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Construction of the Leray-Schauder degree for elliptic operators in unbounded domains

by

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ABSTRACT. — This paper is devoted to a construction of the Leray-Schauder degree for quasilinear elliptic operators in unbounded domains. The main problem here is that such operators can not be reduced to compact ones and the usual theory (*see* [1]) can not be applied. In [2, 3] the Leray-Schauder degree was constructed in one dimensional case. To do this certain lower estimates of the operators were obtained and the approach of Skrypnik [4] was applied. In this paper we generalize these results to the multidimensional case. When the degree is defined the Leray-Schauder method can be used to prove the existence of solutions [3].

Key words : Elliptic operators, Leray-Schauder degree.

RÉSUMÉ. — Cet article est consacré à la construction du degré de Leray-Schauder pour les opérateurs elliptiques dans les domaines non bornés. Le problème essentiel ici est que ces opérateurs ne peuvent pas être réduits à des compacts. Par conséquent, on ne peut pas utiliser la théorie habituelle [1]. Dans [2, 3] le degré est construit pour les cas monodimensionnel. Pour cette construction, certaines estimations des opérateurs sont

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obtenues, et l'approche de Skrypnik [4] est utilisée. Dans cet article, nous généralisons ces résultats pour le cas multidimensionnel. Quand le degré est déterminé, la méthode de Leray-Schauder peut être appliquée pour prouver l'existence de solutions [3].

1. INTRODUCTION

We consider the quasilinear elliptic system of equations

$$a\,\Delta w+(r(x')+c)\frac{\partial w}{\partial x_1}+F(w, x)=0 \tag{1.1}$$

in the infinite cylinder $\Omega\subset R^m$, with the boundary condition

$$\left.\frac{\partial w}{\partial n}\right|_{\partial\Omega}=0 \tag{1.2}$$

and the conditions at the infinity

$$\lim_{x_1\to\pm\infty} w(x)=w_{\pm}, \qquad w_+\neq w_-. \tag{1.3}$$

Here $\Omega=D\times R^1$, D is a bounded domain in R^{m-1} with a smooth boundary, x_1 is a coordinate along the axis of the cylinder, $x'=(x_2, \ldots, x_m)$, $w=(w_1, \ldots, w_n)$, $F=(F_1, \ldots, F_n)$, a is a constant symmetric positive-definite matrix, $r(x')$ is a scalar function, w_+ and w_- are constant vectors, c is a constant which can be given or unknown. In the last case the value of the parameter c for which (1.1)-(1.3) has a solution is to be found. In particular, it is the case for traveling wave solutions where c is a wave velocity. There is a large number of works devoted to traveling waves in onedimensional case (*see* the bibliography in [5]). In the multidimensional case the problems of this type are studied in [6]-[8] (*see* also references there).

It is well known (*see*, for example, [9]) that if (1.1) is considered in a bounded domain then the corresponding vector field can be reduced to a compact one. Indeed, let A be the operator corresponding to the left hand side of (1.1), acting in the space C^α with the domain $C^{2+\alpha}$. Due to the compact imbedding of $C^{2+\alpha}$ into C^α it can be represented in the form

$$A=L+B=L(I+L^{-1}B). \tag{1.4}$$

where L has a compact inverse, and the equality $Lu=0$ implies that $u=0$, B is a bounded operator. Thus we can consider the completely continuous vector field $I+L^{-1}B$ and apply the usual definition of the Leray-Schauder degree. If we consider unbounded domains then there is no the compact imbedding of $C^{2+\alpha}$ into C^{α}, and this approach can not be used. Nevertheless the reduction to compact vector fields can be fulfilled even for unbounded domains. In [10] a weighted Sobolev space with strong weights, for example $\exp(x^2)$, is considered. Such weights lead to a compact imbedding of H^{k+1} into H^k, where H^m is a space of functions for which m derivatives are integrable with a square. As above the representation (1.4), where L^{-1} is compact, can be obtained. It is important to note that this construction is possible only in the case when the weight function growth is more fast than the exponential one. Hence, exponentially decreasing functions do not belong to these spaces, and it is a strong restriction on the application of this approach. In particular, it can not be used to the study of the problem formulated above.

We also consider weighted Sobolev spaces but with weak weights [2, 3]. In this case exponentially decreasing functions belong to them but there is no compact imbedding of the spaces which leads to (1.4) with a compact operator L^{-1}. Hence we can not use the Leray-Schauder theory known for the completely continuous vector fields, and the aim of this work is to construct the degree for the operators corresponding to the left hand side of (1.1). To do this we obtain certain lower estimates of the operators which give a possibility to apply the method of Skrypnik of the degree construction [4] (see also [13]). In this method the degree for the original operator is defined as the degree for some finite-dimensional operators, and it is possible due to the compactness of the set of zeros of the operator. For the operators under consideration this compactness follows from the estimates mentioned above.

The contents of the paper is the following. We introduce function spaces and operators and formulate the main results in Section 2. Lower estimates of linear operators are presented in Section 3. We study the nonlinear operators in Section 4.

2. FORMULATION OF RESULTS

We introduce the function space and the operator which correspond to the left hand side of (1.1).

We consider the weighted Sobolev space $W^1_{2,\mu}(\Omega)$ of the vector-valued functions, defined in the cylinder Ω, with the inner product

$$[u, v]_\mu = \int_\Omega \left(\sum_{k=1}^{m} \left(\frac{\partial u}{\partial x_k}, \frac{\partial v}{\partial x_k} \right) + (u, v) \right) \mu \, dx,$$

where $u, v : \Omega \to \mathbb{R}^p$, $(\partial u/\partial n)|_{\partial\Omega} = 0$. The norm in this space is denoted by $\|\cdot\|_\mu$.

The weight function μ depends on x_1 only, and it is supposed to satisfy the following conditions:

1. $\mu(x_1) \geqq 1$, $\mu(x_1) \to \infty$ as $|x_1| \to \infty$,

2. μ'/μ and μ''/μ are continuous functions which tend to zero as $|x_1| \to \infty$.

For example we can take $\mu(x_1) = 1 + x_1^2$ or $\mu(x_1) = \ln(1 + x_1^2)$.

We emphasize again that this weight is weak in the sense that exponentially decreasing at infinity functions belong to $W^1_{2,\mu}(\Omega)$. Contrary to the weight spaces with a strong weight there is no a compact imbedding of $W^1_{2,\mu}(\Omega)$ into a space $L_{2,\mu}(\Omega)$ of square-summable functions with the weight μ.

We define the operator $A(u)$ acting from $W^1_{2,\mu}(\Omega)$ into the conjugate space $(W^1_{2,\mu}(\Omega))^*$ by the following formula:

$$\langle A(u), v \rangle = \int_\Omega \sum_{k=1}^{m} \left(a \frac{\partial u}{\partial x_k}, \frac{\partial (v\mu)}{\partial x_k} \right) dx - \int_\Omega \left(a \frac{\partial^2 \psi}{\partial x_1^2} + (r+c) \frac{\partial}{\partial x_1}(u+\psi) + F(u+\psi, x), v \right) \mu \, dx,$$

where $u, v \in W^1_{2,\mu}(\Omega)$, the notation $\langle f, v \rangle$ means the action of the functional $f \in W^1_{2,\mu}(\Omega)$ on the element v, ψ is a twice continuously differentiable function of x_1 of the form

$$\psi(x_1) = w_- \, \omega(x_1) + w_+ (1 - \omega(w_1)).$$

Here $\omega(x_1)$ is a monotone sufficiently smooth function which is equal to zero for $x_1 \geqq 1$ and to unity for $x_1 \leqq -1$.

We suppose that $r(x')$ is a bounded function, and the function $F(w, x)$ satisfies the following conditions:

1. $F(\psi, x) \in W^1_{2,\mu}(\Omega)$.

For example, if the function F does not depend on x explicitly, and

$$F(w_+) = F(w_-) = 0,$$

then, obviously, this condition is satisfied.

2. The function $F(w, x)$ and the matrices $F'(w, x)$ and $F_i''(w, x)$, $i = 1, \ldots, n$ are uniformly bounded for all $w \in R^n$ and $x \in \Omega$.

3. There exist uniform limits

$$b_\pm = \lim_{x_1 \to \pm\infty} F'(w_\pm, x), \qquad x' \in D.$$

where $b_\pm$ are constant matrices.

We note that for twice continuously differentiable functions u and compactly supported functions v (2.1) can be rewritten in the form

$$\langle A(u), v \rangle = -\int_\Omega \left(a \Delta (u+\psi) + (r+c) \frac{\partial}{\partial x_1}(u+\psi) + F(u+\psi, x), v\right) \mu \, dx. \quad (2.2)$$

If we put $w=u+\psi$ then (1.1) has the form

$$a\Delta(u+\psi)+(r+c)\frac{\partial}{\partial x_1}(u+\psi)+F(u+\psi, x)=0, \qquad (2.3)$$

and the connection between (2.2) and (2.3) is clear.

Any solution $u\in W^1_{2,\mu}(\Omega)$ of the equation

$$A(u)=0 \qquad (2.4)$$

has continuous second derivatives and satisfies (2.3). Conversely, every solution u of (2.3) having continuous second derivatives, satisfying condition (1.2) and belonging to $W^1_{2,\mu}(\Omega)$ is a solution of (2.4).

We should make some remarks about the case when the function F does not depend on x explicitly. In this case some additional difficulties appear. First of all, along with a solution $w(x)$ of (1.1) there are also solutions $w(x+h)$, where h is an arbitrary number. It means that for each solution $u(x)$ of

$$a\Delta(u+\psi)+(r+c)\frac{\partial}{\partial x_1}(u+\psi)+F(u+\psi)=0, \qquad (2.5)$$

there is one-parameter family of solutions

$$u_h(x)=u(x_1+h, x_2, \ldots, x_m)+\psi(x_1+h)-\psi(x_1). \qquad (2.6)$$

This nonisolatedness of solutions complicates further investigations. Moreover, we know from the investigations in the one-dimensional case [3] that (2.5) can have solutions for isolated values of c. It means that a small changing of the system (2.5) can lead to the disappearance of the solution. There is no a contradiction with the homotopy invariance of the degree since the system linearized on the solution has zero eigenvalue, and the index of the stationary point can be equal to zero. But the construction of the degree does not have sense in this case.

Thus if the function F does not depend on x explicitly the constant c should not be considered as given. We have the following formulation of the problem: to find c for which (2.5) has a solution. So the constant c is unknown along with the function $u(x)$, and we have to consider the operator defined on the space $W^1_{2,\mu}(\Omega)\times R$. To avoid this complication and nonisolatedness of solutions we apply the method of a functionalization of a parameter. It means that instead of unknown constant c we consider a functional $c(u)$. It is supposed to be given and to satisfy the following conditions:

1. For each $u\in W^1_{2,\mu}(\Omega)$ the function $c(u_h)$, where u_h is defined by (2.6), is monotone in h,

2. $c(u_h)\to\mp\infty$ as $h\to\pm\infty$.

In this case (2.5) is equivalent to the equation

$$a\,\Delta(u+\psi)+(r+c(u))\frac{\partial}{\partial x_1}(u+\psi)+\mathrm{F}(u+\psi)=0 \tag{2.7}$$

in the following sense. Let the constant $c=c_0$ and the family $u_h(x)$, $-\infty<h<+\infty$ be a solution of (2.5). We choose $h=h_0$ to satisfy the equality $c(u_h)=c_0$. It follows from the conditions 1 and 2 that such h exists for any c_0, and it is unique. Obviously, the function $u(x)=u_{h_0}(x)$ satisfies (2.7). Conversely, let $u(x)$ be a solution of (2.7). Then the constant $c=c(u)$ and the family $u_h(x)$, defined by (2.6), satisfy (2.5).

We now construct a functional $c(u)$ which satisfies the conditions above. Let $\sigma(x_1)$ be a monotone increasing function such that $\sigma(x_1)\to 0$ as $x_1\to-\infty$, $\sigma(x_1)\to 1$ as $x_1\to+\infty$,

$$\int_{-\infty}^{0}\sigma(x_1)\,dx_1<\infty.$$

We put

$$\rho(u)=\left(\int_{\Omega}|u+\psi-w_+|^2\,\sigma(x_1)\,dx\right)^{1/2} \tag{2.8}$$

and

$$c(u)=\ln\rho(u). \tag{2.9}$$

ASSERTION 2.1. – *The functional* $c(u)$ *defined on* $\mathrm{W}^1_{2,\mu}(\Omega)$ *by* (2.8), (2.9) *satisfies Lipschitz condition on every bounded set of* $\mathrm{W}^1_{2,\mu}(\Omega)$, *and has the following properties:* $c(u_h)$ *is a monotone decreasing function of* h, $c(u_h)\to\mp\infty$ *as* $h\to\pm\infty$. *Here* u_h *is defined by* (2.6), $u\in\mathrm{W}^1_{2,\mu}(\Omega)$.

The properties of the operator A which are formulated below are the same for the cases when c is a constant and a functional, and when F depends on x explicitly and does not depend. So, if the contrary is not pointed out, we consider all these cases.

ASSERTION 2.2. – *The operator* $\mathrm{A}(u)$ *satisfies Lipschitz condition on every bounded set of* $\mathrm{W}^1_{2,\mu}(\Omega)$.

CONDITION 2.1. – *All eigenvalues of the matrices* $b_\pm-a\,\xi^2$ *are in the left half-plane for all real* ξ.

We note that this condition is connected with the location of the continuous spectrum of the linearized operator [11].

THEOREM 2.1. – *Let Condition* 2.1 *be satisfied. Then there exists a linear bounded symmetric positive definite operator* S, *acting in the space* $\mathrm{W}^1_{2,\mu}(\Omega)$, *such that the inequality*

$$\langle \mathrm{A}(u)-\mathrm{A}(u_0),\ \mathrm{S}(u-u_0)\rangle \geqq \|u-u_0\|_\mu^2+\phi(u,u_0) \tag{2.10}$$

takes place for any $u,\ u_0\in\mathrm{W}^1_{2,\mu}(\Omega)$. *Here* $\phi(u_n,u_0)\to 0$ *as* $u_n\to u_0$ *weakly.*

From Theorem 2.1 it follows that a condition similar to Condition α of Skrypnik [4] is valid (compare with class $(S)_+$ mappings in [13]):

if u_n *is a sequence in* $W^1_{2,\mu}(\Omega)$ *which converges weakly to an element* $u_0 \in W^1_{2,\mu}(\Omega)$ *and if*

$$\lim_{n \to \infty} \langle A(u_n), S(u_n - u_0) \rangle \leqq 0 \tag{2.11}$$

then u_n *converges to* u_0 *strongly in* $W^1_{2,\mu}(\Omega)$.

From this we conclude that the Leray-Schauder degree (the rotation of a vector field) $\gamma(A, \mathcal{M})$ can be constructed similar to [4] for any bounded set $\mathcal{M}$ in $W^1_{2,\mu}(\Omega)$. The degree does not depend on the arbitrariness in the choice of the operator S satisfying the conditions of the Theorem 2.1.

The Leray-Schauder degree is proved to possess two usual properties: the principle of nonzero rotation and the homotopy invariance. The principle of nonzero rotation means that if $\gamma(A, \mathcal{M}) \neq 0$ then (2.4) has a solution in $\mathcal{M}$.

We define now the homotopy of the operators under consideration. We consider families of matrices a_τ, functions $F_\tau(u, x)$, $r_\tau(x')$ and constants c_τ, in case if they are considered as given, depending on the parameter $\tau \in [0, 1]$. The following conditions are supposed to be satisfied:

1. The matrices a_τ are symmetric positive definite and continuous in τ,
2. The functions $F_\tau(\psi_\tau, x)$ are continuous in τ in the $W^1_{2,\mu}(\Omega)$ norm,
3. The matrices $F'_\tau(w, x)$ and $F''_{u_i, \tau}(w, x)$, $i = 1, \ldots, n$ are uniformly bounded for all $w \in R^p$, $x \in \Omega$, $\tau \in [0, 1]$. The matrix $F'_\tau(w, x)$ satisfies Lipschitz condition in $w \in R^p$, $\tau \in [0, 1]$ uniformly in $x \in \Omega$.
4. For each $\tau \in [0, 1]$ there exist uniform limits

$$b_\pm(\tau) = \lim_{x_1 \to \pm\infty} F'(w_\pm(\tau), x), \qquad x' \in D.$$

All eigenvalues of the matrices

$$b_\pm(\tau) - a_\tau \xi^2$$

lie in the left half plane for all real ξ.

5. The functions $r_\tau(x')$ are bounded and continuous in τ uniformly in $x' \in D$. If c_τ is a given constant then it is continuous in τ.

Here $w_+(\tau)$ and $w_-(\tau)$ are given vector-valued functions which are supposed to be continuous,

$$\psi_\tau(x_1) = w_-(\tau)\,\omega(x_1) + w_+(\tau)(1 - \omega(x_1)).$$

In the case when F_τ does not depend on x explicitly, and c_τ is a functional, it is defined by the formulas

$$\rho_\tau(u) = \left(\int_\Omega |u + \psi_\tau - w_+(\tau)|^2 \sigma(x_1)\, dx \right)^{1/2} \tag{2.12}$$

$$c_\tau(u) = \ln \rho_\tau(u). \tag{2.13}$$

The operator $A_\tau(u): W^1_{2,\mu}(\Omega) \to (W^1_{2,\mu}(\Omega))^*$ is defined by the equality

$$\langle A_\tau(u), v \rangle = \int_\Omega \sum_{k=1}^m \left(a_\tau \frac{\partial u}{\partial x_k}, \frac{\partial (v\mu)}{\partial x_k} \right) dx - \int_\Omega (a_\tau \psi''_\tau + (r_\tau + c_\tau)\left(\frac{\partial u}{\partial x_1} + \psi'_\tau \right) + F_\tau(u + \psi_\tau, x), v)\, \mu\, dx,$$

THEOREM 2.2. — *Let $\mathcal{M}$ be a bounded domain in the space $W^1_{2,\mu}(\Omega)$ with the boundary Γ, and $A_\tau(u) \neq 0$ for $u \in \Gamma$, $\tau \in [0, 1]$. If the conditions* 1-5 *are satisfied then*

$$\gamma(A_0, \mathcal{M}) = \gamma(A_1, \mathcal{M}).$$

Let $u_0 \in W^1_{2,\mu}(\Omega)$ be an isolated stationary point of the operator $A(u)$:

$$A(u_0) = 0,$$

and $A(u) \neq 0$ for $u \neq u_0$ in some neighbourhood of the point u_0. Then the index of the stationary point u_0 is defined in the usual way: it is the rotation of the vector field $A(u)$ on a sphere with the center u_0 of sufficiently small radius.

We linearize the operator $A(u)$ at the stationary point u_0. The linearized operator $A'(u_0): W^1_{2,\mu}(\Omega) \to (W^1_{2,\mu}(\Omega))^*$ is defined by the equality

$$\langle A'(u_0)u, v \rangle = \int_\Omega \sum_{k=1}^m \left(a \frac{\partial u}{\partial x_k}, \frac{\partial (v\mu)}{\partial x_k} \right) dx - \int_\Omega \left(c'(u) \frac{\partial (u_0 + \psi)}{\partial x_1} + (r(x') + c(u_0)) \frac{\partial u}{\partial x_1} + F'(u_0 + \psi)u, v \right) \mu\, dx,$$

where

$$c'(u) = \frac{\int_\Omega (u_0(x) + \psi - w_+, u(x))\, \sigma(x_1)\, dx}{\int_\Omega |u_0 + \psi - w_+|^2 \sigma(x_1)\, dx}$$

If c is a given constant then the term with $c'(u)$ in (2.15) should be omitted.

THEOREM 2.3. — *Let Condition* 2.1 *be satisfied. Then there exists a linear symmetric positive definite bounded operator* S *acting in the space $W^1_{2,\mu}(\Omega)$ such that for any $u \in W^1_{2,\mu}(\Omega)$*

$$\langle A'(u_0)u, Su \rangle \geqq \| u \|^2_\mu + \theta(u),$$

where $\theta(u)$ is a functional defined on $W^1_{2,\mu}(\Omega)$ and satisfying the condition: $\theta(u_n) \to 0$ as $u_n \to$ weakly in $W^1_{2,\mu}(\Omega)$.

From this theorem it follows, in particular, that the operators are Fredholm (*see* Theorem 3.4).

We introduce an operator $J: W^1_{2,\mu}(\Omega) \to (W^1_{2,\mu}(\Omega))^*$ by the equality

$$\langle J u, v \rangle = \int_\Omega (u, v)\, \mu\, dx$$

THEOREM 2.4. — *Let Condition* 2.1 *be satisfied. Then for all* $\lambda \geqq 0$ *the operator* $A'(u_0) + \lambda J$ *is Fredholm. For all* $\lambda \geqq 0$, *except perhaps a finite number, it has a bounded inverse defined on the whole* $(W^1_{2,\mu}(\Omega))^*$.

We use this theorem to investigate the isolatedness of a stationary point and to calculate its index.

THEOREM 2.5. — *Let* u_0 *be a stationary point of the operator* $A(u)$, *and suppose that the equation*

$$A'(u_0)\, u = 0$$

has no solutions except zero. Then the stationary point u_0 *is isolated, and the absolute value of its index is equal to* 1.

We show that the sign of the index is connected with the multiplicity of eigenvalues similarly to completely continuous vector fields (*see*, for example, [1]). To do this we map $(W^1_{2,\mu}(\Omega))$ into $(W^1_{2,\mu}(\Omega))^*$ by means of the operator J, and denote $(W^1_{2,\mu}(\Omega))^*_0 = J(W^1_{2,\mu}(\Omega))$. We consider the operator $A_* = A'(u_0) J^{-1}$ acting in $(W^1_{2,\mu}(\Omega))^*$ with the domain $(W^1_{2,\mu}(\Omega))^*_0$. It follows from Theorem 4.1 that the operator A_* has no more than a finite number of negative eigenvalues, and all other negative numbers are its regular points. We note that the real eigenvalues λ of the operator A_* satisfy the equality

$$\langle A'(u_0)\, u, v \rangle = \lambda \int_\Omega (u, v)\, \mu\, dx$$

for some $u \neq 0$, $u \in W^1_{2,\mu}(\Omega)$ and for all $v \in W^1_{2,\mu}(\Omega)$. Here $J u$ is an eigenfunction of the operator A_* which corresponds to the eigenvalue λ. In fact, we are speaking about the usual definition of eigenvalues and eigenfunctions for differential operators in the class of generalized solutions in $W^1_{2,\mu}(\Omega)$.

THEOREM 2.6. — *Under the conditions of Theorem* 5.1 *the index of a stationary point* u_0 *is equal to* $(-1)^\nu$, *where* ν *is the sum of the multiplicities of the negative eigenvalues of the operator* A_*.

3. LOWER ESTIMATES OF LINEAR OPERATORS

3.1. In this section we consider a linear operator

$$L: \quad W^1_2(\Omega) \to (W^1_2(\Omega))^*$$

of the form

$$\langle Lu, v \rangle = \int_\Omega \left(\sum_{k=1}^{m} \left(a \frac{\partial u}{\partial x_k}, \frac{\partial v}{\partial x_k} \right) - (bu, v) \right) dx. \tag{3.1}$$

Here $W_2^1(\Omega)$ is the Sobolev space of vector-valued functions defined in Ω and satisfying the boundary condition

$$\left. \frac{\partial u}{\partial n} \right|_{\partial\Omega} = 0 \tag{3.2}$$

with the inner product

$$[u, v] = \int_\Omega \left(\sum_{k=1}^{m} \left(\frac{\partial u}{\partial x_k}, \frac{\partial v}{\partial x_k} \right) + (u, v) \right) dx;$$

the triangular brackets $\langle f, v \rangle$ mean the action of the functional $f \in (W_2^1(\Omega))^*$ on the element $v \in W_2^1(\Omega)$, $(W_2^1(\Omega))^*$ is a conjugate space; a is a constant symmetric positive definite matrix, b is a constant matrix.

We suppose that the following condition is satisfied.

CONDITION 3.1. — *All eigenvalues of the matrix* $b - a\xi^2$ *are in the left half plane for all real* ξ.

THEOREM 3.1. — *Let* D *be a bounded domain with a smooth boundary. Then there exists a symmetric bounded positive definite linear operator* T, *acting in the space* $W_2^1(\Omega)$, *such that for any* $u \in W_2^1(\Omega)$

$$\langle Lu, Tu \rangle = \|u\|^2, \tag{3.3}$$

where $\| . \|$ *is the norm in* $W_2^1(\Omega)$.

Remark. — The assumption that the boundary is smooth is done for simplicity. It is clear that the results are valid for more general domains. We believe even that domains with a finite perimeter can be considered (*see* [12]).

We present first some auxiliary results.

We consider the eigenvalue problem in the domain D

$$-\Delta' g = \lambda g, \qquad \left. \frac{\partial g}{\partial n} \right|_{\partial D} = 0. \tag{3.4}$$

Here Δ' is the Laplace operator in the section of the cylinder

$$\Delta' = \sum_{k=2}^{m} \frac{\partial^2}{\partial x_k^2}.$$

It is known (*see*, for example, [9]) that the problem (3.4) has a sequence of eigenvalues λ_i,

$$\lambda_i \leq \lambda_{i+1}, \qquad i = 1, 2, \ldots$$

and a corresponding sequence of eigenvalues g_i which satisfy the condition

$$\int_D g_i g_j \, dx' = 0, \quad i \neq j. \tag{3.5}$$

These eigenfunctions form a complete orthogonal sequence in $L^2(D)$. It means that each function square-integrable on D can be represented as Fourier series with respect to these eigenfunctions, converging in $L^2(D)$.

As known, if $u \in W_2^1(\Omega)$ then for every x_1 fixed $u \in L^2(D)$ (changed if necessary on a set of measure zero). So any function $u \in W_2^1(\Omega)$ can be represented in the form

$$u(x) = \sum_{i=1}^{\infty} v_i(x_1) g_i(x'), \tag{3.6}$$

where v_i are the coefficients of the expansion for each x_1 fixed. Assuming that

$$\int_D g_i^2(x') \, dx' = 1,$$

we have from (3.6)

$$v_i(x_1) = \int_D u(x) g_i(x') \, dx'. \tag{3.7}$$

The functions v_i belong to $L^2(R)$. Indeed,

$$\int_{-\infty}^{\infty} v_i^2 \, dx_1 \leqq \int_{-\infty}^{\infty} dx_1 \left(\int_D |u|^2 \, dx' \right) \left(\int_D |g_i|^2 \, dx' \right) \leqq \|u\|^2.$$

Similarly,

$$\| v_i' \|_{L^2(R)} \leqq \| u \|.$$

Consequently, $v_i \in W_2^1(R)$.

We introduce linear operators T_i, acting in $W_2^1(\Omega)$ by the formula

$$\widetilde{T_i w} = R_i(\xi) \tilde{w}(\xi). \tag{3.8}$$

Here $\sim$ denotes the Fourier transformation, $R_i(\xi)$ is a symmetric positive definite matrix which satisfies the equality

$$((a(\xi^2 + \lambda_i) - b)p, R_i(\xi)p) = ((1 + \xi^2 + \lambda_i)p, p) \tag{3.9}$$

for any vector p. If the condition 3.1 is satisfied and $\lambda_i \geqq 0$ then such matrix exists, and the operator T_i is bounded symmetric and positive definite [3]. We note that the problem (3.4) does not have negative eigenvalues. So the operators (3.8) are defined for all its eigenvalues.

We define now the operator T which appears in Theorem 3.1:

$$T u = \sum_{i=1}^{\infty} T_i(v_i) g_i(x'), \tag{3.10}$$

where u is given by (3.6).

LEMMA 3.1. — *The operator* T *is a linear bounded symmetric and positive definite operator in* $W_2^1(\Omega)$.

Proof. — The linearity of T is obvious. We show now its boundedness. From (3.5) we have

$$\int_\Omega |u|^2\,dx = \sum_{i=1}^{\infty}\int_{-\infty}^{+\infty}|v_i(x_1)|^2\,dx_1,$$

$$\int_\Omega \left|\frac{\partial u}{\partial x_1}\right|^2 dx = \sum_{i=1}^{\infty}\int_{-\infty}^{+\infty}|v_i'(x_1)|\,dx_1,$$

$$\sum_{j=2}^{m}\frac{\partial g_i}{\partial x_j}\frac{\partial g_k}{\partial x_j}\,dx' = -\int_D \Delta' g_i g_k\,dx' = \lambda_i\,\delta_i^k,$$

where $\delta_i^k = 1$ for $i = k$, and $\delta_i^k = 0$ for $i \neq k$,

$$\sum_{j=2}^{m}\int_\Omega \left|\frac{\partial u}{\partial x_j}\right|^2 dx = \sum_{i=1}^{\infty}\lambda_i\int_{-\infty}^{+\infty}|v_i|^2\,dx_1.$$

From these equation and the estimation

$$\max_{\xi,\,i}\|R_i(\xi)\| \leqq M$$

(*see* [3]) we obtain

$$\int_\Omega |Tu|^2\,dx = \frac{1}{2\pi}\sum_{i=1}^{\infty}\int_{-\infty}^{+\infty}|R_i(\xi)\,\tilde{v}_i(\xi)|^2\,d\xi \leqq M^2\sum_{i=1}^{\infty}\int_{-\infty}^{+\infty}|v_i(x_1)|^2\,dx_1,$$

$$\int_\Omega \left|\frac{\partial Tu}{\partial x_1}\right|^2 dx = \frac{1}{2\pi}\sum_{i=1}^{\infty}\int_{-\infty}^{+\infty}|i\xi\,R_k\,\tilde{v}_k|^2\,dx \leqq M^2\sum_{i=1}^{\infty}\int_{-\infty}^{+\infty}|v_i'(x_1)|^2\,dx_1,$$

$$\sum_{j=2}^{m}\int_\Omega \left|\frac{\partial Tu}{\partial x_j}\right|^2 dx \leqq M^2\sum_{i=1}^{\infty}\lambda_i\int_{-\infty}^{+\infty}|v_i|^2\,dx_1.$$

Thus

$$\|Tu\| \leqq M\,\|u\|.$$

We show now that the operator T is symmetric. Let $w \in W_2^1(\Omega)$ and

$$w = \sum_{i=1}^{\infty}\omega_i g_i,\quad Tw = \sum_{i=1}^{\infty}T_i(\omega_i)\,g_i$$

We have

$$\int_\Omega (Tu,\,w)\,dx = \frac{1}{2\pi}\sum_{i=1}^{\infty}\int_{-\infty}^{+\infty}(R_i\,\tilde{v}_i,\,\tilde{\omega}_i)\,d\xi,$$

$$\int_\Omega (u,\,Tw)\,dx = \frac{1}{2\pi}\sum_{i=1}^{\infty}\int_{-\infty}^{+\infty}(\tilde{v}_i,\,R_i\,\tilde{\omega}_i)\,d\xi,$$

$$\int_\Omega \left(\frac{\partial Tu}{\partial x_1},\,\frac{\partial w}{\partial x_1}\right)dx = \frac{1}{2\pi}\sum_{i=1}^{\infty}\int_{-\infty}^{+\infty}(i\xi\,R_k\,\tilde{v}_k,\,\overline{i\xi\tilde{\omega}_k})\,d\xi,$$

$$\int_\Omega \left(\frac{\partial u}{\partial x_1}, \frac{\partial \mathrm{T} w}{\partial x_1}\right) dx = \frac{1}{2\pi} \sum_{i=1}^{\infty} \int_{-\infty}^{+\infty} (i\xi \tilde{v}_k, \overline{i\xi \mathrm{R}_k \tilde{\omega}_k})\, d\xi,$$

$$\sum_{j=2}^{m} \int_\Omega \left(\frac{\partial \mathrm{T} u}{\partial x_j}, \frac{\partial w}{\partial x_j}\right) dx = \sum_{i=1}^{\infty} \lambda_i \int_{-\infty}^{+\infty} (\mathrm{T}_i(v_i), \omega_i)\, dx_1,$$

$$\sum_{j=2}^{m} \int_\Omega \left(\frac{\partial u}{\partial x_j}, \frac{\partial \mathrm{T} w}{\partial x_j}\right) dx = \sum_{i=1}^{\infty} \lambda_i \int_{-\infty}^{+\infty} (v_i, \mathrm{T}_i(\omega_i))\, dx_1,$$

Since the matrices R_i are symmetric and real, the operators T_i are symmetric, we obtain

$$[\mathrm{T} u, w] = [u, \mathrm{T} w].$$

It means that the operator T is symmetric in $\mathrm{W}_2^1(\Omega)$.

To prove that the operator T is positive definite we note that there is such positive number μ that all eigenvalues of the matrices $\mathrm{R}_i(\xi)$ for all i and ξ satisfy the inequality $\lambda \geqq \mu$. It means that

$$(\mathrm{R}_i(\xi) p, p) \geqq \mu (p, p) \qquad \text{for all } i, \xi.$$

We have

$$\int_\Omega (\mathrm{T} u, u)\, dx = \frac{1}{2\pi} \sum_{i=1}^{\infty} \int_{-\infty}^{+\infty} (\mathrm{R}_i \tilde{v}_i, \tilde{v}_i)\, d\xi \geqq \mu \int_\Omega |u|^2\, dx,$$

$$\sum_{i=1}^{m} \int_\Omega \left(\frac{\partial \mathrm{T} u}{\partial x_j}, \frac{\partial u}{\partial x_j}\right) dx \geqq \mu \sum_{j=1}^{m} \int_\Omega \left|\frac{\partial u}{\partial x_j}\right|^2 dx.$$

Thus

$$[\mathrm{T} u, u] \geqq \mu [u, u].$$

The lemma is proved.

Remark. – We note that the operator T is bounded symmetric and positive definite in $\mathrm{L}^2(\Omega)$ also.

Proof of Theorem 3.1. – To prove the theorem we should show only that (3.3) is valid. We have

$$\langle \mathrm{L} u, \mathrm{T} u \rangle = \int_\Omega \left(\sum_{k=1}^{m} \left(a \frac{\partial u}{\partial x_k}, \frac{\partial \mathrm{T} u}{\partial x_k} \right) - (bu, \mathrm{T} u) \right) dx$$
$$= \frac{1}{2\pi} \sum_{i=1}^{\infty} \int_{-\infty}^{+\infty} ((ai\xi \tilde{v}_k, \overline{i\xi \mathrm{R}_k \tilde{v}_k}) + \lambda_k (a\tilde{v}_k, \mathrm{R}_k \tilde{v}_k) - (b\tilde{v}_k, \mathrm{R}_k \tilde{v}_k))\, d\xi$$
$$= \frac{1}{2\pi} \sum_{i=1}^{\infty} \int_{-\infty}^{+\infty} ((1+\xi^2+\lambda_i) \tilde{v}_i, \tilde{v}_i)\, d\xi = \|u\|^2.$$

The theorem is proved.

3.2. – In this section we consider the operator L

$$\mathrm{L}: \quad \mathrm{W}_2^1(\Omega) \to (\mathrm{W}_2^1(\Omega))^*,$$

defined by the equality

$$\langle \mathrm{L}u, v\rangle = \int_\Omega \left(\sum_{k=1}^{m} \left(a\frac{\partial u}{\partial x_k}, \frac{\partial v}{\partial x_k}\right) - (b(x)u, v)\right) dx,$$

where a is a constant symmetric positive definite matrix, $b(x)$ is a continuous matrix. We suppose that there exist limits

$$b_1 = \lim_{x_1 \to -\infty} b(x), \qquad b_2 = \lim_{x_1 \to +\infty} b(x)$$

uniformly with respect to x', and the constant matrices b_1 and b_2 satisfy the condition 3.1.

THEOREM 3.2. — *There exists linear, bounded, symmetric, positive definite operator* S_0 *acting in* $\mathrm{W}_2^1(\Omega)$ *such that the inequality*

$$\langle \mathrm{L}u, \mathrm{S}_0 u\rangle \geqq \|u\|^2 + \theta(u)$$

takes place. Here $\theta(u)$ *is a functional defined on* $\mathrm{W}_2^1(\Omega)$ *which satisfies the condition:* $\theta(u_n) \to 0$ *as* $u_n \to 0$ *weakly.*

Proof. — We consider first the case when

$$b(x) = b_0(x),$$

where

$$b_0(x) = \phi_1(x_1)b_1 + \phi_2(x_1)b_2$$

$\phi^i(x_1)$ are smooth functions, $0 \leqq \phi_i(x_1) \leqq 1$,

$$\begin{aligned} \phi_1(x_1) + \phi_2(x_1) &\equiv 1, \\ \phi_1(x_1) = 0 \quad &\text{for} \quad x_1 > 1, \\ \phi_2(x_1) = 0 \quad &\text{for} \quad x_1 < -1. \end{aligned}$$

We denote by $\mathrm{T}^{(i)}$ the operator which is defined for the matrix b_i as it was done in Theorem 3.1, and

$$\mathrm{T}^{(0)} = \sum_{i=1}^{2} \phi_i \mathrm{T}^{(i)} \phi_i.$$

We have

$$\int_\Omega \sum_{k=1}^{m} \left(a\frac{\partial u}{\partial x_k}, \frac{\partial \mathrm{T}^{(0)} u}{\partial x_k}\right) dx = \sum_{i=1}^{2} \int_\Omega \sum_{k=1}^{m} \left(a\frac{\partial (\phi_i u)}{\partial x_k}, \frac{\partial \mathrm{T}^{(i)} \phi_i u}{\partial x_k}\right) dx + \sum_{i=1}^{2} \int_\Omega \left(\left(a\frac{\partial u}{\partial x_1}, \phi_i' \mathrm{T}^{(i)} \phi_i u\right) - \left(au\phi_i', \frac{\partial \mathrm{T}^{(i)} \phi_i u}{\partial x_1}\right)\right) dx. \quad (3.11)$$

The second summand in the right hand side of (3.11) tends to zero as $u \to 0$ weakly (*see* lemma 3.2 below). We have further

$$\int_\Omega (bu, T^{(0)} u)\, dx = \sum_{i=1}^{2} \int_\Omega (\phi_i b_i u, \phi_i T^{(i)} \phi_i u)\, dx + \theta(u)$$
$$= \sum_{i=1}^{2} \int_\Omega (\phi_i b_i u, T^{(i)} \phi_i u)\, dx + \theta(u).$$

Here $\theta(u)$ denotes all functionals which satisfy the condition in the formulation of the theorem. Thus

$$\langle Lu, T^{(0)} u\rangle = \sum_{i=1}^{2} \int_\Omega \left(\sum_{k=1}^{m} \left(a \frac{\partial(\phi_i u)}{\partial x_k}, \frac{\partial T^{(i)} \phi_i u}{\partial x_k} \right) - (b_i \phi_i u, T^{(i)} \phi_i u) \right) dx + \theta(u).$$

From Theorem 3.1 it follows that

$$\langle Lu, T^{(0)} u\rangle = \|\phi_1 u\|^2 + \|\phi_2 u\|^2 + \theta(u) \geqq \frac{1}{2} \|u\|^2 + \theta(u). \quad (3.12)$$

Let now $b(x)$ be an arbitrary matrix satisfying the conditions above. Then from Lemma 3.2 it follows that

$$\int_\Omega ((b - b_0) u, u)\, dx = \theta(u)$$

To complete the proof of the theorem we should show that

$$T^{(0)} = \frac{1}{2} S^{(0)} + K, \quad (3.13)$$

where $S^{(0)}$ is a symmetric positive definite operator, K is a compact operator in $W_2^1(\Omega)$.

We note first that the operator T_i defined by (3.8) satisfies the equality

$$\frac{\partial(T_i u)}{\partial x_1} = T_i \frac{\partial u}{\partial x_1}$$

and, consequently, the similar equality is valid for the operator T (*see* (3.6), (3.10)) and for the operators $T^{(i)}$. Since the operators $T^{(i)}$ are symmetric in L^2 we have

$$\int_\Omega (\phi_i T^{(i)} \phi_i u, v)\, dx = \int_\Omega (u, \phi_i T^{(i)} v)\, dx.$$

$$\int_\Omega \left(\phi_i T^{(i)} \phi_i \frac{\partial u}{\partial x_1}, \frac{\partial v}{\partial x_1} \right) dx = \int_\Omega \left(\frac{\partial u}{\partial x_1}, \phi_i T^{(i)} \phi_i \frac{\partial v}{\partial x_1} \right) dx.$$

$$\sum_{k=2}^{m} \int_\Omega \left(\frac{\partial}{\partial x_k} (\phi_i T^{(i)} \phi_i u), \frac{\partial v}{\partial x_k} \right) dx = \sum_{k=2}^{m} \int_\Omega \left(\frac{\partial u}{\partial x_k}, \frac{\partial}{\partial x_k} (\phi_i T^{(i)} \phi_i v) \right) dx.$$

It is easy to verify now that

$$[T^{(0)} u, v] - [u, T^{(0)} v] = \sum_{i=1}^{2} \int_{\Omega} \left(\left(\phi_i' T^{(i)} \phi_i u + \phi_i T^{(i)} \phi_i' u, \frac{\partial v}{\partial x_1} \right) - \left(\frac{\partial u}{\partial x_1}, \phi_i' T^{(i)} \phi_i v + \phi_i T_i^{(i)\prime} v \right) \right) dx.$$

Denote the right hand side of this equality by $\Phi(u, v)$. This is a bilinear bounded functional in $W_2^1(\Omega)$, so

$$\Phi(u, v) = [u, K^{(0)} v], \ u, \qquad v \in W_2^1(\Omega)$$

where K^0 is linear bounded operator. We claim that K^0 is compact. Let $v_n \to 0$ weakly in $W_2^1(\Omega)$. Denote $y_n = K^0 v_n$. Then

$$\| y_n \|^2 = [y_n, K^0 v_n] = \Phi(y_n, v_n).$$

From Lemma 3.2 below it follows that $\Phi(y_n, v_n) \to 0$ since $v_n, y_n \to 0$ weakly in $W_2^1(\Omega)$ and $\partial v_n/\partial x_1$, $\partial y_n/\partial x_1$ are uniformly bounded in $L^2(\Omega)$. It proves the compactness of the operators K^0. Thus it is proved that

$$T^{(0)*} - T^{(0)} = K^0 \tag{3.14}$$

is a compact operator.

We have further

$$[T^{(0)} u, v] + [u, T^{(0)} v] = \Phi_1(u, v) + \Phi_2(u, v)$$

where

$$\Phi_1(u, v) = \sum_{i=1}^{2} \int_{\Omega} \sum_{k=1}^{m} \left(\left(\frac{\partial}{\partial x_k} (T^{(i)} \phi_i u), \frac{\partial}{\partial x_k} (\phi_i v) \right) + \left(\frac{\partial}{\partial x_k} (\phi_i u), \frac{\partial}{\partial x_k} (T^{(i)} \phi_i v) \right) \right) dx + 2 \sum_{i=1}^{2} \int_{\Omega} (T^{(i)} \phi_i u, \phi_i v) \, dx,$$

$$\Phi_2(u, v) = \sum_{i=1}^{2} \int_{\Omega} \left(\left(\phi_i' T^{(i)} \phi_i u, \frac{\partial v}{\partial x_1} \right) - \left(\frac{\partial}{\partial x_1} (T^{(i)} \phi_i u), \phi_i' v \right) + \left(\frac{\partial u}{\partial x_1}, \phi_i' T^{(i)} \phi_i v \right) - \left(\phi_i' u, \frac{\partial}{\partial x_1} (T^{(i)} \phi_i v) \right) \right) dx.$$

Since Φ_1 and Φ_2 are bounded bilinear functionals in $W_2^1(\Omega)$ then

$$\Phi_1(u, v) = [S_0 u, v], \qquad \Phi_2(u, v) = [B u, v],$$

where S_0 and B are bounded linear operators. The operator S_0 is symmetric and positive definite since the functional $\Phi_1(u, v)$ is symmetric and

$$[S_0 u, u] = \Phi_1(u, u) = \sum_{i=1}^{2} ([T^{(i)} \phi_i u, \phi_i u] + [\phi_i u, T^{(i)} \phi_i u])$$
$$\geqq 2\mu \sum_{i=1}^{2} \| \phi_i u \|^2 \geqq \mu \| u \|^2.$$

As above, from Lemma 3.2 it follows that the operator B is compact.

The equality (3.13) with $K = (B - K^0)/2$ follows now from (3.14) and the equality

$$T^0 + (T^0)^* = S_0 + B.$$

The theorem is proved.

We used the following lemma.

LEMMA 3.2. — *Let a sequence of vector-valued functions f_n be uniformly bounded in $L^2(\Omega)$ and a sequence g_n converges weakly to zero in $W_2^1(\Omega)$. Let, further, $\psi(x)$ be a bounded smooth function which tends to zero as $x_1 \to \pm\infty$ uniformly with respect to x'. Then*

$$\int_\Omega \psi(x)(f_n(x), g_n(x))\, dx \to 0, \; n \to \infty.$$

Proof. — We have

$$\left| \int_\Omega \psi(f_n, g_n)\, dx \right| \leqq \left(\int_\Omega |f_n|^2\, dx \right)^{1/2} \left(\int_\Omega \psi^2 |g_n|^2\, dx \right)^{1/2},$$

and it is sufficient to show that the second integral in the right hand side of this inequality tends to zero. We represent it in the form

$$\int_\Omega \psi^2 |g_n|^2\, dx = \int_{|x_1| \geq R} \int_D \psi^2 |g_n|^2\, dx_1\, dx' + \int_{|x_1| < R} \int_D \psi^2 |g_n|^2\, dx_1\, dx'. \quad (3.15)$$

For any given $\varepsilon > 0$ we can choose R such that the first integral in the right hand side of (3.15) is less than $\varepsilon/2$ since $\psi \to 0$ as $x_1 \to \pm\infty$. For R fixed the second integral tends to zero as $n \to \infty$ since weak convergence of g_n in $W_2^1(\Omega)$ implies strong convergence of ϕg_n in $L^2(\Omega)$ for any finitary smooth ϕ and hence strong convergence of g_n in $L^2(\Omega_R)$. Here $\Omega_R = D \times \{ |w_1| < R \}$. Thus there is an integer number N such that $\| \psi g_n \|^2_{L^2(\Omega)} \leqq \varepsilon$ for $n \geqq N$. The lemma is proved.

3.3. — We consider now the weighted space $W^1_{2,\mu}(\Omega)$ with the inner product

$$[u, v]_\mu = \int_\Omega \left(\sum_{k=1}^m \left(\frac{\partial u}{\partial x_k}, \frac{\partial v}{\partial x_k} \right) + (u, v) \right) \mu(x_1)\, dx.$$

The weight function μ is supposed to satisfy the conditions formulated in Section 2. We consider the operator $L : W^1_{2,\mu} \to (W^1_{2,\mu})^*$ given by

$$\langle Lu, v \rangle = \int_\Omega \left(\sum_{k=1}^m \left(a \frac{\partial u}{\partial x_k}, \frac{\partial v}{\partial x_k} \right) - (bu, v) \right) \mu\, dx,$$

where $u, v \in W^1_{2,\mu}(\Omega)$, a and b are the same as in Section 3.2.

THEOREM 3.3. — *Let the matrices b_1 and b_2 satisfy Condition* 3.1. *Then there exists a symmetric, bounded, linear, positive definite operator* S *acting in $W^1_{2,\mu}(\Omega)$ such that for any $u \in W^1_{2,\mu}(\Omega)$,*

$$\langle Lu, Su \rangle \geqq \| u \|^2_\mu + \theta_\mu(u),$$

where $\| . \|_\mu$ is the norm in $W^1_{2,\mu}(\Omega)$, $\theta_\mu(u)$ is a functional defined in $W^1_{2,\mu}(\Omega)$ and satisfying the condition $\theta_\mu(u_n) \to 0$ as $u_n \to 0$ weakly.

Proof. — We put

$$T = \omega^{-1} T^{(0)} \omega,$$

where $\omega = \sqrt{\mu}$ and T^0 is defined in the previous section. Introducing the notation $w = \omega u$, we obtain

$$\langle Lu, Tu \rangle = \int_\Omega \left(\sum_{k=1}^m \left(a \frac{\partial w}{\partial x_k}, \frac{\partial}{\partial x_k} (T^{(0)} w) \right) - (bw, T^{(0)} w) \right) dx + I,$$

where

$$I = \int_\Omega \left(\left(a \omega^{-1} \omega^{-1\prime} \frac{\partial w}{\partial x_1}, T^{(0)} w \right) + (a \omega^{-1\prime} w, \omega^{-1\prime} T^{(0)} w) + \left(a \omega^{-1\prime} w, \omega^{-1} \frac{\partial}{\partial x_1} (T^{(0)} w) \right) \right) \omega^2\, dx.$$

It is easy to verify that the operator of multiplication by ω is a bounded operator from $W^1_{2,\mu}(\Omega)$ into $W^1_2(\Omega)$, and the operator of multiplication by ω^{-1} is a bounded operator from $W^1_2(\Omega)$ into $W^1_{2,\mu}(\Omega)$.

Hence $w \in W^1_2(\Omega)$, and from (3.12) we have

$$\int_\Omega \left(\sum_{k=1}^m \left(a \frac{\partial w}{\partial x_k}, \frac{\partial}{\partial x_k} (T^{(0)} w) \right) - (bw, T^{(0)} w) \right) dx \geqq \frac{1}{2} \| w \|^2 + \theta(w) \geqq c \| u \|^2_\mu + \theta_\mu(u),$$

where $c = N^{-2}/2$, N is the norm of the multiplication operator ω^{-1}.

From Lemma 3.2 it follows that $I \to 0$ as $w \to 0$ weakly in $W_2^1(\Omega)$. Thus we have shown that

$$\langle Lu, Tu \rangle \geqq c \| u \|_\mu^2 + \theta_\mu(u).$$

We use the notation θ_μ here for all functionals which satisfy the condition of the theorem.

For the complete proof of the theorem it is sufficient to show that

$$T = cS + K \tag{3.16}$$

where S is a symmetric positive definite operator, K is a compact operator in $W_{2,\mu}^1(\Omega)$. For this we construct the operators in $W_{2,\mu}^1(\Omega)$ from operators defined in $W_2^1(\Omega)$ in the following way. To each linear bounded operator A, acting in $W_2^1(\Omega)$, we assign a corresponding linear operator A_μ in $W_{2,\mu}^1(\Omega)$ by means of

$$[u, A_\mu v]_\mu = [\omega u, A \omega v], \qquad u, v \in W_{2,\mu}^1(\Omega), \tag{3.17}$$

where, as above, $[.,.]_\mu$ and $[.,.]$ are the inner products in $W_{2,\mu}^1(\Omega)$ and $W_2^1(\Omega)$, respectively. Going over in (3.13) to operators in $W_{2,\mu}^1(\Omega)$ by this rule, we obtain $T_\mu^{(0)} = \frac{1}{2} S_\mu^{(0)} + K_\mu$. From Lemma 3.3 below it follows that $S_\mu^{(0)}$ is a bounded, symmetric, positive definite operators, K_μ is a compact operator. By the same lemma and equation $T = \omega^{-1} T^{(0)} \omega$ we have $T_\mu^{(0)} = T + B$, where B is a compact operator in $W_{2,\mu}^1(\Omega)$. Hence we obtain (3.16), where $S = \frac{1}{2c} S_\mu^{(0)}$, $K = K_\mu - B$. The theorem is proved.

LEMMA 3.3. — *Let an operator* A_μ *acting in* $W_{2,\mu}^1(\Omega)$ *be defined according to the operator* A *in* $W_2^1(\Omega)$ *by the equality* (3.17). *Then:*

1. A_μ *is a bounded linear operator in* $W_{2,\mu}^1(\Omega)$ *if* A *is bounded and linear,*
2. A_μ *is a compact operator in* $W_{2,\mu}^1(\Omega)$ *if* A *is compact in* $W_2^1(\Omega)$,
3. A_μ *is a symmetric positive definite operator in* $W_{2,\mu}^1(\Omega)$ *if* A *is symmetric positive definite in* $W_2^1(\Omega)$.
4. $A_\mu = \omega^{-1} A \omega + B$, *where* B *is a linear compact operator* $W_{2,\mu}^1(\Omega)$.

Proof. — Assertions 1-3 can be verified directly. We shall prove assertion 4. Denote $\tilde{A} = \omega^{-1} A \omega$. This is a bounded operator in $W_{2,\mu}^1(\Omega)$. We have for $u, v \in W_{2,\mu}^1(\Omega)$

$$[u, \tilde{A} v]_\mu = \int_\Omega \left(\sum_{k=1}^m \left(\frac{\partial u}{\partial x_k}, \frac{\partial (\tilde{A} v)}{\partial x_k} \right) + (u, \tilde{A} v) \right) \mu \, dx.$$

Denote $y = \omega u$, $z = \omega v$. Then we obtain

$$[u, \tilde{A} v]_\mu = [y, A z] + \Phi(y, z),$$

where

$$\Phi(y, z)=\int_\Omega\Big(((\omega^{-1})'\,\omega\, y, (\omega^{-1})'\,\omega\, A z) + \Big(\frac{\partial y}{\partial x_1}, (\omega^{-1})'\,\omega\, A z\Big)+\Big((\omega^{-1})'\,\omega\, y, \frac{\partial A z}{\partial x_1}\Big)\Big)\,dx.$$

$\Phi(y, z)$ is a bilinear bounded functional in $W_2^1(\Omega)$, so

$$\Phi(y, z)=[y, K z],$$

where K is a linear bounded operator in $W_2^1(\Omega)$. From Lemma 3.2 it follows, as above, that K is compact. Therefore (*see* assertion 2) K_μ is compact in $W_{2,\mu}^1(\Omega)$. We have further

$$[u, \tilde{A} v]_\mu=[u, A_\mu v]_\mu+[u, K_\mu v]_\mu.$$

It means that $\tilde{A}=A_\mu+K_\mu$. The lemma is proved.

4. ESTIMATIONS OF NONLINEAR OPERATORS

Proof of Assertion 2.1. — For any $u_1, u_2 \in W_{2,\mu}^1(\Omega)$ we have

$$|\rho(u_1)-\rho(u_2)| \leqq \Big(\int_\Omega |u_1-u_2|^2\,\sigma\, dx\Big)^{1/2} \leqq \|u_1-u_2\|_\mu. \tag{4.1}$$

We obtain now the lower estimation of the functional $\rho(u)$. For any $N>1$ we have

$$\begin{aligned}\rho(u) &\geqq \Big(\int_{-N}^{-1}\int_D |u+w_- - w_+|^2\,dx_1\,dx'\Big)^{1/2}\sqrt{\sigma(-N)}\\ &\geqq \Big[\Big(\int_{-N}^{-1} dx_1 \int_D |w_- - w_+|^2\,dx'\Big)^{1/2} - \Big(\int_{-N}^{-1} dx_1\int_D |u|^2\,dx'\Big)^{1/2}\Big]\sqrt{\sigma(-N)}\\ &\geqq (|w_- - w_+|\sqrt{N-1}\ \text{mes}\ D - \|u\|_\mu)\sqrt{\sigma(-N)}.\end{aligned}$$

Let u lie in the ball $\|u\|_\mu \leqq R$. Choosing N to satisfy the condition

$$|w_- - w_+|\sqrt{N-1}\ \text{mes}\ D - R > 1,$$

we obtain

$$\rho(u) > \sigma(-N).$$

From this and (4.1) it follows that the functional $c(u)$ satisfies Lipschitz condition on every bounded set.

It is easy to verify that $\rho(u_h)$ is monotone in h (*see* [3]) and $\rho(u_h) \to 0$ as $h \to +\infty$, and $\rho(u_h) \to +\infty$ as $h \to -\infty$. Hence the functional $c(u)$ satisfies the conditions of the Assertion. The Assertion is proved.

We consider now the operator $A(u)$ defined by (2.1). First of all we show that the integral in the right hand side of (2.1) exists. Obviously, we should verify only the existence of the integral

$$I = \int_\Omega (F(u+\psi, x), v)\,\mu\,dx.$$

We have

$$F(u+\psi, x) - F(\psi, x) = b(x)\,u,$$

where

$$b(x) = \int_0^1 F'(tu(x)+\psi(x_1), x)\,dt.$$

So

$$I = \int_\Omega (b(x)\,u, v)\,\mu\,dx + \int_\Omega (F(\psi, x), v)\,\mu\,dx.$$

The first integral in the right hand side of this equality exists since $F'(w, x)$ is bounded for all $w \in R^n$ and $x \in \Omega$, and the second integral exists since $F(\psi, x) \in W^1_{2,\mu}(\Omega)$.

Proof of Assertion 2.2. – Let $u_1, u_2 \in W^1_{2,\mu}(\Omega)$, $\|u_i\|_\mu \leqq R$, $i=1, 2$ where R is a positive number. We have for $v \in W^1_{2,\mu}(\Omega)$.

$$\langle A(u_1) - A(u_2), v \rangle = \int_\Omega \sum_{k=1}^{m} \left(a \frac{\partial (u_1 - u_2)}{\partial x_k}, \frac{\partial (v\mu)}{\partial x_k} \right) dx$$
$$- \int_\Omega \left(c(u_1)\left(\frac{\partial u_1}{\partial x_1} + \psi'\right) - c(u_2)\left(\frac{\partial u_2}{\partial x_1} + \psi'\right) + r \frac{\partial (u_1 - u_2)}{\partial x_1} \right.$$
$$\left. + F(u_1+\psi, x) - F(u_2+\psi. x), v \right) \mu\,dx. \quad (4.2)$$

We estimate each summand in the right hand side of this equality in the usual way. Since μ'/μ and r are bounded functions we obtain

$$|\langle A(u_1) - A(u_2), v \rangle| \leqq K \|u_1 - u_2\|_\mu \|v\|_\mu. \quad (4.3)$$

The Assertion is proved.

Proof of Theorem 2.1. – For the operator S in the formulation of the theorem we take the operator constructed in Theorem 3.3. Let $u_n \to u_0$ weakly. Denote $v_n = u_n - u_0$. We have

$$\langle A(u_n), S v_n \rangle = \int_\Omega \left[\sum_{k=1}^{m} \left(a \frac{\partial u_n}{\partial x_k}, \frac{\partial (S v_n)}{\partial x_k} \right) + \left(a \frac{\partial u_n}{\partial x_1}, \nu S v_n \right) \right.$$
$$\left. - \left(a\psi'' + (c+r)\left(\frac{\partial u_n}{\partial x_1} + \psi'\right) + F(u_n+\psi, x), S v_n \right) \right] \mu\,dx, \quad (4.4)$$

where $\nu=\mu'/\mu$. We consider the first term in the right hand side of (4.4):

$$\int_\Omega \left(\sum_{k=1}^m \left(a\frac{\partial u_n}{\partial x_k}, \frac{\partial (\mathrm{S}\, v_n)}{\partial x_k}\right)\right)\mu\, dx = \int_\Omega \left(\sum_{k=1}^m \left(a\frac{\partial v_n}{\partial x_k}, \frac{\partial (\mathrm{S}\, v_n)}{\partial x_k}\right)\right)\mu\, dx + \phi(v_n),$$

where

$$\phi(v_n) = \int_\Omega \left(\sum_{k=1}^m \left(a\frac{\partial u_0}{\partial x_k}, \frac{\partial (\mathrm{S}\, v_n)}{\partial x_k}\right)\right)\mu\, dx.$$

Since $\phi \in (\mathrm{W}^1_{2,\mu}(\Omega))^*$, $\phi(v_n)\to 0$ as $n\to\infty$. We consider now the second summand in the right hand side of (4.4). Since $\mathrm{S}\, v_n \sqrt{\mu}\to 0$ weakly in $\mathrm{W}^1_2(\Omega)$, functions $\partial u_n/\partial x_1 \cdot \sqrt{\mu}$ are uniformly bounded in $\mathrm{L}^2(\Omega)$, and the function ν is bounded, continuous, and tends to zero as $x_1\to\pm\infty$, it follows from Lemma 3.2 that this integral tends to zero.

Further, we have, obviously

$$\int_\Omega (a\psi'' + (c(u_n)+r)\psi', \mathrm{S}\, v_n)\,\mu\, dx \to 0.$$

We show now that

$$\int_\Omega (c+r)\left(\frac{\partial u_n}{\partial x_1}, \mathrm{S}\, v_n\right)\mu\, dx \to 0. \tag{4.5}$$

Denoting by ϕ all terms which tend to zero as $n\to\infty$, we have

$$\begin{aligned}\int_\Omega (c+r)\left(\frac{\partial u_n}{\partial x_1}, \mathrm{S}\, v_n\right)\mu\, dx &= \int_\Omega (c+r)\left(\frac{\partial v_n}{\partial x_1}, \mathrm{S}\, v_n\right)\mu\, dx + \phi(v_n)\\ &= \int_\Omega (c+r)\left(\frac{\partial v_n}{\partial x_1}, \frac{1}{c_1}(\mathrm{T}-\mathrm{K})\, v_n\right)\mu\, dx + \phi(v_n)\\ &= \int_\Omega \xi\left(\frac{\partial v_n}{\partial x_1}, \mathrm{T}\, v_n\right)\mu\, dx + \phi(v_n)\\ &= \int_\Omega \xi\left(\frac{\partial w_n}{\partial x_1}, \mathrm{T}^{(0)} w_n\right)dx + \int_\Omega \xi\omega(\omega^{-1})'(w_n, \mathrm{T}^{(0)} w_n)\, dx + \phi(v_n). \end{aligned}\tag{4.6}$$

Here

$$w_n = \omega\, v_n, \qquad \xi = \frac{c(u_n)+r(x')}{c_1}.$$

The seconds integral in (4.6) tends to zero by Lemma 3.2. The first integral can be represented in the form

$$\int_\Omega \xi\left(\frac{\partial w_n}{\partial x_1}, \mathrm{T}^{(0)} w_n\right)dx = \sum_{i=1}^2 \left(\int_\Omega \xi\left(\frac{\partial(\phi_i w_n)}{\partial x_1}, \mathrm{T}^{(i)}\phi_i w_n\right)dx - \int_\Omega \xi(\phi_i' w_n, \mathrm{T}^{(i)}\phi_i w_n)\, dx\right).$$

The second summand here tends to zero also by Lemma 3.2. We show that the first term in the right hand side of the last equality is equal to zero. Indeed, since the operator $T^{(i)}$ is symmetric in $L^2(\Omega)$ and

$$\frac{\partial (T^{(i)} u)}{\partial x_1} = T^{(i)} \frac{\partial u}{\partial x_1}.$$

we have

$$\int_D \xi \, dx' \int_{-\infty}^{+\infty} \left(\frac{\partial (\phi_i w_n)}{\partial x_1}, T^{(i)} \phi_i w_n \right) dx_1$$
$$= - \int_D \xi \, dx' \int_{-\infty}^{+\infty} \left(\phi_i w_n, \frac{\partial}{\partial x_1} (T^{(i)} \phi_i w_n) \right) dx_1$$
$$= - \int_D \xi \, dx' \int_{-\infty}^{+\infty} \left(T^{(i)} \phi_{(i)} w_n, \frac{\partial (\phi_i w_n)}{\partial x_1} \right) dx_1.$$

It remains to consider the last term in the right hand side of (4.4). We show that

$$\int_\Omega (F(u_n + \psi, x), S v_n) \mu \, dx = \int_\Omega (b(x) v_n, S v_n) \mu \, dx + \varepsilon_n,$$

where $\varepsilon_n \to 0$, $b(x) = F'(\psi, x)$. We note that

$$F(u_n + \psi, x) = F(\psi, x) + F'(\psi, x) u_n + y, \qquad (4.7)$$

where y is a vector,

$$y = \sum_{k, l} R_{k, l}(u_n) u_n^k u_n^l, \quad u_n = (u_n^1, \ldots, u_n^p), \quad R_{kl}(u) = \int_0^1 (1-t) \frac{\partial^2 F(tu + \psi, x)}{\partial u^k \partial u^l} dt. \qquad (4.8)$$

Thus

$$\int_\Omega (F(u_n + \psi, x), S v_n) \mu \, dx = \int_\Omega (F(\psi, x), S v_n) \mu \, dx$$
$$+ \int_\Omega (b(x) v_n, S v_n) \mu \, dx + \int_\Omega (b(x) u_0, S v_n) \mu \, dx + \int_\Omega (y, S v_n) \mu \, dx. \qquad (4.9)$$

The first and the third integral in the right hand side of (4.9) tend to zero as $n \to \infty$ since they are bounded linear functionals on $W^1_{2, \mu}(\Omega)$. We consider the last term. It follows from (4.7) and (4.8) that $|y(x)| \leqq K_1 + K_2 |u_n(x)|$, $|y(x)| \leqq K_3 |u_n(x)|^2$ where K_1, K_2, K_3 do not depend on $u_n(x)$. Hence

$$|y(x)| \leqq K |u_n(x)|^{1+\alpha}, \qquad 0 < \alpha < 1. \qquad (4.10)$$

Denote $S v_n = w_n$. We have

$$\left|\int_\Omega (y, S v_n)\, \mu\, dx\right| = \left|\int_\Omega (y, w_n)\, \mu\, dx\right| \leqq K \int_\Omega |u_n|^{1+\alpha} |w_n|\, \mu\, dx$$
$$\leqq K \left(\int_\Omega |u_n \omega|^{2(1+\alpha)} dx\right)^{1/2} \left(\int_\Omega \left|\frac{w_n \omega}{\omega^\alpha}\right|^2 dx\right)^{1/2}, \quad (4.11)$$

where $\omega = \sqrt{\mu}$. We choose α:

$$0 < \alpha < \min\left(1, \frac{2}{m-2}\right) (m > 2) \quad (4.12)$$

For $m=2$ the proof is similar. Since $u_n \to u_0$ weakly in $W^1_{2,\mu}(\Omega)$, then $\omega u_n \to \omega u_0$ weakly in $W^1_2(\Omega)$ and the functions ωu_n are uniformly bounded in $W^1_2(\Omega)$. From Lemma 4.1 below it follows that they are uniformly bounded in $L^{2(1+\alpha)}$. Similarly, $\omega w_n \to 0$ weakly in $W^1_2(\Omega)$. From Lemma 3.2 it follows that the last integral in the right hand side of (4.11) tends to zero.

Thus we have proved the following equality

$$\langle A(u_n), S v_n \rangle = \int_\Omega \left(\sum_{k=1}^m \left(a \frac{\partial v_n}{\partial x_k}, \frac{\partial (S v_n)}{\partial x_k}\right) - (b(x) v_n, S v_n)\right) \mu\, dx + \varepsilon_n,$$

where $\varepsilon_n \to 0$. Since there exist uniform limits

$$b_\pm = \lim_{x_1 \to \pm\infty} F'(\psi, x), \quad x' \in D$$

and Condition 2.1 is satisfied, we have by Theorem 3.3

$$\langle A(u_n), S v_n \rangle \geqq \| u_n - u_0 \|^2_\mu + \varepsilon_n.$$

From this and the following convergence

$$\langle A(u_0), S(u_n - u_0) \rangle \to 0, \quad n \to \infty$$

it follows the validity of the theorem. The theorem is proved.

Remark. – In the proofs of Assertion 2.2 and Theorem 2.1, we considered the case when c is a functional. The case when c is a constant is similar and even more simple.

LEMMA 4.1. – *If* $\frac{2m}{m-2} > p \geqq 2$ *and* $u \in W^1_2(\Omega)$ *then*

$$\| u \|_{L^p(\Omega)} \leqq K \| u \|_{W^1_2(\Omega)}. \quad (4.13)$$

(*We remind that m is the dimension of the space*, $\Omega \subset R^m$.)

Proof. – If we consider a bounded domain then (4.13) follows from imbedding theorems (*see*, for example, [9]). We show that the same estimation is valid in the case of the infinite cylinder also.

We represent Ω as the union of finite cylinders

$$\Omega_i = \mathrm{D} \times (i, i+1],$$

Then we have

$$\left(\int_{\Omega_i} |u|^p\, dx\right)^{1/p} \leqq \mathrm{K}\left(\int_{\Omega_i}\left(\sum_{k=1}^{m}\left|\frac{\partial u}{\partial x_k}\right|^2 + |u|^2\right) dx\right)^{1/2}$$

The constant K obviously does not depend on i. Hence

$$\int_{\Omega} |u|^p\, dx = \sum_i \int_{\Omega_i} |u|^p\, dx \leqq \mathrm{K}^p\left(\int_{\Omega}\left(\sum_{k=1}^{m}\left|\frac{\partial u}{\partial x_k}\right|^2 + |u|^2\right) dx\right)^{p/2}$$

The lemma is proved.

LEMMA 4.2. — *If* $\dfrac{2m}{m-2} > p$ *and* $u_n \to 0$ *weakly in* $\mathrm{W}_2^1(\Omega)$ *then*

$$\int_{\Omega} |\psi(x_1)\, u_n(x)|^p\, dx \to 0,$$

where $\psi(x_1)$ *is a bounded continuous function which tends to zero as* $|x_1| \to \infty$.

Proof. — We have

$$\int_{\Omega} |\psi(x_1)\, u_n(x)|^p\, dx = \int_{|x_1| \geqq \mathrm{R}} dx_1 \int_{\mathrm{D}} |\psi u_n|^p\, dx' + \int_{|x_1| \leqq \mathrm{R}} dx_1 \int_{\mathrm{D}} |\psi u_n|^p\, dx'.$$

For any $\varepsilon > 0$ the first integral in the right hand side of this equality is less then $\varepsilon/2$ for all n if R is sufficiently large since $\psi \to 0$ as $|x_1| \to \infty$. The second integral tends to zero as $n \to \infty$ due to the compact imbedding of W_2^1 into L^p in bounded domains. Hence it is less than $\varepsilon/2$ for n sufficiently large. The lemma is proved.

Proof of Theorem 2.2. — It can be verified directly that if conditions 1-5 are satisfied then the following inequality holds:

$$\begin{aligned} |\langle \mathrm{A}_{\tau_1}(u_1) - \mathrm{A}_{\tau_1}(u_2) - \mathrm{A}_{\tau_2}(u_1) \\ + \mathrm{A}_{\tau_2}(u_2), v \rangle| \leqq k_{\mathrm{R}}(\tau_1, \tau_2)\, \|u\|_{\mu}\, \|v\|_{\mu}, \qquad u = u_1 - u_2 \end{aligned} \tag{4.14}$$

for any $u_1, u_2 \in \mathrm{E} : \|u_1\|_{\mu} \leqq \mathrm{R}$, $\|u_2\|_{\mu} \leqq \mathrm{R}$, where R is an arbitrary given positive number. Here $k_{\mathrm{R}}(\tau_1, \tau_2)$ is a function of variables $\tau_1, \tau_2 \in [0, 1]$ bounded and satisfying the condition:

$$\lim_{\tau_2 \to \tau_1} k_{\mathrm{R}}(\tau_2, \tau_1) = 0.$$

It follows from Theorem 2.1 that for any $\tau \in [0, 1]$ there is a bounded symmetric positive definite operator S_τ in the space E such that

$$\langle \mathrm{A}_\tau(u_n), \mathrm{S}_\tau(u_n - u_0) \rangle \geqq \|u_n - u_0\|_{\mu}^2 + \varepsilon_{\tau_n}, \tag{4.15}$$

where u_n is an arbitrary sequence in E which converges weakly to u_0, $\varepsilon_{\tau_n} \to 0$.

Let τ_0 be an arbitrary number from the interval [0, 1]. We show that in some neighbourhood Δ of the point τ_0 the following estimate takes place:

$$\langle A_\tau(u_n), S_\tau(u_n - u_0) \rangle \geqq \frac{1}{2} \| u_n - u_0 \| - \mu^2 + \varepsilon_{\tau_n},$$

where $\varepsilon_{\tau_n} \to 0$ as $n \to \infty$ uniformly in $\tau \in \Delta$. We denote

$$\phi_\tau = \langle A_\tau(u_0), S_{\tau_0}(u - u_0) \rangle.$$

We have

$$\langle A_\tau(u_n), S_{\tau_0}(u_n - u_0) \rangle = \langle A_\tau(u_n) - A_\tau(u_0) - A_{\tau_0}(u_n) + A_{\tau_0}(u_n),$$
$$S_{\tau_0}(u_n - u_0) > + \phi_\tau(u_n) + \langle A_{\tau_0}(u_n), S_{\tau_0}(u_n - u_0) \rangle - \phi_{\tau_0}(u_n).$$

For Δ sufficiently small we have now from (4.14) and (4.15):

$$\langle A_\tau(u_n), S_\tau(u_n - u_0) \rangle \geqq \frac{1}{2} \| u_n - u_0 \|_\mu^2 + \varepsilon_{\tau_0 n} + \phi_\tau(u_n) - \phi_{\tau_0}(u_n). \quad (4.16)$$

We show that $\phi_\tau(u_n) \to 0$ uniformly in τ as $u_n \to u_0$ weakly. Indeed, let us assume that it is not so. Then there exist a positive ε, subsequence $\{u_{n_k}\}$ and sequence τ_k, which we can consider as converging to some τ^*, such that

$$|\phi_{\tau_k}(u_{n_k})| > \varepsilon. \quad (4.17)$$

We have

$$\phi_{\tau_k}(u_{n_k}) = \langle A_{\tau_k}(u_0) - A_{\tau^*}(u_0) - A_{\tau_k}(0) + A_{\tau^*}(0), S_{\tau_0}(u_{n_k} - u_0) \rangle$$
$$+ \langle A_{\tau_k}(0) - A_{\tau^*}(0), S_{\tau_0}(u_{n_k} - u_0) \rangle + \phi_{\tau^*}. \quad (4.18)$$

From the inequality (4.14) it follows that the first term tends to zero as $k \to \infty$. The convergence of the other terms to zero can be easily verified directly. Thus $\phi_{\tau_k}(u_{n_k}) \to 0$ which contradicts (4.17).

We show that $\gamma(A_\tau, D)$ is independent of τ for $\tau \in \Delta$. Since $\gamma(A_\tau, D)$ does not depend on the arbitrariness in the choice of the operator S_τ which is supposed to satisfy the conditions of Theorem 2.1, then as such operator the operator $2 S_{\tau_0}$ can be taken. For the operator $2 S_{\tau_0}^* A_\tau(u)$ condition α') [4] on the interval Δ is satisfied. This means that for any sequence $\tau_n \to \tau^*$ and for any sequence u_n which converges weakly to u_0 from the inequality

$$\lim_{n \to \infty} \langle 2 S_{\tau_0}^* A_{\tau_n}(u_n), u_n - u_0 \rangle \leqq 0,$$

follows the strong convergence $u_n \to u_0$. This follows directly from (4.16) and the uniform convergence of $\phi_\tau(u_n)$ to zero.

It can be verified directly that the operator $A_\tau(u)$ is jointly continuous in $\tau \in [0, 1]$, $u \in E$. Thus the operator $2 S_{\tau_0}^* A_\tau(u)$ realizes the homotopy and, consequently, $\gamma(A_\tau, D)$ does not depend on τ on the interval Δ.

Considering the corresponding interval Δ as a neighbourhood of each point $\tau_0 \in [0, 1]$ and taking a finite covering, we obtain $\gamma(A_0, D) = \gamma(A_1, D)$. The theorem is proved.

We prove now Theorem 2.3. Together with Theorem 2.1 they determine the properties of the operators, and Theorems 2.4-2.6 follow from them. The proofs of these theorems are similar to those in [3] since the concrete form of the operators is not essential here, and we do not give the proofs in this paper.

Proof of Theorem 2.3. – We denote $b(x) = F'(\psi, x)$. Let L be the operator, acting from $W^1_{2,\mu}(\Omega)$ into $(W^1_{2,\mu}(\Omega))^*$, constructed in Theorem 3.3. Then

$$A'(u_0) = L + K,$$

where K is determined by the equality

$$\begin{aligned}\langle K u, v \rangle = \int_\Omega \left(a \frac{\partial u}{\partial x_1}, v \right) \mu' \, dx \\ - \int_\Omega \left(c'(u) \frac{\partial (u_0 + \psi)}{\partial x_1} + (c + r) \frac{\partial u}{\partial x_1}, v \right) \mu \, dx \\ - \int_\Omega ((F'(u_0 + \psi, x) - F'(\psi, x)) u, v) \mu \, dx.\end{aligned}$$

If c is a given constant then the term with $c'(u)$ in (4.19) should be omitted. We show that

$$\langle K u, S u \rangle \to 0$$

as $u \to 0$ weakly in $W^1_{2,\mu}(\Omega)$. Consider the first summand in the right hand side of (4.19):

$$\begin{aligned}\left| \int_\Omega \left(a \frac{\partial u}{\partial x_1}, v \right) \mu' \, dx \right| \leqq \left| \int_\Omega \left(au, \frac{\partial v}{\partial x_1} \right) \mu' \, dx \right| + \left| \int_\Omega (au, v) \mu'' \, dx \right| \\ \leqq \| a \| \| v \|_\mu \left(\left(\int_\Omega \left(\frac{\mu'}{\mu} \right)^2 | u |^2 \mu \, dx \right)^{1/2} + \left(\int_\Omega \left(\frac{\mu''}{\mu} \right)^2 | u |^2 \mu \, dx \right)^{1/2} \right).\end{aligned}$$

From Lemma 3.2 it follows that the integrals in the right hand side of this inequality tend to zero.

We have, further

$$\int_\Omega c'(u) \left(\frac{\partial (u_0 + \psi)}{\partial x_1}, S u \right) \mu \, dx \to 0$$

since $c'(u)$ is a bounded functional, and $S u \to 0$ weakly. The convergence

$$\int_\Omega (c + r) \left(\frac{\partial u}{\partial x_1}, S u \right) \mu \, dx \to 0$$

was proved in the proof of Theorem 2.1. It remains to consider the last integral in the right hand side of (4.19). Denote

$$y(x) = F'(u_0 + \psi, x) - F'(\psi, x).$$

Then for the norm of the matrix we have

$$|y(x)| \leqq K_0, \ |y(x)| \leqq K_1 |u_0(x)|, \tag{4.20}$$

where K_0 and K_1 are constants wich do not depend on u_0. It follows from (4.20) that

$$|y(x)| \leqq K |u_0(x)|^\beta \tag{4.21}$$

for any $\beta : 0 \leqq \beta \leqq 1$, where $K = \max(K_0, K_1)$. Now we have

$$\left| \int_\Omega ((F'(u_1 + \psi, x) - F'(\psi, x)) u, v) \mu \, dx \right|$$

$$\leqq K \int_\Omega |u_0(x)|^\beta |u(x)| |v(x)| \omega^2 \, dx$$

$$\leqq K \left(\int_\Omega |u_0(x) \omega|^{\beta\lambda_1} \right)^{1/\lambda_1} \left(\int_\Omega |u(x) \omega|^{\lambda_2} \right)^{1/\lambda_2} \left(\int_\Omega \frac{|v(x) \omega|^{\lambda_3}}{\omega^{\beta\lambda_3}} dx \right)^{1/\lambda_3},$$

where

$$\lambda_i > 0, \qquad \textstyle\sum_i \lambda_i^{-1} = 1, \qquad i = 1, 2, 3.$$

We choose

$$0 < \beta < \min\left(1, \frac{4}{m-2}\right) (m > 2),$$

$$0 < \beta < 1 \ (m = 2), \qquad \beta\lambda_1 = \lambda_2 = \lambda_3 = \beta + 2.$$

Then

$$2 < \beta + 2 < \frac{2m}{m-2},$$

and it remains to use Lemmas 4.1 and 4.2. Thus we proved the equality

$$\langle A'(u_0) u, S u \rangle = \langle L u, S u \rangle + \theta(u),$$

where $\theta(u) \to 0$ as $u \to 0$ weakly. To complete the proof of the theorem it remains to apply Theorem 3.3. The theorem is proved.

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