On the modified Enskog equation for elastic and inelastic collisions. Models with spin

by

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ABSTRACT. — Under appropriate assumptions on the collision kernel we prove the existence of global solutions of the Enskog equation with elastic or inelastic collisions. We consider also this equation with spin, that is, the case when the angular velocities of the colliding particles are taken into account. In this case we also prove global existence results.

Key words : Modified Enskog equation, elastic collisions, inelastic collisions, spin, renormalized solutions.

RÉSUMÉ. — Nous démontrons que sous des hypothèses appropriées sur le noyau de collision il existe une solution globale de l'équation d'Enskog avec collisions élastiques ou inélastiques. Nous considérons aussi le modèle avec spin, où la vitesse angulaire des particules n'est pas négligée dans la description des collisions. Dans ce cas nous prouvons aussi des résultats d'existence de solutions globales en temps.

Dedicated to Ron DiPerna

0. INTRODUCTION

In this paper we prove the global existence of solutions for the modified Enskog equation modeling elastic or inelastic collisions. This equation is a kinetic equation involving a collision kernel Q of Boltzmann type but taking into account the delocalization of collisions due to the finite size of the colliding particles and the transformation of translation energy into internal energy during the collision. This leads us to look for the microscopic density $f(t, x, v)$, $x, v \in \mathbb{R}^3$, $t \geqq 0$, solution of the initial value problem

$$\left.\begin{array}{c} \partial_t f + v . \nabla_x f = Q(f) \\ f(0, x, v) = f_0(x, v) \end{array}\right\} \qquad (0.1)$$

where the collision kernel Q is given by the following set of notations

$$Q(f) = Q^+(f) - Q^-(f), \qquad (0.2)$$

$$Q^+(f) = a^2 \int_{\mathbb{R}^3 \times S^2} \frac{\langle \lambda, v_1 - v \rangle_\gamma}{(1 - 2\varepsilon)^2} Y(n, n_+) f(v^*) f_+(v_1^*) \, dv_1 \, d\lambda, \qquad (0.2')$$

$$Q^-(f) = a^2 \int_{\mathbb{R}^3 \times S^2} \langle \lambda, v_1 - v \rangle_\gamma Y(n, n_-) f(v) f_-(v_1) \, dv_1 \, d\lambda, \qquad (0.2'')$$

where by n, n_+ and n_- we denote

$$\left.\begin{array}{c} n(t, x) = \int_{\mathbb{R}^3} f(t, x, v) \, dv, \\ n_+ = n(t, x + a\lambda), \qquad n_- = n(t, x - a\lambda), \end{array}\right\} \qquad (0.3)$$

and $f(u)$, $f_+(u)$ and $f_-(u)$ hold for

$$\left.\begin{array}{c} f(u) = f(t, x, u), \\ f_+(u) = f(t, x + a\lambda, u), \qquad f_-(u) = f(t, x - a\lambda, u). \end{array}\right\} \qquad (0.3')$$

Finally we define the velocities $(v^*, v_1^*) = T(v, v_1)$ by the (linear) formula

$$\left.\begin{array}{c} v^* = v + \dfrac{\varepsilon\lambda}{2\varepsilon - 1} \langle \lambda, v_1 - v \rangle, \\ v_1^* = v_1 - \dfrac{\varepsilon\lambda}{2\varepsilon - 1} \langle \lambda, v_1 - v \rangle. \end{array}\right\} \qquad (0.4)$$

Notice that the operator T can be easily inverted and give rise to the formula $(v', v_1') = T^{-1}(v, v_1)$ with

$$v' = v + \varepsilon\lambda \langle \lambda, v_1 - v \rangle, \qquad v_1' = v_1 - \varepsilon\lambda \langle \lambda, v_1 - v \rangle; \qquad (0.4')$$

$a > 0$ is a parameter (related to the size of particles) and $\varepsilon \in (1/2, 1]$ is the elasticity coefficient; when $\varepsilon = 1$ the collisions are totally elastic and $v' = v^*$, $v_1' = v_1^*$.

In the section IV of this paper we will consider a model with spin which generalizes (0.1) in which the density f depends also on the spin of the particles, ω. In this case one has to introduce also the spins before and after collision, ω^*, ω', and the formulas (0.4) and (0.4') have to be modified. As we will see below the introduction of the spin changes a little bit the analysis of the problem.

In (0.2) we have used the notation $\langle \,.,.\,\rangle_\gamma$ which for a given nonnegative γ means the following

$$\langle \lambda, u \rangle_\gamma = \begin{cases} 0 & \text{if } \langle \lambda, u \rangle < \gamma, \\ \langle \lambda, u \rangle & \text{if } \langle \lambda, u \rangle \geqq \gamma, \end{cases} \qquad (0.5)$$

and $\langle \,.,.\,\rangle$ denotes the usual inner product in $\mathbb{R}^3$. Finally the local rate of collisions Y is a continuous function that we assume to satisfy

$$Y \geqq 0, \qquad Y(\theta, \sigma) = Y(\sigma, \theta) \quad \text{for all } \theta, \sigma \in \mathbb{R}^+, \qquad (0.6)$$

$Y \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^+)$ and

or
$$\sqrt{\sigma\theta}\, |Y(\theta, \sigma) - Y^\infty| \leqq C\sqrt{\log\theta\log\sigma} \quad \text{for } \theta, \quad \sigma \geqq 2, \qquad (0.6')$$

$$\sqrt{\sigma\theta}\, |Y(\theta, \sigma) - Y^\infty| \leqq C \quad \text{for } \theta, \quad \sigma \geqq 0, \quad Y^\infty \in \mathbb{R}^+. \qquad (0.6'')$$

There exist more general models for this kind of collision phenomena which define the local rate of collisions Y as a non local function of the densities n, n_+ or n_- (see [22], [23]). We can also consider such general Y's but we will not do it here to simplify the presentation of our results. On the other hand, in the case of models with spin, assumptions (0.6') or (0.6'') can be improved, as we will see in section IV below.

Our main concern here is to give the (first) mathematical treatment of the inelastic Enskog equation. We will also slightly improve the existing results for the elastic equation. Under the above assumptions we will prove the existence of global solutions of (0.1) for arbitrarily large initial data.

Taking into account the recent progress in the analysis of Boltzmann equation and in particular the renormalization method of DiPerna and Lions ([13], [15]) (that we will use mainly as a "compactness" technique) and the averaging lemmas in Golse, Lions, Perthame, Sentis [17] (initiated in [18]), the main difficulty of equation (0.1) lies on the obtention of an a priori bound on f implying the weak compactness in L^1 of any family of solutions. For the Boltzmann equation this is usually achieved through Boltzmann's H-Theorem which asserts that the total entropy

$$\int_{\mathbb{R}^6} f(t, x, v) \log f(t, x, v)\, dx\, dv$$

is nonincreasing in time. With such a bound the collision operator is not well defined in L^1, but has a sense in the renormalized fashion:

$Q(f)/(1+\delta f) \in L^1$ for every $\delta > 0$. We are going to show that this is the situation for (0.1) when $\varepsilon = 1$ under assumptions (0.6)-(0.6″) and $\gamma = 0$ (which is physically satisfactory). The same result has been proved by several authors under slightly less general assumptions. See for example the works of Arkeryd ([2], [3]), Arkeryd and Cercignani [5], Cercignani [10] and Polewczak ([22], [23]).

For inelastic collisions ($\varepsilon < 1$) we cannot give such a general result and we will assume either (0.6)-(0.6′) with $Y^\infty = \gamma = 0$ or (0.6) with $\gamma > 0$ in (0.5). In both cases it turns out that the collision operator $Q(f)$ is in L^1 and thus the solutions of (0.1) will be usual solutions in the distributional sense. Both sets of assumptions are not entirely physically satisfactory, because in the first case we assume that for high densities the gas undergoes to few particle collisions, while in the second case we neglect the grazing collisions (which are known to be singular and are generally treated separately by a Landau-Fokker-Planck operator [12] in plasma physics).

We would like also to give further references of the physical background of (0.1). Our main motivation to study inelastic collisions in (0.1) comes from astrophysical models of collisions in a planetary ring. *See* Araki, Tremaine [1], Goldreich, Tremaine [19] and Hornung, Pellat, Barge [20]). Further references for models with inelastic collisions and spin can be found in Cercignani [11]. Moreover a derivation of the Enskog equation can be found in Resibois [24].

From a mathematical viewpoint different ideas have been used so far to treat Enskog equation. Toscani and Bellomo [25] prove global existence near the vacuum and in the limit $\alpha \to 0$, Bellomo and Lachowicz ([6], [7], [8]) recover the Boltzmann equation. Concerning general initial data, renormalization is used by Arkeryd and Cercignani [4] to treat the case $\gamma = -\infty$ (*i.e.* the λ-integration is performed over the complete sphere) which is physically irrelevant. Arkeryd [3], Arkeryd and Cercignani [5] introduce new ideas which allow to treat the case $\varepsilon = 1$, $Y = Y^\infty$, $\gamma = 0$ and in this case our results will be a mere extension of those in ([3], [5]). Polewczak ([22], [23]) proves existence of renormalized solutions for $\varepsilon = 1$ either when $\gamma > 0$ in (0.5) or when Y decays rapidly to 0 at infinity (assuming a condition which is slightly less general than (0.6″)). Finally, for further general references on the Enskog model we address the reader to ([7], [23]).

The plan of this paper is as follows. In Section I we state our main results for the model without spin and prove the main auxiliary results. Section II deals with the proof of Theorem 1 while in section III we prove Theorem 2. Finally in Section IV we introduce the model with spin and give the main results for this case.

I. MAIN RESULTS

In this section we give a precise statement of our main results and we also show how the main entropy and energy estimates can be obtained. The proofs of the Theorems 1 and 2 will be given in the next two sections.

THEOREM 1 (*Elastic or inelastic collisions*). — *Let the initial data f_0 satisfy*

$$\int_{\mathbb{R}^6} f_0 (1 + |x|^2 + |v|^2 + |\log f_0|)\, dx\, dv < +\infty, \qquad (\mathrm{I}.1)$$

take $\varepsilon \in (1/2, 1]$ and assume (0.6) and either (i) *or* (ii) *with*
  (i) (0.6′) *holds with $Y^\infty = 0$ and $\gamma = 0$ in (0.5)*,
  (ii) $\gamma > 0$ *in (0.5)*.
Then, problem (0.1)-(0.2) has a solution $f \in C([0, T]; L^1(\mathbb{R}^6))$ such that for every $T \geq 0$ there exists a constant $C(T)$ with:

$$\int_{\mathbb{R}^6} f(1 + |x|^2 + |v|^2 + |\log f|)\, dx\, dv \leq C(T) \qquad for \quad t \leq T. \qquad (\mathrm{I}.2)$$

Moreover for all $T > 0$,

$$Q^+(f), Q^-(f) \in L^\infty([0, T]; L^1(\mathbb{R}^6)). \qquad (\mathrm{I}.3)$$

THEOREM 2 (*Elastic collisions*). — *Let the initial data f_0 satisfy (I.1), assume (0.6), (0.6″) and take $\gamma = 0$ in (0.5). Then the modified Enskog equation for elastic collisions i.e. (0.1)-(0.2) with $\varepsilon = 1$, has a renormalized global solution $f \in C([0, T]; L^1(\mathbb{R}^6))$ which satisfies (I.2).*

We will recall the meaning of "renormalized" solution in the proof of theorem 1 in section III.

Remarks. — (1) The results of theorem 1 (i) and theorem 2 were announced in [16]; here the condition (0.6′) is improved. Let us emphasize that this unphysical limitation on Y is the main point which should be improved. On the other hand, as we already said in the introduction, we could easily treat, following [22], kernels Y which depend nonlocally on f, but we prefer to skip this case for the purpose of clarity.

(2) Both assumptions (i) and (ii) show the necessity of truncation (in order to get an estimate for the total entropy, which in turn implies an L^1-bound for Q^+ and Q^-).

(3) (ii) was introduced by Polewczak in [23]. Our method however greatly simplifies the proof in [23]; in [23] renormalized solutions are obtained while here we obtain standard $\mathscr{D}'$-solutions.

(4) As we will show in section IV, a theorem similar to theorem 1 can be stated for the models with spin. In that case (i) will remain the same, while (ii) will be improved.

Let us now give some calculations on Q^+ and Q^- which show why (I.2) holds. Following Cercignani [12] and Truesdell and Muncaster [26], for every function $\psi(x, v)$ in $C^\infty(\mathbb{R}^6)$ we have:

$$\int_{\mathbb{R}^3} Q^+(f)\psi\,dv = a^2 \int_{\mathbb{R}^6 \times S^2} \frac{Y(n, n_+)}{(1-2\varepsilon)^2} f(v^*)f_+(v_1^*)\psi(v)$$
$$\langle \lambda, v_1 - v \rangle_\gamma\,dv\,dv_1\,d\lambda$$
$$= a^2 \int_{\mathbb{R}^6 \times S^2} \frac{Y(n, n_+)}{(2\varepsilon-1)} f(v^*)f_+(v_1^*)\psi(v)$$
$$\times \langle \lambda, v_1 - v \rangle_\gamma\,dv^*\,dv_1^*\,d\lambda,$$

because the change of variables $(v, v_1) \to (v^*, v_1^*)$ has a jacobian given by $|\det T|^{-1} = (2\varepsilon-1)$. Then we change the notations, replacing (v^*, v_1^*) by (v, v_1), $v_1 - v$ (resp. v) thus become $v_1' - v'$ (resp. v') [see (0.4')]. Noticing that $< \lambda, v_1' - v' > = (1-2\varepsilon)\langle \lambda, v_1 - v \rangle$, we also perform a change of variables $\lambda \to -\lambda$ which yields

$$\int_{\mathbb{R}^3} Q^+(f)\psi\,dv = a^2 \int_{\mathbb{R}^6 \times S^2} Y(n, n_-) f(v)f_-(v_1)\psi(v')$$
$$\times \langle \lambda, v_1 - v \rangle_\gamma\,dv\,dv_1\,d\lambda. \quad (I.4)$$

Choosing $\psi \equiv 1$ and $\psi \equiv |x|^2$ successively, we obtain, for smooth solutions,

$$\int_{\mathbb{R}^6} f(t, x, v)\,dx\,dv = \int_{\mathbb{R}^6} f_o(x, v)\,dx\,dv \quad \text{for all } t. \quad (I.5)$$

$$\frac{d}{dt}\int_{\mathbb{R}^6} |x|^2 f(t, x, v)\,dx\,dv = 2\int_{\mathbb{R}^6} \langle x, v \rangle f(t, x, v)\,dx\,dv. \quad (I.6)$$

Notice that (I.4) alone yields the conservation of the total mass, (I.5), and (I.6), which we will use below to obtain estimates for $\int |x|^2 f(t, x, v)\,dx\,dv$.

Next we will push further our computations to obtain equivalent expressions for $\int Q^+(f)\psi\,dx\,dv$ and then we analyse the integrals $\int Q^-(f)\psi\,dx\,dv$.

We proceed one step further by performing in (I.4) a change of variables $(v, v_1, \lambda) \to (v_1, v, -\lambda)$ and integrating (I.4) in x. Then we change x in $y = x + a\lambda$ and by using the symmetry of Y we obtain,

$$\int_{\mathbb{R}^6} Q^+(f)\psi\,dx\,dv = a^2 \int_{\mathbb{R}^6 \times S^2} Y(n, n_-) f(v)f_-(v_1)\psi_-(v_1')$$
$$\times \langle \lambda, v_1 - v \rangle_\gamma\,dx\,dv\,dv_1\,d\lambda. \quad (I.7)$$

Finally, by using the same kind of changes of variable with Q^- we have

$$\int_{\mathbb{R}^6} Q^-(f) \psi \, dx \, dv = a^2 \int_{\mathbb{R}^9 \times S^2} Y(n, n_-) f(v) f_-(v_1) \psi_-(v_1)$$
$$\times \langle \lambda, v_1 - v \rangle_\gamma \, dx \, dv \, dv_1 \, d\lambda. \quad (\mathrm{I}.8)$$

Putting together (0.2), $(\mathrm{I}.4)$, $(\mathrm{I}.7)$ and $(\mathrm{I}.8)$ we have proved the

PROPOSITION 3. $-$ *Let f be a smooth solution of (0.1)-(0.2) with $Y \in L^\infty(\mathbb{R}^2)$. Then for every smooth function $\psi(x, v)$ we have*

$$\int_{\mathbb{R}^6} Q(f) \psi \, dx \, dv = \frac{a^2}{2} \int_{\mathbb{R}^9 \times S^2} Y(n, n_-) \langle \lambda, v_1 - v \rangle_\gamma f(v) f_-(v_1)$$
$$\times [\psi(v') + \psi_-(v_1') - \psi(v) - \psi_-(v_1)] \, dx \, dv \, dv_1 \, d\lambda.$$

We choose now $\psi(x, v) \equiv |v|^2$. Then the equality

$$|v|^2 + |v_1|^2 + 2 \varepsilon(\varepsilon - 1) |\langle \lambda, v_1 - v \rangle|^2 = |v'|^2 + |v_1'|^2 \quad (\mathrm{I}.9)$$

gives

$$\frac{d}{dt} \int_{\mathbb{R}^6} |v|^2 f \, dx \, dv = \varepsilon(\varepsilon - 1) a^2 \int_{\mathbb{R}^9 \times S^2} Y(n, n_-)$$
$$\times \langle \lambda, v_1 - v \rangle_\gamma^3 f(v) f_-(v_1) \, dx \, dv \, dv_1 \, d\lambda, \quad (\mathrm{I}.10)$$

which shows that

$$\int_{\mathbb{R}^6} |v|^2 f \, dx \, dv \leqq \int_{\mathbb{R}^6} |v|^2 f_0 \, dx \, dv, \quad \text{for all } t, \quad (\mathrm{I}.11)$$

$$a^2 \varepsilon(1 - \varepsilon) \int_0^{+\infty} \int_{\mathbb{R}^9 \times S^2} Y(n, n_-) \langle \lambda, v_1 - v \rangle_\gamma^3$$
$$\times f(v) f_-(v_1) \, dx \, dv \, dv_1 \, d\lambda \, dt \leqq \int_{\mathbb{R}^6} f_0 |v|^2 \, dx \, dv. \quad (\mathrm{I}.12)$$

Our next choice of test function is (classically) $\psi(x, v) = \langle x, v \rangle$ which yields

$$\frac{d}{dt} \int_{\mathbb{R}^6} \langle x, v \rangle f \, dx \, dv - \int_{\mathbb{R}^6} |v|^2 f \, dx \, dv$$
$$= \frac{\varepsilon a^3}{2} \int_{\mathbb{R}^9 \times S^2} Y(n, n_-) \langle \lambda, v_1 - v \rangle_\gamma^2$$
$$\times f(v) f_-(v_1) \, dx \, dv \, dv_1 \, d\lambda \quad (\mathrm{I}.13)$$

which provides the estimate

$$\int_0^T \int_{\mathbb{R}^9 \times S^2} Y(n, n_-) \langle \lambda, v_1 - v \rangle_\gamma^2 f(v) f_-(v_1) \, dx \, dv \, dv_1 \, d\lambda \, dt \leqq C'(T). \quad (\mathrm{I}.14)$$

Indeed by using $\psi = |x - tv|^2$ in Proposition 3 and (I.11) we see that

$$\int_{\mathbb{R}^6} f |x|^2 \, dx \, dv \leq 2 t \int_{\mathbb{R}^6} f \langle x, v \rangle \, dx \, dv + \int_{\mathbb{R}^6} f_0 |x|^2 \, dx \, dv$$

and then using again (I.11) an Cauchy-Schwarz's inequality we obtain

$$\int_{\mathbb{R}^6} f |x|^2 \, dx \, dv$$

$$\leq 2 t \left(\int_{\mathbb{R}^6} f_0 |v|^2 \, dx \, dv \right)^{1/2} \left(\int_{\mathbb{R}^6} f |x|^2 \, dx \, dv \right)^{1/2} + \int_{\mathbb{R}^6} f_0 |x|^2 \, dx \, dv$$

which implies

$$\int_{\mathbb{R}^6} f |x|^2 \, dx \, dv, \qquad \int_{\mathbb{R}^6} f \langle x, v \rangle \, dx \, dv \leq C(T), \quad \forall \, 0 \leq t \leq T. \quad (I.15)$$

Estimate (I.14) is reminiscent of [23] and enables us to considerably simplify some proofs in [23].

Our final *a priori* estimate deals with the entropy and is obtained by choosing $\psi = \log f$ and using the inequality $y (\log z - \log y) \leq z - y$. Then Proposition 3 yields

$$\frac{d}{dt} \int_{\mathbb{R}^6} f \log f \, dx \, dv \leq \int_{\mathbb{R}^9 \times S^2} \frac{1}{(1 - 2\varepsilon)^2} Y(n, n_+) \langle \lambda, v_1 - v \rangle_\gamma$$

$$\times f(v) f_+ (v_1) \, dx \, dv \, dv_1 \, d\lambda - \int_{\mathbb{R}^9 \times S^2} Y(n, n_-) \langle \lambda, v_1 - v \rangle_\gamma$$

$$\times f(v) f_- (v_1) \, dx \, dv \, dv_1 \, d\lambda. \quad (I.16)$$

All these estimates will be used in the proofs of theorems 1 and 2 below. They are actually one of the main tools that we use to prove these results.

II. PROOF OF THEOREM 1

To prove theorem 1 we use some of the renormalization ideas introduced by DiPerna and Lions in [13] to obtain the global existence of solutions for the Boltzmann equation with arbitrarily large initial data. In the proof of theorem 1 we will not try to find renormalized solutions but standard solutions in the sense of distributions. However in order to get some intermediate compactness results, we will follow here a first part of the renormalization program.

II. 1. Entropy and energy bounds

LEMMA 4. − *Under the assumptions of Theorem* 1 *any smooth solution of* (0.1)-(0.2) *satisfies the estimate* (I.2).

Proof. − First we notice that formulae (I.5), (I.11) and (I.15) imply that this lemma will be proved as soon as we obtain an estimate for $\int f |\log f| \, dx \, dv$.

Moreover, as it can be seen, for instance, in [13],

$$\int_{\mathbb{R}^6} f |\log f| \, dx \, dv \leqq \int_{\mathbb{R}^6} f \log f + 2 f(|x|^2 + |v|^2) \, dx \, dv + C,$$

for some constant C independent of t and of f. Therefore we only have to prove an estimate on the entropy function $H(t) = \int f \log f \, dx \, dv$.

As in [9] we set $g(t, x, v) = f(t, x + tv, v)$ and we perform the change of variables $(\lambda, t) \to z = x + (v_1 - v) t + a\lambda$ to deduce that

$$I = \int_0^T \int_{\mathbb{R}^9 \times S^2} \langle \lambda, v_1 - v \rangle_\gamma f(v) f_+ (v_1) \, dx \, dv \, dv_1 \, d\lambda \, dt$$

$$\leqq \frac{1}{a^2} \int_{\mathbb{R}^{12}} g(t, x, v) g(t, z, v_1) \, dy \, dz \, dv \, dv_1$$

$$\leqq \frac{1}{a^2} \left(\int_{\mathbb{R}^6} \sup_{0 \leqq t \leqq T} g(t, x, v) \, dx \, dv \right)^2.$$

Hence, using (0.1), (I.4), (I.14) we find

$$I^{1/2} \leqq \frac{1}{a} \int_{\mathbb{R}^6} \sup_{0 \leqq t \leqq T} \int_0^t Q(s, x - vs, v) \, ds \, dx \, dv$$

$$\leqq \frac{1}{a} \int_0^T \int_{\mathbb{R}^6} Q^+ (t, x, v) \, dx \, dv \, dt \leqq \frac{C(T)}{a \gamma}.$$

Finally from (I.16) and the above inequality we infer that

$$\int_{\mathbb{R}^6} f \log f \, dx \, dv \leqq \frac{T C(T)^2 \|Y\|_\infty}{2 \gamma^2 (1 - 2\varepsilon)^2} + \int_{\mathbb{R}^6} f_0 \log f_0 \, dx \, dv = C'(T), \quad \forall 0 \leqq t \leqq T.$$

In the case of assumption (i) we proceed differently because now γ is equal to 0. Considering separately values of $\langle \lambda, v_1 - v \rangle_0$ larger or smaller

than 1 we find

$$\frac{1}{(1-2\,\epsilon)^2} \int_{\mathbb{R}^9 \times S^2} Y(n, n_+) \langle \lambda, v_1 - v \rangle_0 f(v) f_+(v_1)\, dx\, dv\, dv_1\, d\lambda$$

$$\leq \frac{1}{(1-2\,\epsilon)^2} \int_{\mathbb{R}^9 \times S^2} Y(n, n_+) < \lambda, v_1 - v \rangle_0^2 f(v) f_+(v_1)\, dx\, dv\, dv_1\, d\lambda$$

$$+ \frac{C}{(1-2\,\epsilon)^2} \int_{\mathbb{R}^3} (\log n \log n_+)^{1/2} (n\, n_+)^{1/2}\, dx + \frac{C}{(1-2\,\epsilon)^2} \int_{\mathbb{R}^3} (n_+ + n)\, dx$$

by (0.6′), and the r. h. s. of the above inequality is less than or equal to

$$\frac{C}{(1-2\,\epsilon)^2} \int_{\mathbb{R}^3} n \log n\, dx + g(t),$$

where $g(t)$ belongs to $L^1_{loc}(\mathbb{R}^+)$ by (I.14). Therefore from (I.16) we obtain

$$\frac{d}{dt} \int_{\mathbb{R}^6} f \log f\, dx\, dv \leq C \int_{\mathbb{R}^3} n \log n\, dx + g(t), \qquad (II.2)$$

where C is independent of f. Now we use a result in [14] which implies

$$\int_{\mathbb{R}^3} n \log n\, dx \leq C_1 \int_{\mathbb{R}^6} f \log f + C_2 \qquad (II.3)$$

where C_1 is independent of f and C_2 depends only on

$$\int f (1 + |x|^2 + |v|^2)\, dx\, dv.$$

Finally from the Gronwall lemma and (II.2)-(II.3) we find that for every $T > 0$ there exists a constant $C(T)$ such that

$$\int_{\mathbb{R}^6} f \log f\, dx\, dv \leq C(T), \quad \text{for all } t \leq T \qquad (II.4)$$

and thus Lemma 5 is proved. $\square$

Notice that the above argument immediately gives (I.3).

II.2. Truncated equation

Following ([5], [13]) we can easily build a solution f^k to the truncated equation

$$\left. \begin{array}{l} \partial_t f^k + v \cdot \nabla_x f^k = Q_k(f^k), \quad t \leq 0; \qquad x, v \in \mathbb{R}^3 \\ f^k(0, x, v) = (f_0(x, v))_k \end{array} \right\} \qquad (II.5)$$

where $(t)_k = \inf(k, t)$ for all $t \in \mathbb{R}$ and

$$Q_k = Q_k^+ - Q_k^-, \qquad (II.6)$$

$$Q_k^+(f) = a^2 \int_{\mathbb{R}^3 \times S^2} \frac{\langle \lambda, v_1 - v \rangle_\gamma}{(1 - 2\varepsilon)^2}$$
$$\times \chi_k^* Y(n, n_+) (f(v^*))_k (f_+(v_1^*))_k \, dv_1 \, d\lambda \quad (II.6')$$

$$Q_k^-(f) = a^2 \int_{\mathbb{R}^3 \times S^2} \langle \lambda, v_1 - v \rangle_\gamma Y(n, n_+) \chi_k (f(v))_k (f_-(v_1))_k \, dv_1 \, d\lambda \quad (II.6'')$$

with

$$\chi_k^* = \begin{cases} 0 & \text{if } |v_1^*| + |v^*| \geq k \\ 1 & \text{otherwise} \end{cases} \quad (II.7)$$

and χ_k is defined in a similar way by simply dropping the asterisk.

Since the kernel $Q_k(f)$ is bounded in $L^\infty([0, T]; L^p(\mathbb{R}^6))$ for every $p \in (1, +\infty)$, it is easy to find a solution f^k to (II.5). Moreover a careful examination of the changes of variables performed in section I shows that the estimate (I.2) still holds for f^k and uniformly in k. Furthermore (I.14) holds also, with $f(v), f_-(v_1)$ replaced by $(f^k(v))_k, (f_-^k(v_1))_k$

II.3. Passing to the limit

Here we will show that the sequence $\{f^k\}$ is relatively compact in some sense and that at the limit we find a solution of (0.1)-(0.2).

LEMMA 5. — *Under the assumptions of theorem 1 and extracting sub-sequences of $\{f_k\}$ that we still denote by $\{f^k\}$, we have: there exists f such that*

(a) *for all $T < +\infty$ and for all ψ such that*

$$\psi/(1 + |v|) \in L^\infty((0, T) \times \mathbb{R}_x^3 \times \mathbb{R}_v^3),$$

$$\int_{\mathbb{R}^3} f^k(t, x, v) \psi(t, x, v) \, dv \xrightarrow{k} \int_{\mathbb{R}^3} f(t, x, v) \psi(t, x, v) \, dv \quad (II.8)$$

in $L^p(0, T; L^1(\mathbb{R}^3))$ for all $p < +\infty$.

(b) *For all $\psi \in \mathscr{D}(\mathbb{R}^3)$*

$$\int_{\mathbb{R}^3} Q_k^\pm(f^k) \psi \, dv \xrightarrow{k} \int_{\mathbb{R}^3} Q^\pm(f) \psi \, dv$$

in $L^p(0, T; L^1(\mathbb{R}^3))$ for all $p < +\infty$.

(c) *$Q^\pm(f) \in L^\infty(0, T; L^1(\mathbb{R}^3 \times \mathbb{R}^3))$ and the limit f satisfies (0.1)-(0.2) in $\mathscr{D}'$.* □

We recall that the estimate in (c) is already clear from the argument of section II.1.

Proof. — The proof of (a) requires the use of some ideas related to the renormalization theory (*see* [13]). We define $g_\delta^k = \frac{1}{\delta} \log(1 + \delta f^k)$ for all k

and for all $\delta > 0$. Then from (II.5),

$$\partial_t g_\delta^k + v \cdot \nabla_x g_\delta^k = \frac{Q(f^k)}{1 + \delta f^k}, \qquad (II.9)$$

and now we fix δ and let k go to $+\infty$. It is classical that $Q_k^- (f^k)/(1 + \delta f^k)$ is weakly compact in $L^1((0, T) \times \mathbb{R}^6)$ and then, following [13], the weak compactness of $Q_k^+ (f^k)/(1 + \delta f^k)$ is obtained through a gain-loss inequality similar to that of [2]. It was extended to Enskog models (*see* [5], [22]) for $\varepsilon = 1$, and here, for $\varepsilon < 1$, the argument has again to be adapted.

Consider separately the points (t, x, v, v_1, λ) such that

$$f(v^*) f_+(v_1^*) \geqq K f(v) f_+(v_1) \qquad \text{or} \qquad f(v^*) f_+(v_1^*) \leqq K f(v) f_+(v_1)$$

for every $K > 1$; we have (dropping the k's)

$$Q^+ (f) \leqq \frac{K a^2}{(1-2\varepsilon)^2} \int_{\mathbb{R}^3 \times S^2} Y(n, n_+) \langle \lambda, v_1 - v \rangle_\gamma f(v) f_+(v_1) \, dv_1 \, d\lambda + \frac{a^2}{(1-2\varepsilon)^2 \log K}$$

$$\times \int_{\mathbb{R}^3 \times S^2} Y(n, n_+) \langle \lambda, v_1 - v \rangle_\gamma f(v^*) f_+(v_1^*) \left| \log \frac{f(v^*) f_+(v_1^*)}{f(v) f_+(v_1)} \right| dv_1 \, d\lambda. \quad (II.10)$$

Denoting A et B the two integral terms in (II.10), we obtain that $A/(1 + \delta f^k)$ is weakly compact [exactly as $Q_k^-/(1 + \delta f^k)$], while a change of variable $(v, v^*) \to (v', v)$ and $\lambda \to -\lambda$ gives

$$\int_{\mathbb{R}^3} B \, dv = (2\varepsilon - 1) \int_{\mathbb{R}^6 \times S^2} Y(n, n_-) \langle \lambda, v_1 - v \rangle_\gamma$$

$$\times f(v) f_-(v_1) \left| \log \frac{f(v) f_-(v_1)}{f(v') f_-(v_1')} \right| dv \, dv_1 \, d\lambda.$$

Since this term, appart from the absolute value, is the entropy loss term, we conclude that B is bounded in $L^1([0, T] \times \mathbb{R}^6)$. Finally, this shows that $Q_k^+ (f^k)/(1 + \delta f^k)$ is weakly compact in $L^1((0, T) \times \mathbb{R}^6)$.

The end of the proof of (*a*) uses principally the averaging lemma in [17], [18]. Since $Q_k(f^k)/(1 + \delta f^k)$ is weakly compact in L^1 and since $|v|^2 g_\delta^k$ is bounded in $L^\infty(0, T; L^1(\mathbb{R}^6))$, we obtain that [with the notations in (*a*)] $\int f_\delta^k \psi \, dv$ is compact in $L^p(0, T; L^1(\mathbb{R}^6))$ for all $p < +\infty$. [In fact we could allow any function ψ such that $\psi(1 + |v|^{1+\alpha}) \in L^\infty$ for any $\alpha > 1$.] Finally, by using the weak compactness of $\{f^k\}$ which is deduced from the entropy estimate, we see that the inequality (35) in [13] shows that $\int f^k \psi \, dv$ is compact in $L^p(0, T; L^1(\mathbb{R}^6))$ for all $p < +\infty$ and (*a*) is proved.

We prove (*b*) only for Q^-, since the same ideas applied to Q^+ and (*a*) show that

$$n^k, n_-^k \xrightarrow[k]{} n, n_- \quad \text{in } L^p(0, T; L^1(\mathbb{R}^3)).$$

Thus

$$Y(n^k, n^k_-) \underset{k}{\to} Y(n, n_-) \quad \text{a. e. in } \mathbb{R}^+ \times \mathbb{R}^3.$$

A simple extension of the averaging technique used to prove (a) shows that for every $R > 0$,

$$\int_{\mathbb{R}^6} \langle \lambda, v_1 - v \rangle_\gamma (f^k(v))_R (f^k_-(v_1))_R \psi(v) \, dv_1 \, dv$$

$$\underset{k}{\to} \int_{\mathbb{R}^6} \langle \lambda, v_1 - v \rangle_\gamma (f(v))_R (f^k_-(v_1))_R \psi(v) \, dv_1 \, dv$$

in $L^p(0. T; L^1(\mathbb{R}^3))$ and this for every $\lambda \in S^2$. Since $Y(n^k, n^k_-)$ is bounded in L^∞, we also have that for every $R > 0$,

$$\int_{\mathbb{R}^6 \times S^2} Y(n^k, n^k_-) \langle \lambda, v_1 - v \rangle_\gamma (f^k(v))_R (f^k_-(v_1))_R \psi(v) \, dv_1 \, dv \, d\lambda$$

$$\underset{k}{\to} \int_{\mathbb{R}^6 \times S^2} Y(n, n_-) \langle \lambda, v_1 - v \rangle_\gamma (f(v))_R (f_-(v_1))_R \psi(v) \, dv_1 \, dv \, d\lambda \quad \text{(II.11)}$$

in $L^p(0, T; L^1(\mathbb{R}^3_x))$. Now, since both integrals in (II.11) are nondecreasing in R, they both converge to a limit as R goes to $+\infty$, and this limit is in $L^\infty(0, T; L^1(\mathbb{R}^3_x))$. (Indeed, the L^∞ estimate is proved as in subsection II.1.) Moreover from (II.11) we obtain

$$\int_{\mathbb{R}^6 \times S^2} Y(n^k, n^k_-) \langle \lambda, v_1 - v \rangle_\gamma f^k(v) f^k_-(v_1) \, dv_1 \, dv \, d\lambda$$

$$\underset{k}{\to} \int_{\mathbb{R}^6 \times S^2} Y(n, n_-) \langle \lambda, v_1 - v \rangle_\gamma f(v) f_-(v_1) \, dv_1 \, dv \, d\lambda \quad \text{(II.12)}$$

in $L^p(0, T; L^1(\mathbb{R}^3_x))$. And with this (b) is proved. $\square$

A first consequence of (II.12) is that $Q^-_k(f^k) \underset{k}{\to} Q^-(f)$ [resp. $Q^+_k(f^k) \underset{k}{\to} Q^+(f)$] in the distributional sense, and thus equation (0.1)-(0.2) holds in $\mathscr{D}'$ dans (c) is proved.

Finally the proof of theorem 1 is completed by noting that f satisfies a transport equation with a source terme in $L^\infty((0, R); L^1(\mathbb{R}^6))$ which implies that $f \in C([0, T]; L^1(\mathbb{R}^6))$.

III. PROOF OF THEOREM 2

As pointed out above, Theorem 1 was proved by a partial use of the renormalization method. Theorem 2 gives the existence of renormalized solutions and its proof will follow exactly the proof performed in [13] and

adapted to Enskog model in [5], [22] and we do not repeat it here. Thus we only need to prove an entropy bound.

III.1. Entropy bound

LEMMA 6. − *Under the assumptions of Theorem 2, any smooth solution f of (0.1)-(0.2) satisfies*

$$\int_{\mathbb{R}^6} f \log f \, dx \, dv \leqq C_0, \qquad (III.1)$$

where C_0 depends only on f_0.

Proof. − From $(0.6'')$ we deduce that

$$Y(\theta, \sigma) = Y^\infty + \frac{Z(\theta, \sigma)}{1 + \sqrt{\theta\sigma}} \qquad (III.2)$$

where $z(.,.)$ is a bounded function. Then consider the right hand side of $(I.16)$. Following [9] we have

$$Y^\infty \int_0^T \int_{\mathbb{R}^9 \times S^2} \langle \lambda, v_1 - v \rangle_0$$
$$\times f(v)(f_+(v_1) - f_-(v_1)) \, dx \, dv \, dv_1 \, d\lambda \, dt$$
$$\leqq Y^\infty \left(\int_{\mathbb{R}^3} n(x) \, dx \right)^2 \leqq C Y^\infty, \quad \text{by } (I.5). \quad (III.3)$$

On the other hand

$$\left| \int_{\mathbb{R}^3_x \times S^2} \frac{z(n, n_+)}{1 + \sqrt{nn_+}} \int_{\mathbb{R}^3_v \times \mathbb{R}^3_{v_1}} \langle \lambda, v_1 - v \rangle_0 \, f(v) f_+(v_1) \, dv \, dv_1 \, dx \, d\lambda \right|$$
$$\leqq \|z\|_\infty \int_{\mathbb{R}^3_x \times S^2} \frac{dx \, d\lambda}{1 + \sqrt{nn_+}} \left[n_+ \left(n \int_{\mathbb{R}^3} f |v|^2 \, dv \right)^{1/2} \right.$$
$$\left. + n \left(n_+ \int_{\mathbb{R}^3} f_+ |v|^2 \, dv \right)^{1/2} \right]$$
$$\leqq C \|z\|_\infty \int_{\mathbb{R}^3_x} (n_+)^{1/2} \left(\int_{\mathbb{R}^3} f |v|^2 \, dv \right)^{1/2} dx$$
$$\leqq C' \|z\|_\infty \int_{\mathbb{R}^6} f_0 (1 + |v|^2) \, dx \, dv = C_0'$$

where C and C' are independent of everything; therefore C_0' only depends on f_0. This and $(III.3)$ prove the lemma. □

COROLLARY 7. − *Under the assumptions of Theorem* 2, *any smooth solution of* (0.1)-(0.2) *satisfies* (I.2).

Proof. − This is classically deduced from (III.1), (I.5) and (I.11) (*see* [9], [13], [26]).   □

Remarks. − 1. Lemma 6 is the equivalent of the H-theorem for (0.1)-(0.2). It seems that the first H-theorem proved for the Enskog equation is due to Résibois [24], and it was proved rigorously, under different assumptions, by Bellomo and Lachowicz [7] and by Polewczak [22] in the case $\varepsilon = 1$.

2. We could weaken assumption (0.6″) to assume only that

$$\sqrt{\theta\sigma}\,\big|\,Y(\theta,\,\sigma) - Y^{\infty}\,\big| \leq C \min(\sqrt{\log\theta},\,\sqrt{\log\sigma}) \quad \text{for } \theta,\,\sigma \geq 2.$$

Theorem 2 is also valid under this weaker assumption.

III.2. Renormalized solutions

Following [13], we say that f is a renormalized solution of (0.1)-(0.2) if

$$\frac{1}{1+f}Q^{\pm}(f) \in L^1_{\text{loc}}((0,\,+\infty) \times \mathbb{R}^3 \times \mathbb{R}^3)$$

and $g = \log(1+f)$ solves

$$\partial_t g + v.\nabla_x g = \frac{Q(f)}{1+f}.$$

It is by now well known ([4], [13]) that this definition is equivalent to three other definitions: those of mild, exponential multiplier and iterated form solutions.

In Theorem 2 we seek renormalized solutions of (0.1)-(0.2). Hence, one first uses Lemma 6 and the truncation procedure of section II. This provides us with the weak compactness which is necessary to pass the limit. And this time we will not be able to obtain distributional solutions, but renormalized solutions. Lemma 6 together with proofs which follow very closely those of [13] enable us to apply the renormalization method to our situation and prove Theorem 2. For more details on the way to apply renormalization to the Enskog equation and how lemma 6 suffices to do it our case, *see* [3], [9], [22], [23].

IV. MODELS WITH SPIN

The Enskog equation is a generalization of the Boltzmann equation which includes finite particle size. Another possible generalization consists

in taking into account the spin degrees of freedom. This is done when the spin of the particle is not negligible during the collision. Considering spin models modifies the physics of the collisions. Furthermore in this case the total energy will include not only the translational energy due to the particles linear velocities but also the spin energy due to the angular velocities

The Enskog equation for perfectly smooth spherical particles with spin (see [1], [11]) is still given by (0.1), but now $Q(f) = Q^+(f) - Q^-(f)$ is defined as follows

$$Q^+(f) = a^2 \int_{\mathbb{R}^6 \times S^2} \frac{\langle \lambda, v_1 - v \rangle_0}{(1 - 2\varepsilon)^2} H_\gamma(|W|)$$
$$\times Y(n, n_+) f(v^*, \omega^*) f_+(v_1^*, \omega_1^*) \, d\omega_1 \, dv_1 \, d\lambda \quad \text{(IV.1)}$$

$$Q^-(f) = a^2 \int_{\mathbb{R}^6 \times S^2} \langle \lambda, v_1 - v \rangle_0 H_\gamma(|W|)$$
$$\times Y(n, n_-) f(v, \omega) f_-(v_1, \omega_1) \, d\omega_1 \, dv_1 \, d\lambda \quad \text{(IV.2)}$$

where by n we denote

$$n(t, x) = \int_{\mathbb{R}^6} f(t, x, v, \omega) \, dv \, d\omega \quad \text{(IV.3)}$$

and n_+, n_- are given by (0.3). Moreover $f(u, \omega)$, $f_+(u, \omega)$, $f_-(u, \omega)$ stand for

$$f(u, \omega) = f(t, x, u, \omega), \qquad f_\pm(u, \omega) = f(t, x \pm a\lambda, u, \omega). \quad \text{(IV.4)}$$

The velocities v', v_1' and the spins ω', ω_1' are defined by

$$v' = v + \varepsilon W_n + \eta W_t, \qquad v_1' = v' - \varepsilon W_n - \eta W_t,$$
$$\omega' = \omega - \eta \mu \lambda \times W_t, \qquad \omega_1' = \omega' - \eta \mu \lambda \times W_t, \quad \text{(IV.5)}$$

where $\eta \in (0, 1)$ and $\mu > 0$ is a physical constant satisfying

$$\frac{1}{2\eta} \neq \frac{a}{2}\mu + 1 < \frac{1}{\eta}.$$

Also we have set

$$W = v_1 - v + \frac{a}{2}(\omega + \omega_1) \times \lambda,$$
$$W_n = \langle v_1 - v, \lambda \rangle \lambda, \qquad W_t = W - W_n,$$

note that W_t is orthogonal to λ. Finally, as in Section I, $(v^*, v_1^*, \omega^*, \omega_1^*)$ are obtained by inverting the operator T: $(v, v_1, \omega, \omega_1) \to (v', v_1', \omega', \omega_1')$. The restriction on μ implies that T is one to one and

$$|\det T| = (2\varepsilon - 1)\left[1 - 2\eta\left(1 + \frac{a}{2}\mu\right)\right]^2 \neq 0.$$

Finally $H_\gamma(r) = 0$ if $r < \gamma$ and $H_\gamma(r) = r$ if $r \geq \gamma$.

Note that when $\gamma > 0$, using $\langle u, \lambda \rangle_0 H_\gamma(|W|)$ to "kill" the grazing collisions is less restrictive than considering $\langle u, \lambda \rangle_\gamma$ as in sections I to III. Hence the consideration of the spin degrees of freedom as a non negligible phenomenon in the collisions enables us to weaken the technical restrictive conditions we have to assume in order to prove the existence of solutions. Of course, we would like to get rid of this restriction on the grazing collisions by always having $\langle u, \lambda \rangle_0$ alone in the collision kernel. This is still an open problem.

IV.1. Estimates

As in the case without spin (see section 1) we can obtain estimates for the total mass, translational energy and other moments by studying the value of integrals of the form $\int_{\mathbb{R}^6} Q^\pm (f) \psi \, dv \, d\omega$ for every smooth function $\psi(x, v, \omega): \mathbb{R}^9 \to \mathbb{R}$.

By performing the same kind of variable changes as in section I we can prove the following

PROPOSITION 8. — Let f be a smooth solution of (0.1)-$(IV.1)$-$(IV.2)$. Then for every function $\psi(x, v, \omega)$ in $C^\infty(\mathbb{R}^9)$ we have:

$$
\int_{\mathbb{R}^9} Q(f) \psi \, dx \, dv \, d\omega = \frac{a^2}{2} \int_{\mathbb{R}^{15} \times S^2} Y(n, n_-) \langle \lambda, v_1 - v \rangle_0 H_\gamma(|W|)
$$
$$
\times f(v, \omega) f_-(v_1, \omega_1) [\psi(v', \omega') + \psi_-(v_1', \omega_1') - \psi(v, \omega)
$$
$$
- \psi_-(v_1, \omega_1)] \, dx \, dv \, dv_1 \, d\omega \, d\omega_1 \, d\lambda. \quad (IV.6)
$$

By choosing now different functions ψ we can obtain some estimates as we did in section I. In particular, by choosing $\psi \equiv 1$ we obtain the conservation of the total mass:

$$
\int_{\mathbb{R}^9} f \, dx \, dv \, d\omega = \int_{\mathbb{R}^9} f_0 \, dx \, dv \, d\omega, \qquad \forall t. \quad (IV.7)
$$

Then we consider $\psi \equiv |x|^2$, $\psi = |x - tv|^2$ and $\psi \equiv \langle x, v \rangle$ to obtain a bound for $\int_{\mathbb{R}^9} f |x|^2 \, dx \, dv \, d\omega$ as a function of $\int_{\mathbb{R}^9} f_0 (1 + |x|^2 + |v|^2) \, dx \, dv \, d\omega$ and $(I.14)$ still holds under the form

$$
\int_0^T \int_{\mathbb{R}^{15} \times S^2} Y(n, n_-) \langle \lambda, v_1 - v \rangle_0^2 H_\gamma(|W|)
$$
$$
\times f(v, \omega) f_-(v_1, \omega_1) \, dx \, dv \, dv_1 \, d\omega \, d\omega_1 \, dt \, d\lambda \leq C(T). \quad (IV.8)
$$

We have

$$|v'|^2 + |v'_1|^2 + \frac{a}{2\mu}(|\omega'|^2 + |\omega'_1|^2) = |v|^2 + |v_1|^2 + \frac{a}{2\mu}(|\omega|^2 + |\omega_1|^2)$$

$$- 2\varepsilon(1-\varepsilon)|W_n|^2 + 2\eta\left(\left(\frac{a}{2}\mu+1\right)\eta - 1\right)|W_t|^2$$

$$\leq |v|^2 + |v_1|^2 + \frac{a}{2\mu}(|\omega|^2 + |\omega_1|^2) - \nu|W|^2$$

For some constant $\nu > 0$ depending only on ε, η, μ. By using $\psi \equiv |v|^2 + \frac{a}{2\mu}|\omega|^2$ as a test function in (IV.6), we obtain

$$\frac{d}{dt} \int_{\mathbb{R}^9} f\left(|v|^2 + \frac{a}{2\mu}|\omega|^2\right) dv\, d\omega\, dx$$

$$\leq -\nu \int_{\mathbb{R}^{15} \times S^2} Y(n, n_-) \langle \lambda, v_1 - v \rangle_0$$

$$\times H_\gamma(|W|^3) f(v, \omega) f_-(v_1, \omega_1)\, dx\, dv\, dv_1\, d\omega\, d\omega_1\, d\lambda,$$

which implies an estimate for the total energy at any time t:

$$\int_{\mathbb{R}^9}\left(|v|^2 + \frac{a}{2\mu}|\omega|^2\right) f\, dx\, dv\, d\omega \leq \int_{\mathbb{R}^9}\left(|v|^2 + \frac{a}{2\mu}|\omega|^2\right) f_0\, dx\, dv\, d\omega \quad \text{(IV.9)}$$

and at the same time we also obtain

$$\varepsilon(\varepsilon-1) \int_0^{+\infty} \int_{\mathbb{R}^{15} \times S^2} Y(n, n_-) \langle \lambda, v_1 - v \rangle_0 H_\gamma(|W|^3)$$

$$\times f(v, \omega) f_-(v_1, \omega_1)\, dx\, dv\, dv_1\, d\omega\, d\omega_1\, d\lambda\, dt$$

$$\leq \int_{\mathbb{R}^9} f_0\left(|v|^2 + \frac{a}{2\mu}|\omega|^2\right) dx\, dv\, d\omega. \quad \text{(IV.10)}$$

Our last estimate, as in the case of the Enskog equation without spin, deals with the total entropy

$$\int_{\mathbb{R}^9} f \log f\, dx\, dv\, d\omega.$$

As in section I, we consider $\psi = \log f$ in (IV.6) and we apply Proposition 8. We obtain so an inequality similar to (I.16) (but this time integrated in ω). This inequality will be sufficient to obtain an estimate for the entropy under assumptions (0.6)-(0.6″) when $\gamma = 0$, $\varepsilon = 1$; and either under assumptions (0.6)-(0.6′), $Y^\infty = \gamma = 0$ or $\gamma > 0$ in the definition of H_γ.

All the above estimates and the use of the renormalization theory should enable us to prove two results which are equivalent to Theorems 1 and 2 for the model with spin.

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