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How to blow infinitely large soap bubbles with a fixed boundary

by

Rugang YE (1)

Department of Mathematics, Stanford University,
Stanford, CA 94305, U.S.A.

ABSTRACT. — Consider (area minimizing) soap bubbles (*i.e.* surfaces of constant mean curvature) spanning a Jordan curve Γ in $\mathbb{R}^3$. We show that they converge to infinitely large minimal surfaces spanning Γ as their volumes approach infinity.

RÉSUMÉ. — On considère des bulles de savon (*i.e.* des surfaces de courbure moyenne constante) à bord d'une courbe Jordan $\Gamma \subset \mathbb{R}^3$. Nous démontrons qu'ils convergent vers des surfaces minimales infiniment larges à bord Γ si leur volume tend vers l'infini.

Mots clés : Surfaces de courbure moyenne constante.

1. INTRODUCTION

In order to find soap bubbles (*i.e.* surfaces of constant mean curvature) spanning an oriented, rectifiable Jordan curve Γ in $\mathbb{R}^3$, one minimizes

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area (or the Dirichlet energy) under a volume constraint. Let $D_R = \{ z \in \mathbb{C} : |z| \leqq R \}$ for $R > 0$, $D = D_1$ and

$H_\Gamma = \{ u \in H_2^1 (D, \mathbb{R}^3) \cap C^0 (D, \mathbb{R}^3)$:

$u_{|\partial D}$ is a weakly monotonic representation of $\Gamma \}$.

For $u \in H_2^1 (\Omega, \mathbb{R}^3)$, $\Omega \subset \mathbb{C}$ one defines the Dirichlet energy

$$E(u) = \frac{1}{2} \iint_\Omega |\nabla u|^2 \, dx \, dy,$$

the area

$$A(u) = \iint_\Omega |u_x \wedge u_y| \, dx \, dy$$

and the oriented volume

$$V(u) = \frac{1}{3} \iint_\Omega u . u_x \wedge u_y \, dx \, dy.$$

We also use the notation $E(u, \Omega') := E(u_{|\Omega'})$ for $\Omega' \subset \Omega$.

The following solution of the *volume constrained Plateau problem* is due to H. Wente [23] (the immersion property was established in [19]).

EXISTENCE THEOREM. — *Let* V *be a given constant and*

$$H_{\Gamma, V} = \{ u \in H_\Gamma : V(u) = V \}.$$

Then there exists in $H_{\Gamma, V}$ *an element which minimizes the Dirichlet energy and area. The mapping* $u_{|\mathring{D}}$ *is a real analytic conformal immersion of constant mean curvature.* [Such a u will called a "minimizer of volume V" (*spanning* Γ).]

In 1980 Wente [23] established

SOAP BUBBLE THEOREM. — *Let* Γ_k *be a sequence of oriented rectifiable Jordan curves in* $\mathbb{R}^3$ *with* $0 \in \Gamma_k$ *and length* $(\Gamma_k) \to 0$ *as* $k \to \infty$. *Let* u_k *be minimizers of volume* $\frac{4}{3}\pi$ *spanning* Γ_k. *Then a subsequence of* u_k *(still denoted by* u_k*) converges to a round sphere* S_0 *of radius* 1 *and containing the origin in the following sense: there exist radii* $r_k \to \infty$ *and conformal maps* $\varphi_k : D_{r_k} \to D$ *such that* $u_k \circ \varphi_k$ *and their first derivatives converge to* v_0 *and its first derivatives on any disk* D_R, *where* v_0 *is a conformal representation of* S_0 *with* $v_0(\infty) = 0$. (*A more general study in this direction was later on taken by Brezis and Coron* [3].)

Formulated in another way, this theorem says that for a fixed Γ minimizers u_V of volume V become more and more spherical as $V \to \infty$ in the sense that after suitable rescaling u_V converge to round spheres.

In this paper we are concerned with the following question raised by Wente: if we do not rescale u_V, can we get infinitely large minimal surfaces spanning Γ in the limit? Or: if we blow larger and larger soap bubbles with a fixed boundary, can we get an "infinitely large soap bubble"?

We prove.

MAIN THEOREM. – *Let u_k be a sequence of minimizers of volume $V_k \to \infty$ spanning Γ, which satisfy the classical three point condition of the Plateau problem* [4]. *Then a subsequence of u_k (still denoted by u_k) converges to an infinitely large minimal surface u spanning Γ. More precisely, we have* ($D^* := D \setminus \{0\}$, $D_r^* := D_r \setminus \{0\}$).

A. (properties of u).

1. $u \in C^0(D^*)$, $u \in H_2^1(D \setminus D_r)$ for every $r > 0$, $u_{|\partial D}$ parametrizes Γ topologically and $u_{|\mathring{D}^*}$ is a real analytic conformal minimal immersion;

2. $\lim_{z \to 0} |u(z)| = \infty$, $\lim_{z \to 0} n(z)$ exists, where n denotes a continuous unit normal of u;

3. for some $r > 0$, $u_{|D_r^*}$ is an embedding of finite total Gaussian curvature and $u(D_r^*)$ is a graph over a plane;

4. u minimizes area under constant volume constraint (for precise meaning *see* Proposition 2.2).

B. (Convergence of u_k) There is a conformal transformation $\varphi : D \to D$ such that

1. $u_k \circ \varphi$ converge to u uniformly on $D \setminus \mathring{D}_r$ and strongly in $H_2^1(D \setminus D_r)$ for earch $r > 0$;

2. $u_k \circ \varphi$ converge to u locally smoothly in $\mathring{D}^*$;

3. if Γ is a regular Jordan curve of class $C^{l, \alpha}$ for some $l \geqq 1$ and $\alpha \in (0, 1)$, then $u_k \circ \varphi$ also converge to u in the C^l norm on $D \setminus \mathring{D}_r$ for each $r > 0$.

We remark that this theorem also provides a resolution of the exterior Plateau problem: find a minimal surface spanning Γ, which has the conformal type of the punctured disk and stretches out to infinity. This problem has been solved in [21] by employing a sequence of expanding minimal annuli, where one additionally prescribes the asymptotical normal at infinity. In the present paper we encounter the same analytical difficulty as in [21], *i.e.* we have to deal with sequences of mappings whose Dirichlet energies approach infinity. This kind of difficulty seems to be new, and we hope that our arguments (in this paper and in [21]) can be useful to other problems also. We note that the situation in the present paper is more complicated than in [21]. One major reason is that the size of a minimal surface is well-controlled by its boundary via the maximum principle or isoperimetric inequality whereas in general this is not the case for surfaces of nonzero mean curvature. This presents additional difficulties for estimating the area of certain subsets of our soap bubbles. We appeal to the monotonicity formula for area, Wente's Soap Bubble Theorem, his diameter estimate for soap bubbles in [23] and Heinz' isoperimetric inequality for "small" surfaces of constant mean curvature [10].

An additional remark: the Main Theorem of this paper is a counterpart to Wente's Soap Bubble Theorem, but a subtlety should be recognized. The round sphere limit of the rescaled bubbles disappears at infinity during the convergence of the unscaled bubbles u_V as $V \to \infty$, while the minimal surface limit of the unscaled bubbles shrinks to a point during the rescaled convergence. Thus the two theorems capture different parts of the expanding soap bubbles in the limit.

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2. CONVERGENCE OF MINIMIZERS OF LARGE VOLUME

We fix a least area disk u_0 spanning Γ and define $a_0 = A(u_0)$, $V_0 = V(u_0)$. Put $d_0 = \operatorname{diam} \Gamma$, $l_0 = \text{length}(\Gamma)$, $a(V) = \inf\{E(u) : u \in H_{\Gamma, V}\}$ and

$$\mathcal{H}(V) = \{H : H \text{ is the mean curvature of some minimizer } u \text{ of volume } V \text{ spanning } \Gamma\}.$$

Here the sign of H is determined by the equation $\Delta u = 2 H u_x \wedge u_y$. The mean curvature of u will be denoted by $H(u)$. We assume w.l.o.g. that $0 \in \Gamma$.

LEMMA 2.1. – *We have*

1. *for all* $V \in \mathbb{R}$:

$$-a_0 \leqq a(V) - \sqrt[3]{36\pi}\,|V - V_0|^{2/3} \leqq a_0;$$

2. *for all* $V \in \mathbb{R}$ *and* $H \in \mathcal{H}(V)$:

$$-a_0 - \frac{1}{2} d_0 l_0 \leqq -3\,VH - \sqrt[3]{36\pi}\,|V - V_0|^{2/3} \leqq a_0 + \frac{1}{2} d_0 l_0.$$

We refer to [17] for a proof. This lemma says that the area and mean curvature of a minimizer of volume V approximately equal those of a round sphere of the same volume if V is large.

For a continuous mapping u defined on a closed plane domain whose boundary contains $S^1 := \partial D$ we define $\Omega(u, \rho) = u^{-1}(B_\rho)$, $\Omega_0(u, \rho)$ = the connected component of S^1 in $\Omega(u, \rho)$, where $\rho > 0$, $B_\rho = \{x \in \mathbb{R}^3 : |x| \leqq \rho\}$. We shall abbreviate $E(u, \Omega(u, \rho))$ to $E_\rho(u)$.

We also define $\Omega_0^*(u, \rho)$ to be the annulus bounded by S^1 and the outside boundary of $\Omega_0(u, \rho)$, provided that $\Omega_0(u, \rho)$ is a bounded C^1 domain contained in $\mathbb{C}\setminus\mathring{D}$. Thus, $\Omega_0^*(u, \rho)$ is obtained from $\Omega_0(u, \rho)$ by filling in holes between the inside and outside boundaries of $\Omega_0(u, \rho)$. $E_\rho^*(u)$ will stand for $E(u, \Omega_0^*(u, \rho))$.

In the sequel we only treat minimizers of positive volumes, since negative volumes merely involve a switch in orientation. Let $\varepsilon_0 \in \left(0, \frac{1}{2}\right)$ be the number satisfying $\frac{1}{\sqrt{1-\varepsilon_0^2}} e^{2\varepsilon_0} = \frac{6}{5}$. Put $R_0 = \max\left\{d_0, \sqrt{\frac{2 d_0 l_0}{\pi}} e^{\varepsilon_0}\right\}$.

LEMMA 2.2. — *There exists* $\tilde{R}_0 > \varepsilon_0^{-1} R_0$ *such that*

$$E_\rho(u) \leqq \frac{3}{2} \pi \rho^2$$

for all $\rho \in (R_0, \varepsilon_0 R)$, *whenever* u *is a minimizer of volume* $\frac{4}{3}\pi R^3$ *with* $R \geqq \tilde{R}_0$.

Proof. — Assume the contrary. Then there are minimizers v_k of volume $\frac{4}{3}\pi R_k^3$ with $R_k \to \infty$ and radii $\rho_k \in (R_0, \varepsilon_0 R_k)$ such that

$$E_{\rho_k}(v_k) > \frac{3}{2}\pi\rho_k^2. \tag{2.1}$$

We apply the Soab Bubble Theorem to $u_k = \frac{1}{R_k} v_k$ to obtain

$$\begin{aligned} E(u_k, \Omega(u_k, \varepsilon_0)^c) &\geqq 4\pi - \text{area}(S_0 \cap B_{\varepsilon_0}) - o(1) \\ &\geqq 4\pi - \frac{1}{\sqrt{1-\varepsilon_0^2}} \pi\varepsilon_0^2 - o(1), \end{aligned}$$

or

$$E(v_k, \Omega(v_k, \varepsilon_0 R_k)^c) \geqq \left(4\pi - \frac{1}{\sqrt{1-\varepsilon_0^2}} \pi\varepsilon_0^2 - o(1)\right) R_k^2.$$

Hence by Lemma 2.1

$$E_{\varepsilon_0 R_k}(v_k) \leqq a_0 + \left(\frac{1}{\sqrt{1-\varepsilon_0^2}} \pi\varepsilon_0^2 + o(1)\right) R_k^2. \tag{2.2}$$

On the other hand, the monotonicity formula for area [16] implies for $r<1$ and $\rho>d_0$

$$\frac{d}{d\rho}(\rho^{-2}\mathrm{E}_\rho(v_{k\,|\mathrm{D}_r})\geqq -|\mathrm{H}_k|\,\rho^{-2}\mathrm{E}_\rho(v_{k\,|\mathrm{D}_r})-\rho^{-3}\int_{\partial \mathrm{D}_r}|v_k|\,ds,$$

where $\mathrm{H}_k=\mathrm{H}(u_k)$ and ds denotes the arclength element in the metric induced by v_k. But $\int_{\partial \mathrm{D}_r} ds \to \text{length}(\Gamma)$ as $r\to 1$ by [10], hence we obtain

$$(2.3)\qquad \frac{d}{d\rho}\rho^{-2}\mathrm{E}_\rho(v_k))\geqq -|\mathrm{H}_k|\,\rho^{-2}\mathrm{E}_\rho(v_k)-\rho^{-3}d_0\,l_0.$$

Multiplying (2.3) by $e^{|\mathrm{H}_k|\rho}$ and then integrating it yield

$$(2.4)\quad \sigma^{-2}e^{|\mathrm{H}_k|\sigma}\,\mathrm{E}_\sigma(v_k)\leqq\rho^{-2}e^{|\mathrm{H}_k|\rho}\,\mathrm{E}_\rho(v_k)+d_0\,l_0\int_\sigma^\rho \rho^{-3}\,e^{|\mathrm{H}_k|\rho}\,d\rho$$

for $\rho\geqq\sigma>d_0$.

But Lemma 2.1 implies

$$(2.5)\qquad |\mathrm{H}_k|\leqq\frac{1}{\mathrm{R}_k}+\frac{1}{4\,\pi\,\mathrm{R}_k^3}\left(a_0+\frac{1}{2}\,d_0\,l_0\right).$$

Choosing $\rho=\varepsilon_0\,\mathrm{R}_k$ in (2.4) we deduce on account of (2.2) and (2.5)

$$\mathrm{E}_\sigma(v_k)\leqq\sigma^2\left(\pi\,\frac{1}{\sqrt{1-\varepsilon_0^2}}\,e^{2\,\varepsilon_0}+\frac{1}{2}\,d_0\,l_0\,e^{2\,\varepsilon_0}\,\sigma^{-2}\right)$$
$$\leqq\sigma^2\left(\pi\,\frac{1}{\sqrt{1-\varepsilon_0^2}}\,e^{2\,\varepsilon_0}+\frac{1}{4}\,\pi\right)=\frac{29}{20}\pi\sigma^2$$

if R_k is large enough and $\sigma\geqq\mathrm{R}_0$. Applying this to $\sigma=\rho_k$ we get a contradiction to (2.1).

Q.E.D.

The following lemma is an easy corollary of Wente's Soap Bubble Theorem.

LEMMA 2.3. – *There exists a number* $\bar{\mathrm{R}}_0\geqq\tilde{\mathrm{R}}_0$ *such that the following is true. For each minimizer u of volume* $\frac{4}{3}\pi\,\mathrm{R}^3$ *with* $\mathrm{R}\geqq\bar{\mathrm{R}}_0$, *there is a smooth, simply connected domain* $\Omega\subset\mathring{\mathrm{D}}$ *such that* $u_{|\Omega'}$ *is an embedding and* $\mathrm{E}(u,\Omega)\geqq\frac{31}{8}\pi\,\mathrm{R}^2$. *Moreover, there is a round sphere* S_0 *of radius* 1 *and containing the origin, such that the embedded surface* $\frac{1}{\mathrm{R}}u(\bar{\Omega})$ *is within* *distance* $\frac{1}{1{,}000}$ *from the spherical cap* $S_0 \setminus B_{1/100}$ *in the* C^1 *sence. We shall call* Ω *a* sphere domain *for* u. *We choose* $z_0 \in \Omega$ *such that* $\frac{1}{R} u(z_0)$ *is a closest point on* $\frac{1}{R} u(\bar{\Omega})$ *to the north pole of* S_0 (*the origin is thought of as the south pole of* S_0). *We call* z_0 *a* north pole *of* u.

For each minimizer u of volume no less than $\frac{4}{3}\pi \bar{R}_0^3$, we define a *nomalization* for u to be a conformal transformation $\varphi : D \to D$ such that φ maps the origin to a north pole of u. If the origin is itself a north pole of u, then we say that u is *normalized.*

For a map $u : D \to \mathbb{R}^3$, we define $\hat{u} : \mathbb{C} \setminus \mathring{D} \to \mathbb{R}^3$ via $\hat{u}(z) = u\left(\frac{1}{z}\right)$.

LEMMA 2.4. – *Let* u *be a normalized minimizer of volume* $\frac{4}{3}\pi R^3$ $(R \geqq \bar{R}_0)$. *Then*

$$\operatorname{dist}(\partial\Omega_0^*(\hat{u}, \rho) \setminus S^1, \partial\Omega_0^*(\hat{u}, 2\rho) \setminus S^1) \geqq e^{-24\pi^2},$$

provided that $\rho \in \left(R_0, \frac{1}{2}\varepsilon_0 R\right)$ *and* ρ, 2ρ *are both regular values of the function* $|\hat{u}|$.

Proof. – Since u is normalized, the assumption about ρ implies that $\Omega_0^*(\hat{u}, \rho)$ and $\Omega_0^*(\hat{u}, 2\rho)$ are well-defined annuli with common inside boundary S^1. Consider $z_0 \in \partial\Omega_0^*(\hat{u}, \rho) \setminus S^1$ such that

$$\operatorname{dist}(z_o, \partial\Omega_0(\hat{u}, 2\rho) \setminus S^1) = r := \operatorname{dist}(\partial\Omega_0^*(\hat{u}, \rho) \setminus S^1, \partial\Omega_0^*(\hat{u}, 2\rho) \setminus S^1).$$

Set

$$D_r(z_0) = \{ |z - z_0| \leqq r \},$$
$$D_r(z_0)' = D_r(z_0)' = D_r(z_0) \cap \Omega_0(\hat{u}, 2\rho)$$

and

$$\partial' D_r(z_0) = \partial D_r(z_0) \cap \Omega_0(\hat{u}, 2\rho).$$

If $r \geqq 1$, we are done. Otherwise, by the Courant-Lebesgue lemma [4] there exists $r_1 \in (r, \sqrt{r})$ such that

$$\left(\int_{\partial' D_{r_1}(z_0)} \left| \frac{\partial \hat{u}}{\partial \theta} \right| d\theta \right)^2 \leqq \frac{4\pi E(\hat{u}, D_{r_1}(z_0)')}{\log(1/r)}. \tag{2.6}$$

Since $r_1 < 1$ and $\partial\Omega_0^*(\hat{u}, \rho) \setminus S^1$, $\partial\Omega_0^*(\hat{u}, 2\rho) \setminus S^1$ surround S^1, we can find an arc in $\partial' D_{r_1}(z_0)$ connecting $\partial\Omega_0^*(\hat{u}, \rho) \setminus S^1$ and $\partial\Omega_0^*(\hat{u}, 2\rho) \setminus S^1$.

Hence

$$\int_{\partial' D_{r_1}(z_0)} \left|\frac{\partial \hat{u}}{\partial \theta}\right| d\theta \geqq \rho.$$

Then the desired inequality follows from (2.6) and Lemma 2.2.

Q.E.D.

A word about the strategy of our arguments: in order to estimate the Dirichlet energy of $\hat{u}$ on compact regions, we look at certain "good" regions on which the Dirichlet energy of $\hat{u}$ is under control and try to show that these regions exhaust the whole domain of $\hat{u}$. Now the region $\Omega_0^*(\hat{u}, \rho)$ have the exhausting property by the above lemma, and Lemma 2.2 provides an estimate of the Dirichlet energy of $\hat{u}$ on the subregions $\Omega_0(\hat{u}, \rho)$. $\Omega_0^*(\hat{u}, \rho) \setminus \Omega_0(\hat{u}, \rho)$ is the union of several disk type domains, we call their images under the inversion $z \to \frac{1}{z}$ $\rho-$*bubble disks.* It remains to control the Dirichlet energy of u on these disks. Let $L_\rho(u)$ denote the 1-dimensional Hausdorff measure of the set $u^{-1}(\partial B_\rho)$ in the metric induced by u. Note that for a regular value $\rho > d_0$ of $|u|$, $L_\rho(u)$ is just the length of the curve $u_{|\partial\Omega(u, \rho) \setminus S^1}$.

LEMMA 2.5. – *There is a positive constant $c > 0$ such that*

$$\text{(2.7)} \qquad E_\rho^*(\hat{u}) \leqq c\rho^2,$$

whenever u is a normalized minimizer of volume $\frac{4}{3}\pi R^3$ with $R \geqq \bar{R}_0$ and $\rho \in \left(R_0, \frac{\varepsilon_0}{1{,}000} R\right)$ is a regular value of $|u|$ satisfying

$$\text{(2.8)} \qquad L_\rho(u) \leqq 8\pi\rho.$$

Proof. – We claim that there is a constant $R_1 \geqq \bar{R}_0$ such that the following is true. If u is a normalized minimized of volume $\frac{4}{3}\pi R^3$ with $R \geqq R_1$ and $\rho \in \left(R_0, \frac{\varepsilon_0}{1{,}000} R\right)$ is a regular value of $|u|$, then

$$\text{(2.9)} \qquad |u(z)\, H(u)| \leqq \frac{1}{4},$$

whenever z lies in a ρ-bubble disk. For later use we note that (2.9) also holds on $u^{-1}(B_\rho)$ for $\rho \in \left(R_0, \frac{\varepsilon_0}{1{,}000} R\right)$ provided that $R \geqq R_2$ for some R_2. This is a direct consequence of Lemma 2.1. We may assume $R_1 \geqq R_2$.

Whenever the above claim is established, the desired energy estimate (2.7) follows immediately. In fact, we deduce from (2.9) and Heinz' isoperimetric inequality [10] that

$$E(u, \Omega') \leqq \frac{1}{4\pi} \frac{1+1/4}{1-1/4} (\text{length}(u|_{\partial\Omega'}))^2$$

for each ρ-bubble disk Ω'. Hence

$$E(\hat{u}, \Omega_0^*(\hat{u}, \rho) \setminus \Omega_0(\hat{u}, \rho)) \leqq 32\pi\rho^2,$$

which together with Lemma 2.2 implies (2.7). (This covers all $R \geqq R_1$. But (2.7) is automatic for $R \in [\bar{R}_0, R_1]$ on account of Lemma 2.1, if we choose c suitably.)

Now we prove the claim. Consider a sphere domain Ω for u given by Lemma 2.3. Since u is normalized, it is readily seen that Ω must contain the origin; in particular, Ω is not a ρ-bubble disk. Since $\rho \leqq \frac{\varepsilon_0}{1{,}000} R \leqq \frac{1}{1{,}000} R$ and the distance of $u(\Omega)$ from the origin is at least $\frac{1}{200} R$, it follows that Ω is disjoint from the ρ-bubble disks. Hence Lemma 2.1 and Lemma 2.4 imply for any ρ-bubble disk Ω'

$$(2.10)\quad E(u, \Omega') \leqq a_0 + 4\pi R^2 \left|1 - \frac{3}{4\pi R^3} V_0\right|^{2/3} - \frac{31}{8}\pi R^2 \leqq \frac{\pi}{4} R^2,$$

provided that R is large enough in comparison with a_0 and V_0. According to Theorem 4.1 in [23] we have for $z \in \Omega'$

$$(2.11)\quad |u(z)| \leqq \rho + \frac{1}{2\pi} E(u, \Omega') |H(u)|.$$

Combining (2.10), (2.11) with Lemma 2.1 then yields the estimate (2.9), provided that R is large enough in comparison with a_0, d_0, l_0 and V_0.

Q.E.D.

LEMMA 2.6. – *There is a constant $c > 0$ such that*

$$(2.12)\quad E(u, D \setminus D_r) \leqq \rho^{c/r}$$

for any normalized minimized u of orbitrary volume and $r \in (0, 1)$.

Proof. – In view of Lemma 2.1 we only need to consider minimizers u of volume no less than $\frac{4}{3}\pi \bar{R}_0^3$. We first observe

$$(2.13)\quad \text{dist}(0, \partial\Omega_0^*(\hat{u}, \rho) \setminus S^1) \geqq c \log \rho$$

for $\rho > 0$ and some universal constant $c > 0$. One shows this by writing $\rho = 2^m + \varepsilon$ for a natural number m and $\varepsilon \in (0, 2)$, applying Lemma 2.3 to radii 2^k, $k<m$ and summing up. (We actually apply Lemma 2.3 to regular values of $|\hat{u}|$ which approximate the radii 2^k.) The desired estimate follows from (2.13) and Lemma 2.5, because for any $\rho'>2d_0$ the coarea formula and Lemma 2.2 imply

$$\int_{\rho'/2}^{\rho'} \mathrm{L}_\rho(u)\, d\rho \leqq \frac{3}{2}\pi(\rho')^2,$$

and hence for a regular value $\rho\in[\rho'/2, \rho']$ of $|u|$ the condition (2.8) is satisfied.

Q.E.D.

We fix once for all three distinct points $z_1, z_2, z_3 \in \mathrm{S}^1$ and three distinct points $p_1, p_2, p_3 \in \Gamma$. Recall the classical *three point condition* for $u\in \mathrm{H}_\Gamma$: u maps $\{z_1, z_2, z_3\}$ bijectively to $\{p_1, p_2, p_3\}$. (We do not specify which z_i is mapped to which p_i, because we need to switch orientation to treat negative volumes, see the comment preceeding Lemma 2.2). We define

$$\mathscr{F} = \Big\{ \varphi : \varphi \text{ is a normalization for a minimizer of volume} \geqq \frac{4}{3}\pi \mathrm{R}_1^3 \text{ which satisfies the three point condition} \Big\}.$$

(R_1 was defined in the proof of Lemma 2.5.)

LEMMA 2.7. — *$\mathscr{F}$ is compact in the C^∞ topology.*

We defer the proof till the end of this section. Now we are in a position to show the convergence of minimizers with volume blowing up.

PROPOSITION 2.1. — *Let v_k be a sequence of minimizers of volume $\mathrm{V}_k \to \infty$ spanning Γ, which satisfy the three point condition. Then there are a subsequence of v_k (still denoted by v_k), a conformal diffeomorphism $\varphi : \mathrm{D} \to \mathrm{D}$ and a map $u\in \mathrm{C}^0(\mathrm{D}^*, \mathbb{R}^3) \cap \mathrm{C}^\infty(\mathring{\mathrm{D}}^*, \mathbb{R}^3)$ such that*

1. *The mappings $u_k := v_k \circ \varphi$ converge to u uniformly on $\mathrm{D}\setminus\mathring{\mathrm{D}}_r$ and strongly in $\mathrm{H}_2^1(\mathrm{D}\setminus D_r)$ for each $r\in(0,1)$, u_k also converge to u locally smoothly in $\mathring{\mathrm{D}}^*$; moreover, if Γ is a regular Jordan curve of class $\mathrm{C}^{l,\alpha}$ for some $l\geqq 1$ and $\alpha\in(0,1)$, then u_k converge to u in the C^l norm on $\mathrm{D}\setminus\mathring{\mathrm{D}}_r$ for each $r\in(0,1)$;*
2. $u_{|\mathrm{S}^1}$ parametrizes Γ *topologically and $u_{|\mathring{\mathrm{D}}^*}$ is a conformal branched minimal immersion;*
3. $\mathrm{E}_\rho(u) \leqq \frac{3}{2}\pi\rho^2$ for $\rho > \mathrm{R}_0$.

Proof. — We may assume that all v_k have volume $\geqq \frac{4}{3}\pi \mathrm{R}_1^3$. For each v_k we choose a normalization φ_k. By Lemma 2.7 we may assume by passing to a subsequence that φ_k converge to a conformal diffeomorphism

$\varphi : D \to D$. Then the energy estimate (2.12) holds for $u_k = v_k \circ \varphi$, possibly with a different constant c. By passing to a subsequence we deduce that u_k converge to a map $u \in \bigcap_{0<r<1} H_2^1(D \setminus D_r)$ weakly in $H_2^1(D \setminus D_r)$ for each $r \in (0, 1)$. To show the uniform convergence of u_k away from the origin, we first observe that the three point condition and the energy estimate (2.12) imply that $u_{k\,|\,S^1}$ are equicontinuous, *cf.* [4]. Since u_k satisfy the equation $\Delta u_k = 2\,H(u_k) \dfrac{\partial u_k}{\partial x} \wedge \dfrac{\partial u_k}{\partial y}$ and we have the estimate (2.9), we can then apply Theorem 4.1 and Theorem 4.3 in [11] to conclude that $u_{k\,|\,D \setminus D_r}$ are equicontinuous for each fixed $r \in (0, 1)$. The desired uniform convergence follows.

Once the equicontinuity is established, the local smooth convergence is standard. For the additional C^l convergence on $D \setminus \mathring{D}_r$ we refer to [20]. Taking limit in the equations $\Delta u_k = 2\,H(u_k) \dfrac{\partial u_k}{\partial x} \wedge \dfrac{\partial u_k}{\partial y}$ yields $\Delta u = 0$, hence u has zero mean curvature (u is conformal since so are u_k). That $u_{|\,S^1}$ parameterizes Γ topologically is standard. The estimate 3 follows from Lemma 2.2.

We finally show the strong convergence of u_k in $H_2^1(D \setminus D_r)$ for $r \in (0, 1)$. Passing to a subsequence we may assume that $\lim_{k \to \infty} E(u_k, D \setminus D_{1/2})$ exists. It suffices to show $E(u, D \setminus D_{1/2}) \geqq \lim_{k \to \infty} E(u_k, D \setminus D_{1/2})$. For large k we can connect $u_{k\,|\,D_{1/2}}$ with $u_{|\,D \setminus \mathring{D}_{1/2}}$ along $\partial D_{1/2}$ by a (parameterized) thin strip of area $\varepsilon_k \to 0$ and volume $\delta_k \to 0$ to obtain a map $\tilde{u}_k$ defined on D. Then

$$V(\tilde{u}_k) = V(u_k) + (V(u_{|\,D \setminus D_{1/2}}) - V(u_{k\,|\,D \setminus D_{1/2}})) + \delta_k,$$
$$V(\tilde{u}_k) = A(u_k) + (A(u_{|\,D \setminus D_{1/2}}) - A(u_{k\,|\,D \setminus D_{1/2}})) + \varepsilon_k.$$

In order to achieve $V(\tilde{u}_k) = V(u_k)$ we modify $\tilde{u}_k$ in the following way. First observe that by the weak and uniform convergence of u_k we have $\lim_{k \to \infty} V(u_{k\,|\,D \setminus D_{1/2}}) = V(u_{|\,D \setminus D_{1/2}})$, *cf.* Theorem 2.3 in [20]. Now fix k and choose a point $z \in \mathring{D}_{1/2}$, $z \neq 0$. Choose an oriented round sphere S near $\tilde{u}_k(z)$ whose volume is roughly $V(u_{k\,|\,D \setminus D_{1/2}}) - V(u_{|\,D \setminus D_{1/2}}) - \delta_k$, cut a small disk S′ from S and a small disk $D_\rho(z) \subset D_{1/2} \setminus \{ 0 \}$ from D, and then connect $S \setminus S'$ with $\tilde{u}_{k\,|\,D \setminus D_\rho(z)}$ along $\partial(S \setminus S')$ and $\partial D_\rho(z)$ by a thin tube T. The resulting map will be denoted by $\tilde{\tilde{u}}_k$. For suitable choice of S, S′, r and T we can make sure that $V(\tilde{\tilde{u}}_k) = V(u_k)$ and $A(\tilde{\tilde{u}}_k) = A(\tilde{u}_k) + \varepsilon'_k$ with $\varepsilon'_k \to 0$ as $k \to \infty$. It follows that $A(\tilde{\tilde{u}}_k) \geqq A(u_k)$. Hence

$A(u_{|D\setminus D_{1/2}}) - A(u_{k|D\setminus D_{1/2}}) + \varepsilon'_k \geqq 0$. Taking limit yields

$$E(u_{|D\setminus D_{1/2}}) \geqq \lim_{k\to\infty} E(u_{k|D\setminus D_{1/2}}).$$

Q.E.D.

We shall call u a "limit minimal surface" (spanning Γ).

PROPOSITION 2.2. – *Let u be a limit minimal surface spanning* Γ. *Then u minimizes area under constant volume constraint in the following sense: if* $v \in \bigcap_{0<r<1} H_2^1(D\setminus D_r)$, $v \in C^0(D^*, \mathbb{R}^3)$, $v_{|S^1}$ *is a weakly monotonic representation of* Γ *and* $v_{|D_r\setminus\{0\}} = u_{|D_r\setminus\{0\}}$, $V(v_{|D\setminus D_r}) = V(u_{|D\setminus D_r})$ *for some* $r \in (0, 1)$, *then* $A(u_{|D\setminus D_r}) \leqq A(v_{|D\setminus D_r})$.

It follows from a classical argument of R. Osserman ([14], [6]) that u has no true branch points in the interior, *i. e.* for each $z \in \mathring{D}^*$, u maps a neighborhood of z onto an embedded surface.

Proof of the Proposition. – Let $u_k \to u$ be as in Proposition 2.1. For large k we connect $u_{k|D_r}$ with $v_{|D\setminus\mathring{D}_r}$ along ∂D_r by a thin strip of area $\varepsilon_k \to 0$ and volume $\delta_k \to 0$ to obtain a map $\tilde{u}_k$ defined on D. Then

$$V(\tilde{u}_k) = V(u_k) + V(u_{|D\setminus D_r}) - V(u_{k|D\setminus D_r})) + \delta_k,$$
$$A(\tilde{u}_k) = A(u_k) + (A(v_{|D\setminus D_r}) - A(u_{k|D\setminus D_r})) + \varepsilon_k.$$

By the same argument as in the last part of the proof of Proposition 2.1 we can modify $\tilde{u}_k$ to obtain a suitable comparison map $\tilde{\tilde{u}}_k$. Taking limit in the inequality $A(\tilde{\tilde{u}}_k) \geqq (u_k)$ yields $A(v_{|D\setminus D_r}) \geqq A(u_{|D\setminus D_r})$.

Q.E.D.

Proof of Lemma 2.7. – Consider the three chosen points z_1, z_2, z_3 on S^1. It suffices to show $\inf_{\varphi\in\mathscr{F}} \operatorname{dist}(\varphi^{-1}(z_j)) > 0$ for each pair i, j, $i \neq j$. Assume the contrary, say we have a sequence of minimizers $u_k \in \mathscr{F}$, a normalization φ_k for each u_k such that $\operatorname{dist}(\varphi_k^{-1}(z_1), \varphi_k^{-1}(z_2)) \to 0$ as $k \to \infty$. Then the boundary values $\tilde{u}_{k|S^1}$ of $\tilde{u}_k = u_k \circ \varphi_k$ are not equicontinuous. But the convergence argument in the proof of Proposition 2.1 for the sequence u_k applies to $\tilde{u}_k$ here and shows that $\tilde{u}_{k|S^1}$ are equicontinuous.

Q.E.D.

3. THE ASYMPTOTICAL BEHAVIOR OF LIMIT MINIMAL SURFACES

We fix a limit minimal surface u and a sequence $u_k \to u$ as in Proposition 2.1.

LEMMA 3.1. – $\Omega_0(\hat{u}, \rho)$ *is a bounded subset of* $\mathbb{C}$ *for every* $\rho > 0$. *On the other hand,* $\operatorname{dist}(0, \partial\Omega_0(\hat{u}, \rho)\setminus S^1) \geqq c \log \rho$ *for some constant* $c > 0$.

Proof. – We first show $\Omega_0(\hat{u}, \rho) \neq \mathbb{C} \setminus \mathring{D}$. Otherwise, u would have finite Dirichlet energy by Proposition 2.1. Hence the origin is a removable singularity of u. In particular, $|u|_{L^\infty} \leqq d_0$ and u has finite volume. Moreover, using the Courant-Lebesgue lemma we can find radii $r_l \to 0$ and positive numbers $\varepsilon_l \to 0$ such that

$$\int_{\partial D_{r_l}} \left|\frac{\partial u}{\partial \theta}\right|^2 d\theta \leqq \varepsilon_l.$$

Then

$$\int_{\partial D_{r_l}} \left|\frac{\partial u_k}{\partial \theta}\right|^2 d\theta \leqq 2\,\varepsilon_l \qquad \text{and} \qquad |u_k(z)| \leqq 2\,d_0$$

on D_{r_l} whenever $k \geqq k_l$ for some k_l depending on l. Henceforth we only consider $k \geqq k_l$ for each l. Let v_k be the harmonic extension of $u_{k\,|\,D_{r_l}}$ to D_{r_l}. Then

$$E(v_k) \leqq \int_{D_{r_l}} \left|\frac{\partial u_k}{\partial \theta}\right|^2 d\theta \leqq 2\,\varepsilon_l \quad (cf.\ [4])$$

and

$$|V(v_k)| \leqq d_0\,\varepsilon_l.$$

Hence, if we define $\tilde{u}_k = u_k$ on $D \setminus D_{r_l}$ and $\tilde{u}_k = v_k$ on D_{r_l}, then

$$A(\tilde{u}_k) + A(u_{k\,|\,D_{r_l}}) \leqq A(u_k) + 2\,\varepsilon_l,$$
$$|V(\tilde{u}_k) + V(u_{k\,|\,D_{r_l}}) - V(u_k)| \leqq 2\,d_0\,\varepsilon_l.$$

Now we apply the isoperimetric inequality [18] to get

$$(3.1) \qquad 36\,\pi\,(V(v_k) - V(u_{k\,|\,D_{r_l}}))^2 \leqq (A(v_k) + A(u_{k\,|\,D_{r_l}}))^2.$$

Choose an oriented round sphere S of volume $V(u_{k\,|\,D_{r_l}}) - V(v_k)$ (we suppress the dependence of S on k, l). Then by (3.1)

$$\text{area}(S) \leqq A(v_k) + A(u_{k\,|\,D_{r_l}}) \leqq 2\,\varepsilon_l + A(u_{k\,|\,D_{r_l}}).$$

In order to correct the volume of $\tilde{u}_k$, we are going to "attach" S to it. Consider a disk $D_r(z) \subset \mathring{D} \setminus D_{r_l}$ such that $u_{|\,D_r(z)}$ is an embedding. Then $u(\mathring{D}_r(z))$ contains a graph G defined over a plane disk Q_ρ of radius $\rho > 0$ and contained in the cylinder $Z_\rho = Q_\rho \times (-c\,\rho^2, c\,\rho^2)$ for some $c > 0$. While keeping c fixed we shall determine ρ later on. By the convergence of u_k towards u, there is some k_0 such that if $k \geqq k_0$, then $u_{k\,|\,D_r(z)}$ is also an embedding and for some $\Omega_k \subset D_r(z)$, $u_k(\Omega_k)$ is a graph G_k over Q_ρ with $G_k \subset Z_\rho$.

For fixed l, observe

$$|V(u_{k\,|\,D \setminus D_{r_l}})| \leqq |V(u_{k\,|\,D \setminus D_{r_l}})| + 1$$

for k large enough. But $V(u_k) \to +\infty$, thus S has volume $> \frac{3}{4}\pi$ (say) for large k. We translate S so that it touches the center p of the top of Z_ρ from above. Here we assume that the normal of u_k points upwards. (For later applications we remark that, if S had negative volume, then we should bring S to the bottom of Z_ρ.) Denote by S_0 the spherical cap of S lying over Q_ρ and centered at p, and by G'_k the region of the cylinder $Q_\rho \times \mathbb{R}^1$ lying between S_0 und G_k. We replace $S_0 \cup G_k$ by G'_k and use $\tilde{u}_{k \mid D \setminus \Omega_k}$, $S \setminus S_0$ and G'_k to construct a new map $\tilde{\tilde{u}}_k : D \to \mathbb{R}^3$. Easy computations show

$$|V(\tilde{\tilde{u}}_k) - V(u_k)| \leqq O(\rho^4) + 2 d_0 \varepsilon_l,$$
$$A(\tilde{\tilde{u}}_k) \leqq A(u_k) + 4 \varepsilon_l + O(\rho^3) - O(\rho^2).$$

Now we fix an l with $\varepsilon_l \leqq O(\rho^4)$.

Then

$$|V(\tilde{\tilde{u}}_k) - V(u_k)| \leqq O(\rho^4),$$
$$A(\tilde{\tilde{u}}_k) \leqq A(u_k) + O(\rho^3) - O(\rho^2).$$

Since the volume of S has a uniform positive lower bound, we can modify it to make the volume of $\tilde{\tilde{u}}_k$ equal $V(u_k)$, while the area of $\tilde{\tilde{u}}_k$ increases at most by $O(\rho^4)$. Hence $A(\tilde{\tilde{u}}_k) < A(u_k)$, provided that ρ has been chosen small enough (independent of k, l). This contradicts the minimizing property of u_k, however.

Next we assume that $\Omega_0(\hat{u}, \rho)$ is unbounded for some ρ and let $\rho' > \rho$ be a regular value for the function $|\hat{u}|^2$. Then $\Omega_0(\hat{u}, \rho')$ is also unbounded. The maximum principle, the implicit function theorem and the fact $\Omega_0(\hat{u}, \rho') \neq \mathbb{C} \setminus \mathring{D}$ imply that $\partial\Omega_0(\hat{u}, \rho') \setminus S^1$ consist of properly embedded analytic curves running to infinity in both directions such that each compact subset of $\mathbb{C}$ meets only finitely many of them. By the Courant-Lebesgue lemma and Proposition 2.1, 3, we can find arbitrarily small circles $C_r = \partial D_r$ around the origin such that $\int_{C_r \cap \Omega_0(u, \rho')} \left|\frac{\partial u}{\partial \theta}\right| d\theta$ and hence $\int_{C_{1/r} \cap \Omega_0(u, \rho')} \left|\frac{\partial \hat{u}}{\partial \theta}\right| d\theta$ also become arbitrarily small. Choosing r small enough we can achieve that $C_{1/r}$ meets $\partial\Omega_0(\hat{u}, \rho') \setminus S^1$ at least once. Then it follows that any component of $C_{1/r} \cap \Omega_0(\hat{u}, \rho')$ has a boundary point on $\partial\Omega_0(\hat{u}, \rho') \setminus S^1$. Hence $|\hat{u}|^2$ differs arbitrarily little from $(\rho')^2$ on $C_{1/r} \cap \Omega_0(\hat{u}, \rho')$. Consequently $|\hat{u}|^2 > \rho^2$ on the outside boundary of $\Omega_0(\hat{u}, \rho') \cap D_{1/r}$ for a suitably chosen r.

This implies $\Omega_0(\hat{u}, \rho) \subset D_{1/r}$, contradicting the assumption about $\Omega_0(\hat{u}, \rho)$. Hence $\Omega_0(\hat{u}, \rho)$ is bounded.

Finally the estimate $\mathrm{dist}(0, \partial\Omega_0(\hat{u}, \rho)\setminus S^1) \geq c \log \rho$ follows from (2.7) and a limit procedure, where we may assume that ρ is a regular value of $|\hat{u}|^2$.

Q.E.D.

COROLLARY 3.1. — *Let $\rho > d_0$ be a regular value of $|u|^2$. Then $\Omega_0(u, \rho)$ is an annulus and each component of $\Omega(u, \rho)\setminus\Omega_0(u, \rho)$ is a disk.*

Proof. — The maximum principle implies that $\Omega_0(\hat{u}, \rho)$ is an annulus and each bounded component of $\partial\Omega(\hat{u}, \rho)\setminus\partial\Omega_0(\hat{u}, \rho)$ bounds a disk in $\Omega(\hat{u}, \rho)$. On the other hand, $\partial\Omega(\hat{u}, \rho)$ has no unbounded component, otherwise it would intersect $\partial\Omega_0(\hat{u}, \rho')$ for some large $\rho' > \rho$ by the lemma. The assersion follows.

Q.E.D.

LEMMA 3.2. — *For some $r_0 \in (0, 1)$, $u_{|D^*_{r_0}}$ is a stable minimal immersion of finite total Gaussian curvature.*

Proof. — Choose $0 < R < r < 1$ such that u has no branch point along the circles C_r, C_R. Let $z_1, \ldots, z_N$ be all branch points of u in $D_r\setminus D_R$ with orders $m_1, \ldots, m_N$. By the Gauss-Bonnet formula for branched surfaces [13] we have

$$2\pi \sum_{i=1}^{N} m_i - \iint_{D_r\setminus D_R} K\, d\omega = \int_{C_r} \kappa_g\, ds + \int_{C_R} \kappa_g\, ds$$

where K = the Gaussian curvature of u, $d\omega$ = the area form induced by u and κ_σ, ds denote the geodesic curvature and arclength element in the metric induced by u respectively. Letting the superscript k indicate that a geometric quantity is determined by u_k, we have

$$\begin{aligned}\int_{C_r} \kappa_g\, ds + \int_{C_R} \kappa_g\, ds &= \lim_{k\to\infty}\left(\int_{C_r} \kappa_g^k\, ds^k + \int_{C_R} \kappa_g^k\, ds^k\right)\\ &= \lim_{k\to\infty}\int_{D_r\setminus D_R} (-K^k)\, d\omega^k\\ &\leq \limsup_{k\to\infty}\int_{D_r} (-K^k)\, d\omega^k + \limsup_{k\to\infty}\int_{K^k>0} K^k\, d\omega^k.\end{aligned}$$

Clearly

$$\limsup_{k\to\infty}\int_{D_r} (-K^k)\, d\omega^k = \lim_{k\to\infty}\int_{C_r} \kappa_g^k\, ds^k - 2\pi = \int_{C_r} \kappa_g\, ds - 2\pi.$$

On the other hand, if $K^k(z) > 0$, then $K^k(z) \leq H(u_k)^2$. Hence by Lemma 2.1, $\limsup_{k\to\infty}\int_{K^k>0} K^k\, d\omega^k \leq 4\pi$.

We deduce

$$2\pi \sum_{i=1}^{N} m_i - \iint_{D_r \setminus D_R} K\, d\omega \leqq \int_{C_r} \kappa_g\, ds + 2\pi.$$

Letting $R \to +\infty$ we conclude that $\iint_{D_r^*} K\, d\omega$ is finite and u has no branch points on D_r^*, for some $r' \in (0, r)$. If the minimal immersion $u_{|D_{r'}^*}$ is stable, we are done. Otherwise there is a test function φ with supp $\varphi \subset \mathring{D}_r^*$, such that the second variation of area $\iint (|\nabla \varphi|^2 + 2K\varphi^2)\, d\omega$ is negative. Choose $r_0 \in (0, r')$ such that supp $\varphi \cap D_{r_0}^* = \varnothing$. Then $u_{|D_{r_0}^*}$ is stable.

In fact, if this were wrong, we would be able to find ψ with

$$\operatorname{supp} \psi \subset \mathring{D}_{r_0}^* \qquad \text{and} \qquad \iint (|\nabla \psi|^2 + 2K\psi^2)\, d\omega < 0.$$

But then $\iint (|\nabla \tilde{\varphi}|^2 + 2K\tilde{\varphi}^2)\, d\omega < 0$ for a linear combination $\tilde{\varphi}$ of φ, ψ with $\iint \tilde{\varphi}\, d\omega = 0$. This contradicts the minimizing property of u (Prop. 2.2), *cf.* [2].

Q.E.D.

The immersion $u_{|D_{r_0}^*}$, considered as an immersed surface (an equivalence class of immersions as well as an oriented integral varifold in the sense of [12]), will be denoted by Σ. According to Lemma 3.1, Σ is complete away from its boundary $\partial\Sigma \cong u_{|\partial D_{r_0}}$.

By Proposition 2.1, 3, Σ has quadratic area growth

$$\text{area}\,(\Sigma \cap B_\rho) \leqq \frac{3}{2}\pi\rho^2 \qquad \text{for} \quad \rho > r_0.$$

Consider the surfaces $\Sigma_\rho = \frac{1}{\rho}(\Sigma \cap B_\rho)$ contained in the unit ball B_1. It follows from Theorem 3.6.3 in [12] that any sequence Σ_{ρ_k}, $\rho_k \to \infty$ subconverges to an oriented integral 2-varifold Σ_∞ which is supported in B_1 and stationary in $\mathring{B}_1 \setminus \{0\}$. Moreover (M means *mass* [12], [16])

$$M(\Sigma_\infty \llcorner B_\rho) \leqq \frac{3}{2}\pi\rho^2 \qquad \text{for} \quad 0 < \rho \leqq 1, \tag{3.2}$$

in particular $M(\Sigma_\infty) \leqq \frac{3}{2}\pi$.

A standard cut-off function argument shows that Σ_∞ is stationary in $\mathring{B}_1$. The varifolds Σ_∞ obtained in this fashion will be called *asymptotic varifolds*.

Let Σ_∞ be an asymptotic varifold. The proof of Theorem 1.1 in [1] implies that Σ_∞ is a cone. Note that contrary to the situation in [1] we have here $\partial\Sigma_\rho \not\subset \partial B_1$. But this does not affect the argument, because the length of $\partial\Sigma_\rho \setminus \partial B_1$ converges to zero as $\rho \to \infty$. On the other hand, the curvature estimates for stable minimal surfaces obtained by R. Schoen in [15] and (3.2) imply that $\Sigma_\infty \cap (\mathring{B}_1 \setminus \{0\})$ is a properly immersed minimal surface. It follows that Σ_∞ is a cone on the sum of finitely many great circles in ∂B_1. By the area estimate (3.2), Σ_∞ must be a flat disk of multiplicity 1.

Hence we can apply the results of L. Simon [17] to obtain.

LEMMA 3.3. — *Σ_∞ is the unique asymptotic varifold and Σ_ρ converges to Σ_∞ smoothly as $\rho \to \infty$. In particular, $\Sigma \setminus B_{\rho_0}$ for some $\rho_0 > 0$ is a graph (of multiplicity* 1) *over the plane containing Σ_∞ and the unit normal of Σ has an asymptotical limit at infinity.*

This lemma, the next one and Lemma 3.1 establish the asymptotic behavior of u as stated in the Main Theorem.

LEMMA 3.4. — *We have*

1. *if $\rho > 0$ is large enough, then $\Omega(u, \rho) = \Omega_0(u, \rho)$;*
2. *if $r > 0$ is small enough, then $u_{|D_r^*}$ is an embedding.*

Proof. — 1. If $\rho > \rho_0$ is large enough, then ρ is a regular value of $|u|^2$ and image $(u) \cap \partial B_\rho$ is a smooth Jordan curve γ according to Lemma 3.1, 3.2 and 3.3. Consequently, u maps the inside boundary of $\Omega_0(u, \rho)$ onto γ. If $\Omega(u, \rho)$ has a component $\Omega \neq \Omega_0(u, \rho)$, then u also maps $\partial\Omega$ onto γ. This contradicts the fact that $\Sigma \setminus B_{\rho_0}$ is a graph (with multiplicity 1).

2. This follows from 1, Lemma 3.1 and the fact that $\Sigma \setminus B_{\rho_0}$ is an embedded surface of multiplicity 1.

Q.E.D.

To give a complete proof of the Main Theorem, there only remains the following last step.

PROPOSITION 3.1. — *u is an immersion in the interior.*

Proof. — Choose a regular value $\rho > d_0$ of $|u|^2$ such that u is an embedding on $D^* \setminus \mathring{\Omega}_0(u, \rho)$. Recall that $M = \Omega_0(u, \rho)$ is a smooth annulus region with outside boundary S^1. Assume that u has branch points in $\mathring{M}$. According to [7] and the fact that u has no true branch points in the interior (Prop. 2.2) there is an orientation preserving branched covering

$\pi : M \to M'$ together with a commutative diagram,

$$\begin{array}{ccc} M & \xrightarrow{u} & \mathbb{R}^3 \\ \downarrow & \nearrow & \\ M' & & u' \end{array}$$

where the quotient space M' is a compact and oriented surface possibly with boundary and u' is an immersion in $\mathring{M}'$ and continuous on $\bar{M}'$.

Let L denote the inside boundary of M. Since $u(L)$ is extreme ($u(L) \subset \partial B_\rho$ and $\rho > d_0$) and $u_{|L}$ is an embedding, $\pi_{|L}$ is a diffeomorphism onto a boundary component L' of M'. We distinguish between two cases.

Case 1. – $\partial M'$ has a second component.

In this case M' is an annulus and π maps ∂M onto $\partial M'$. Hence the generalized Riemann-Hurwitz formula [8] $\chi(M) + O_\pi = \deg \pi . \chi(M')$ implies $O_\pi = 0$, where O_π denotes the total branch order of π.

We arrive at a contradiction.

Case 2. – $\partial M' = L'$.

Then M' is a disk. The Jordan curve $L'' = \pi(S^1)$ bounds a (closed) dis Ω_1 in $\mathring{M}'$. Since π is an embedding along L, we have $\deg(\pi, \mathring{M}, z) = 1$ for any $z \in \mathring{\Omega}_2$, where $\Omega_2 = M' \setminus \mathring{\Omega}_1$. Because L'' must have preimage points in $\mathring{M}$, we obtain $\deg(\pi, \mathring{M}, z) = 2$ for all $z \in \mathring{\Omega}_1$. It follows that

$$V(u_{|M}) = V(u'_{|\Omega_2}) + 2V(u'_{|\Omega_1}),$$
$$A(u_{|M}) = A(u'_{|\Omega_2}) + 2A(u'_{|\Omega_1}).$$

Now we choose an orientation preserving diffeomorphism $\varphi : M \to \Omega_2$ with $\varphi_{|L} = \pi_{|L}$ and define $\tilde{u} : D^* \to \mathbb{R}^3$ via $\tilde{u}_{|D^* \setminus M} = u_{|D^* \setminus M}$, $\tilde{u}_{|M} = u'_{|\Omega_2} \circ \varphi$. Then $\tilde{u}_{|S^1} = u'_{|L''} \circ \varphi_{|S^1}$ is a topological representation of Γ. Moreover,

$$V(\tilde{u}_{|M}) = V(u_{|M}) - 2V(u'_{|\Omega_1}), \tag{3.3}$$
$$A(\tilde{u}_{|M}) = A(u_{|M}) - 2A(u'_{|\Omega_1}). \tag{3.4}$$

Next we choose a smooth disk $v : D \setminus \mathring{M} \to \mathbb{R}^3$ with $v_{|L} = u_{|L}$ and define v_1, $v_2 : D \to \mathbb{R}^3$ via $v_{1\,|D \setminus M} = v$, $v_{1\,|M} = u_{|M}$; $v_{2\,|D \setminus M} = v_{2\,|M} = \tilde{u}_{|M}$. According to [19] there is an integral 3-current I with $\partial I = J_{v_1} - J_{v_2}$ and $V(I) = V(v_1) - V(v_2)$, where J_{v_i} denotes the integral 2-current induced by v_i and V(I) is the *oriented volume* of I, defined as $I(dx_1 \wedge dx_1 \wedge dx_3)$. By the construction of v_1, v_2 and the fact about the degree of π_1 we have

$$V(I) = V(u_{|M}) - V(u'_{|\Omega_2}) = V(u_{|M \setminus \pi^{-1}(\Omega_2)}) = 2V(u'_{|\Omega_1}),$$
$$\partial I = J_{u_{|M}} - J_{u'_{|\Omega_2}} = J_{u_{|M}} - J_{u_{|\pi^{-1}(\Omega_2)}} = J_{u_{|M \setminus \pi^{-1}\Omega_2)}}.$$

Hence the mass of ∂I, $M(\partial I)$ satisfies

$$M(\partial I) \leqq A(u_{|M \setminus \pi^{-1}(\Omega_2)}) = 2A(u'_{|\Omega_1}).$$

Note $V(u'_{|\Omega_1}) \neq 0$, otherwise (3.3) and (3.4) would contradict the minimizing property of u. Let S be an oriented round sphere of volume V(I). The classical isoperimetric inequality [5], 4.5.9(31), implies $A(S) \leqq 2A(u'_{|\Omega_1})$. Following the arguments in the proof of Lemma 3.1 we attach S to $\tilde{u}_{|M}$ to make the volume of $\tilde{u}_{|M}$ equal to $V(u_{|M})$ and its area strictly less than $A(u_{|M})$. We again arrive at a contradiction.

Q.E.D.

The above arguments and the techniques in [9] and [6] also give the following additional information.

PROPOSITION 3.2. – *u has no* ramification point (cf. [18] *for definition*) *along* S^1. *If* Γ *is real analytic, then u is immersed up to boundary.*

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