entire function of z, and that $|w(z)|$ is bounded in the strip $0 \leqq \operatorname{Re}(z) \leqq 1$. It therefore follows from Lindelöf's theorem (*see* Donaghue [1], p. 18, for a short, self-contained proof) that the function

$$G(\xi) = \sup_\eta |w(\xi + i\eta)|$$

is log convex on [0, 1]; that is, that

$$G(\theta) \leqq G(0)^{1-\theta} G(1)^\theta, \qquad 0 \leqq \theta \leqq 1. \tag{2.27}$$

We estimate G(0) and G(1) as follows. From the hypothesis (2.21) for $j=1$, we have that

$$\begin{aligned} |w(0+i\eta)| &\leqq M_1 \big(\sum \lambda_k^{(\alpha-\alpha_1)/2+(\alpha_1-\alpha_2/2)\,i\eta} a_k \varphi_k\big|_{\alpha_1} \big|\sum \lambda_j^{(\beta-\beta_1)/2+(\beta_1-\beta_2/2)\,i\eta} b_j \varphi_j\big|_{\beta_1} \\ &= M_1 \big(\sum \lambda_k^\alpha |a_k|^2\big)^{1/2} \big(\sum \lambda_j^\beta |b_j|^2\big)^{1/2} \\ &= M_1 |g|_\alpha |g^*|_\beta, \end{aligned}$$

so that

$$G(0) \leqq M_1 |g|_\alpha |g^*|_\beta.$$

We obtain in a similar way from the $j=2$ case of (2.21) that

$$G(1) \leqq M_2 |g|_\alpha |g^*|_\beta.$$

Applying these two estimates in (2.27), we then conclude that

$$\begin{aligned} |B(g, g^*)| = |w(\theta)| &\leqq G(\theta) \\ &\leqq M_1^{1-\theta} M_2^\theta |g|_\alpha |g^*|_\beta, \end{aligned}$$

which proves (*a*).

To prove (*b*) we first fix a time $\bar{t} \in (t_1, t_2)$, a trigonometric polynomial g as in (2.26), and a function $\chi \in L^2([-1, 1] \times [t_1, \bar{t}])$. We then define

$$w(z) = \int_{t_1}^{\bar{t}} \int_{-1}^{1} (t_2 - t)^{(1-z)/2} S\big(\sum \lambda_k^{(\alpha-z)/2} a_k \varphi_k\big) \chi \, dx \, dt.$$

A simple difference quotient argument shows that w is entire, and w is clearly bounded on the strip $0 \leqq \operatorname{Re}(z) \leqq 1$. The function

$$G(\xi) = \sup_\eta |w(\xi + i\eta)|$$

is therefore log convex:

$$G(\alpha) \leqq G(0)^{1-\alpha} G(1)^\alpha. \tag{2.28}$$

The first hypothesis in (*b*) shows that

$$\begin{aligned}|w(i\eta)| &\leqq \iint (t_2-t)^{1/2}\,|S(\textstyle\sum \lambda_k^{(\alpha-i\eta)/2} a_k \varphi_k)|\,|\chi|\,dx\,dt\\ &\leqq M_0\,|\textstyle\sum \lambda_k^{(\alpha-i\eta)/2} a_k\varphi_k|_0 \left(\iint \chi^2\right)^{1/2}\\ &= M_0\,|g|_\alpha \left(\iint \chi^2\right)^{1/2},\end{aligned}$$

so that

$$G(0) \leqq M_0\,|g|_\alpha \left(\iint \chi^2\right)^{1/2}.$$

The second hypothesis of (*b*) shows in a similar way that

$$G(1) \leqq M_1\,|g|_\alpha \left(\iint \chi^2\right)^{1/2}.$$

Applying (2.28), we thus obtain that

$$\begin{aligned}\left|\int_{t_1}^{\bar t}\int_{-1}^{1} (t_2-t)^{(1-\alpha)/2}\,(Sg)\,\chi\,dx\,dt\right| &= |w(\alpha)|\\ &\leqq G(\alpha) \leqq M_0^{1-\alpha} M_1^{\alpha}\,|g|_\alpha \left(\iint \chi^2\right)^{1/2},\end{aligned}$$

so that

$$\left(\int_{t_1}^{\bar t}\int_{-1}^{1} (t_2-t)^{1-\alpha}\,|Sg|^2\,dx\,dt\right)^{1/2} \leqq M_0^{1-\alpha} M_1^{\alpha}\,|g|_\alpha.$$

We then let $\bar t \uparrow t_2$ and extend to $g \in H_0^1$ to complete the proof. □

3. APPLICATION TO FINITE DIFFERENCE APPROXIMATIONS

We now apply Theorem 1.4 to obtain specific rates of convergence for certain finite difference approximations to solutions of (1.1)−(1.4). We consider only the special case in which $F_1 = F_2 = 0$. Our approximations are generated by the same difference scheme used in [9], but our application of Theorem 1.4 here yields convergence rates in H^α norm which are twice the order of the L^2 rates previously obtained.

We define

$$\Lambda(v)=\varepsilon \ln v,$$
$$\tilde{v}(t)=\frac{1}{2}\int_{-1}^{1} v(x, t)\,dx,$$
$$\Gamma(v, \tilde{v})=\int_{\tilde{v}}^{v} [p(\tilde{v})-p(s)]\,ds,$$

and we assume that there are constants $C_2, \ldots, C_6$, and M_1, M_2 such that

$$\int_{-1}^{1}\left[\frac{1}{2}u_0(x)^2+\psi(v_0(x), \tilde{v}(0))\right]dx \leqq C_2, \tag{3.1}$$

$$\sup_{0\leqq t\leqq T}|u_l(t)|+\sup_{0\leqq t\leqq T}|u_r(t)|\leqq C_3, \tag{3.2}$$

$$\sup_{0\leqq t\leqq T}|\dot{u}_l(t)|+\sup_{0\leqq t\leqq T}|\dot{u}_r(t)|\leqq C_4, \tag{3.3}$$

$$\int_{0}^{\infty}(|u_l(t)|+|u_r(t)|)\,dt\leqq C_5, \tag{3.4}$$

$$\int_{0}^{\infty}(|\dot{u}_l(t)|+|\dot{u}_r(t)|)\,dt\leqq C_6, \tag{3.5}$$

$$M_1\leqq\tilde{v}(0)+\frac{1}{2}\int_{0}^{t}[u_r(s)-u_l(s)]\,ds\leqq M_2, \quad 0\leqq t\leqq T. \tag{3.6}$$

By transforming to Eulerian coordinates, it may be seen that this last condition simply prevents the pistons from either colliding or becoming arbitrarily far apart in finite time.

In addition, we assume that one of the following cases holds:

Case I. – $v_0\in H^1(y_i, y_{i+1})$ and $u_0\in C^0(y_i, y_{i+1})$ for $i=0, 1, \ldots, J$, where $-1=y_0<\ldots<y_{J+1}=1$, and there exists a constant C_7 such that

$$TV(u_0)+\|(v_0)_x\|+\sum_{i=1}^{J}|v_0(y_i+0)+v_0(y_i-0)|\leqq C_7,$$

where $\|.\|$ denotes the piecewise L^2 norm

$$\|w\|^2=\sum_{i=0}^{J}\int_{y_i}^{y_{i+1}} w(x)^2\,dx.$$

We also assume in this case that $C_2, \ldots, C_7$ are sufficiently small, depending on $\underline{v}$ and $\bar{v}$.

Case II. – This is the same as Case I except that no smallness assumption is imposed, but

$$u_l(t)\leqq 0\leqq u_r(t), \qquad 0\leqq t\leqq T,$$
$$p(v)=kv^{-\gamma}, \qquad k>0, \qquad \frac{3}{2}<\gamma<3.$$

Case III. – v_0, $u_0 \in H^1(-1, 1)$, and there exists a constant C_7 such that

$$\|(u_0)_x\| + \|(v_0)_x\| \leqq C_7.$$

We also assume in this case that

$$u_l(t) \leqq 0 \leqq u_r(t), \qquad 0 \leqq t \leqq T,$$
$$p(v) = kv^{-\gamma}, \qquad k > 0, \qquad \gamma > 1.$$

Approximate solutions (v^h, u^h) to (1.1)-(1.4) are constructed by the procedure described in ([8]-[9]), which we now summarize.

Let Δx and Δt be fixed increments in x and t and set $x_k = k\,\Delta x$ for $k = 0, \pm 1, \ldots, \pm K$, where $K\,\Delta x = 1$; $x_j = j\,\Delta x$ for

$$j = \pm \frac{1}{2}, \ldots \pm \left(K - \frac{1}{2}\right);$$

and $t^n = n\,\Delta t$ for $n = 0, 1, \ldots$ We denote by v_j^n and u_k^n approximations to $v(x_j, t^n)$ and $u(x_k, t^n)$ and we form initial sequences $\{v_j^0\}$ and $\{u_k^0\}$ by pointwise evaluation of v_0 and by integral averages of u_0 over intervals of length Δx. We set $u_{-K}^0 = u_l(0)$ and $u_K^0 = u_r(0)$. We also let x_{k_i} be the value of x_k nearest to y_i for $i = 1, \ldots, J$ with $x_{k_0} = 0$ and $x_{k_{J+1}} = 1$. For $n = 1, 2, \ldots, \{v_j^n\}$ and $\{u_k^n\}$ are then computed from the scheme

$$\frac{v_j^{n+1} - v_j^n}{\Delta t} - \delta u_j^{n+1} = 0, \qquad j = -K + \frac{1}{2}, \ldots, K - \frac{1}{2} \tag{3.7}$$

$$\frac{u_k^{n+1} - u_k^n}{\Delta t} + \delta p_k^{n+1} = \delta(\lambda^{n+1/2}\,\delta u^{n+1})_k, \qquad k = -K + 1, \ldots, K - 1 \tag{3.8}$$

$$u_{-K}^n = \frac{1}{\Delta t}\int_{t^{n-1}}^{t^n} u_l(t)\,dt, \qquad u_K^n = \frac{1}{\Delta t}\int_{t^{n-1}}^{t^n} u_r(t)\,dt. \tag{3.9}$$

Here δ is the difference operator $\delta w_l = (w_{l+1/2} - w_{l-1/2})/\Delta x$ for $l = j$ or k, and $\lambda_j^{n+1/2}$ is the divided difference

$$\lambda_j^{n+1/2} = \Lambda[v_j^n, v_j^{n+1}] = \begin{cases} \dfrac{\Lambda_j^{n+1} - \Lambda_j^n}{v_j^{n+1} - v_j^n}, & v_j^n \neq v_j^{n+1} \\[2ex] \Lambda'(v_j^n) = \dfrac{\varepsilon}{v_j^n}, & v_j^n = v_j^{n+1} \end{cases}$$

where $\Lambda_j^n = \Lambda(v_j^n)$.

It was established in [9] that, under appropriate constraints on Δt and Δx, this scheme can be solved up to any fixed time $T > 0$. In particular, we assume the existence of a constant C such that

$$\frac{\varepsilon\,\Delta t}{\Delta x^2} \leqq C, \quad \text{as } \Delta x \to 0. \tag{3.10}$$

Thus, given sequences $\{v_j^n\}$ and $\{u_k^n\}$ satisfying (3.7)-(3.9), we construct approximate solutions to (1.1) by interpolating the sequence $\{u_k^n\}$ to a function $u^h(x, t)$ which is bilinear on rectangles of the form

$$S_j^n = [x_{j-1/2}, x_{j+1/2}] \times [t^n, t^{n+1}], \qquad j = -K + \frac{1}{2}, \ldots, K - \frac{1}{2},$$

and by interpolating $\{v_j^n\}$ to a function which is bilinear on rectangles of the form

$$S_k^n = [x_{k-1/2}, x_{k+1/2}] \times [t^n, t^{n+1}], \qquad k = -K+1, \ldots, K-1; \qquad k \neq k_i,$$

with appropriate extensions near the lines of discontinuity $x = x_{k_i}$ in Cases I and II. Approximate functions $p^h(x, t)$ and $\Lambda^h(x, t)$ are constructed in a similar way from the sequences $\{p_j^n\} = \{p(v_j^n)\}$ and $\{\Lambda_j^n\} = \{\Lambda(v_j^n)\}$. The construction is such that v^h, p^h, and Λ^h are continuous on $(x_{k_i}, x_{k_{i+1}}) \times [0, t^N]$, $i = 0, \ldots, J$, but are discontinuous along lines $x = x_{k_i}$, $i = 1, \ldots, J$.

The following regularity properties were established in ([8]-[9]), for $r > \frac{1}{4}$:

$$0 < \underline{v} \leqq v^h(x, t) \leqq \bar{v}, \qquad \text{for some } [\underline{v}, \bar{v}] \in (0, \infty) \tag{3.11}$$

$$\sup_{0 \leqq t \leqq T} |u_t^h(-1, t)| + \sup_{0 \leqq t \leqq T} |u_t^h(1, t)| \leqq C \tag{3.12}$$

$$\sup_{0 \leqq t \leqq T} \{ \| u^h(., t) \| + t^r (\| v_t^h(., t) \| + \| u_x^h(., t) \|) \} + \int_0^T \int_{-1}^1 u_x^{h2} \, dx \, dt \leqq C \tag{3.13}$$

$$\sup_{0 \leqq t \leqq T} \| v_x^h(., t) \| + \int_0^T \int_{-1}^1 v_x^{h2} \, dx \, dt \leqq C \tag{3.14}$$

$$\| u_x^h(., t) \|, \qquad \| v_t^h(., t) \| \leqq C t^{-1/4}, \qquad 0 < t \leqq T \tag{3.15}$$

$$\int_0^T \int_{-1}^1 t^{2r} u_t^{h2} \, dx \, dt, \qquad \int_0^T \int_{-1}^1 t^{2r} u_{xx}^{h2} \, dx \, dt \leqq C \tag{3.16}$$

$$\int_0^T \int_{-1}^1 t^{2r} \Lambda_{xt}^{h2} \, dx \, dt, \qquad \int_0^T \int_{-1}^1 t^{2r} p_{xt}^{h2} \, dx \, dt \leqq C \tag{3.17}$$

In addition, we'll need the following properties of the sequence $\{u_k^n\}$, which were also derived in [8]-[9]:

$$\sum_{n=0}^{N-1} \sum_{k=-K+1}^{K-1} (t^{n+1})^{2r} \left(\frac{u_k^{n+1} - u_k^n}{\Delta t} \right)^2 \Delta x \, \Delta t \leqq C, \tag{3.18}$$

$$\sum_{n=0}^{N-1} \| \delta u^{n+1} \|_\infty \Delta t \leqq C. \tag{3.19}$$

The following theorem gives the actual error bound which is obtained by applying Theorem 1.4 to the finite difference approximations constructed as above.

THEOREM 3.1. — *Assume that* $v_0(x)$, $u_0(x)$, $u_l(t)$, *and* $u_r(t)$ *satisfy* (3.1)-(3.6), *and assume that the hypotheses of one of Cases I, II, or III are in force. Assume also that* Δt *and* Δx *are chosen so that the scheme* (3.7)-(3.9) *can be solved up to time* $t^N \leqq T$ *and let* (v^h, u^h) *be the functions constructed as above, satisfying* (3.11)-(3.19) *with* $\frac{1}{4} < r < \frac{1}{2}$. *Let* $E_{-\alpha}(t)$ *be as in* (1.24). *Then for any* $\alpha \in (0, 1]$,

(*a*) $\sup_{0 \leqq t \leqq T} E_{-\alpha}(t) \leqq C(\alpha)[E_{-\alpha}(0) + O(\Delta x^{1/2})]$,

in Cases I and II;

(*b*) $\sup_{0 \leqq t \leqq T} E_{-\alpha}(t) \leqq C(\alpha)[E_{-\alpha}(0) + O(\Delta x)]$,

in Case III.

where $C(\alpha)$ *may become unbounded as* $\alpha \downarrow 0$.

Proof. — By Theorem 1.4 (*a*), we need only prove the following:

LEMMA 3.2. — *Let* $\mathscr{L}_1$ *and* $\mathscr{L}_2$ *be as in* (1.20) *and* (1.21) *and let* $0 \leqq t^m < t^N \leqq T$. *Then under the hypotheses of Theorem* 3.1,

$$|\mathscr{L}_1(0, t^N, \phi)| \leqq C(\alpha)\Delta x \,|||\phi, \psi|||_{\alpha, [0, t^N]}, \text{ in all Cases;} \tag{3.20}$$

$$|\mathscr{L}_2(0, t^N, \psi)| \leqq C(\alpha)\Delta x^{1/2} \,|||\phi, \psi|||_{\alpha, [0, t^N]}, \text{ in Cases I and II,} \tag{3.21}$$

and

$$|\mathscr{L}_2(0, t^N, \psi)| \leqq C(\alpha)\Delta x \,|||\phi, \psi|||_{\alpha, [0, t^N]}, \text{ in Case III.}$$

Proof of Lemma 3.2. — We prove only (3.21) for Cases I and II, since the proofs of the other results are much simpler. From (1.21)

$$\mathscr{L}_2(0, t^N, \psi) = \int_0^{t^N}\int_{-1}^{1} \frac{\varepsilon u_x^h}{v^h}\psi_x\,dx\,dt - \int_0^{t^N}\int_{-1}^{1} p(v^h)\psi_x\,dx\,dt + \int_{-1}^{1} u^h\psi\Big|_0^{t^N}dx - \int_0^{t^N}\int_{-1}^{1} u^h\psi_t\,dx\,dt. \tag{3.22}$$

Now consider the expression

$$\int_0^{t^N}\int_{-1}^{1} (\Lambda_{xt}^h - p_x^h - u_t^h)\psi\,dx\,dt = -\sum_{i=1}^{J}\int_0^{t^N} [\Lambda_t^h - p^h]_{k_i}\psi(x_{k_i}, t)\,dt - \int_0^{t^N}\int_{-1}^{1} (\Lambda_t^h - p^h)\psi_x\,dx\,dt - \int_{-1}^{1} u^h\psi\Big|_0^{t^N}dx + \int_0^{t^N}\int_{-1}^{1} u^h\psi_t\,dx\,dt. \tag{3.23}$$

Here, $[w]_k$ denotes $w(x_k+0) - w(x_k-0)$.

Adding (3.23) to (3.22), we obtain

$$\mathscr{L}_2(0, t^N, \psi) = -\int_0^{t^N}\int_{-1}^{1} (\Lambda_{xt}^h - p_x^h - u_t^h)\,\psi\,dx\,dt + \int_0^{t^N}\int_{-1}^{1}\left(\frac{\varepsilon u_x^h}{v^h} - \Lambda_t^h\right)\psi_x\,dx\,dt - \int_0^{t^N}\int_{-1}^{1}(p(v^h) - p^h)\,\psi_x\,dx\,dt - \sum_{i=1}^{J}\int_0^{t^N}[\Lambda_t^h - p^h]_{k_i}\,\psi(x_{k_i}, t)\,dt. \quad (3.24)$$

From the definitions of u^h, p^h, and Λ^h, we find that on rectangles S_k^n, $k \neq k_i$,

$$\Lambda_{xt}^h - p_x^h - u_t^h = -u_{xt}^h(x - x_k) - p_{xt}^h(t - t^{n+1})$$

with similar expressions holding on rectangles $S_{k_i}^n$ and along the boundaries. Using this in the first double integral of (3.24) and integrating by parts, we can bound this term by

$$\begin{aligned}
&\Delta x \int_0^{t^N}\int_{-1}^{1} |u_x^h \psi_t|\,dx\,dt + \Delta t \int_0^{t^N}\int_{-1}^{1} |p_{xt}^h \psi|\,dx\,dt \\
&\leqq \Delta x \left(\int_0^{t^N}\int_{-1}^{1} (t^N - t)^{\alpha - 1} u_x^{h2}\,dx\,dt\right)^{1/2} \left(\int_0^{t^N}\int_{-1}^{1} (t^N - t)^{1-\alpha} \psi_t^2\,dx\,dt\right)^{1/2} \\
&\quad + \Delta t \left(\int_0^{t^N}\int_{-1}^{1} t^{2r} p_{xt}^{h2}\,dx\,dt\right)^{1/2} \left(\int_0^{t^N}\int_{-1}^{1} t^{-2r}\psi^2\,dx\,dt\right)^{1/2} \\
&\leqq C\,\Delta x \left(\int_0^{t^N} (t^N - t)^{\alpha - 1} t^{-1/2}\,dt\right)^{1/2} |||\phi, \psi|||_{\alpha, [0, t^N]} \\
&\quad + C\,\Delta t \sup_{0 \leqq t \leqq t^N} \|\psi(., t)\| \left(\int_0^{t^N} t^{-2r}\,dt\right)^{1/2} \quad \text{by (3.13), (3.17)} \\
&\leqq C(\Delta x + \Delta t)\, |||\phi, \psi|||_{\alpha, [0, t^N]}, \quad (3.25)
\end{aligned}$$

where we have also used the assumption that

$$\frac{1}{4} < r < \frac{1}{2}. \quad (3.26)$$

For the second term in (3.24) we again use the form of Λ^h and triangulate to obtain

$$\begin{aligned}
\left|\int_0^{t^N}\int_{-1}^{1}\left(\frac{\varepsilon u_x^h}{v^h} - \Lambda_t^h\right)\psi_x\,dx\,dt\right| &\leqq \left|\sum_{n=0}^{N-1}\sum_j \int_{S_j^n}\int\left(\lambda_j^{n+1/2} - \frac{\varepsilon}{v^h}\right) u_x^h \psi_x\,dx\,dt\right| \\
&\quad + \left|\sum_{n=0}^{N-1}\sum_j \int_{S_j^n}\int \Lambda_{xt}^h (x - x_j)\,\psi_x\,dx\,dt\right| \\
&\quad + \left|\sum_{n=0}^{N-1}\sum_j \int_{S_j^n}\int \lambda_j^{n+1/2}(\delta u_j^{n+1} - u_x^h)\,\psi_x\,dx\,dt\right|. \quad (3.27)
\end{aligned}$$

By using the definition of $\lambda_j^{n+1/2}$ and the form of v^h, the first term on the right-hand side of (3.27) can be bounded by

$$C\,\Delta x\int_0^{t^N}\|u_x^h(\,.\,,t)\|_\infty\left(\int_{-1}^1 (v_t^{h2}+v_x^{h2})\,dx\right)^{1/2}\left(\int_{-1}^1 \psi_x^2\,dx\right)^{1/2} dt$$
$$\leqq C\,\Delta x\,|||\,\phi,\,\psi\,|||_{\alpha,\,[0,\,t^N]}\int_0^{N}(t^N-t)^{(\alpha-1)/2}\,t^{-1/4}\,\|u_x^h(\,.\,,t)\|_\infty\,dt.$$

(by (1.23), (3.13), (3.15))

By a Sobolev inequality, this is bounded by

$$C\,\Delta x\,|||\,\phi,\,\psi\,|||_{\alpha,\,[0,\,t^N]}\int_0^{t^N}\Bigg[(t^N-t)^{(\alpha-1)/2}\,t^{-1/4-r/2}\left(\int_{-1}^1 u_x^{h2}\,dx\right)^{1/4}\left(t^{2r}\int_{-1}^1 u_{xx}^{h2}\,dx\right)^{1/4}\Bigg]dt$$
$$\leqq C\,\Delta x\,|||\,\phi,\,\psi\,|||_{\alpha,\,[0,\,t^N]}\left(\int_0^{t^N}(t^N-t)^{\alpha-1}\,t^{-1/2-r}\,dt\right)^{1/2}\left(\int_0^{t^N}\int_{-1}^1 u_x^{h2}\,dx\,dt\right)^{1/4}$$
$$\times\left(\int_0^{t^N}\int_{-1}^1 t^{2r}\,u_{xx}^{h2}\,dx\,dt\right)^{1/4}$$
$$\leqq C\,\Delta x\,|||\,\phi,\,\psi\,|||_{\alpha,\,[0,\,t^N]} \tag{3.28}$$

where we have used (3.14), (3.16), and (3.26). The second term on the right-hand side of (3.27) is

$$\left|\sum_{n=0}^{N-1}\sum_j\int_{S_j^n}\int \Lambda_{xt}^h\,(x-x_j)\,\psi_x\,dx\,dt\right|$$
$$\leqq \Delta x\left(\int_0^{t^N}\int_{-1}^1 t^{2r}\,\Lambda_{xt}^{h2}\,dx\,dt\right)^{1/2}\left(\int_0^{t^N}t^{-2r}\int_{-1}^1 \psi_x^2\,dx\,dt\right)^{1/2}$$
$$\leqq C\,\Delta x\left(\int_0^{t^N}t^{-2r}\int_{-1}^1 \psi_x^2\,dx\,dt\right)^{1/2}\quad\text{(by (3.17))}.$$

But by (1.23) and (3.26)

$$\left(\int_0^{t^N}t^{-2r}\int_{-1}^1 \psi_x^2\,dx\,dt\right)^{1/2}\leqq C\,|||\,\phi,\,\psi\,|||_{\alpha,\,[0,\,t^N]}\left(\int_0^{t^N}t^{-2r}\,(t^N-t)^{\alpha-1}\,dt\right)^{1/2}$$
$$\leqq C\,|||\,\phi,\,\psi\,|||_{\alpha,\,[0,\,t^N]}. \tag{3.29}$$

Thus, the second term in (3.27) is bounded by

$$C\,\Delta x\,|||\,\phi,\,\psi\,|||_{\alpha,\,[0,\,t^N]}. \tag{3.30}$$

Next, we use (3.11) and the form of u^h to bound the third term in (3.27) by

$$C\Delta t\int_0^{t^N}\int_{-1}^1 |u_{xt}^h \psi_x|\,dx\,dt$$

$$\leq C\Delta t\left(\int_0^{t^N}\int_{-1}^1 t^{2r} u_{xt}^{h2}\,dx\,dt\right)^{1/2}\left(\int_0^{t^N}\int_{-1}^1 t^{-2r}\psi_x^2\,dx\,dt\right)^{1/2}$$

$$\leq C\Delta t\,|||\phi,\psi|||_{\alpha,[0,t^N]}\left(\int_0^{t^N}\int_{-1}^1 t^{2r} u_{xt}^{h2}\,dx\,dt\right)^{1/2}. \quad \text{(by (3.29))}$$

But it follows from the construction of u^h that

$$\int_0^{t^N}\int_{-1}^1 t^{2r} u_{xt}^{h2}\,dx\,dt \leq \frac{C}{\Delta x^2}\int_0^{t^N}\int_{-1}^1 t^{2r} u_t^{h2}\,dx\,dt,$$

$$\leq \frac{C}{\Delta x^2} \quad \text{(by (3.16))}.$$

The third term in (3.27) is therefore bounded by

$$C\Delta x\,|||\phi,\psi|||_{\alpha,[0,t^N]}, \tag{3.31}$$

where we have also used (3.10). From (3.28), (3.30), and (3.31), the second term in (3.24) is therefore also bounded by

$$C\Delta x\,|||\phi,\psi|||_{\alpha,[0,t^N]}. \tag{3.32}$$

Next, the third term in (3.24) is bounded by

$$\left(\int_0^{t^N}\int_{-1}^1 \psi_x^2\,dx\,dt\right)^{1/2}\left(\int_0^{t^N}\int_{-1}^1 (p(v^h)-p^h)^2\,dx\,dt\right)^{1/2},$$

which by (1.23) and the bilinear form of v^h and p^h, is bounded by

$$C\,|||\phi,\psi|||_{\alpha,[0,t^N]}\left(\Delta x\left(\int_0^{t^N}\int_{-1}^1 v_x^{h2}\,dx\,dt\right)^{1/2}+\Delta t\left(\int_0^{t^N}\int_{-1}^1 v_t^{h2}\,dx\,dt\right)^{1/2}\right)$$

$$\leq C\,|||\phi,\psi|||_{\alpha,[0,t^N]}(\Delta x+\Delta t). \tag{3.33}$$

Finally, we consider the fourth term in (3.24) which, after a lengthy but straightforward argument based on (3.7), (3.8), and the definitions of Λ^h and p^h, can be bounded by

$$C\Delta t\sum_{n=0}^{N-1}\left(\sum_{i=1}^{J}\left|\frac{u_{ki}^{n+1}-u_{ki}^n}{\Delta t}\right|^2\Delta x\right)^{1/2}\int_{t^n}^{t^{n+1}}\|\psi(.,t)\|_\infty\,dt$$

$$+C\Delta t\sum_{n=0}^{N-1}\|\delta u^{n+1}\|_\infty\int_{t^n}^{t^{n+1}}\|\psi(.,t)\|_\infty\,dt$$

$$+C\Delta t^{1/2}\left(\sum_{n=0}^{N-1}(t^{n+1})^{-\mu}\left(\int_{t^n}^{t^{n+1}}\|\psi(.,t)\|_\infty\,dt\right)^2\right)^{1/2}+B, \tag{3.34}$$

where

$$B=\Delta x^3 \sum_{n=0}^{N-1} \sum_{i=1}^{J} \left| \delta^2 \left(\frac{u_{k_i}^{n+1}-u_{k_i}^n}{\Delta t} \right) \right| \int_{t^n}^{t^{n+1}} \| \psi(.,t) |_\infty \, dt. \quad (3.35)$$

But by a simple Sobolev inequality,

$$\begin{aligned} \int_{t^n}^{t^{n+1}} \| \psi(.,t) \|_\infty \, dt & \leqq C \int_{t^n}^{t^{n+1}} \left(\int_{-1}^{1} \psi^2 \, dx \right)^{1/4} \left((t^N - t) \int_{-1}^{1} \psi_x^2 \, dx \right)^{1/4} (t^N - t)^{-1/4} \, dt \\ & \leqq C \||\phi, \psi\||_{\alpha,[0,t^N]} \int_{t^n}^{t^{n+1}} (t^N - t)^{-1/4} \, dt \quad \text{(by (1.23))} \\ & \leqq C \||\phi, \psi\||_{\alpha,[0,t^N]} [(t^N - t^n)^{3/4} - (t^N - t^{n+1})^{3/4}] \\ & \leqq C \||\phi, \psi\||_{\alpha,[0,t^N]} (t^N - t^n)^{-1/4} \Delta t \quad (3.36) \\ & \leqq C \||\phi, \psi\||_{\alpha,[0,t^N]} \Delta t^{3/4}. \quad (3.37) \end{aligned}$$

So from (3.34) and the last inequality in (3.37), the fourth term in (3.24) is bounded by

$$\begin{aligned} & C \||\phi, \psi\||_{\alpha,[0,t^N]} \Delta t^{3/4} \left(\sum_{n=0}^{N-1} \sum_{k=-K+1}^{K-1} (t^{n+1})^{2r} \left(\frac{u_k^{n+1}-u_k^n}{\Delta t} \right)^2 \Delta x \, \Delta t \right)^{1/2} \\ & \qquad \times \left(\sum_{n=0}^{N-1} (t^{n+1})^{-2r} \Delta t \right)^{1/2} \\ & \quad + C \||\phi, \psi\||_{\alpha,[0,t^N]} \Delta t^{3/4} \sum_{n=0}^{N-1} \| \delta u^{n+1} \|_\infty \Delta t \\ & \quad + C \||\phi, \psi\||_{\alpha,[0,t^N]} \Delta t^{3/4} + B \\ & \leqq C \||\phi, \psi\||_{\alpha,[0,t^N]} \Delta x^{3/2} + B \quad \text{(by (3.10), (3.18), (3.19))}. \quad (3.38) \end{aligned}$$

We now use (3.36) to bound B as follows:

$$\begin{aligned} B & \leqq C \||\phi, \psi\||_{\alpha,[0,t^N]} \Delta x^3 \sum_{n=0}^{N-1} \sum_{i=1}^{J} \left| \delta^2 \left(\frac{u_{k_i}^{n+1}-u_{k_i}^n}{\Delta t} \right) \right| (t^N - t^n)^{-1/4} \Delta t \\ & \leqq C \||\phi, \psi\||_{\alpha,[0,t^N]} \Delta x^{5/2} \left(\sum_{n=0}^{N-1} (t^N - t^n)^{-1/2} (t^{n+1})^{-2r} \Delta t \right)^{1/2} \\ & \qquad \times \left(\sum_{n=0}^{N-1} \sum_{k=-K+1}^{K-1} (t^{n+1})^{2r} \left(\delta^2 \left(\frac{u_k^{n+1}-u_k^n}{\Delta t} \right) \right)^2 \Delta x \, \Delta t \right)^{1/2} \\ & \leqq C \||\phi, \psi\||_{\alpha,[0,t^N]} \Delta x^{1/2} \left(\sum_{n=0}^{N-1} \sum_{k=-K+1}^{K-1} (t^{n+1})^{2r} \left(\frac{u_k^{n+1}-u_k^n}{\Delta t} \right)^2 \Delta x \, \Delta t \right)^{1/2} \\ & \leqq C \||\phi, \psi\||_{\alpha,[0,t^N]} \Delta x^{1/2} \quad \text{(by (3.18))}. \end{aligned}$$

Using this in (3.38) and combining this with (3.25), (3.28), (3.32), and (3.33), we obtain

$$|\mathscr{L}_2(0, t^N, \psi)| \leqq C \||| \phi, \psi \|||_{\alpha, [0, t^N]} \Delta x^{1/2}, \text{ in Cases I and II.}$$

In Case III, the fourth term in (3.24) is 0, so that

$$|\mathscr{L}_2(0, t^N, \psi)| \leqq C \||| \phi, \psi \|||_{\alpha, [0, t^N]} \Delta x.$$

This establishes (3.21), and the proof of (3.20) is similar.

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