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Surfaces of constant Gauβ curvature and of arbitrary genus

by

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ABSTRACT. — We construct a class of topologically non-trivial surfaces with singularities, immersed in $\mathbb{R}^3$, with gauss curvature $K \equiv 1$.

Key words : Surfaces, constant curvature, riemannian geometry.

RÉSUMÉ. — On construit une classe de surfaces dans $\mathbb{R}^3$, présentant des singularités, topologiquement non triviales, dont la courbure gaussienne est constante et positive.

We will construct a class of surfaces $\mathscr{F}$ immersed into $\mathbb{R}^3$ with Gauss curvature $K \equiv 1$. Such a surface $\mathscr{F}$ is strictly convex near any point locally but not necessarily globally convex. The surfaces shall be topologically not trivial; therefore they will have singularities, namely a finite set of branch points $p_1, \ldots, p_k$. This assumption is reasonable since a regular manifold with handles in $\mathbb{R}^3$ cannot have positive curvature in all points. The notion of Gauss curvature of $\mathscr{F}$ is intrinsic, but it can be defined as well by means of the second fundamental form; then $K = \det \mathrm{II}/\det \mathrm{I}$. This extrinsic definition of K shall be adopted here.

For example, if $\mathscr{F}$ denotes a 2-leaf cover of $S^2 \subset \mathbb{R}^3$ with two simple branch points in the poles, the Gauss curvature of K can be defined as $K \equiv 1$ in all points of $\mathscr{F}$. But since $\mathscr{F}$ is topologically trivial and area $(\mathscr{F}) = 8\pi$, whereas area $(S^2) = 4\pi$, we conclude from the Gauss-Bonnet

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theorem that each branch point should carry a Dirac measure of curvature multiplied with (-2π). More generally $\mathscr{F}$ will denote a 2-leaf cover of $S^2 \subset \mathbb{R}^3$ with $(2g+2)$ simple branch points. Then we can understand $\mathscr{F}$ as some concrete Riemann surface, if we use a classical notion (*cf.* [1]). The singularities of $\mathscr{F}$ again are branch points, and these singularities are fairly weak.

The result not only shows an example surface, but we prove an existence theorem for solutions of a boundary value problem giving surfaces with Gauss curvature $K \equiv 1$. Namely we look at the Dirichlet problem for the equation $K \equiv 1$ in the space of sections of the normal bundle of S^2 or of $\mathscr{F}$. This equation is of Monge-Ampère type and the boundary value problem could be solved. The branch points have to be respected, where the equation $K \equiv 1$ always is singular. We did calculate the commutator of the degenerating metric on $\mathscr{F}$ with the linearization of the curvature K in order to obtain a Fredholm operator on the sections in the normal bundle of $\mathscr{F}$ using elliptic regularity.

Our approach is a local one and uses complex analysis. To see how far away from S^2 the constructed surfaces could be, one may need sub- or supersolutions. In a subsequent paper we will prove a similar result for the equation $K \equiv 1$ and surfaces $\mathscr{F}$ of dimension 3, with $H^2(\mathscr{F}, \mathbb{Z})$ being quite nontrivial. In that case the commutator has less regularity [5].

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1. MULTIPLIERS IN A HÖLDER SPACE

If the metric of a manifold degenerates, where we want to control its curvature, we need a continuity theorem for unbounded multipliers.

LEMMA 1. — *Let* $I := [0, 1] \subset \mathbb{R}^1$ *and* $0 < \alpha < 1$.
If we denote

$$C_{[1]}^{1+\alpha}(I) := \{ f \in C^{1+\alpha}(I, \mathbb{R}) \,|\, f(0) = f'(0) = 0 \}$$

and define $T(f)(t) := \frac{1}{t} f(t)$, *then* $T : C_{[1]}^{1+\alpha}(I) \to C^{\alpha}(I)$ *is bounded.*

Proof. — We have $f(t) = \int_0^t f'(\tau)\, d\tau$ and $|f'(\tau)| \leq M \,.\, \tau^{\alpha}$, since f' is Hölder continuous. Then if $0 < s < t < 1$, we have

$$\left| \frac{f(t)}{t} - \frac{f(s)}{s} \right| \leq N \,|t - s|^{\alpha},$$

since

$$\left|\frac{f(t)}{t}-\frac{f(s)}{s}\right| \leqq \frac{1}{t}\int_s^t |f'(\tau)|\,d\tau + \left|\frac{1}{t}-\frac{1}{s}\right| \int_0^s |f'(\tau)|\,d\tau$$
$$\leqq \frac{1}{t}\mathrm{M}\,(t^{\alpha+1}-s^{\alpha+1})\frac{1}{\alpha+1} + \left|\frac{1}{t}-\frac{1}{s}\right| \mathrm{M}\,s^{\alpha+1}\frac{1}{\alpha+1}$$
$$\leqq \frac{\mathrm{M}}{\alpha+1}\left[\frac{t^{\alpha+1}-s^{\alpha+1}}{|t-s|^{\alpha}\,t} + \frac{1}{st}|t-s|^{1-\alpha}s^{\alpha+1}\right].\,|t-s|^{\alpha}.$$

The function of s and t in the bracket is homogeneous and does not change if we replace (s, t) by $(\lambda s, \lambda t)$ for $\lambda > 0$. So we can assume $0 < s < t = 1$.

The function $g(s) := \dfrac{1-s^{\alpha+1}}{(1-s)^{\alpha}} + s^{\alpha+1}(1-s)^{1-\alpha}$ is continuous for $0 < s < 1$ as well as for $s \to 0$ and for $s \to 1$. It has a maximum n_*. Now $\mathrm{N} := \dfrac{n_*}{\alpha+1}.\,\mathrm{M}$ gives the lemma 1.

If $\mathrm{U} \subset \mathbb{R}^{\mathrm{N}}$ is a ball around $0 \in \mathbb{R}^{\mathrm{N}}$, and if for smooth functions $f: \mathrm{U} \to \mathbb{R}$ we define a vector

$$j_k(f) := (f(0), \nabla f(0), \ldots, \nabla^k f(0)),$$

denoting the k-jet of f at zero, then we denote

$$\mathrm{C}^m_{[k]}(\mathrm{U}) := \{f \in \mathrm{C}^m(\mathrm{U}) \,|\, j_k(f) = 0\} \qquad \text{for} \quad m \geqq k.$$

LEMMA 2. – *For* $\mathrm{N} = 2$ *the mapping* $\mathrm{T}: \mathrm{C}^{1+\alpha}_{[1]}(\bar{\mathrm{U}}) \to \mathrm{C}^{\alpha}(\bar{\mathrm{U}})$, *defined by* $(\mathrm{T}f)(u) := \dfrac{1}{|u|}.\,f(u)$ *for any* $u \in \mathrm{U}$, *is well defined and continuous.*

Proof. – We will imitate the following technique obtaining Lipschitz bounds for functions h in 2 variables.

If

$$|h(x_1, x_2) - h(y_1, x_2)| \leqq \mathrm{N}\,|x_1 - y_1|$$

and

$$|h(y_1, x_2) - h(y_1, y_2)| \leqq \mathrm{N}\,|x_2 - y_2|$$

hold for any (x_1, x_2), (y_1, y_2) in a rectangle in $\mathbb{R}^2$, then these two estimates imply a Lipschitz bound for h by the triangle inequality.

The same holds in polar coordinates too. Here h is Lipschitz on $\mathrm{U} \subset \mathbb{R}^2$, if

$$|h(r_0, \varphi_0) - h(r_1, \varphi_0)| \leqq \mathrm{N}\,|r_0 - r_1|$$

and

$$|h(r_1, \varphi_0) - h(r_1, \varphi_1)| \leqq \mathrm{N}.\,r_1\,|\varphi_0 - \varphi_1|$$

holds for all points (z_0, φ_0) and (r_1, φ_1) in a disk around zero.

This argument shall be used for Hölder norms. We may fix a unit vector $\hat{\underline{t}}$ in $\mathbb{R}^2$.

Then for $\underline{s}=\sigma.\hat{\underline{t}}$ and $\underline{t}=\tau.\hat{\underline{t}}$, σ, $\tau\in\mathbb{R}$, we use lemma 1 obtaining $\left|\frac{f(\underline{t})}{\tau}-\frac{f(\underline{s})}{\sigma}\right|\leqq N\,|\underline{t}-\underline{s}|^\alpha$, if $f\in C^{1+\alpha}_{[1]}(\bar{U})$.

We have to estimate f along circles around 0. In this case $|\underline{s}|=|\underline{t}|=r$, and the size of the angle between $\underline{s}$ and $\underline{t}$ is equivalent to the distance between $\underline{s}$ and $\underline{t}$. Therefore

$$\left|\frac{f(\underline{t})}{r}-\frac{f(\underline{s})}{r}\right|\leqq\frac{1}{r}.\,r^\alpha.\,M_1\,|\underline{s}-\underline{t}|=\left(\frac{|\underline{s}-\underline{t}|}{|\underline{s}|}\right)^{1-\alpha}.\,|\underline{s}-\underline{t}|^\alpha.\,M_1.$$

Since $|\underline{s}-\underline{t}|\leqq 2\,|\underline{s}|$, $\left|\frac{f(\underline{t})}{r}-\frac{f(\underline{s})}{r}\right|\leqq M.\,|s-t|^\alpha$ is also true.

Since we know in $\mathbb{R}^2\mathrel{\hat{=}}\mathbb{C}$ that for $\underline{t}\mathrel{\hat{=}}\tau e^{i\varphi}$ and $\underline{s}\mathrel{\hat{=}}\sigma e^{i\psi}$, the distance $|\underline{t}-\underline{s}|$ is equivalent to $|\tau-\sigma|+\left|\frac{(\tau+\sigma)}{2}\right|\,|e^{i\psi}-e^{i\varphi}|$, we can add up the inequalities above to prove lemma 2 in a way which is analogous to that for Lipschitz bounds given above.

Remark. — It is not difficult to generalize from $N=2$ to arbitrary N.

THEOREM 1. — *If* $\bar{U}$ *is a ball around zero in* $\mathbb{R}^2$ *and* $Tf(u):=\frac{1}{|u|}f(u)$ *and* $T^2f(u):=\frac{1}{|u|^2}f(u)$, *then the mappings* $T:C^{1+\alpha}_{[1]}(\bar{U})\to C^\alpha(\bar{U})$ *and* $T^2:C^{2+\alpha}_{[2]}(\bar{U})\to C^\alpha(\bar{U})$ *are both continuous.*

Proof. — Lemma 2 implies the first estimate. Therefore we only have to check that T maps also $C^{2+\alpha}_{[2]}(\bar{U})\to C^{1+\alpha}_{[1]}(\bar{U})$ which can easily be done.

The theorem is now useful for estimates of elliptic equations with degenerate coefficients.

2. GAUSS CURVATURE OF SECTIONS IN THE NORMAL BUNDLE ON A SPHERE

We assume $U:=\{(u_1, u_2)\in\mathbb{R}^2\,|\,0<|u|<1\}$. S^2 denotes the 2-sphere in $\mathbb{R}^3$ and $\underline{x}:U\to S^2$ is a regular mapping. On the set $V=\underline{x}(U)$, we have a metric $e_{ij}(u)\,du^i\,du^j=\underline{x}_i(u)\cdot\underline{x}_j(u)\,du^i\,du^j$. With any smooth function $p:U\to\mathbb{R}^+$, we get a section in the normal bundle N on V, namely $\underline{Z}(u):=p(u).\underline{x}(u)$, and we may define a surface $\mathscr{X}=\mathscr{X}(p)$ in $\mathbb{R}^3$.

LEMMA 1. — *The first fundamental form of* $\mathscr{X}$ *is*

$$\mathrm{I}(\mathscr{X})=\mathrm{I}(p):=g_{ij}(u)\,du^i\,du^j$$

with

$$g_{ij}=(p_i p_j+p^2 e_{ij})$$

and

$$\det \mathrm{I}(p)=(p^2+|\nabla p|^2)\,p^2 \det(e_{ij}).$$

If $p_0\equiv 1$, then $\det \mathrm{I}(p_0)=\det(e_{ij})$.

We follow the notation and formulas of [3].

LEMMA 2. — *The second fundamental form of* $\mathscr{X}$ *is*

$$\mathrm{II}(\mathscr{X})=\mathrm{II}(p):=b_{ij}(u)\,du^i\,du^i$$

with

$$b_{ij}(u)=(p^2+|\nabla p|^2)^{-1/2}(-p\,\nabla_{ij}p+p^2 e_{ij}+2p_i p_j)$$

and

$$\det \mathrm{II}(p)=(p^2+|\nabla p|^2)^{-1}\det(-p\,\nabla_{ij}p+p^2 e_{ij}+2p_i p_j).$$

If $p_0\equiv 1$, then $\det \mathrm{II}(p_0)=\det(e_{ij})$.

Remark. — The Gauss curvature of $\mathscr{X}$ is here

$$\mathrm{K}(\mathscr{X})=\mathrm{K}(p)=\frac{\det \mathrm{II}(p)}{\det \mathrm{I}(p)}=p^{-2}(p^2+|\nabla p|^2)^{-2}\frac{1}{\det(e_{ij})} \times \det(-p\,\nabla_{ij}p+p^2 e_{ij}+2p_i p_j).$$

Here we must assume, that $\underline{x}(\mathrm{U})=\mathrm{V}\subset \mathrm{S}^2$ has a conformal structure, and that the metric $((e_{ij}))$ has a singularity at $u=0$. We assume $((e_{ij}(u)))=\begin{pmatrix}\mathrm{E}(u) & 0\\ 0 & \mathrm{E}(u)\end{pmatrix}$ and $\mathrm{E}(u)=|u|^2.\eta(u)$ where η is smooth and $\eta(0)=1$ and η does not vanish. Such a map $\underline{x}$ can easily be constructed as the inverse image under stereographic projection from $\mathbb{R}^2$ into S^2 starting with the mapping $(u_1, u_2)\to(u_1^2-u_2^2,\, 2u_1 u_2)$ of $\mathbb{R}^2$ onto itself.

Then the surface $\mathscr{X}_0=\mathscr{X}(p_0\equiv 1)$ is a double cover of S^2 around the pole $p\in\mathrm{S}$. It is obvious that $\mathscr{X}_0$ at the pole p has also Gauss curvature $\mathrm{K}\equiv 1$. We will, therefore, restrict ourselves to perturbations of $p_0\equiv 1$ such that $p=1+q$, $\|q\|_{C^{2+\alpha}}$ is small enough and $q=\mathrm{E}.\rho$. Then $\mathrm{K}(p)=\mathrm{K}(\mathscr{X})$ has value 1 at $u=(0, 0)$. First we should look at the leading term of the Gauss curvature, defining

$$\mathrm{M}(\rho):=\frac{1}{\mathrm{E}^2}\det(-p\,\nabla_{ij}(\mathrm{E}\,\rho)+p^2\,\mathrm{E}\,\delta_{ij}+2(\mathrm{E}\,\rho)_i(\mathrm{E}\,\rho)_j).$$

THEOREM 2. — *If* $\rho\in C^{2+\alpha}_{[2]}(\bar{\mathrm{U}})$, *then* $\mathrm{M}(\rho)\in C^{\alpha}_{[0]}(\bar{\mathrm{U}})$.

Proof. — We simply write the determinant as a $\wedge$-vector product.

$$\mathrm{M}(\rho)=\frac{1}{\mathrm{E}^2}[p^4\,\mathrm{E}^2\,\delta_{1j}\wedge\delta_{2j}+p^2\,\nabla_{1j}(\mathrm{E}\,\rho)\wedge\nabla_{2j}(\mathrm{E}\,\rho)$$
$$+2p^2\,\mathrm{E}\,|\underline{\nabla}(\mathrm{E}\,\rho)|^2-p^3\,\mathrm{E}\Delta(\mathrm{E}\,\rho)$$
$$-p\,\nabla_{1j}(\mathrm{E}\,\rho)\wedge 2(\mathrm{E}\,\rho)_2(\mathrm{E}\,\rho)_1+p\,\nabla_{2j}(\mathrm{E}\,\rho)\wedge 2(\mathrm{E}\,\rho)_1(\mathrm{E}\,\rho)_j].$$

For simplicity we may first assume that $\mathrm{E}(u)=|u|^2$.
Then we get the commutators

$$(\mathrm{E}\,\rho)_j=\mathrm{E}_j\,\rho+\mathrm{E}\,\rho_j,$$
$$|\nabla(\mathrm{E}\,\rho)|^2\leqq\mathrm{E}^2\,|\nabla\rho|^2+|\nabla\mathrm{E}|^2.\rho^2+2\,\mathrm{E}\,|\nabla\mathrm{E}|\,\rho.|\nabla\rho|,$$

and

$$\nabla_{ij}(\mathrm{E}\,\rho)=\mathrm{E}\,\nabla_{ij}(\rho)+\rho\,\nabla_{ij}(\mathrm{E})+v_{ij}^{kl}\,\mathrm{E}_k\,\rho_l.$$

The functions v_{ij}^{kl} are smooth and bounded and $|\nabla\mathrm{E}|\leqq$ const. $|\mathrm{E}|^{1/2}$.
In summation, we conclude from theorem 1, that

$$[\![\mathrm{E}^{-1}\,\nabla_{ij}(\mathrm{E}\,\rho)]\!]_{\mathrm{C}^\alpha}\leqq\text{const. }[\![\rho]\!]_{\mathrm{C}^{2+\alpha}},$$
$$[\![\mathrm{E}^{-1}\,|\underline{\nabla}(\mathrm{E}\,\rho)|^2]\!]_{\mathrm{C}^\alpha}\leqq\text{const. }[\![\rho]\!]^2_{\mathrm{C}^{2+\alpha}}$$

and

$$[\![\mathrm{E}^{-1}\,(\mathrm{E}\,\rho)_i\,(\mathrm{E}\,\rho)_j]\!]_{\mathrm{C}^\alpha}\leqq\text{const. }[\![\rho]\!]^2_{\mathrm{C}^{2+\alpha}},$$

if $\rho\in\mathrm{C}^{2+\alpha}_{[2]}(\bar{\mathrm{U}})$.

After adding all terms up, assuming $[\![\rho]\!]_{\mathrm{C}^{2+\alpha}}\leqq\frac{1}{2}$, we obtain the estimate stated in the theorem.

Corollary 1. — *If we split* $\mathrm{M}(\rho)$ *into three terms, all three have to be homogeneous in* ρ, *of order* 0, 1 *or* 2, *we get:*

$$\mathrm{M}(\rho)=:p^4\,\delta_{1j}\wedge\delta_{2j}-\frac{p^3}{\mathrm{E}}\Delta(\mathrm{E}\,\rho)+\mathrm{M}_1(\rho);$$

then we have the estimate

$$[\![\mathrm{M}_1(\rho)]\!]_{\mathrm{C}^\alpha(\bar{\mathrm{U}})}\leqq\text{const. }[\![\rho]\!]_{\mathrm{C}^{2+\alpha}(\bar{\mathrm{U}})},$$

if $\rho\in\mathrm{C}^{2+\alpha}_{[2]}(\bar{\mathrm{U}})$.

Corollary 2. — *If* $\rho\in\mathrm{C}^{2+\alpha}_{[2]}$ *and* $p=1+\mathrm{E}\,\rho$, *then the Gauss curvature* $K(p)$ *of the surface* $p(u).\underline{x}(u)$ *is*

$$\mathrm{K}(p)=p^{-2}(p^2+|\nabla p|^2)^{-2}\,\mathrm{M}(\rho)\in\mathrm{C}^\alpha(\bar{\mathrm{U}}).$$

Theorem 3. — *The mapping* M *is differentiable on the ball*

$$\mathrm{B}:=\left\{\rho\in\mathrm{C}^{2+\alpha}_{[2]}(\bar{\mathrm{U}})\,\middle|\,[\![\rho]\!]_{\mathrm{C}^{2+\alpha}}\leqq\frac{1}{2}\right\}.$$

Proof. – The inequality $[\![\rho]\!]_{C^{2+\alpha}} \leqq \frac{1}{2}$ excludes the possibility that the denominator vanishes for $u \neq 0$. The multiplier $\kappa(p) = p^{-2}(p^2 + |\nabla p|^2)^{-2}$ acts differentiably on B, and its derivative is

$$\kappa_{*\,|\,1+E\rho}[E\sigma] = -2p^{-3}(p^2+|\nabla p|^2)^{-2}(E\sigma) \\ -2(p^2+|\nabla p|^2)^{-3}p^{-2}(2p(E\sigma)+2\underline{\nabla} p.\underline{\nabla}(E\sigma)).$$

Analogously, we derive from $p = 1 + E\rho$ the formula

$$M_{1,*\,|\,1+E\rho}[E\sigma] = \frac{1}{E^2}[p^2\nabla_{ij}(E\rho) \wedge \nabla_{2j}(E\sigma) \\ + p^2\nabla_{1j}(E\sigma) \wedge \nabla_{2j}(E\rho) + 4p^2 E\underline{\nabla}(E\rho).\underline{\nabla}(E\sigma) \\ -p\nabla_{1j}(E\sigma) \wedge 2(E\rho)_2(E\rho)_j \\ -p\nabla_{1j}(E\rho) \wedge 2(E\sigma)_2(E\rho)_j + (E\rho)_2(E\sigma)_j) \\ +p\nabla_{2j}(E\sigma) \wedge 2(E\rho)_1(E\rho)_j \\ +p\nabla_{2j}(E\rho) \wedge 2(E\sigma)_1(E\rho)_j + (E\rho)_1(E\sigma)_j) - \psi(\rho, D\rho, D^2\rho).E\sigma].$$

Here ψ collects all terms of the function $M_{1,*\,|\,1+E\rho}[E\sigma]$, where p, but not its derivatives, are linearized along $(E\sigma)$. But the continuity of the linear operator $M_{1,*:E}.C^{2+\alpha}_{[2]} \to C^\alpha$ can be proved, if each multiplier on $C^\alpha(\bar{U})$ has a uniform estimate, and if the effect of the denominator E^2 can be cancelled out against $E.C^{2+\alpha}_{[2]}(\bar{U})$. The argument of theorem 2 applies, and here too the terms of M are bounded just as we proved above.

COROLLARY. – *The mapping* $K(1+E\rho) = K_1(\rho)$ *is differentiable on the ball* B.

Proof:

$$K_1(\rho) = \kappa(1+E\rho).M(1+E\rho)$$

and

$$K_{1,*\,|\,\rho}[\sigma] = \kappa_{*\,|\,(1+E\rho)}[E\sigma].M(1+E\rho) + \kappa(1+E\rho).M_{*\,|\,(1+E\rho)}[E\sigma]$$

and by construction

$$M_{*\,|\,(1+E\rho)}[E\sigma] \\ = -p^3 E^{-1}\Delta(E\sigma) + 3p^2(E\sigma)\Delta(E\rho).E^{-1} + M_{1,*\,|\,(1+E\rho)}[E\sigma].$$

Now we use the same estimates as before and apply theorem 1.

We check $K_{1*\,|\,\rho=0} = -4E\sigma - E^{-1}\Delta E\sigma$ for all $\sigma \in C^{2+\alpha}_{[2]}(\bar{U})$.

We define $\mathbb{K} := E^{-1}\Delta E$ and it will denote the leading term of $K_{1,*}$.

This is a mapping $\mathbb{K} : C^{2+\alpha}_{[2]}(\bar{U}) \to C^\alpha(\bar{U})$.

If we restrict the source space, defining

$$\mathring{C}^{2+\alpha}_{[2]}(\bar{U}) := \{\varphi \in C^{2+\alpha}_{[2]}(\bar{U}) \,|\, \varphi \text{ vanishes on } \partial U\},$$

then $\mathbb{K}$ is continuous and injective onto its range $W \subset C^{\alpha}(\bar{U})$. It has a formal inverse $\mathbb{L} := E^{-1} \Delta E$, and $\mathbb{L}$ maps W again onto $C^{2+\alpha}_{[2]}(\bar{U})$.

THEOREM 4. — *On the space* $\mathring{C}^{2+\alpha}(\bar{U})$, *a linear projection* Π *exists with finite corank, such that* $\Pi \circ \mathbb{L} : C^{\alpha}(\bar{U}) \to \mathring{C}^{2+\alpha}(\bar{U})$ *is bounded.*

Proof. — If we assume $E = E(u) = |u|^2$, it is particularly easy to see that the commutator $[E, \Delta] = E\Delta - \Delta E$ is a compact operator from $C^{2+\alpha}_{[2]}(\bar{U})$ into $C^{\alpha}(\bar{U})$, namely $[\Delta, E]\varphi = (2\underline{u}.\underline{\nabla} + 2)\varphi$. This implies

$$\Delta^{-1}[\Delta, E]\Delta^{-1} = E\Delta^{-1} - \Delta^{-1}E = [E, \Delta^{-1}] = \Delta^{-1}(2\underline{u}.\underline{\nabla} + 2)\Delta^{-1},$$

mapping $C^{\alpha}(\bar{U})$ into $\mathring{C}^{2+\alpha}(\bar{U})$, and $[E, \Delta^{-1}]$ is again a compact operator. But now we can apply theorem 1 to see that $\varphi \to E^{-1}[E, \Delta^{-1}]\varphi$ is bounded for any $\varphi \in C^{\alpha}(\bar{U})$ if $j_{[2]}(\Delta^{-2}\varphi) = 0$ and $j_{[1]}(\nabla\Delta^{-2}\varphi) = 0$. This second condition, however, follows from the first. Quite formally we get

$$\begin{aligned}\mathbb{L} = E^{-1}\Delta^{-1}E = E^{-1}E\Delta^{-1} - E^{-1}[E, \Delta^{-1}] \\ = \Delta^{-1} - E^{-1}\Delta^{-1}(2\underline{u}.\underline{\nabla} + 2)\Delta^{-1} \\ = \Delta^{-1} - 4E^{-1}\Delta^{-2} - 2E^{-1}\underline{u}.\underline{\Delta}\Delta^{-2}.\end{aligned}$$

If we now choose a projector P, which projects simultaneously $\mathring{C}^{2+\alpha}(\bar{U})$ onto $\mathring{C}^{2+\alpha}_{[2]}(\bar{U})$, and $\mathring{C}^{4+\alpha}(\bar{U})$ onto $\mathring{C}^{2+\alpha}_{[2]}(\bar{U})$, then $L_1 := E^{-1}P\Delta^{-2}$ and $L_2 := E^{-1}\underline{u}.\underline{\nabla}P\Delta^{-2}$ are bounded on $C^{\alpha}(\bar{U})$. If $P^{\perp}$ denotes the complement projector $P^{\perp} := 1 - P$, then, for any $\varphi \in C^{\alpha}(\bar{U})$, the functions $E^{-1}P^{\perp}\Delta^{-2}\varphi$ and $E^{-1}(u.\nabla)P^{\perp}\Delta^{-2}\varphi$ are meromorphic in a certain finite dimensional space, but this does not exclude that their sum could even be regular. Thus, we need a projection Π of finite corank such that $\Pi P = \Pi$; and $\Pi E^{-1}P^{\perp}\Delta^{-2}\varphi$ and $\Pi E^{-1}(\underline{u}.\underline{\nabla})P^{\perp}\Delta^{-2}\varphi$ vanish for all $\varphi \in C^{\alpha}(\bar{U})$. Then

$$\begin{aligned}\Pi\mathbb{L} = \Pi\Delta^{-1} + 4\Pi E^{-1}P\Delta^{-2} + 4\Pi E^{-1}P^{\perp}\Delta^{-2} \\ + 2\Pi E^{-1}(\underline{u}.\underline{\nabla})P\Delta^{-2} + 2\Pi E^{-1}(\underline{u}.\underline{\nabla})P^{\perp}\Delta^{-2}\end{aligned}$$

is bounded from $C^{\alpha}(\bar{U})$ into $\mathring{C}^{2+\alpha}(\bar{U})$, since all possible unbounded terms equal 0. If we consider more generally $E(u) = |u|^2 . \eta(u)$, $\eta(0) = 1$, η smooth, we have a similar formula too for $[\Delta, E]$, and the argument remains unchanged.

COROLLARY 1. — *The mapping* $\Pi \circ \mathbb{L} \circ K_1(\rho)$ *is differentiable on the ball* B. *Always* $K_1(\rho) = K(1 + E\rho)$ *always equals* $K(1 + E\rho)$.

Proof. — Clearly the mapping $\rho \to K(1 + E\rho)$ from B into $C^{\alpha}(\bar{U})$ is bounded and differentiable with uniformly bounded derivatives. By definition then $\Pi \circ \mathbb{L} : C^{\alpha}(\bar{U}) \to \mathring{C}^{2+\alpha}(\bar{U})$ is linear and bounded. This implies the statement.

COROLLARY 2. — *The mapping* $\mathbb{K} := -E^{-1}\Delta E : \mathring{C}^{2+\alpha}_{[2]}(\bar{U}) \to C^{\alpha}_{[0]}(\bar{U})$ *is injective and a Fredholm operator.*

Proof. — That $\mathbb{K}$ is bounded, follows from theorem 2. It is injective and has a formal inverse

$$\mathbb{K}^{-1}=\mathscr{L}=\mathrm{E}^{-1}\Delta^{-1}\mathrm{E}=\Delta^{-1}+\mathrm{E}^{-1}[\Delta^{-1}, \mathrm{E}],$$

which is bounded if we restrict $\mathbb{L}$ using a projection of finite corank. Therefore $\mathbb{K}$ must be a Fredholm operator.

LEMMA 3. — *The mapping* $\mathrm{K}=\mathrm{K}(p)=\dfrac{\det \mathrm{II}(p)}{\det \mathrm{I}(p)}$ *satisfies at* $p_0\equiv 1$ *the equations* $\mathrm{K}(p_0)\equiv 1$ *and* $\mathrm{K}_{*\,|p_0}[\mathrm{E}\,\sigma]=-4\,\mathrm{E}\,\sigma-\mathbb{K}\,\sigma$ *for any* $\sigma\in \mathrm{C}^{2+\alpha}_{[2]}(\bar{\mathrm{U}})$.

Proof. — The computations are easy because the terms of $\mathrm{M}_1(\rho)$ need not be considered since they are quadratic in ρ just around p_0.

LEMMA 4. — *A constant projection* $\mathrm{Q}:\mathrm{C}^{\alpha}(\bar{\mathrm{U}})\to \mathrm{C}^{\alpha}(\bar{\mathrm{U}})$ *of finite corank exists, such that* $\mathrm{Q}\circ \mathrm{K}_{*\,|p}[\mathrm{E}.]$ *is a surjective Fredholm operator from* $\mathring{\mathrm{C}}^{2+\alpha}_{[2]}(\bar{\mathrm{U}})$ *onto* $\mathrm{QC}^{\alpha}_{[0]}(\bar{\mathrm{U}})$ *for any* $p=1+\mathrm{E}\,\rho$, *if* $\rho\in \mathrm{C}^{2+\alpha}_{[2]}(\bar{\mathrm{U}})$ *is sufficiently small.*

Proof. — This is clear since the space of Fredholm operators is open in the ring of bounded operators, and $\mathrm{K}_{*\,|p_0}[\mathrm{E}.]$ is Fredholm operator indeed.

Handling the corank of Q, we proceed as follows. We increase the space $\mathring{\mathrm{C}}^{2+\alpha}_{[2]}(\bar{\mathrm{U}})$ using a finite dimensional subspace in $\mathrm{C}^{2+\alpha}_{[2]}(\bar{\mathrm{U}})$, such that $\mathrm{K}_{*\,|p_0}$ is surjective on the larger space.

LEMMA 5. — *The mapping* $\mathbb{K}=\mathrm{E}^{-1}\Delta\mathrm{E}$ *maps* $\mathrm{C}^{2+\alpha}_{[2]}(\bar{\mathrm{U}})$ *onto* $\mathrm{C}^{\alpha}_{[0]}(\bar{\mathrm{U}})$, *if the source space carries no restrictions for the boundary values on* $\partial\mathrm{U}$.

Proof. — If π_r is a monomial of degree $r\geqq 3$, then $\Delta\mathrm{E}\,\pi_r$ is a monomial of the same degree. Then $\mathrm{E}^{-1}\Delta\mathrm{E}\,\pi_r\in \mathrm{C}^{r-2+\alpha}(\bar{\mathrm{U}})\subset \mathrm{C}^{\alpha}(\bar{\mathrm{U}})$, $\mathrm{E}(u)=|u|^2$.

If ζ_s is a monomial of degree s, then $\Delta^{-1}\mathrm{E}\,\zeta_s$ can be defined as a monomial of degree $(s+4)$ and in $\mathrm{C}^{\infty}(\bar{\mathrm{U}})$. Then $\mathrm{E}^{-1}\Delta^{-1}\mathrm{E}\,\zeta_s$ is homogeneous of degree $s+2$, and is in $\mathrm{C}^{s+1+\alpha}(\bar{\mathrm{U}})$, and contained in $\mathrm{C}^{2+\alpha}(\bar{\mathrm{U}})$ if $s\geqq 1$. Since $\Delta(\mathrm{E}^2)=$const. E, we have no problems with $s=0$, and any monomial ζ_s must be in the range of $\mathbb{K}:\mathrm{C}^{2+\alpha}_{[2]}(\bar{\mathrm{U}})\to \mathrm{C}^{\alpha}(\bar{\mathrm{U}})$. Since the range of $\mathbb{K}$ is closed, we see that $\mathbb{K}$ is surjective.

LEMMA 6. — *There exists a finite dimensional subspace* $\mathrm{W}_0\subset \mathrm{C}^{2+\alpha}_{[2]}(\bar{\mathrm{U}})$ *such that* $\mathbb{K}:\mathring{\mathrm{C}}^{2+\alpha}_{[2]}(\bar{\mathrm{U}})\oplus \mathrm{W}_0\to \mathrm{C}^{\alpha}(\bar{\mathrm{U}})$ *is surjective. There exists a finite dimensional subspace* $\mathrm{W}\in \mathrm{C}^{2+\alpha}_{[2]}(\bar{\mathrm{U}})$ *such that*

$$\mathrm{K}_{*\,|p_0}[\mathrm{E}.]:\mathring{\mathrm{C}}^{2+\alpha}_{[2]}(\bar{\mathrm{U}})\oplus \mathrm{W}\to \mathrm{C}^{\alpha}(\bar{\mathrm{U}})$$

is surjective as well.

Proof. — Since $\mathbb{K}$ is a Fredholm operator, it is necessary that the dimension of W is larger than or equal to the corank of $\mathbb{K}$.

Since $K_{*\,|p_0}[E\,\sigma] = -\mathbb{K}\,\sigma - 4\,E\,\sigma$, we see that $\sigma \to 4\,E\,\sigma$ is a compact mapping from $C^{2+\alpha}_{[2]}(\bar{U}) \to C^{\alpha}(\bar{U})$. Both mappings $\mathbb{K}$ and $K_{*\,|p_0}$ have the same corank on $C^{2+\alpha}_{[2]}(\bar{U})$, if there is no kernel of $K_{*\,|p_0}\,E$. If, however $K_{*\,|p_0}[E.]$ has a kernel on $C^{2+\alpha}_{[2]}(\bar{U})$, we may simply increase the diameter of U to circumvent the kernel by using perturbation arguments. Then we get surjectivity on a larger ball and the former arguments given above hold again. Lemma 5 implies the existence of such a space W.

The space W makes it possible to define a splitting of $Y := C^{2+\alpha}(\partial U, \mathbb{R})$, namely into $\mathring{W} := \{ w_{|U} \mid w \in W \}$ and some arbitrarily fixed complement Y_1 in Y. For any $y \in Y_1$ we define $v = \mathrm{ext}(y)$ such that v satisfies $\mathbb{K}\,v + 4\,E\,v \equiv 0$ and has boundary values $v_{|U} = y \in Y_1$. But since $\mathbb{K} + 4\,E$ could have a kernel on $C^{2+\alpha}_{[2]}(\bar{U})$, it may be necessary to fix a subspace $Y_2 \subset Y_1$, such that ext (y) can be well defined for $y \in Y_2$.

PROPOSITION 1. — *For any $y_* \in Y_2$, if its norm $\| y_* \|$ is small, we can easily construct $w \in W$ and $\rho \in \mathring{C}^{2+\alpha}_{[2]}(\bar{U})$ such that*

$$K_1(\mathrm{ext}(y_*) + w + \rho) = K(1 + E\,\mathrm{ext}(y_*) + w + \rho) \equiv 1.$$

Therefore we can define a subspace $Y \subset Y_2$ such that

$$v = \mathrm{ext}(y) \in C^{2+\alpha}_{[2]}(\bar{U}),$$

if $y \in Y$. In this case we have $\mathbb{K}\,v + 4\,E\,v = K_{*\,|p_0}[E\,v] = 0$.

Proof. — The mapping K_1 is differentiable on the space $C^{2+\alpha}_{[2]}(\bar{U})$ keeping in mind that $K(1 + E\,\omega) = K_1(\omega)$.

We have

$$K_1(0) = K(1) \equiv 1$$

and

$$K_{1*\,|\,0} = K_{*\,|p_0}[E] = \mathbb{K} + 4\,E.$$

By construction $K_{1*\,|\,0}[\mathrm{ext}(y_*)] \equiv 0$.

Lemma 6 implies that $K_{1*\,|\,0} : C^{2+\alpha}_{[2]} \oplus W \to C^{\alpha}(\bar{U})$ is surjective and a Fredholm operator.

The iteration procedure of the implicit function theorem has to start with $w = 0$, and if the norm of ext (y_*) is too large we may take instead $\varepsilon_* . \mathrm{ext}(y_*)$, $\varepsilon_* \in \mathbb{R}^+$. The choice of w may not be unique, but we can fix a differentiable section among these vectors, such that they depend smoothly on y_*.

THEOREM 5. — *In the space of immersions of class $C^{2+\alpha}$ of ∂U into $\mathbb{R}^3$ near $\underline{x}_0 : \partial U \to S^2$ (winding around twice), there exists a finite codimensional submanifold Y such that for any $\underline{y} \in Y$ there exists a solution F of the nonstandard Plateau problem where $F = F(y)$ satisfies $K \equiv 1$ and has as boundary values the curve y and a branch point near the pole P of $\underline{x} : \partial U \to \mathbb{R}^3$.*

Proof. — If $\underline{y}$ is some smooth section in the normal bundle of S^2 along $\underline{x}_0(\partial U)$, the theorem follows from proposition 1.

Otherwise we simply have to redefine the set U such that $\underline{x}_0$ maps $\tilde{U}$ onto the sphere S^2 with the same pole but such that $\underline{x}_0(\partial\tilde{U})$ parametrizes the curve $\underline{y}$ in both angular variables correctly. We only have to reparametrize $\underline{y}$ such that $\underline{y}=\rho.\underline{x}_{0\,|\,\tilde{U}}$, and ρ is is near the value 1 in $C^{2+\alpha}(\partial\tilde{U}, \mathbb{R})$.

Since we have never used the fact that ∂U is a circle, all arguments remain unchanged.

3. BRANCHED SURFACE WITH GAUSS CURVATURE ≡ 1

Let $\mathscr{R}$ denote a hyperelliptic Riemann surface of genus g.

Let $\underline{x}_1 : \mathscr{R} \to S^2 \subset \mathbb{R}^3$ denote a fixed immersion of $\mathscr{R}$ into the sphere S^2, which is conformal, covers twice almost all points, and has $(2g+2)$ simple branch points.

We will denote them as $\zeta_1, \ldots, \zeta_{2g+2} \in \mathscr{R}$ with $d\underline{x}_0(\zeta_j)=0$ and $\underline{x}_0(\zeta_j)=: P_j$. This notation generalizes the notation of section 2, where $\underline{x}_0 : \mathscr{P}^1 \to S^2$ had a source space of genus $g=0$.

We fix a smooth embedding $\delta_0 : S^1 \to \mathscr{R}$ surrounding all branch points in $\mathscr{R}$ with the exception of ζ_{2g+2} and define $\mathscr{F} := \mathrm{int}(\delta)$, here denoting the interior of δ, such that $\mathscr{F}$ is a manifold of genus g with boundary $\delta(S^1)$. Then $\underline{x}_0(\delta\mathscr{F})$ is an immersion of S^1 into $\mathbb{R}^3$ and bounds a surface of genus g in $\mathbb{R}^3$ which has Gauss curvature 1 everywhere and $(2g+1)$ branch points at $P_1, \ldots, P_{2g+1}$. We may study surfaces near $\underline{x}_0$ but again with Gauss curvature 1. If $u=(u^1, u^2)$ are local coordinates of $\mathscr{F}$ around a branch point ζ_j, then $\underline{x}_0$ induces a metric on $\mathscr{F}$, denoted again as $e_{ij}(u)\,du^i\,du^j$, as a pull back of the metric in $\mathbb{R}^3$. Any surface near $\underline{x}_0$ may be parametrized as a section in the normal bundle N on $\underline{x}_0(\mathscr{F})$ namely $\underline{Z}(u)=p(u).\underline{x}_0(u)$, and as such define a surface $\mathscr{Z}=\mathscr{Z}(p)$ in $\mathbb{R}^3$. In local coordinates, the first and the second fundamental form of $\mathscr{Z}$ can be computed as in section 2.

Now we can similarly use this notation further and define

$$C^{2+\alpha}_{[2]}(\bar{F}) := \{ p \in C^{2+\alpha}(\mathscr{F}) \,|\, j_{[2]}(p)(\zeta_j)=0 \}.$$

For the computation of the 2-jet $j_{[2]}$, we use local coordinates u around ζ_j and we see that $C^{2+\alpha}_{[2]}(\bar{F})$ is a well defined finite codimensional subspace of $C^{2+\alpha}(\bar{F})$.

In local coordinates u around any ζ_j, we again may assume that the metric tensor is diagonal and of the form $(e_{ij}(u))=\begin{pmatrix} E(u) & 0 \\ 0 & E(u) \end{pmatrix}$ and $E(u)=|u|^2.\eta(u)$, where η is smooth and $\eta(0)=1$. We will focus our interest on the surfaces $\mathscr{X}=\mathscr{X}(p)$, where $p=1+\mathrm{E}\rho:\bar{\mathscr{F}}\to\mathbb{R}$ and $\rho\in \mathrm{C}^{2+\alpha}_{[2]}(\bar{\mathscr{F}})$. As an immediate generalization of theorem 1, we get

PROPOSITION 2. — *If* $\rho\in \mathrm{C}^{2+\alpha}_{[2]}(\bar{\mathrm{U}})$ *with* $\|\rho\|_{2+\alpha}\leqq\frac{1}{2}$ *then the Gauss curvature of* $\mathscr{X}$ *is* $\mathrm{K}(\mathscr{X})=\mathrm{K}(p)=\mathrm{K}(1+\mathrm{E}\rho)$ *and is in* $\mathrm{C}^{\alpha}(\bar{\mathrm{U}})$.

Proof. — That the Gauss curvature of $\mathscr{X}$ is in C^{α} is obvious outside the branch points. Near the branch points, however, we can easily apply the theorem 2, since the singular term of the metric is then cancelled out. That the Gauss curvature equals 1 in the branch points is clear from looking at the spherical image around ζ_j.

PROPOSITION 3. — *If* $\mathrm{B}:=\left\{\rho\in \mathrm{C}^{2+\alpha}_{[2]}(\bar{\mathscr{F}})\,\middle|\,\|\rho\|_{2+\alpha}<\frac{1}{2}\right\}$, *then* $\mathrm{K}_1(\rho):=\mathrm{K}(1+\mathrm{E}\rho)$ *is a smooth mapping of* B *into* $\mathrm{C}^{\alpha}(\bar{\mathscr{F}})$.

If $p_0\equiv 1:\mathscr{F}\to\mathbb{R}$, *then* $\mathrm{K}(p_0)\equiv 1$ *and* $\mathrm{K}_{*\,|\,p_0}[\mathrm{E}\sigma]=-\mathrm{E}^{-1}\Delta\mathrm{E}\sigma-4\mathrm{E}\sigma$, *if* $\sigma\in \mathrm{C}^{2+\alpha}_{[2]}(\bar{\mathscr{F}})$ *and* E *is the diagonal term of the metric in the local coordinates.*

Proof. — We do not need a general Laplace-Beltrami operator on $\mathscr{F}$, since Δ is invariant under a change of conformal coordinates. Therefore the explicit formulas can be read in theorem 3, because differentiation is performed locally. We only have to check that the singular multiplier E^{-1} does not induce a weaker regularity than stated.

PROPOSITION 4. — *The mapping* $\mathbb{K}:=\mathrm{E}^{-1}\Delta\mathrm{E}$ *as a map of* $\mathring{\mathrm{C}}^{2+\alpha}_{[2]}(\bar{\mathrm{U}})(\bar{\mathscr{F}})$ *into* $\mathrm{C}^{\alpha}(\bar{\mathscr{F}})$ *is injective and a Fredholm operator.*

Proof. — $\mathbb{K}$ is a product of injective mappings on $\mathrm{C}^{2+\alpha}(\bar{\mathscr{F}})$ and has a formal inverse $\mathbb{L}=\mathrm{E}^{-1}\Delta^{-1}\mathrm{E}$ as in section 2. As $\mathbb{K}$ was well defined with values in $\mathrm{C}^{\alpha}(\bar{\mathrm{U}})$ on the subspace $\mathrm{C}^{2+\alpha}_{[2]}(\bar{\mathscr{F}})$ only, the inverse $\mathbb{L}$ is well defined and bounded again on a space of finite codimension. But for testing the Fredholm character of $\mathbb{K}$, this is enough. We can apply theorem 4.

Since we have defined the boundary $\partial\mathscr{F}$ in $\mathscr{R}$ with a smooth embedding $\delta:\mathrm{S}^1\to\mathscr{R}$, we can now define a space $\mathrm{Y}(\delta)=\mathrm{Y}$ as $\mathrm{C}^{2+\alpha}(\partial\mathscr{F},\mathbb{R}^3)$, such that $\underline{y}\in\mathrm{Y}$ if $\underline{y}=\mathring{p}.\underline{x}_0:\partial\mathscr{F}\to\mathbb{R}^3$ and $\mathring{p}$ denotes the boundary values of some $p\in \mathrm{C}^{2+\alpha}(\bar{\mathscr{F}})$, *i.e.* $\mathring{p}\in \mathrm{C}^{2+\alpha}(\partial\mathscr{F})$. Then we must study the Dirichlet problem for $\underline{y}$ and the equation $\mathrm{K}(1+\mathrm{E}\rho)=\mathrm{K}(p)\equiv 1$ with boundary values $\underline{y}$.

LEMMA 7. — *In* $\mathrm{Y}(\delta)$, *there exists a subspace* $\mathrm{Y}_1(\delta)$ *of finite codimension such that for* $\underline{y}=\mathring{p}.\underline{x}_0:\partial\mathscr{F}(\delta)\to\mathbb{R}^3$, *there exists an extension*

$\operatorname{ext}(\underline{y}) \in C^{2+\alpha}(\overline{\mathcal{F}})$, *ext* $(\underline{y}) = \tilde{p} \cdot \underline{x}_0$, $\tilde{p} \in C^{2+\alpha}_{[2]}(\overline{\mathcal{F}})$, *and*

$$E^{-1} \Delta E \tilde{p} + 4 E \tilde{p} \equiv 0 \textit{ in } \overline{\mathcal{F}}.$$

Proof. — As in theorem 4, we see that

$$\mathbb{K} = E^{-1} \Delta E : \overset{\circ}{C}{}^{2+\alpha}_{[2]}(\overline{\mathcal{F}}) \to C^{\alpha}(\overline{\mathcal{F}})$$

is an injective Fredholm operator. Since $\overset{\circ}{C}{}^{2+\alpha}_{[2]}(\overline{\mathcal{F}}) \oplus Y(\delta)$ is isomorphic to $C^{2+\alpha}_{[2]}(\overline{\mathcal{F}})$, we can see that $\mathbb{K} y \equiv 1$ or $\mathbb{K} \tilde{y} - \mathbb{K} \tilde{y}_0 \equiv 0$ is solvable for boundary values in a finite codimensional subspace of $Y(\delta)$. But the mapping $K_{* \mid p_0}[E.] : \overset{\circ}{C}{}^{2+\alpha}_{[2]}(\overline{U})(\overline{\mathcal{F}}) \to C^{\alpha}(\overline{\mathcal{F}})$ is only a compact perturbation of $\mathbb{K}$. This implies lemma 7.

THEOREM 6. — *There exists a submanifold* $\mathcal{Y}_*(\delta) \subset Y(\delta)$ *of finite codimension such that for any* $\underline{y} \in \mathcal{Y}_*(\delta)$, *there exists an extension* $e(\underline{y}) = \tilde{\rho} \cdot \underline{x}_0 \in C^{2+\alpha}_{[2]}(\overline{\mathcal{F}})$ *such that* $K(p_0 + E \tilde{\rho}) \equiv K(1 + E \tilde{\rho}) \equiv 1$ *on* $\mathcal{F}$. *Therefore the surface* $\mathcal{X} = \mathcal{X}(\tilde{\rho})$ *has the property that its Gauss curvature is identical to* 1.

Proof. — The space $\mathcal{Y}_*(\delta)$ is such that $\underline{x}_0 \in \mathcal{Y}_*(\delta)$, and its tangent space $T\mathcal{Y}_*(\delta)$ at x_0 equals $Y_*(\delta)$. For any $y \in Y_*(\delta)$, we have by definition $K_{* \mid p_0}(1 + E \operatorname{ext}(y)) \equiv 0$. With the same techniques as in proposition 1, we can again prove the surjectivity of $K_{1* \mid p_0} : C^{2+\alpha}_{[2]}(\overline{\mathcal{F}}) \to C^{\alpha}(\overline{\mathcal{F}})$. Namely for the equation

$$K_{1* \mid p_0}(E v) = b \in C^{\alpha}(\overline{\mathcal{F}}),$$

we use a partition of unity $1 = \sum_{j=1}^{2g+1} \varphi_j$ on $\mathcal{F}$ and solve the equation $K_{1* \mid p_0}(E v_j) = b \varphi_j$ in local coordinates. We find a solution v_j in these coordinates with domain of definition U_j. Since we are only interested in a surjectivity, we can take a smooth extension vanishing outside U_j on $\overline{\mathcal{F}}$. Since the mapping

$$K_{1* \mid p_0} : E . C^{2+\alpha}_{[2]}(\overline{\mathcal{F}}) \to C^{\alpha}(\overline{\mathcal{F}})$$

is an isomorphismm from the complement of its kernel onto its range $C^{\alpha}(\overline{\mathcal{F}})$, we can apply the iteration technique which proves the implicit function theorem in Banach spaces. As a result the construction of the higher order terms which change the space $Y_*(\delta)$ into the submanifold $\mathcal{Y}_*(\delta)$ in $Y(\delta)$.

If $\Gamma_0 : S^1 \to S^2$ is an embedding of class $C^{2+\alpha}$, and $\Gamma : S^1 \to S^2$ denotes the double cover of Γ_0, then we can fix $g \geqq 0$ arbitrarily and $2g+1$ points $P_1, \ldots, P_{2g+1}$ in the interior of Γ_0. Together with one point P_{2g+2} in the exterior of Γ_0 we have marked all branch points of a hyperelliptic surface $\mathcal{L}$ on S^2. If $f : \mathcal{R} \to \mathcal{L}$ is a parametrization of $\mathcal{L}$ then $f^{\text{inv}}(\operatorname{int}(\Gamma_0)) = \mathcal{F}$ is an open subset of $\mathcal{R}$ being a manifold with boundary of class $C^{2+\alpha}$ and of genus g. Now we obtain

THEOREM 7. — *We fix an embedding* $\Gamma : S^1 \to S^2 \subset \mathbb{R}^3$ *with its double cover* $\Gamma_2 : S^1 \to S^2 \subset \mathbb{R}^3$, *and a Riemann surface* $\mathscr{F}$, *being hyperelliptic and having the boundary* $\partial \mathscr{F} \cong S^1$. *Then we have a neighborhood* $\mathscr{U}$ *of* Γ_2 *in* $C^{2+\alpha}(S^1, \mathbb{R}^3)$ *and an open submanifold* $\mathscr{Y} \subset \mathscr{U}$ *of finite codimension, such that any* $y \in \mathscr{Y}$ *extends to a surface* $\mathscr{L}(y)$ *with boundary. This boundary* $\partial \mathscr{L}(y)$ *is parametrized by* y, $\mathscr{L}(y)$ *has* $(2g+1)$ *interior branch points, and has Gauss curvature* $K \equiv 1$.

Proof. — The open set $U \in C^{2+\alpha}(S^1, \mathbb{R}^3)$ can be defined such that any $z \in U$ is the restriction of some $z \in C^{2+\alpha}(\mathscr{F}, \mathbb{R}^3)$.

A mapping $z \in C^{2+\alpha}(\mathscr{F}, \mathbb{R}^3)$ can be given as parametrized in polar is coordinate (r, d) of $\mathbb{R}^3$, where $r = r(z) = |z| \in \mathbb{R}^+$ is the radial variable, and $d = d(z) \in S^2$ are the two angular variables. We can reparametrize the surface z such that $(r(z), d(z))$ can be written as $(p(u_1, u_2), u_1, u_2)$, where $r = r(u)$ denotes the radial function and the variables $u = (u_1, u_2)$ are the trivial angular variables on S^2. In fact the variable u can be pulled back from S^2 onto $\mathscr{R}$ using $\underline{x}_0^{\text{inv}}$, and then u defines also a local variable on $\mathscr{F} \subset \mathscr{R}$.

In this way we can easily obtain a situation which was treated in theorem 6. We find a finite codimensional submanifold $\mathscr{Y}_*$ in $C^{2+\alpha}(S^1, \mathbb{R}^3)$ such that for any $y \in \mathscr{Y}_*(\delta)$ the Dirichlet problem of the equation $K(p) \equiv 1$ can be solved for functions $p \in C^{2+\alpha}(\mathscr{F}, \mathbb{R}^3)$ and with boundary values $\tilde{y}$, obtained from $y \in \mathscr{Y}$ by reparametrization. Then the surface $\mathscr{L}(y)$, which is parametrized by $(p(u), u)$, has Gauss curvature $K(\mathscr{L}) = K(y) \equiv 1$ and its boundary $\partial \mathscr{L}(y)$ is parametrized by y given before.

Remark 1. — For simplicity only we did start above with some curve $\Gamma_2 : S^1 \to S^2$, being the double cover of an embedding $\Gamma : S^1 \to S^2$, such that Γ_2 bounds a realization $\mathscr{L}(\Gamma)$ of a hyperelliptic surface $\mathscr{F}$ with boundary, immersed into S^2 and having $2g+1$ branch points, which are fixed. Many other immersions $\tilde{\Gamma}$ different from Γ_2 admit a similar extension to a hyperelliptic surface $\mathscr{F}$ with simple branch points.

Remark 2. — The codimension c_0 of $\mathscr{Y} \subset \mathscr{U}$ can easily estimated, since it depends only on the number of branch points. c_0 could coincide with the number of abelian differentials on $\mathscr{F}$, or c_0 could even vanish similar to the situation for minimal surfaces with $H \equiv 0$ ([2] and [4]).

Remark 3. — In order to get not only one solution $\mathscr{L}(y)$ with $K \equiv 1$ for the Plateau problem with boundary $y \in \mathscr{Y}$, we need a transversal intersection of some manifolds $\{\mathscr{Y}_j\}_{j \in J}$, representing surface of different branch points, such that any curve in this intersection must have $|J|$ different solutions with $K \equiv 1$, comparable to the result of [2].

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