Ann. Inst. Henri Poincaré,

Vol. 11, n° 3, 1994, p. 275-296. *Analyse non linéaire*

Prescribing the Jacobian determinant in Sobolev spaces

by

Dong YE

École Normale Supérieure de Cachan
Centre de Mathématiques et de leurs Applications
Unité associée au C.N.R.S. U.R.A.-1611
61, avenue du Président-Wilson, 94235 Cachan Cedex, France

ABSTRACT. — Let Ω be a bounded domain in R^n with regular boundary. In this paper, we study the equations of the type $\det(\nabla u(x))=f$ in Ω and $u(x)=x$ on $\partial\Omega$ where f lies in some Sobolev spaces. We establish some existence and non-existence results. A discussion of general cases is also included.

Key words : Jacobian determinant, Sobolev spaces, nonlinear PDE.

RÉSUMÉ. — Soit Ω un domaine borné régulier de R^n. Dans cet article, nous allons étudier les équations du type $\det(\nabla u(x))=f$ dans Ω et $u(x)=x$ sur $\partial\Omega$ avec f appartenant à certains espaces de Sobolev. Nous établissons quelques résultats d'existence et de non-existence. Une discussion pour des cas généraux est aussi incluse.

Classification A.M.S. : 35 J 60, 46 E 35.

Annales de l'Institut Henri Poincaré - Analyse non linéaire - 0294-1449
Vol. 11/94/03/$4.00/

1. INTRODUCTION

Let Ω be a bounded domain of R^n with regular boundary and f be a smooth function on Ω. In [1], motivated by his study of volume forms on compact smooth manifolds, J. Moser has considered the following type of equations:

$$\det(\nabla u(x)) = \lambda f \quad \text{in } \Omega,$$
$$u(x)|_{\partial\Omega} = x,$$

where $f(x)$ is a positive C^∞ function on $\bar{\Omega}$, $u(x)$ is a C^∞ diffeomorphism from $\bar{\Omega}$ to itself and λ is given by:

$$\lambda = \operatorname{vol}(\Omega) \Big/ \left(\int_\Omega f(x)\, dx \right).$$

In particular, he proved that given any smooth positive function $f(x)$ on $\bar{\Omega}$, there exists $u(x)$ which solves the equation.

Later on, B. Dacorogna and J. Moser studied the corresponding problem (*see* [2]) in the case where $f \in C^{k,\alpha}(\bar{\Omega})$, $0<\alpha<1$ and $f \in C^k(\bar{\Omega})$. We restate their main results as follows.

THEOREM [DM1]. – *Let $k \geqq 0$ be an integer, $0<\alpha<1$, Ω have a $C^{k+3,\alpha}$ boundary $\partial\Omega$. Let $f \in C^{k,\alpha}(\bar{\Omega})$ with $f>0$ in $\bar{\Omega}$. Then there exists a diffeomorphism φ with φ, $\varphi^{-1} \in C^{k+1,\alpha}(\bar{\Omega})$ and*

$$\det(\nabla \varphi(x)) = \lambda f \quad \text{in } \Omega,$$
$$\varphi(x)|_{\partial\Omega} = x,$$

where $\lambda = \operatorname{vol}(\Omega) \Big/ \left(\int_\Omega f(x)\, dx \right)$.

THEOREM [DM2]. – *Let $k \geqq 0$ be an integer, Ω have a $C^k \cap C^1$ boundary. $f, g \in C^k(\bar{\Omega})$, $f, g>0$ in $\bar{\Omega}$ with*

$$\int_\Omega f(x)\, dx = \int_\Omega g(x)\, dx.$$

Then there exists $\varphi \in C^k\text{-Diff}(\bar{\Omega})$ with $\varphi(x)=x$ on $\partial\Omega$ such that

$$\int_E f(x)\, dx = \int_{\varphi(E)} g(x)\, dx$$

for every open set E of Ω.

Moreover if supp $(f-g)$ is included in Ω, then supp $(\varphi - \mathrm{id})$ is also included in Ω where id stands for the identity map.

It is natural to ask whether there exists such a solution in the case where $f(x)$ is of C^0, or in some Sobolev spaces. To our knowledge, these questions are open. In this paper, we study the case where f would lie in

Sobolev spaces. Certain difficult while interesting aspects would appear. The main difficulties (which occurs also in [1] and [2]) are the strong non uniqueness of the eventual solution and the strong non-linearity of the Jacobian determinant.

In order to simplify our presentation, we will say that such a problem is of the type $\{X, Y\}$, if we study the existence and/or non-existence of $u(x) \in Y$ under the conditions $f(x) \in X$. We use also the following abreviated notation:

- $W^{m,p}(\Omega) = W^{m,p}(\Omega, R)$, $C^m(\bar{\Omega}) = C^m(\bar{\Omega}, R)$,
- $W^{m,p,+}(\Omega) = \{g \in W^{m,p}(\Omega)$, there exists $c > 0$ such that $\mathrm{Inf}_\Omega g(x) \geqq c\}$,
- $W^{m,p,*}(\Omega, R^n) = \{g \in W^{m,p}(\Omega, R^n), \det(\nabla g(x)) \geqq c > 0\}$ where $p \geqq \max(1, n^2/(1+mn))$ (*see* Lemma 10).

Actually, we will provide the existence or non-existence criteria only for certain kinds of Sobolev spaces X and Y. The main existence result can be stated as follows:

THEOREM 1. — *Let $m \geqq 1$ be an integer. Let Ω be a bounded domain in R^n with a C^{m+k+3} boundary, where $k = ([n/p]+2)$. Let $p \in (\max(1, n/m), \infty)$. Then for each $f \in W^{m,p,+}(\Omega)$, there exists u such that u, $u^{-1} \in W^{m+1,p}(\Omega, R^n)$ and*

$$\left.\begin{aligned} \det(\nabla u(x)) &= \lambda f \quad \text{in } \Omega \\ u(x)|_{\partial\Omega} &= x \end{aligned}\right\} \tag{1.1}$$

where $\lambda = \mathrm{vol}(\Omega) \Big/ \left(\int_\Omega f(x)\, dx \right)$.

We will prove this theorem by using some ideas in [2] as well as some special properties of $W^{m,p}(\Omega)$ for $p > n/m$, which are similar to the properties of the space $C^{k,\alpha}(\bar{\Omega})$ (*cf.* § 5). The non-existence theorems and the symmetric case will be treated in sections 6, 7 respectively and we will try to figure out some expectations for the general cases in section 8.

2. PRELIMINARY

Let Ω be a bounded open set of R^n. We recall here some well-known results.

LEMMA 1 (trace theorem). — *Let $m \geqq 1$ be an integer and $p \in (1, \infty]$. If $\partial\Omega$ is of class C^{m+1}, $u \in C^\infty(\bar{\Omega})$, we define the trace as $(\partial^j u/\partial \upsilon_j)$ with $j = 0, 1, \ldots, m-1$, where υ denotes the unit normal vector on $\partial\Omega$.*

Then, it extends to a continuous linear surjective mapping from $W^{m,p}(\Omega)$ into $\Pi_j W^{m-j-1/p,p}(\partial\Omega)$.

LEMMA 2 (regularity of the Neumann problem). — *Let* $m \in \mathrm{N}$, $p \in (1, \infty)$ *and* Ω *have a* C^{m+2} *boundary. For every* $g \in \mathrm{W}^{m+1-1/p, p}(\partial\Omega)$, $f \in \mathrm{W}^{m, p}(\Omega)$ *such that* $\int_\Omega f(x)\,dx = \int_{\partial\Omega} g(x)\,d\sigma$, *if* u *is the solution of*

$$\begin{gathered} -\Delta u(x) = f \quad \text{in } \Omega, \\ \partial u/\partial \upsilon = g \quad \text{on } \partial\Omega, \\ \int_\Omega f(x)\,dx = 0. \end{gathered}$$

Then $u \in \mathrm{W}^{m+2, p}(\Omega)$. *There exists a constant* $\mathrm{C}(\Omega, m, p)$ *such that*

$$\|u\|_{m+2, p} \leqq \mathrm{C}(\Omega, m, p)(\|f\|_{m, p} + \|g\|_{m+1-1/p, p})$$

LEMMA 3 (the Sobolev imbedding theorem). — *Let* m *be a non-negative integer,* $p \in [1, \infty)$ *and* $\partial\Omega$ *is of class* C^m, *then there exist the following imbeddings:*

Case 1: $mp < n$.

$$\mathrm{W}^{m, p}(\Omega) \to \mathrm{L}^q(\Omega), \qquad 1/q = 1/p - m/n.$$

Case 2: $mp = n$.

$$\mathrm{W}^{m, p}(\Omega) \to \mathrm{L}^q(\Omega), \qquad p \leqq q < \infty.$$

Case 3: $mp > n$. *Set* $k = \max\{j, (m-j)p > n\}$,

(i) *If* $(m-k-1)\,p < n$, *then*

$$\mathrm{W}^{m, p}(\Omega) \to \mathrm{C}^{k, \alpha}(\bar\Omega), \qquad 0 < \alpha \leqq m - k - n/p.$$

(ii) *If* $(m-k-1)\,p = n$, *then*

$$\mathrm{W}^{m, p}(\Omega) \to \mathrm{C}^{k, \alpha}(\bar\Omega), \qquad 0 < \alpha < 1.$$

LEMMA 4. — *Let* $v \in \mathrm{C}^1(\bar\Omega, \mathrm{R}^n)$, *suppose that* $\det(\nabla v(x)) > 0$ *in* $\bar\Omega$ *and that* $v(x) = x$ *on* $\partial\Omega$, *then* v *is a* C^1 *diffeomorphism from* $\bar\Omega$ *into itself.*

Sketch of proof. — We work on each connected component of $\bar\Omega$, so we can suppose that $\bar\Omega$ is connected. Since $\det(\nabla v(x)) > 0$ and by the degree argument, we have $v(\bar\Omega) = \bar\Omega$ and $v^{-1}(\partial\Omega) = \partial\Omega$. Set

$$\mathrm{E}_\Omega = \{x \in \bar\Omega, \operatorname{card}[v^{-1}(v(x))] = 1\},$$

one can prove that E_Ω is open and closed in $\bar\Omega$ and $\partial\Omega \in \mathrm{E}_\Omega$ by contradiction. Thus $\mathrm{E}_\Omega = \bar\Omega$.

3. THE LINEARIZED PROBLEM

Denote by $\mathrm{W}^{m, p}(\Omega)/\mathrm{R} = \{f \in \mathrm{W}^{m, p}(\Omega), \int_\Omega f(x)\,dx = 0\}$ where Ω is a bounded domain of R^n. We consider the following linearized version of system (1.1):

$$\left.\begin{aligned} \operatorname{div} v(x)) &= f(x) \quad \text{in } \Omega, \\ v(x)\,|_{\partial\Omega} &= 0. \end{aligned}\right\} \tag{3.1}$$

We have:

THEOREM 2. — *Let* $m \in \mathrm{N}$, $p \in (1, \infty)$ *and* $\partial\Omega$ *be of class* C^{m+3}. *Then there exists a continuous linear mapping* L *from* $\mathrm{W}^{m,p}(\Omega)/\mathrm{R}$ *to* $\mathrm{W}^{m+1,p}(\Omega, \mathrm{R}^n)$ *such that* $v = \mathrm{L}(f)$ *satisfies:*

$$\left.\begin{aligned} \operatorname{div}(v(x)) &= f \quad \text{in } \Omega, \\ v &= 0 \quad \text{on } \partial\Omega. \end{aligned}\right\} \tag{3.2}$$

Proof. — We proceed as in [2].

Step 1. — Consider first the problem

$$\left.\begin{aligned} -\Delta g(x) &= f(x) \quad \text{in } \Omega, \\ \partial g/\partial \upsilon &= 0 \quad \text{on } \partial\Omega, \\ \int_\Omega g(x)\,dx &= 0. \end{aligned}\right\} \tag{3.3}$$

By Lemma 2, we have $g \in \mathrm{W}^{m+2,p}(\Omega)/\mathrm{R}$. Denote $c = -\nabla g(x)$. Clearly $c \in \mathrm{W}^{m+1,p}(\Omega, \mathrm{R}^n)$ and div $(c(x)) = f$. But one has only $\langle c, \upsilon \rangle = 0$ on $\partial\Omega$, where $\upsilon = (\upsilon^i)$ is the unit normal vector on $\partial\Omega$.

In what follows, we will construct a family of functions $b = \{ b_{ij} \}$ such that $b(x) \in \mathrm{W}^{m+2,p}(\Omega, \mathrm{R}^{n\times n})$ and for any $x \in \Omega$, $(b_{ij}(x))_{n\times n}$ is an antisymmetric matrix which satisfies:

$$\nabla(b_{ij}(x)) = c_{ij}\upsilon \quad \text{on } \partial\Omega, \tag{3.4}$$

where $c_{ij} = (-1)^{i+j}(c_j \upsilon^i - c_i \upsilon^j)$.

As in [2], we note $\operatorname{rot}^*(b)_j = \sum_i (-1)^{i+j}(\partial b_{ij}/\partial x_i)$. Then if $x \in \partial\Omega$,

$$\begin{aligned} \operatorname{rot}^*(b)_j &= \sum_i (-1)^{i+j}(\partial b_{ij}/\partial x_i) \\ &= \sum_i (-1)^{i+j} c_{ij}\upsilon^i = \sum_i (c_j\upsilon^i - c_i\upsilon^j)\upsilon^i \\ &= c_j |\upsilon|^2 - \langle c, \upsilon \rangle \upsilon^j = c_j. \end{aligned}$$

On the other hand, $\forall x \in \Omega$, one has:

$$\begin{aligned} \operatorname{div}(\operatorname{rot}^*(b(x)) &= \sum_j \partial_j \Big(\sum_i (-1)^{i+j}(\partial b_{ij}/\partial x_i)\Big) \\ &= \sum_{i,j} (-1)^{i+j}(\partial b_{ij}/\partial x_i\,\partial x_j) = 0. \end{aligned}$$

Thus, $v = -\operatorname{rot}^*(b) - \nabla g$ will be a solution of the equation (3.1).

Step 2. — Construction of $\{b_{ij}\}$.

In order to get the desired regularity and continuity results, we will use a refinement of an argument in [2].

As $\partial\Omega \in C^{m+3}$, one has $\upsilon \in C^{m+3}(\partial\Omega)$ and one can assume that $c_{ij}|_{\partial\Omega} \in W^{m+1-1/p, p}(\partial\Omega)$ by the trace theorem. Let $\{d_{ij}\}$ be the solutions of

$$\left.\begin{aligned} -\Delta d_{ij}(x) &= \int_{\partial\Omega} c_{ij}(x)\, d\sigma/\mathrm{vol}(\Omega) \quad \text{in } \Omega, \\ \partial d_{ij}/\partial\upsilon &= c_{ij} \quad \text{on } \partial\Omega, \\ \int_{\Omega} d_{ij}(x)\, dx &= 0. \end{aligned}\right\} \tag{3.3}$$

By Lemma 2, one has $d_{ij}(x) \in W^{m+2, p}(\Omega)$.

Set $\Psi(x) = x - \omega\, d(x, \partial\Omega)\, \nabla d(x, \partial\Omega)\, \chi(d(x, \partial\Omega)/\varepsilon)$ where $\chi \in C^\infty(R)$ is a cut-off function satisfying $\chi = 1$ in $[-1, 1]$, $\mathrm{supp}(\chi) = [-2, 2]$ and ω, ε are positive constants. We now state a result which will be proved in Step 3.

LEMMA 5. — *The constants* ω *and* ε *can be chosen so that* Ψ *is a* C^{m+2} *diffeomorphism from* R^n *into itself,* $\Psi(x) = x$ *if* $x \in \partial\Omega$ *and* $\Psi(\bar{\Omega}) = \bar{\Omega}$.

Assuming this Lemma, set $b_{ij}(x) = [d_{ij}(x) - \chi(d(x, \partial\Omega)/\varepsilon)\, d_{ij}(\Psi(x))]/\omega$. Clearly, $d_{ij}(x) \in W^{m+2, p}(\Omega)$ implies $d_{ij}(\Psi(x)) \in W^{m+2, p}(\Omega)$ and we observe that $\chi(d(x, \partial\Omega/\varepsilon) \in C^{m+3}(\bar{\Omega})$. Thus $b_{ij}(x) \in W^{m+2, p}(\Omega)$.

Moroever, for any $x \in \partial\Omega$,

$$\begin{aligned} \partial b_{ij}/\partial x_k &= [\partial d_{ij}/\partial x_k - \sum_l ((\partial d_{ij}/\partial x_l)\, \partial \Psi^l/\partial x_k)]/\omega \\ &= [\partial d_{ij}/\partial x_k - \sum_l ((\partial d_{ij}/\partial x_l)(\delta_{kl} - \omega \upsilon^k \upsilon^l))]/\omega \\ &= \sum_l ((\partial d_{ij}/\partial x_l)\, \upsilon^k \upsilon^l) \\ &= (\partial d_{ij}/\partial \upsilon)\, \upsilon^k \\ &= c_{ij} \upsilon^k. \end{aligned}$$

Hence we get $\nabla(b_{ij}(x)) = c_{ij}\upsilon$ on $\partial\Omega$.

Obviously, the above procedure is linear. By Lemma 1 and Lemma 2, one obtains immediately that there is a positive number $C(\Omega, m, p)$ such that

$$\| v \|_{m+2, p} \leqq C(\Omega, m, p) \| f \|_{m, p}.$$

Step 3. — We now prove Lemma 5. Since $\partial\Omega \in C^{m+3}$, then there exists $\varepsilon > 0$ such that $d(x, \partial\Omega) \in C^{m+3}(V_{3\varepsilon})$ where $V_{3\varepsilon} = \{x \in R^n, d(x, \partial\Omega) < 3\varepsilon\}$. We fix this ε.

Since $\Phi(x)=d(x, \partial\Omega)(\nabla d(x, \partial\Omega)\chi(d(x, \partial\Omega)/\varepsilon)\in C^{m+2}(R^n, R^n)$ has a compact support, there exists a sufficiently small $\omega=0$ such that $\|\omega\nabla\Phi(x)\|_\infty<1/2n$. Thus, $\Psi(x)=x-\omega\Phi(x)$ verifies that $\det(\nabla\Psi(x))>0$.

Furthermore, we have $\Psi(x)=x$ on $\partial\Omega$. Proceeding similarly as in the proof of Lemma 4, we set $A=\{y\in R^n, \operatorname{card}[\Psi^{-1}(\Psi(y))]=1\}$. We observe that A is open and closed and that $A\neq\varnothing$ (as $\operatorname{supp}(\Phi(x))$ is compact), from which we get $A=R^n$. Clearly, $\Psi(R^n)=R^n$. So Ψ is a C^{m+2} diffeomorphism. The fact that $\Psi(\bar{\Omega})=\bar{\Omega}$ follows from Lemma 4. ■

4. TECHNICAL LEMMAS

The subtleness of our problem arises not only from the non-linearity of the Jacobian determinant, but also from the fact that the behaviour of the functions in Sobolev spaces are not easy to handle. In particular, the regularity properties after multiplication as well as composition might not be preserved in general.

By trying to solve our problem, we need some properties for the functions in $W^{m,p}(\Omega)$ with $p>n/m$. More precisely, we find that they behave in great similarity with those of the functions in $C^{k,\alpha}(\bar{\Omega})$. We state these properties in the following lemmas. They will play an important role in the proof of our main theorems. After we found the proofs of these lemmas, we learned that some of them are known (*see* [7] and [12]). (Thus we are not sure of the originality of these results.) The proofs are based on the imbedding theorem of Sobolev spaces.

Let Ω be a bounded domain of R^n with regular boundary.

LEMMA 6. — *Let* $m\geqq 1$, $m\in N$, $p\in(n/m, \infty]$, *then for any* $f(x)$ *and* $g(x)$ *in* $W^{m,p}(\Omega)$, *the product* (fg), *lies in* $W^{m,p}(\Omega)$, i.e. $W^{m,p}$ *is an algebra.*

LEMMA 7. — *Let* $m\geqq 1$, $m\in N$, $p\in(n/m, \infty]$. *Let* $f(x)\in W^{m+1,p}(\Omega)$, $g(x)\in W^{m+1,p,*}(\Omega, R^n)$ *and assume that* $g(x)|_{\partial\Omega}=x$. *Then we have that* $f\circ g\in W^{m+1,p}(\Omega)$.

LEMMA 8. — *Let m be a positive integer,* $p\in[1, \infty]$, $f(x)\in W^{m,p}(\Omega)$. *Let* $g(x)\in W^{m,q,*}(\Omega, R^n)$ *where* $q=\infty$ *if* $m=1$; $q\geqq p$, $q\in(n/(m-1), \infty]$, *if* $m>1$. *Let* $g(x)=x$ *on* $\partial\Omega$, *then we have* $f\circ g\in W^{m,p}(\Omega)$.

Proof of Lemma 6. — We use the notation $\partial_h v=\prod_k \partial_{x_k h_k}(v)$ with $h\in N^n$ and $|h|=\sum_k h_k$.

We consider $\partial_h(fg)$ where $|h|=m$. As $\partial_h(fg)=\sum_{k\leqq h} C_{k,h}(\partial_k f)(\partial_{h-k} g)$, it is sufficient to prove that $(\partial_k f)(\partial_{h-k} g)\in L^p(\Omega)$. The case $p=\infty$ is immediate, then we consider for $p<\infty$.

Case 1. – $p\geqq n/(m-|k|)$.

In this case, one has $\partial_k f\in L^q(\Omega)$ for any $q\in R$ and $\partial_{h-k} g\in L^{p'}(\Omega)$ with $p'\geqq p$. Observe that $(\partial_k f)(\partial_{h-k} g)\in L^p(\Omega)$ holds if $p'>p$ and the case $p'=p$ occurs only for $|h-k|=m$, so $|k|=0$ and one gets $(\partial_k f)(\partial_{h-k} g)=f(\partial_h g)\in L^p(\Omega)$, as $f\in C^0(\bar{\Omega})$ by $p>n/m$.

Case 2. – $p\geqq n/(m-|h-k|)$.

By changing the roles of g and f, the proof is then the same as in Case 1.

Case 3. – Otherwise, one has $\partial_k f\in L^{q_1}(\Omega)$, $\partial_{h-k} g\in L^{q_2}(\Omega)$ where $1/q_1=1/p-(m-|k|)/n$, $1/q_2=1/p-(m-|h-k|)/n$. Their product will lie in $L^q(\Omega)$ with $1/q=1/q_1+1/q_2=1/p+(1/p-m/n)<1/p$. ■

As an immediate consequence, we have:

COROLLARY 1. – *If* $u\in W^{m+1,p}(\Omega, R^n)$ *with* $m\geqq 1$, $p\in(n/m, \infty]$, *then we have* $\det(\nabla(x))\in W^{m,p}(\Omega)$.

Remarks. – 1. By the proof, $W^{m,n/m}(\Omega)\cap L^\infty(\Omega)$ is also an algebra.

2. $W^{m,n/m}(\Omega)$ is not an algebra in general, (if $n=m=2$, $W^{2,1}(\Omega)$ is !) but we have that for any f, $g\in W^{m,n/m}(\Omega)$, then $(fg)\in W^{m,p}(\Omega)$, $\forall p\in[1, n/m)$.

Proof of Lemma 7. – By Lemma 4, we have $g\in C^1\text{-Diff}(\bar{\Omega})$, then there is a constant $c>0$ such that $\det(\nabla g(x))\geqq c$ and $\det(\nabla g^{-1}(x))\geqq c$. Thus $f\circ g\in L^q(\Omega)$ if and only if $f\in L^q(\Omega)$. By induction, one knows that: if $|h|=m+1$,

$\partial_h(f\circ g)=\sum_{k\leqq h}(\partial_k f)\circ g\{\Sigma_{\alpha,\beta} C_{k,\alpha,\beta}(\prod_j \partial_{\alpha_j} g^{\beta_j})\}$ where the last sum occurs for $1\leqq j\leqq|k|$, $\alpha_j\in N^n$, $\sum_j|\alpha_j|=m+1$, $|\alpha_j|\geqq 1$ and $\beta_j\in\{1, \ldots, n\}^{|k|}$. It is sufficient to prove that $(\partial_k f)\circ g\prod_j \partial_{\alpha_j} g^{\beta_j}\in L^p(\Omega)$ where $\sum_j|\alpha_j|=m+1$, $|\alpha_j|\geqq 1$. Obviously the case $p=\infty$ holds. Thus we consider $p<\infty$.

Case 1. – $p\geqq n/(m+1-|k|)$.

This means $(\partial_k f)\circ g\in L^q(\Omega)$ for any $q\in[1, \infty)$.

We denote $I=\{j, p\geqq n/(m+1-|\alpha_j|)\}$, $J=\{1, 2, \ldots, |k|\}\setminus I$ and card[ ] as the cardinal number of set. We have $(\prod_{j\in I}\partial_{\alpha_j} g^{\beta_j})\in L^q(\Omega)$ for any $q\in[1, \infty)$.

If $J=\varnothing$, we get that $(\partial_k f)\circ g\prod_j \partial_{\alpha_j} g^{\beta_j}\in L^p(\Omega)$. Otherwise, $(\prod_{j\in J}\partial_{\alpha_j} g^{\beta_j})$ $\in L^{q'}(\Omega)$ where q' satisfies:

$$\begin{aligned}1/q' &= \sum_{j\in J}(1/p-(m+1-|\alpha_j|)/n)\\ &= 1/p+(\operatorname{card}[J]-1)(1/p-m/n)+(\sum_{j\in J}|\alpha_j|-m-\operatorname{card}[J])/n\\ &= \leqq 1/p,\end{aligned}$$

because that $p\geqq m$, $\operatorname{card}[J]\geqq 1$ and $\sum_{j\in J}|\alpha_j|\leqq m+1$.

If $q'>p$, $(\partial_k f)\circ g\prod_j \partial_{\alpha_j} g^{\beta_j}\in L^p(\Omega)$ holds obviously. The case $q'=p$ occurs if and only if $\operatorname{card}[J]=1$ and $\sum_{j\in J}|\alpha_j|=m+1$, that is $\operatorname{card}[I]=0$, $|k|=1$. Thus we can rewrite our terms as $(\partial_{x_j} f)\circ g\,\partial_h g^i$, which is also in $L^p(\Omega)$, by the fact that $(\partial_{x_j} f)\circ g\in C^0(\bar\Omega)$ as $p>n/m$.

Case 2. – $p<m/(m+1-|k|)$. We adopt the same notation for I and J. $(\partial_k f)\circ g\prod_{j\in J}\partial_{\alpha_j} g^{\beta_j}\in L^{q'}(\Omega)$ where q' verifies:

$$\begin{aligned}1/q' &= 1/p-(m+1-|k|)/n+\sum_{j\in J}(1/p-(m+1-|\alpha_j|)/n)\\ &= 1/p+\operatorname{card}[J](1/p-m/n)+(\sum_{j\in J}|\alpha_j|-m-1-\operatorname{card}[J]+|k|)/n.\end{aligned}$$

$$\begin{aligned}\text{Since } \sum_{j\in J}|\alpha_j| &= m+1-\sum_{j\in I}|\alpha_j|\\ &\leqq m+1-\operatorname{card}[I]\\ &\leqq m+1-(|k|-\operatorname{card}[J]),\end{aligned}$$

we have $1/q'\leqq 1/p$. The case $q'>p$ is simple while the case $q'=p$ occurs only when $\operatorname{card}[J]=0$, $|k|=m+1$. So in this case $|\alpha_j|=1$ for every $\alpha_j\in I$. Since $g\in C^1(\bar\Omega)$, then we get

$$(\partial_k f)\circ g\prod_j \partial_{\alpha_j} g^{\beta_j}=(\partial_k f)\circ g\prod_j \partial_{x_j} g^{\beta_j}\in L^p(\Omega).\quad\blacksquare$$

Proof of lemma 8. – One needs only to take a little care in the case $m=1$. Otherwise, by changing $1/p-(m+1-|\alpha_j|)/n$ into $1/q-(m-|\alpha_j|)/n$, the proof is almost the same as that of Lemma 7. Details are left to the interested readers. ■

By using these properties, we can prove the following interesting result.

PROPOSITION 1. – *Let* $b\in W^{n+1,p,*}(\Omega, R^n)$ *with* $m\geqq 1$, $m\in N$, $p>n/m$ *and assume that* $b(x)=x$ *on* $\partial\Omega$, *then* b^{-1} *exists and* $b^{-1}\in W^{m+1,p}(\Omega, R^n)$.

Proof. — Clearly, $b \in C^1(\bar{\Omega}, R^n)$. By Lemma 4, b is a C^1 diffeomorphism from $\bar{\Omega}$ to itself. One has then $\nabla b^{-1}(x) = (\nabla b(x))^{-1} \circ b^{-1}(x)$.

Since $(\nabla b)^{-1} = (adj \nabla b)/\det(\nabla b)$, where $(adj \nabla b)$ denotes the adjoint matrix of ∇b. By Lemma 6, we know $(adj \nabla b(x))$, $\det(\nabla b(x))$ as well as $(\nabla b(x))^{-1} \in W^{m,p}$.

Set

$$k_0 = \max\{k, p > n/(m-k)\}.$$

By induction, one has $\nabla b^{-1}(x) \in C^{k_0}(\bar{\Omega})$, so $b^{-1}(x) \in C^{k_0+1}(\bar{\Omega})$,

Case 1. — $p \neq n/(m-k_0-1)$.

We define p_k by $1/p_k = 1/p - (m-k)/n$. Then $(\nabla b(x))^{-1} \in W^{k_0+1, p_{k_0+1}}$ and b^{-1} is a C^{k_0+1} diffeomorphism from $\bar{\Omega}$ to itself. We see that

$$\nabla b^{-1}(x) \in W^{k_0+1, p_{k_0+1}}, \text{ then } b^{-1}(x) \in W^{k_0+2, p_{k_0+1}}(\Omega, R^n).$$

We are now in the special situation of Lemma 8:

$$p = p_{k_0+2} > n/(k_0+2) \qquad \text{where} \quad 1/q = 1/p - 1/n,$$

i.e. $q = p_{k_0+1}$, $(\nabla b(x))^{-1} \in W^{k_0+2, p_{k_0+2}}$, $b^{-1}(x) \in W^{k_0+2, p_{k_0+1}}(\Omega, R^n)$ and b^{-1} is a C^{k_0+1} diffeomorphism. Thus, $\nabla b^{-1}(x) \in W^{k_0+2, p_{k_0+2}}$ which means $b^{-1}(x) \in W^{k_0+3, p_{k_0+2}}(\Omega, R^n)$. Proceeding continuously, we arrive at $b^{-1}(x) \in W^{m+1, p}(\Omega, R^n)$.

Case. — $p = n/(m-k_0-1)$.

Since $(\nabla b(x))^{-1} \in W^{m,p}$, we have $(\nabla b(x))^{-1} \in W^{k_0+1, q}$, $\forall q \in [1, \infty)$. By the same argument as above, we have $b^{-1}(x) \in W^{k_0+2, q}(\Omega, R^n)$ for any $q \in R$. Choosing $q \geqq n$, we can continue to assume that $\nabla b^{-1}(x) \in W^{k_0+2, p_{k_0+2}}$ by Lemma 8, because $p_{k_0+2} = n$. Now we can proceed in the sameway as in Case 1 to get finally $b^{-1}(x) \in W^{m+1, p}(\Omega, R^n)$. ■

Remark. — In Lemma 7 and 8 we can also take some weaker conditions for f and g. For example, we can change the condition $f, g \in W^{m+1, p}(\Omega)$, by $f, g \in W^{m+1, n/m}(\Omega) \cap W^{1, \infty}(\Omega)$ in Lemma 7.

5. PROOF OF THEOREM 1

We use some idea in [2]. First we prove a lemma for f near the constant mapping of Ω and then we will prove Theorem 1.

LEMMA 9. — *Let Ω, m, p be as in Theorem 1 and $f(x) \in W^{m, p, +}(\Omega)$ satisfying that $\int_\Omega f(x)\,dx = \mathrm{vol}(\Omega)$. Then there exist $\varepsilon = \varepsilon(\Omega, m, p)$ and $C(\Omega, m, p)$ such that for any f verifying $\| f - 1 \|_{m,p} \leqq \varepsilon$, then there exist u,*

$u^{-1} \in W^{m+1,p}(\Omega, R^n)$ *such that* u *is a solution of the equation:*

$$\left.\begin{aligned} \det(\nabla u(x)) = f \quad \text{in } \Omega, \\ u(x)|_{\partial\Omega} = x. \end{aligned}\right\} \tag{5.1}$$

and $\| u - \mathrm{id} \|_{m+1,p} \leq C(\Omega, m, p) \| f - 1 \|_{m,p}$.

Proof. – By Theorem 2, we have a continuous linear mapping L from $W^{m,p}(\Omega)/R$ to $W^{m+1,p}(\Omega, R^n)$ such that for any $f \in W^{m,p}(\Omega)$, we have $\operatorname{div}(L(f)) = f$ in Ω and $L(f) = 0$ on $\partial\Omega$.

If ξ be a $(n \times n)$ matrix, let $\operatorname{tr}(\xi)$ be the trace of ξ. Define $Q(\xi) = \det(I + \xi) - 1 - \operatorname{tr}(\xi)$ where I stands for the identity matrix.

By using the proof of Lemma 6, we see that for any $f, g \in W^{m,p}(\Omega)$

$$\| fg \|_{m,p} \leq C_1 \| f \|_{m,p} \| g \|_{m,p}.$$

Then we obtain: for any $w, v \in W^{m+1,p}(\Omega, R^n)$,

$$\| Q(\nabla w) - Q(\nabla v) \|_{m,p} \leq C_2 \sum_{1 \leq j \leq n-1} (\| w \|_{m+1,p} + \| v \|_{m+1,p})^j \| w - v \|_{m+1,p}.$$

If $u(x)$ is the solution of (5.1), we set $v(x) = u(x) - x$. Thus, the equation (5.1) becomes:

$$\left.\begin{aligned} \operatorname{div}(v(x)) = f - 1 - Q(\nabla v) \quad \text{in } \Omega, \\ v(x) = 0 \quad \text{on } \partial\Omega. \end{aligned}\right\} \tag{5.2}$$

Define $N(v) = f - 1 - Q(\nabla v)$. One will consider the following problem: $v \in W^{m+1,p}(\Omega, R^n)$ such that $LN(v) = v$ and $v(x)|_{\partial\Omega} = 0$.

Step 1. – We prove that $LN(v)$ is well-defined.

Since $\int_\Omega f(x)\,dx = \operatorname{vol}(\Omega)$, we have:

$$\int_\Omega N(v)\,dx = \int_\Omega (f - 1 - Q(\nabla v))\,dx = \int_\Omega (f - \det(\nabla v + I))\,dx + \int_\Omega \operatorname{div}(v)\,dx$$

$$= \operatorname{vol}(\Omega) - \int_\Omega \det(\nabla(v + \mathrm{id}))\,dx = 0.$$

Thus for $v \in W^{m+1,p}(\Omega, R^n)$ and $v(x) = 0$ on $\partial\Omega$, $N(v)$ is in $W^{m,p}(\Omega)/R$.

Step 2. – Let C_0 be the constant in Theorem 2, *i.e.*

$$\| L(g) \|_{m+1,p} \leq C_0 \| g \|_{m,p}$$

We choose $\varepsilon = \min(1/(8C_0), (4C_0)^{-2}(C_2)^{-1})$, $r = 2C_0 \| f - 1 \|_{m,p} < 1/4$, and $B_r = \{ v \in W^{m+1,p}(\Omega, R^n), \| v \|_{m+1,p} \leq r \text{ and } v(x) = 0 \text{ on } \partial\Omega \}$.

Then we have

$$\begin{aligned}\| \mathrm{LN}(v_1)-\mathrm{LN}(v_2)\|_{m+1,p} &\leqq C_0 \| \mathrm{N}(v_1)-\mathrm{N}(v_2)\|_{m,p}\\ &\leqq C_0 \| Q(\nabla v_1)-Q(\nabla v_2)\|_{m,p}\\ &\leqq C_0 C_2 \sum_{1\leqq j\leqq n-1} (\| v_1\|_{m+1,p}+\| v_2\|_{m+1,p})^j \| v_1-v_2\|_{m+1,p}\\ &\leqq 4(C_0)^2 C_2 \| f-1\|_{m,p} \| v_1-v_2\|_{m+1,p}/(1-4C_0\| f-1\|_{m,p})\\ &\leqq 8(C_0)^2 C_2 \| v_1-v_2\|_{m+1,p}\\ &\leqq \| v_1-v_2\|_{m+1,p}/2.\end{aligned}$$

On the other hand,

$$\| \mathrm{LN}(0)\|_{m+1,p}\leqq C_0 \| \mathrm{N}(0)\|_{m,p}\leqq C_0 \| f-1\|_{m,p}\leqq r/2.$$

Using the fixed point theorem, we then obtain the existence of $v\in B_r$ such that $\mathrm{LN}(v)=v$, so $u(x)=v(x)+x$ is a solution of (5.1). More precisely, we have that $\| u-\mathrm{id}\|_{m+1,p}=\| v\|_{m+1,p}\leqq r=2C_0\| f-1\|_{m,p}$ and the fact $u^{-1}\in W^{m+1,p}(\Omega, R^n)$ follows from Proposition 1. ■

Proof of Theorem 1. — We need only prove for f satisfying $\int_\Omega f(x)\,dx=\mathrm{vol}(\Omega)$ and $f\geqq c>0$. By density of $C^\infty(\bar\Omega)$ in $W^{m,p}(\Omega)$, we can choose $g_1\in C^\infty(\bar\Omega)$ such that $g_1\geqq c_1>0$ and $\| f/g_1-1\|_{m,p}\leqq \varepsilon(\Omega, m, p)$ where ε is the constant in Lemma 9. We can also assume that:

$$\int_\Omega f(x)/g_1(x)\,dx=\mathrm{vol}(\Omega).$$

We then define $u_1(x)$ the solution of

$$\left.\begin{aligned}\det(\nabla u_1(x))=f/g_1 \quad \text{in } \Omega,\\ u_1(x)|_{\partial\Omega}=x.\end{aligned}\right\}\tag{5.3}$$

By Lemma 9, such a solution exists and satisfies

$$u_1,\ u_1^{-1}\in W^{m+1,p}(\Omega, R^n).$$

If in what follows, we would have a solution u_2 of the following equation

$$\left.\begin{aligned}\det(\nabla u_2(x))=f_1=g_1\circ u_1^{-1} \quad \text{in } \Omega,\\ u_2(x)|_{\partial\Omega}=x,\\ u_2, u_2^{-1}\in W^{m+1,p}(\Omega, R^n).\end{aligned}\right\}\tag{5.4}$$

$u=u_2\circ u_1$ would be a desired solution of (5.1).

But actually we do not know the existence of u_2 for (5.4). On the other hand, we observe that $\det(\nabla u_1^{-1}(x))=(g_1/f)\circ u_1^{-1}\geqq c>0$, thus

$$f_1=g_1\circ u_1^{-1}\in W^{m+1,p,+}(\Omega)$$

by Lemma 7. By the same reason we can and we will take $g_2 \in C^\infty(\bar{\Omega})$ such that $g_2 \geq c_2 > 0$, $\| f_1/g_2 - 1 \|_{m,p} \leq \varepsilon(\Omega, m+1, p)$ and $\int_\Omega f_1/g_2\, dx = \mathrm{vol}(\Omega)$. Then we have a solution $u_2(x)$ of:

$$\left.\begin{aligned} \det(\nabla u_2(x)) &= f_1/g_2 \quad \text{in } \Omega, \\ u_2(x)|_{\partial\Omega} &= x, \\ u_2, u_2^{-1} &\in W^{m+2,p}(\Omega, R^n). \end{aligned}\right\} \tag{5.5}$$

Similarly, it is now sufficient to solve the equation:

$$\left.\begin{aligned} \det(\nabla u_3(x)) = f_3 = g_2 \circ u_2^{-1} &\in W^{m+2,p,+}(\Omega) \quad \text{in } \Omega. \\ u_3(x)|_{\partial\Omega} &= x, \\ u_3, u_3^{-1} &\in W^{m+1,p}(\Omega, R^n). \end{aligned}\right\} \tag{5.6}$$

Proceeding inductively, we get some u_j, $u_j^{-1} \in W^{m+j,p,*}(\Omega, R^n)$ satisfying

$$\left.\begin{aligned} \det(\nabla u_j(x)) = f_j &\in W^{m+j-1,p,+}(\Omega) \quad \text{in } \Omega, \\ u_j(x)|_{\partial\Omega} &= x. \end{aligned}\right\} \tag{5.7}$$

and we arrive at step k where we consider the following equation:

$$\left.\begin{aligned} \det(\nabla u_k(x)) = f_k = g_{k-1} \circ u_{k-1}^{-1} &\in W^{m+k-1,p,+}(\Omega) \quad \text{in } \Omega, \\ u_k(x)|_{\partial\Omega} &= x. \end{aligned}\right\} \tag{5.8}$$

with $p > n/(k-1)$. Thus, $f_k \in C^{m,\alpha}(\bar{\Omega})$ for some α in $(0, 1)$. By Theorem DM1 stated in section 1, we have a solution $u_k(x) \in C^{m+1,\alpha}(\bar{\Omega})$.

Set $u = u_k \circ u_{k-1} \circ u_{k-2} \ldots \circ u_2 \circ u_1$. We see that $\forall j$:

$$\left.\begin{aligned} \det(\nabla u_j(x)) &= f_j \geq c > 0, \\ u_j(x) &= x \quad \text{on } \partial\Omega. \end{aligned}\right\}$$

and $u_j(x)$ is a C^1-diffeomorphism from $\bar{\Omega}$ to itself and u_j, $u_j^{-1} \in W^{m+1,p,*}(\Omega, R^n)$.

By Lemma 7, we have then $u \in W^{m+1,p}(\Omega, R^n)$, $u(x) = x$ on $\partial\Omega$ and $\det(\nabla u(x)) = f$ holds by construction. Finally, $u^{-1} \in W^{m+1,p}(\Omega, R^n)$ is a direct consequence of Proposition 1. ■

Remark. — This theorem is independent on that in [2]. For example, if $f \in H^{2m}(\Omega)$ where Ω is in R^2 and $m \geq 1$, we have $f \in C^{m-1,\alpha}(\bar{\Omega})$, $0 < \alpha < 1$. So by their result, we have only a solution in $C^{m,\alpha}(\bar{\Omega}, R^n)$.

COROLLARY 2. — *Let m, p, Ω be as in Theorem* 1, $f(x)$, $g(x) \in W^{m,p,+}(\Omega)$ *and assume that*

$$\int_\Omega f(x)\, dx = \int_\Omega g(x)\, dx.$$

Then there exists $u, u^{-1} \in W^{m+1, p}(\Omega, R^n)$ *such that*

$$g(u(x)) \det(\nabla u(x)) = f(x) \quad \text{in } \Omega,$$
$$u(x) = x \quad \text{on } \partial\Omega.$$

Proof. – First we can suppose that $\int_\Omega f(x)\,dx = \int_\Omega g(x)\,dx = \mathrm{vol}(\Omega)$. Then we solve $\det(\nabla u_1) = f$, $u_1(x) = x$ on $\partial\Omega$ and $\det(\nabla u_2) = g$, $u_1(x) = x$ on $\partial\Omega$ by Theorem 1, then $u = u_2^{-1} \circ u_1$ is the desired solution. ■

COROLLARY 3. – *Let* Ω *be as in Theorem* 1. *If* m *is a positive integer,* $f(x) \in C^{m-1, 1}(\Omega) = W^{m, \infty}(\Omega)$, $f \geqq a > 0$ *and* $\int_\Omega f(x)\,dx = \mathrm{vol}(\Omega)$. *Then* $\forall q \in [1, \infty)$, *we have* u_q, $u_q^{-1} \in W^{m+1, q}(\Omega, R^n)$ *such that* $u_q(x) = x$ *on* $\partial\Omega$ *and* $\det(\nabla u_q(x)) = f$ *in* Ω.

COROLLARY 4. – *Let* Ω *be as in Theorem* 1. *Let* $m > n$, $m \in N$, $f(x)$ *be a positive function in* $W^{m, 1, +}(\Omega)$ *and* $\int_\Omega f(x)\,dx = \mathrm{vol}(\Omega)$. *Then there exists* $u, u^{-1} \in W^{m, n/(n-1)}(\Omega, R^n)$ *such that* $\det(\nabla u(x)) = f$ *in* Ω *and* $u(x)|_{\partial\Omega} = x$.

Remark. – These are the immediate consequences of Theorem 1 and these are the limit cases for $p \in (\max(1, n/m), \infty)$.

6. NON-EXISTENCE THEOREMS

Let Ω be a bounded domain of R^n with regular boundary ($n \geqq 2$). In section 5, we prove that the problem of the type $\{W^{m, p, +}, W^{m+1, p}\}$ is well-posed when $m \geqq 1$ and $p \in (\max(1, n/m), \infty)$.

For the problem $\det(\nabla u(x)) = f$ with or without some boundary conditions, it is natural to ask if the question of the type $\{W^{m, p}, W^{m+1, np}\}$ is well-posed. We say that the problem $\{X, Y\}$ is not well-posed if there exists some $f \in X$ such that such a solution $u \in Y$ does not exist.

In [8], R. R. Coifman, P. L. Lions, Y. Meyer and S. Semmes proved that if $u \in W^{1, n}(\Omega, R^n)$, then $\det(\nabla u(x))$ will be in $\mathcal{H}^1$ the Hardy space. This shows that the problem $\{L^1, W^{1, n}\}$ is not well-posed, because $L^1 \setminus \mathcal{H}^1 \neq \emptyset$. In fact, we see that the answer is always negative for $m \geqq 1$ and $p \in [1, \infty)$.

THEOREM 4. – *The problem of the types* $\{W^{m, p}(\Omega), W^{m+1, np}(\Omega, R^n)\}$ *is not well-posed if* m *is a positive integer and* $p \in [1, \infty)$.

Proof. – If $p > 1$ or $m > 1$, by Corollary 1, $u \in W^{m+1, np}(\Omega, R^n)$ implies $\det(\nabla u(x)) \in W^{m, np}(\Omega)$. If $p = 1$ and $m = 1$, then by the Remark after Lemma 6, $u \in W^{2, n}(\Omega, R^n)$ implies $\det(\nabla u(x)) \in W^{1, q}(\Omega)$, $\forall q \in [1, n)$. ■

Remark. — More precisely, let $m \geqq 1$, $m \in \mathrm{N}$, $p \in [1, \infty)$, $p' > p$ and $p' \geqq n/m$. Then the equation of the type $\{W^{m,p}, W^{m+1,p'}\}$ is not well-posed.

In general, we have:

LEMMA 10. — *Let m be a positive integer, $p \geqq \max(1, n^2/(1+mn))$ and $\partial\Omega \in C^{m+1}$. Then there exists a unique continuous map T from $W^{m,p}(\Omega, R^n)$ to $D'(\Omega)$ such that $\forall u \in C^\infty(\bar{\Omega}, R^n)$, $T(u) = \det(\nabla u(x))$.*

Proof. — Let $u \in C^\infty(\bar{\Omega}, R^n)$ and $\varphi \in D(\Omega)$.

$$\langle T(u), \varphi \rangle = \int_\Omega \det(\nabla u(x))\, \varphi\, dx = \int_\Omega [\sum_j (\partial_j u_1)(adj \nabla u)_{1,j}]\, \varphi\, dx$$

$$= -\int_\Omega u_1 \sum_j \partial_j [(adj \nabla u)_{1,j}\, \varphi]\, dx.$$

Since $\sum_j \partial_j (adj \nabla u)_{1,j} = 0$, we have

$$\langle T(u), \varphi \rangle = -\int_\Omega u_1 \sum_j [\partial_j \varphi\, (adj \nabla u)_{1,j}]\, dx.$$

One has then $\| u_1 \sum_j (adj \nabla u)_{1,j} \|_{L^1} \leqq C(\| u \|_{m,p})^n$ by the imbedding theorem. Define $\langle T(u), \varphi \rangle = -\int_\Omega u_1 \sum_j [\partial_j \varphi\, (adj \nabla u)_{1,j}]\, dx, \forall u \in W^{m,p}(\Omega, R^n)$.

The continuity and uniqueness of T are clear. ■

DEFINITION. — We define $\det(\nabla u(x)) = T(u)$ in $W^{m,p}(\Omega, R^n)$ where $m \geqq 1$, $p \geqq \max(1, n^2/(1+mn))$ and T is determined in Lemma 10.

Remark. — In [14], S. Müller proved that if $u \in W^{m,p}(\Omega, R^n)$ with $p \geqq \max(1, n^2/(1+mn))$ and $T(u)$ lies in $L^1(\Omega)$, then $T(u) = \det(\nabla u(x))$, the classical Jacobian determinant.

PROPOSITION 3. — *Let $m \in \mathrm{N}$, $p' > p \geqq 1$, $p' \geqq n^2/(1+mn)$. Then there is no hope to find an estimate for the problem $\{W^{m,p}(\Omega), W^{m+1,p'}(\Omega, R^n)\}$. More precisely, there are no positive numbers ε and C such that:*

For any $f \in W^{m,p}(\Omega)$ with $\| f - 1 \|_{m,p} \leqq \varepsilon$, we have $u \in W^{m+1,p'}(\Omega, R^n)$ such that $\det(\nabla u(x)) = f$ and $\| u - \mathrm{id} \|_{m+1,p'} \leqq C \| f - 1 \|_{m,p}$.

Proof. — Remark first that the condition $p' \geqq n^2/(1+mn)$ comes from Lemma 10, just for well define $\det(\nabla u(x))$. Suppose that the assertion of Theorem is not true, then $\forall f \in W^{m,p}(\Omega)$, define $f_k = 1 + (f-1)/k$. Then for k sufficiently large, we have $\| f_k - 1 \|_{m,p} \leqq \varepsilon$, thus there is some u_k such that $\det(\nabla u_k(x)) = f_k$ and $\| u_k - \mathrm{id} \|_{m+1,p'} \leqq C \| f_k - 1 \|_{m,p} = C \| f - 1 \|_{m,p}/k$.

Define $v_k = k(u_k - \mathrm{id})$. Then $\| v_k - \mathrm{id} \|_{m+1,p'} \leqq C \| f - 1 \|_{m,p}$. In choosing a subsequence denoted also by v_k, we have that v_k converges to $v(x)$

weakly in $W^{m+1,p'}(\Omega, R^n)$. Hence, $u_k = \mathrm{id} + (v + w_k)/k$ where w_k converges to 0 weakly in $W^{m+1,p'}(\Omega, R^n)$. Then $\det(\nabla u_k(x)) = 1 + \operatorname{div}(v(x))/k + R_k$ where $k R_k$ converges to 0 in $D'(\Omega)$. So when k tends to ∞, we obtain that $\operatorname{div}(v(x)) = f - 1$, which implies $f \in W^{m,p'}(\Omega)$. This contradicts with $p' > p$. ■

7. SYMMETRIC CASE

Let $\Omega = B^n$ be the unit ball in R^n, $f = f(r)$ where $r = \|x\|$. We consider the axially symmetric solutions of (5.1). Set

$$S^{m,p}(\Omega) = \{ g \in W^{m,p}(\Omega),\ g = g(r) \text{ is symmetric} \},$$
$$S^{m,p,+}(\Omega) = \{ g \in S^{m,p}(\Omega), \text{ there exists } c > 0,\ \operatorname{Inf}_\Omega g(x) \geqq c \},$$
$$S^{m,p}(\Omega, R^n) = \{ v \in W^{m,p}(\Omega, R^n),\ v = h(r)\, x/r \text{ is equivariant} \}.$$

First, if $u \in S^{m,p}(\Omega, R^n)$ and $u = g(r)\, x = h(r)\, x/r$ where $h(r) = r g(r)$, by simple calculation, $\det(\nabla u(x)) = g^n(r) + r g^{n-1}(r)\, g'(r) = (h^n(r))'/(n r^{n-1})$.

On the other hand,

$$\int_\Omega f(r)\, dx = \mathrm{vol}(\Omega) \quad \text{is equivalent to} \quad \int_0^1 \mathrm{n} r^{n-1} f(r)\, dr = 1,$$

and $u(x)|_{\partial\Omega} = x$ requires $h(1) = g(1) = 1$. As $\det(\nabla u(x)) = f$ and $u(x) = x$ on $\partial\Omega$, one gets:

$$h(r) = \left(\int_0^r n s^{n-1} f(s)\, ds \right)^{1/n}.$$

THEOREM 5. – *Let m, p be as in Theorem 1 and $f \in S^{m,p,+}(\Omega)$ then there exists a unique solution $u \in S^{m+1,p}(\Omega, R^n)$ such that:*

$$\left.\begin{aligned} \det(\nabla u(x)) &= f \quad \text{in } \Omega, \\ u(x) &= x \quad \text{on } \partial\Omega. \end{aligned}\right\} \tag{7.1}$$

Sketch of proof. – We modify the operator $L(f)$ in Theorem 2 by considering:

$$\left.\begin{aligned} &\Delta g(x) = f(r) \quad \text{in } \Omega, \\ &\partial g/\partial \upsilon = 0 \quad \text{on } \partial\Omega, \\ &\int_\Omega f(x)\, dx = 0. \end{aligned}\right\} \tag{7.2}$$

Then the uniqueness of g implies that $g \in S^{m+2,p}(\Omega)$, so $v = \nabla g(x)$ satisfies that $v \in S^{m+1,p}(\Omega, R^n)$, $\operatorname{div}(v(x)) = f(r)$. And $\langle v, \upsilon \rangle = 0$ on $\partial\Omega$ is equivalent to say $v(x) = 0$ on $\partial\Omega$. Define $L(f) = v$. On the other hand, we see that $N(v)$ defined in Lemma 9 will lie in $S^{m,p}(\Omega)$. Thus, Theorem 2 and Lemma 9 work for the case $\{ S^{m,p,+}(\Omega), S^{m+1,p}(\Omega, R^n) \}$.

Moreover, since the smooth symmetric functions are dense in $S^{m,p}(\Omega)$ and the constructions in Theorem DM1 can be chosen to be symmetric, then the proof of Theorem 1 is valid. ■

THEOREM 6. — *Let* $p \geqq (n-1)$, *then the problem* $\{S^{0,p,+}(\Omega), S^{1,p}(\Omega, R^n)\}$ *is well posed. On the other hand,* $\forall p' > p$, *the problem* $\{S^{0,p,+}(\Omega), S^{1,p'}(\Omega, R^n)\}$ *is not well-posed.*

Proof. — First, if $u \in S^{m,p}(\Omega, R^n)$ and $u = g(r)x$, then

$$\|\nabla u(x)\|^2 = \sum_{i,j} |\partial_i u^j|^2 = (h'(r))^2 + (n-1) g^2(r)$$

where $h(r) = rg(r)$. It is sufficient to prove that $h'(r)$ and $g(r) \in L^p(\Omega)$ with:

$$h(r) = \left(\int_0^r ns^{n-1} f(s)\, ds\right)^{1/n}.$$

Define the maximal function $M_f(x) = \sup_{k>0}\left(\int_{x-k}^{x+k} |f|\, ds\right)/2k$ as in [16]. Set $w(s) = f(s)\, s^{(n-1)/p}$ then $w \in L^p(0, 1)$. Thus,

$$g(r) = \left(\int_0^r ns^{n-1} f(s)\, ds\right)^{1/n} / r \leqq C\, (M_w(r/2))^{1/n}\, r^{(1-n)/np}.$$

If $p > 1$, then $w \in L^p(0, 1)$ implies that $M_w(r) \in L^p(0, 1)$ and we have that

$$\int_\Omega |g(x)|^p\, dx = 2\pi \int_0^1 nr^{n-1} g^p(r)\, dr \leqq 2\pi C' \int_0^1 (M_w(r/2))^{p/n}\, dr < \infty.$$

If $p = 1$,

$$\int_\Omega |g(x)|\, dx = 2\pi \int_0^1 nr^{n-1} g(r)\, dr \leqq 2\pi \int_0^1 nr^{n-2}\, dr < \infty.$$

By $h(r) \geqq r\, \mathrm{Inf}_\Omega f(x)$, we have also

$$h'(r) = r^{n-1} f / (h(r))^{n-1} \leqq C f(r) \in L^p(\Omega).$$

Let $p' > p$, we see that $h'(r) \geqq C f(r)$ when $r \geqq 1/2$. We choose f with a singularity on $(1/2, 1)$ such that $f \notin L^{p'}(\Omega)$. Then $\|\nabla u(x)\| \notin L^{p'}(\Omega)$, as we have that $\|\nabla u(x)\| \geqq h'(r) \geqq C f$. ■

Remark. — 1. In the symmetric case, we have always the uniqueness of the solution.

2. In Theorem 6, the condition $p \geqq (n-1)$ is weaker than the condition $p \geqq n^2/(n+1)$ in Lemma 10 for defining $\det(\nabla u(x))$, because here we have *a priori* $u \in L^\infty(\Omega)$. More generally, we can consider Theorem 6 for $p \geqq 1$, because we can set $\det(\nabla u(x)) = g^n(r) + rg^{n-1}(r)\, g'(r)$ where $u = g(r)x \in S^{m,p}(\Omega, R^n)$.

3. In general, the problem $\{S^{m,p,+}(\Omega), S^{m+1,p'}(\Omega, R^n)\}$ is always not well-posed if $p'>p$ for any $m\in N$.

8. GENERAL SITUATIONS AND OPEN PROBLEMS

In this framework of research, there are a lot of interesting questions which are not solved. We collect here a list of open problems and some general discussions for the interested readers.

Let Ω be a connected open set in R^n with smooth boundary, $m\in N$ and $p\in[1, \infty)$. We consider the following equation:

$$\left.\begin{aligned} \det(\nabla u(x)) &= f \quad \text{in } \Omega, \\ u(x)|_{\partial\Omega} &= x. \end{aligned}\right\} \tag{8.1}$$

QUESTION 1 (appeared in [2]). – Let

$$f(x)\in C(\bar\Omega), \qquad f\geqq c>0 \qquad \text{and} \qquad \int_\Omega f(x)\,dx = \text{vol}(\Omega).$$

Does there exist a solution u of (8.1) such that u is a diffeomorphism from $\bar\Omega$ to itself?

QUESTION 2. – Let $f\in W^{m,\infty,+}(\Omega)$ and $m\geqq 1$. Does there exist a solution $u\in W^{m+1,\infty}(\Omega, R^n)$ for the equation (8.1)?

If the answer is negative, can we have a solution in $\bigcap_{1\leqq q<\infty} W^{m+1,q}$?

Remark. – These two questions are the limit cases of Theorem DM1 with $\alpha=0$ and $\alpha=1$.

QUESTION 3. – Is the equation of the type $\{L^{p,+}(\Omega), W^{1,np}(\Omega, R^n)\}$ well-posed for $p>1$?

QUESTION 4. – The same problem for the case $\{W^{m,p,+}, W^{m+1,p}\}$ or for the case $\{S^{m,p,+}, S^{m+1,p}\}$ with $p\leqq n/m$ and $m\geqq 1$.

Remark. – We conjecture that the problem $\{W^{m,p,+}, W^{m+1,p'}\}$ is not well-posed if $p'>p$. The reason is that the problem of type $\{S^{m,p,+}, S^{m+1,p'}\}$ are not true, and the symmetric case gives higher regularities in general.

In section 6, we proved that the problem $\{W^{m,p,+}, W^{m+1,p'}\}$ with $p'>p$ is not well-posed in the sense of estimates (clearly in Proposition 3 we can replace $W^{m,p}$ by $W^{m,p,+}$) and we ask if an estimate is always possible when $p'=p$. More precisely:

QUESTION 5. – If the problem of the type $\{W^{m,p,+}, W^{m+1,p}\}$ is well-posed, does there exist some continuous functions h, from $[0, \infty)\times(0, \infty)$

to $(0, \infty)$ such that for any f in $W^{m,p,+}(\Omega)$, there exists a solution $u(x)$ of (8.1) verifying that $\| u - \mathrm{Id} \|_{m+1,p} \leqq h(\| f-1 \|_{m,p}, \mathrm{Inf}_\Omega f)$?

Remark. – It seems impossible to answer this question from our proof, and the similar problem can be posed for the $C^{k,\alpha}$ case.

Up to now, we have considered the equation always under the hypothesis $\mathrm{Inf}_\Omega f > 0$. What will happen if f admits some zeros or $\mathrm{Inf}_\Omega f < 0$? We state here an example to indicate the complexity of such situations.

Let $\Omega = B^2$ be the unit ball in R^2 and $f(z) = 2r^2$ where $r = |z|$. Then we have $\int_\Omega f(x)\,dx = \pi = \mathrm{vol}(B^2)$. One finds that the symmetric solution is $u_s(z) = rz$, thus u_s is not in C^2!

But does there exist a C^∞ solution of (8.1)? Yes. We construct $u_1(x, y) = (x, 2x^2 y + 2y^3/3)$. We observe that u_1 is a C^∞ homeomorphism from B^2 into $u_1(B^2)$ with a single singularity at the origin, so $u_1(B^2)$ is also diffeomorphic to B^2 and $\mathrm{vol}(B^2) = \mathrm{vol}(u_1(B^2))$. Let $v(x)$ be a C^∞ diffeomorphism from $u_1(B^2)$ into B^2 such that:

$$\left.\begin{aligned} \det(\nabla v(z)) &= 1 \quad \text{in } u_1(B^2), \\ v(z) &= u_1^{-1}(z) \quad \text{on } \partial u_1(B^2). \end{aligned}\right\} \tag{8.2}$$

The existence of such a v is clear (*see* the discussion for Question 7 and 8). Thus $u = v \circ u_1$ will be the desired solution. This means that it is very difficult to work with general functions.

Question 6. – Can we obtain some general results for (8.1) only under the condition $\mathrm{Inf}_\Omega f \geqq 0$?

Now we will consider the volume preserving diffeomorphisms with given boundary data. This problem is important in the study of incompressible fluid and in the study of incompressible material in elasticity (*cf.* [15] and [12], for example). We consider the following equation for $u \in \mathrm{Diff}(\bar{\Omega})$:

$$\left.\begin{aligned} \det(\nabla u(x)) &= 1 \quad \text{in } \Omega, \\ u(x) &= \gamma \quad \text{on } \partial\Omega. \end{aligned}\right\} \tag{8.3}$$

where γ is a diffeomorphism preserving the orientation from $\partial\Omega$ into itself.

As indicated in [2], the system (8.3) admits a solution if and only if $D_\gamma \neq \varnothing$ where $D_\gamma = \{ v \in \mathrm{Diff}_+(\bar{\Omega}),\ v(x)|_{\partial\Omega} = \gamma \}$. When is D_γ non empty? We give here a sufficient condition.

Theorem 7. – *If* $\mathrm{Diff}_+(\partial\Omega)$ *is connected, then* $\forall \gamma \in \mathrm{Diff}_+(\partial\Omega)$, $D_\gamma \neq \varnothing$.

Sketch of proof. – Since $\partial\Omega$ is a compact smooth manifold without boundary, then we have a tubular neighbourhood of $\partial\Omega$. More precisely there exists a diffeomorphism ϕ from V_ε into $\partial\Omega \times [-\varepsilon, \varepsilon]$ where

$V_\varepsilon = \{ x \in R^n, d(x, \partial\Omega) \leqq \varepsilon \}$ and $\phi(V_\varepsilon \cap \Omega) = \partial\Omega \times [0, \varepsilon]$. On the other hand, since $\mathrm{Diff}_+(\partial\Omega)$ is connected, we can construct a C^∞ function $F(\delta, t)$ from $\partial\Omega \times [0, \varepsilon]$ into itself such that $F(\delta, t) \in \mathrm{Diff}_+(\partial\Omega)$, $\forall t \in [0, \varepsilon]$, $F(\delta, 0) = \gamma(\delta)$ and $F(\delta, t) = \mathrm{id}_{\partial\Omega}$ for any $t \in [\varepsilon/2, \varepsilon]$. Then we define $u(x) = \phi^{-1} \circ (F(\delta, t), t) \circ \phi(x)$ on $V_\varepsilon \cap \Omega$ and $u(x) = x$ on $\Omega \setminus V_\varepsilon$.

Remark. – If Ω lies in R^2, then $\mathrm{Diff}_+(\partial\Omega)$ is connected.

Then we ask when $\mathrm{Diff}_+(\partial\Omega)$ is connected? This becomes a very difficult topological problem. For example, we do not know a general result for $\partial\Omega = S^n$, the unit sphere of R^{n+1} (Kervaire and Milnor have proved that $\mathrm{card}[\pi_0(\mathrm{Diff}_+(S^n))]$ is always finite). One can find the values of $\mathrm{card}[\pi_0(\mathrm{Diff}_+(S^n))]$ for $5 \leqq n \leqq 17$ and some more discussions in [13].

We state here a simple counter-example: Let $T^2 \cong S^1 \times S^1$ be the standard torus in R^3, Ω = the bounded domain defined by T^2. We take $\gamma(\theta, \eta) = \gamma(\eta, \theta)$. Clearly, this defines a diffeomorphism from T^2 into itself. Then, we have:

LEMMA 11. – $\mathrm{Diff}(T^2)$ *is not connected and* D_γ *is empty.*

Sketch of proof. – γ induces a mapping on the first homology group of T^2 which exchanges the two generators, while Id_{T^2} induces the identity mapping. By using topological argument, we see that γ and Id_{T^2} are not homotopic.

Furthermore, if $D_\gamma \neq \varnothing$, by considering the image of a circle with degree = 1, we will obtain a contradiction.

QUESTION 7 (appeared in [15]). – Let

$$\mathrm{Diff}_1(\bar\Omega) = \{ u(x) \in \mathrm{Diff}(\bar\Omega),\ u(x) = x \text{ on } \partial\Omega \text{ and } \det(\nabla u(x)) = 1 \text{ in } \bar\Omega \}.$$

When is $\mathrm{Diff}_1(\bar\Omega)$ connected?

Remark. – In the general case of manifolds, it is not true for $\dim \geqq 2$. Consider the torus $T^2 \cong S^1 \times S^1$ associated with the scalar product $d\theta \otimes d\eta$. We see that $\gamma \in \mathrm{Diff}_1(T^2)$ is not homotopic to Id_{T^2}.

If we have a solution u of (8.1), then we have infinite solutions, so we would like to find a "best" solution in certain sense. Suppose that there is a solution $u \in W^{1,2}$. Naturally, we think about

$$\mathrm{Inf}\left\{ \int_\Omega \| \nabla u(x) \|^2 dx,\ u \text{ verifies } (8.1) \right\}. \tag{8.4}$$

First, if Ω is in R^2, the minimum is achieved (*cf.* [4] or Lemma 10 in § 6).

QUESTION 8. — Does the minimiser of (8.4) possess some higher regularity properties?

Some *a priori* estimations for dimension two and the volume preserving mappings are considered in [6]. We state here a result of F. Hélein:

THEOREM (appeared in [10]). — *Let* Ω *be a bounded domain in* $\mathbb{R}^n$ *with regular boundary and* $f \in W^{1,p}(\Omega)$ *satisfy* $\operatorname{Inf}_\Omega f > 0$ *such that a* $W^{2,p}$ *minimizer* $u(x)$ *of* (8.4) *exists where* $p > n$. *Then there exists* $Q \in W^{1,p}(\Omega)$ *such that u is a critical point of:*

$$J(v) = \int_\Omega \{ \| \nabla v(x) \|^2 - f\, Q(v) \}\, dx \quad \text{in } \{ v \in W^{1,2},\ v(x)|_{\partial\Omega} = x \}. \qquad (8.5)$$

Sketch of proof. — We consider $v(x)$ a smooth function with compact support in Ω. $\varphi = \mathrm{Id} + \lambda v$. Then we have that

$$\det(\nabla(u \circ \varphi)) = f + \lambda \operatorname{div}(fv) + o(\lambda)$$

and

$$\int_\Omega \| \nabla(u \circ \varphi) \|^2\, dx = \int_\Omega \| \nabla u \|^2\, dx + \lambda \int_\Omega \langle \operatorname{div}(S_{i,j}),\ v \rangle\, dx + o(\lambda)$$

where:

$$S_{i,j} = 2 \langle \partial_i u,\ \partial_j u \rangle - \| \nabla u \|^2 \delta_{ij}, \qquad \text{for} \quad i, j = 1, 2, n.$$

Thus $\operatorname{div}(fv) = 0$ implies $\displaystyle\int_\Omega \langle \operatorname{div}(S_{i,j}),\ v \rangle\, dx = \int_\Omega \langle \operatorname{div}(S_{i,j})/f,\ fv \rangle\, dx = 0.$
Then there exists $G \in W^{1,p}(\Omega)$ such that $\operatorname{div}(S_{i,j})/f = \nabla G$. Since u is a diffeomorphism from $\bar{\Omega}$ to itself, we define $Q = G \circ u^{-1}$, so we have:
$\operatorname{div}(S_{i,j}) = f\, \nabla Q(u)$, which is equivalent to that u is a critical point of (8.5).

QUESTION 9 (F. Hélein). — Let Ω and f be symmetric, does the symmetric solution of (8.1) be the minimiser of (8.4)?

QUESTION 10. — Consider the Jacobian problem without boundary data, can we obtain some higher regularity results in general?

Remark. — Even in this case, we do not know if the problem of type $\{ W^{m,p}(\Omega),\ W^{m+1,p}(\Omega, \mathbb{R}^n) \}$ is well-posed.

ACKNOWLEDGEMENTS

I would thank Professor Frédéric Hélein for suggesting me this interesting and rich subject, also for his constant encouragement and help.

REFERENCES

[1] J. MOSER, On the Volume Elements on a Manifold, *Trans. Amer. Math. Soc.*, Vol. **120**, 1965, pp. 286-294.

[2] B. DACOROGNA and J. MOSER, On a Partial Differential Equation Involving the Jacobian Determinant, *Ann. I.H.P. Analyse Non Linéaire*, Vol. **7**, 1990, pp. 1-26.

[3] R. A. ADAMS, *Sobolev Spaces*, Academic Press, 1975.

[4] J. M. BALL, Convexity Conditions and Existence Theorems in Nonlinear Elasticity, *Arch. Rational Mech. Anal.*, No. 63, 1977, pp. 337-403.

[5] J. M. BALL, Global Invertibility of Sobolev Functions and the Interpenetration of Matter, *Proceeding of the Royal Society of Edinburgh*, 88A, 1987, pp. 315-328.

[6] P. BAUMAN, N. C. OWEN and D. PHILLIPS, Maximum Principles and *a priori* Estimates for an Incompressible Material in Nonlinear Elasticity, *preprint*.

[7] J.-P. BOURGUIGNON and H. BREZIS, Remarks on the Euler Equation, *Journal of Function Analysis*, 15, 1974, pp. 341-363.

[8] R. R. COIFMAN, P. L. LIONS, Y. MEYER and S. SEMMES, Compacité par compensation et Espace de Hardy, *C. R. Acad. Sci. Paris*, 309, Série I, 1989, pp. 945-949.

[9] G. GEYMONAT and P. GRISVARD, Problèmes aux limites elliptiques dans L^p, *Publications Math. Orsay*, 1964.

[10] F. HÉLEIN, Private Communication, 1992.

[11] J. L. LIONS, Problèmes aux limites dans les équations aux dérivées partielles, *Séminaire de Math. Sup.*, Université de Montréal, 1962.

[12] P. LE TALLEC and J. T. ODEN, Existence and Characterization of Hydrostatic Pressure in Finite Deformations of Incompressible Elastic Bodies, *J. Elasticity*, No. 11, 1981, pp. 341-358.

[13] J. MILNOR, Remarks on Infinite Dimensional Lie Groups, *Relativité, Groupes et Topologie II*, Les Houches, Session XL, B. S. DEWITT and P. STORA Eds., 1983, pp. 1007-1057.

[14] S. MÜLLER, Det = det. A Remark on the Distributional Determinant, *C. R. Acad. Sci. Paris*, 311, Série I, 1990, pp. 13-17.

[15] A. I. SHNIREL'MAN, On the Geometry of the Group of Diffeomorphisms and the Dynamics of an Ideal Incompressible Fluid, *Math. USSR Sbornik*, Vol. **56**, No. 1, 1987, pp. 79-105.

[16] E. M. STEIN, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1970.

(*Manuscript received March* 2, 1993;
accepted September 30, 1993.)