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On the existence of multiple geodesics in static space-times (1)

by

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ABSTRACT. — In this paper we study the problem of the existence of geodesics in static space-times which are a particular case of Lorentz manifolds. We prove multiplicity results about geodesics joining two given events and about periodic trajectories having a prescribed period.

Key words : Lorentz metrics, geodesics, critical point theory.

RÉSUMÉ. — Dans ce papier on étudie le problème de l'existence de géodésiques dans les espaces-temps statiques qui représentent un cas particulier de variétés de Lorentz. On démontre ici des résultats de multiplicité pour les géodésiques qui relient deux events fixés et pour les trajectoires périodiques ayant une période fixée.

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1. INTRODUCTION AND STATEMENT OF THE RESULTS

Let $\mathfrak{M}_0$ be a connected n-dimensional manifold of class C^k $(k \geqq 3)$ and set

$$\mathfrak{M} = \mathfrak{M}_0 \times \mathbb{R}. \tag{1.1}$$

Let $\langle .,. \rangle$ be a C^k (positive definite) Riemannian metric on $\mathfrak{M}_0$ and β be a positive, smooth scalar field on $\mathfrak{M}_0$. We set

$$g = \langle .,. \rangle \oplus (-\beta)\, dt^2$$

that is

$$g(P)\left[\begin{pmatrix}\xi\\ \tau\end{pmatrix}, \begin{pmatrix}\xi\\ \tau\end{pmatrix}\right] = \langle \xi, \xi \rangle - \beta(P)\tau^2, \qquad \xi \in T_P\mathfrak{M}_0, \quad \tau \in \mathbb{R}, \tag{1.2}$$

where $T_P\mathfrak{M}_0$ denotes the tangent space to $\mathfrak{M}_0$ at P. g is a Lorentz metric, *i. e.* a pseudo-Riemannian metric with index 1.

The manifold $\mathfrak{M}$ equipped with the Lorentz metric g is called a static spacetime (for many of the definitions given here we refer to [18] and its references).

$\mathfrak{M}$ is called complete if each geodesic $\gamma(s)$ ([2]) can be extended (in $\mathfrak{M}$) for all real values of the affine path parameter s.

$\mathfrak{M}$ is called geodesically connected if given two events $Q_1, Q_2 \in \mathfrak{M}$ there exists a geodesic $\gamma: [0,1] \to \mathfrak{M}$ such that

$$\gamma(0) = Q_1, \qquad \gamma(1) = Q_2. \tag{1.3}$$

If γ is a geodesic there exists a constant E_γ such that

$$E_\gamma = g(\gamma(s))[\dot\gamma(s), \dot\gamma(s)] \quad \text{for all } s. \tag{1.4}$$

A geodesic γ is called space-like, null or time-like if E_γ is respectively greater, equal or less then zero.

γ is called non degenerate if $\gamma(0)$ and $\gamma(1)$ are not conjugate points (*i. e.* there does not exist a non-zero Jacobi field J along γ which vanishes for $s=0$ and $s=1$).

If $(P(s), t(s))$ is a geodesic joining (P_1, t_1), since the metrice tensor g is independent of t, $(P(s), t(s)+\tau)$ is a geodesic joining $(P_1, t_1+\tau)$ and $(P_2, t_2+\tau)$. Therefore the number of geodesics joining two events (P_1, t_1) and (P_1, t_2) depends only on P_1, P_2 and $|t_2 - t_1|$.

We shall denote with

$$N(P_1, P_2, |t_2 - t_1|)$$

([2]) We recall that $\gamma(s)$ is a geodesic if $\nabla_s \dot\gamma(s) = 0$ for all s, where $\dot\gamma(s)$ is the tangent vector to γ at $\gamma(s)$, and $\nabla_s \dot\gamma(s)$ is the covariant derivative of $\dot\gamma(s)$ in the direction of $\dot\gamma(s)$, with respect the Lorentz metric g.

the number of *time-like* geodesics joining (P_1, t_1) and (P_2, t_2).

Unlike the situation for positive definite Riemannian spaces, there are complete, static spacetime manifolds which are not geodesically connected (Anti-de Sitter space [15], [21]).

In this paper we shall prove the following theorem:

THEOREM 1.1. – *Assume the Riemannian manifold $\mathfrak{M}_0$ to be connected and complete, and β to be a C^2 scalar field on $\mathfrak{M}_0$ satisfying the following assumption:*

$$\text{there exist } \nu, M>0 \text{ such that } M \geqq \beta(P) \geqq \nu \text{ for all } P \in \mathfrak{M}_0. \tag{1.5}$$

Then $\mathfrak{M}$, equipped with the metric g defined by (1.2), *is geodesically connected.*

Moreover, setting $\Delta = |t_2 - t_1|$, we have:

(i) *for every $P_1, P_2 \in \mathfrak{M}_0$ with $P_1 \neq P_2$, there exists Δ sufficiently small such that $N(P_1, P_2, \Delta) = 0$.*

(ii) *if all the time-like geodesics are non-degenerate, then $N(P_1, P_2, \Delta)$ is finite for all $P_1, P_2 \in \mathfrak{M}_0$, for all $\Delta \in \mathbb{R}$.*

Now assume that $\mathfrak{M}_0$ is not contractible in itself and there exists a retraction of $\mathfrak{M}_0$ on a compact subset.

Then, for any two given points $Q_1 = (P_1, t_1)$, $Q_2 = (P_2, t_2) \in \mathfrak{M}$, there exists a sequence $\{\gamma_n\}_{n \in \mathbb{N}}$ of geodesics joining Q_1 and Q_2 such that

$$\lim_{n \to +\infty} E_{\gamma_n} = +\infty$$

Moreover

(iii) $\lim_{\Delta \to +\infty} N(P_1, P_2, \Delta) = +\infty$ for every $P_1, P_2 \in \mathfrak{M}_0$.

Remark 1.2. – Some results of theorem 1.1 have been proved in [6], [7], [8] for stationary spacetime manifolds. Moreover (ii) can be also obtained by standard techniques of Lorentzian Geometry (*see* e. g. [4]).

The proof we give here holds for static space-times and it uses a variational principle, stated in section 2, which permits to overcome the difficulties arising from the indefiniteness of the metric g.

Cases in which $\mathfrak{M}_0$ is not complete are considered in [9].

If $\gamma : [0,1] \to \mathfrak{M}$ is a smooth curve and $Q_1, Q_2 \in \mathfrak{M}$, we set

$$l(\gamma) = \int_0^1 \sqrt{|g(\gamma(s))[\dot{\gamma}(s), \dot{\gamma}(s)]|}\, ds \tag{1.6}$$

If $\mathfrak{M}$ is geodesically connected we define

$$d(Q_1, Q_2) = \inf\{\, l(\gamma) : \gamma \text{ is a geodesic joining } Q_1, Q_2 \,\}. \tag{1.7}$$

$d(Q_1, Q_2)$ is called the geodesic distance between the events Q_1, Q_2.

The same estimates used in proving Theorem 1.1 permit to deduce the following result:

THEOREM 1.3. – *Let* $\mathfrak{M}$ *and* g *be as in Theorem* 1.1. *Then for any* $Q_1, Q_2 \in \mathfrak{M}$ *there exists a geodesic* γ *joining* Q_1 *and* Q_2 *such that* $d(Q_1, Q_2) = l(\gamma)$.

Let us now give the following definition:

DEFINITION 1.4. – *Let* $\gamma(s)$, $s \in [0,1]$ *be a smooth curve on* $\mathfrak{M} = \mathfrak{M}_0 \times \mathbb{R}$. *We denote by* $x(s)$ and $t(s)$ *the components of* γ *on* $\mathfrak{M}_0$ *and* $\mathbb{R}$ *respectively.*

γ *is called a* T-*periodic trajectory if it is a geodesic for* g *and it satisfies the conditions*

$$\left.\begin{array}{ccc} x(0)=x(1), & \dot{x}(0)=\dot{x}(1), & \\ t(0)=0, & t(1)=T, & \dot{t}(0)=\dot{t}(1). \end{array}\right\} \tag{1.8}$$

We say that γ *is non trivial if* $x(s)$ *is not constant.*

Moreover two periodic trajectories $\gamma_1(s)$ *and* $\gamma_2(s)$ *are said to be geometrically distinct if the supports of the curves* γ_1 *and* γ_2 *are different* (i. e. $\gamma_1([0,1]) \neq \gamma_2([0,1])$).

It is easy to see that if $Q \in \mathfrak{M}_0$ is a stationary point for β then $\gamma(s) = (Q, sT)$ is a trivial T-periodic trajectory.

The existence of one timelike non trivial T-periodic trajectory has been recently proved in [7], [13], [14]. Here we shall prove the existence of infinitely many, geometrically distinct, T-periodic trajectories having energy arbitrarily large.

Moreover we prove the analogous of (ii) of Theorem 1.1, essentially in the same way.

More precisely we have the following result:

THEOREM 1.5. – *Let* $\mathfrak{M}$ *and* g *be as in Theorem* 1.1. *Moreover assume that* $\mathfrak{M}_0$ *is compact and* $\pi_1(\mathfrak{M}_0)$ *is finite. Then for every* $T>0$ *there exists a sequence* $\{\gamma_n\}_{n \in \mathbb{N}}$ *of non trivial, geometrically distinct,* T-*periodic trajectories on* $\mathfrak{M}$ *such that*

$$\lim_{n \to +\infty} E_{\gamma_n} = +\infty$$

Moreover if all the T-*periodic trajectories are non degenerate, only finitely many are time-like.*

Of course in the periodic case the concept of nondegeneracy needs a small modification: a periodic orbit γ is called nondegenerate if there does not exist a non zero 1-periodic Jacobi field along γ.

When $\mathfrak{M}_0$ is not compact the results of theorem 1.5 do not hold in general. In fact if $\mathfrak{M}$ is the Minkowskii space-time [*i. e.* $\mathfrak{M} \equiv \mathbb{R}^{n+1}$ and in (1.2) $\langle\ ,\ \rangle$ is the euclidean metric and $\beta(P) \equiv 1$] the only T-periodic trajectories are the trivial ones $\gamma = (P, sT)$, $P \in \mathbb{R}^n$).

Nevertheless the following multiplicity result holds:

THEOREM 1.6. — *Let* $\mathfrak{M}_0 = \mathbb{R}^n$ *and assume that the Riemannian metric*

$$\langle \xi, \eta \rangle = \sum_{ij} \alpha_{ij}(x)\, \xi_i\, \eta_j, \qquad i, j = 1, \ldots, n$$

on $\mathfrak{M}_0$ *and the function* β *satisfy the following assumptions:*

$$\begin{gathered} \alpha_{ij}, \beta \in C^2(\mathbb{R}^n, \mathbb{R}); \\ \forall\, i, j, k \in \{1, \ldots, n\}, \qquad \lim_{|x| \to +\infty} \frac{\partial \alpha_{ij}}{\partial x_k}(x) = 0 \\ \exists\, \nu_0 > 0\colon \ \sum_{i,j} \alpha_{ij}(x)\, \xi_i\, \xi_j \geqq \nu_0 \cdot \big(\sum_i \xi_i^2\big); \end{gathered} \tag{1.9}$$

$$\lim_{|x| \to +\infty} \beta(x) = \sup_{\mathfrak{M}_0} \beta := d_0, \qquad \lim_{|x| \to +\infty} \operatorname{grad} \beta = 0; \tag{1.10}$$

$$c_0 := \sup_{\mathbb{R}^n} \{ \beta(x) : \operatorname{grad} \beta(x) = 0 \} < d_0; \tag{1.11}$$

where $|\ |$ *is the Euclidean norm and* $\operatorname{grad}\beta(x)$ *is the gradient of* β *with respect the Euclidean scalar product in* $\mathbb{R}^n$.

Then for any $m \in \mathbb{N}$ *there exists* $\bar{T} = \bar{T}(m)$ *such that for all* $T \geqq \bar{T}$ *there exist at least m non trivial, time-like, geometrically distinct,* T-*periodic trajectories* $\gamma_1, \ldots, \gamma_m$ *such that, for every* $i = 1, \ldots, m$,

$$-d_0 T^2 < E_{\gamma_i} < -c_0 T^2.$$

Remark 1.7. — These assumptions are e. g. satisfied by a universe containing a unique massive star whose radius is larger than the Schwartzchild radius.

In the literature there are not many global results about the existence of geodesics for Lorentz manifolds. The only results we know, besides the ones we have quoted earlier, are due to Avez [2], Galloway ([11], [12]), Seifert [24], Tipler [26] and Uhlenbeck [27].

The papier is organized as follows:

In section 2 we introduce a variational principle (*see* Theorems 2.1 and 2.2) which will be systematically used in the subsequent sections. This variational principle allows the use of the "classical" Liusternik-Schnirelmann theory for infinite dimensional manifolds (*see* e. g. [23]) in Theorem 1.1, the cohomology of the "free loop space" on $\mathfrak{M}_0$ (*see* e. g. [28]) in Theorem 1.5, and the use of more recent techniques of the critical points theory (*see* e. g. [22]) in Theorem 1.6.

In section 3 we introduce the functional setting for the study of the critical points of the action integral related to the metric *g*. In sections 4 and 5 we prove theorems 1.1, 1.3, 1.5, 1.6.

2. THE VARIATIONAL PRINCIPLE

Consider the static spacetime manifold $\mathfrak{M}$ equipped with the Lorentz metric g [*see* (1.1) and (1.2)] and the action functional

$$f(\gamma)=\int_0^1 g(\gamma(s))[\dot\gamma(s),\dot\gamma(s)]\,ds$$
$$=\int_0^1 (\langle \dot x(s),\dot x(s)\rangle_{x(s)}-\beta(x(s))(\dot t(s))^2)\,ds \qquad (2.1)$$

where $\gamma(s)=(x(s),\,t(s))$, $s\in[0,1]$, is a smooth curve on $\mathfrak{M}$.

Let $Q_1=(P_1,\,t_1)$, $Q_2=(P_2,\,t_2)$ be two points in $\mathfrak{M}$. The geodesics joining Q_1 and Q_2 are the critical points of f with the condition (1.3); namely they are the smooth curves $\gamma:[0,1]\to\mathfrak{M}$ joining Q_1 and Q_2 and satisfying

$$\int_0^1 g(\gamma(s))\left[\dot\gamma(s),\nabla_s\begin{pmatrix}v(s)\\ \tau(s)\end{pmatrix}\right]ds=0, \qquad (2.2)$$

for any C_0^∞ vector field $\begin{pmatrix}v\\ \tau\end{pmatrix}$ along γ. Here $\nabla_s\begin{pmatrix}v\\ \tau\end{pmatrix}$ denotes the covariant derivative of $\begin{pmatrix}v\\ \tau\end{pmatrix}$ with respect the metric g and along γ.

Analogously the T-periodic trajectories (*cf.* Definition 1.4) are the stationary points of f with the condition (1.8), namely they are the smooth curves $\gamma(s)=(x(s),\,t(s))$ on $\mathfrak{M}$ which verify (1.8) and satisfy (2.2) for any C^∞ vector field $\begin{pmatrix}v\\ \tau\end{pmatrix}$ along γ, when v is 1-periodic and $\tau(0)=0$, $\tau(1)=0$.

Since the metric g is indefinite, the action functional (2.1) is unbounded both from below and from above and this causes difficulties in the research of the geodesics for g. Nevertheless the study of the stationary points of f can be reduced to the study of the stationary points of a suitable functional which is bounded from below when β is bounded.

Infact let $Q_1=(P_1,\,t_1)$, $Q_2=(P_2,\,t_2)$ be two points in $\mathfrak{M}$ and consider the functional

$$J(x)=\int_0^1 \langle \dot x(s),\dot x(s)\rangle_{x(s)}\,ds-\Delta^2\left[\int_0^1 \sigma(x(s))\,ds\right]^{-1} \qquad (2.3)$$

where $\sigma=\dfrac{1}{\beta}$, $\Delta=t_2-t_1$ and $x:[0,1]\to\mathfrak{M}_0$ is a smooth curve on $\mathfrak{M}_0$ with $x(0)=P_1$, $x(1)=P_2$ and $\dot x(s)$ is the tangent vector to x at $x(s)$.

Observe that (2.3) is bounded from below if β is bounded.

The following theorems holds.

THEOREM 2.1. — *Let* $\gamma(s)=(x(s), t(s))$, $s\in[0,1]$, *be a smooth curve on* $\mathfrak{M}$ *satisfying* (1.3).

Then the following statements are equivalents:

(i) γ *is a stationary point of f with the condition* (1.3);

(ii) x *is a stationary point for* J *with the condition* (1.3), i. e.

$$J'(x)[v]=\int_0^1 2\langle \dot{x}, \nabla_s v\rangle_{x(s)}\, ds + \int_0^1 \left(\Delta^2\left[\int_0^1 \sigma(x)\, ds\right]^{-2}\right)\sigma'(x)[v]\, ds=0 \quad (^3) \tag{2.4}$$

for any C_0^∞ *vector field* v *along* x, *and* $t=t(s)$ *solves the problem*

$$\left.\begin{aligned} \dot{t}=\Delta\left[\int_0^1 \sigma(x)\, ds\right]^{-1} \sigma(x) \\ t(0)=t_1 \end{aligned}\right\} \tag{2.5}$$

Moreover, if (i) [or (ii)] *is satisfied, we have*

$$f(\gamma)=J(x). \tag{2.6}$$

Proof. — (i) $\Rightarrow$ (ii). Let

$$\gamma(s)=(x(s), t(s))$$

be a stationary point of f with respect the condition (1.3); then

$$f'(\gamma)\left[\begin{pmatrix} v \\ \tau \end{pmatrix}\right]=\int_0^1 \langle \dot{x}, \nabla_s v\rangle_x - \dot{t}^2\, \beta'(x)[v] - 2\,\beta(x)\,\dot{t}\,\dot{\tau}=0 \tag{2.7}$$

for all C_0^∞ vector field $\begin{pmatrix} v \\ \tau \end{pmatrix}$ along γ.

Taking $v=0$ in (2.7) we have

$$\int_0^1 \beta(x)\,\dot{t}\,\dot{\tau}\, ds=0 \quad \text{for all } \tau\in C_0^\infty([0,1], \mathbb{R}), \tag{2.8}$$

then there exists a constant c such that

$$\dot{t}(s)=\frac{c}{\beta(x(s))} \quad \text{for all } s\in[0,1]. \tag{2.9}$$

Integrating in [0,1] we get

$$c=\Delta\left[\int_0^1 \sigma(x(s))\, ds\right]^{-1} \quad \text{where } \sigma=\frac{1}{\beta}, \qquad \Delta=t_2-t_1. \tag{2.10}$$

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(3) Here $\sigma'(\varphi(s))$ denotes the gradient of σ at $\varphi(s)$ and ∇_s the covariant derivative with respect the Riemannian metric $\langle\ ,\ \rangle$.

By (2.9) and (2.10) we deduce that $t=t(s)$ solves (2.5). Now, if we substitute (2.5) in (2.7) and choose $\tau=0$, we see that (2.4) is satisfied.

(ii) ⇒ (i). Suppose that x solves (2.4) and defines c as in (2.10). Then, since t solves (2.5), we get (2.9) and consequently (2.8).

Now if in (2.4) we add (2.8) and substitute $\Delta^2\left[\int_0^1 \sigma x)\,ds\right]^{-2}$ by (2.5), we see that $\gamma=(x, t)$ satisfies (2.7), namely it is a stationary point of f.

Finally (2.6) is immediately checked. ■

An analogous result holds for the T-periodic trajectoires.

THEOREM 2.2. – *Let* $\gamma(s)=(x(s), t(s))$, $s\in[0,1]$, *be a smooth curve on* $\mathfrak{M}$ *satisfying* (1.8). *Then the following statements are equivalent:*

(i) γ *is a stationary point of* f *with respect the condition* (1.8).

(ii) x *is a stationary point for the functional* J (with $\Delta=T$) *on the smooth curves* $x(s)\subset\mathfrak{M}_0$ *which satisfy* (1.8). *Moreover* $t=t(s)$ *solves the problem:*

$$\left.\begin{array}{c} t=T\left[\int_0^1 \sigma(x)\,ds\right]^{-1}\sigma(x) \\ t(0)=0 \end{array}\right\} \tag{2.11}$$

Moreover, if (i) *or* (ii) *is satisfied, we have*

$$f(\gamma)=J(x).$$

The proof of Theorem 2.2 is the same as the proof of Theorem 2.1.

3. THE FUNCTIONAL FRAMEWORK

By a well known result of Nash (*see* [17]) $\mathfrak{M}_0$, equipped with the Riemannian metric $\langle\ ,\ \rangle$, is isometrically embeddable in $\mathbb{R}^N$ with N sufficiently large. This means that there exists a C^k mapping

$$\Psi_0:\ \mathfrak{M}_O\to\mathbb{R}^N \tag{3.1}$$

with injective differential $\Psi'_0(P)$ for all $P\in\mathfrak{M}_0$, such that

$$\langle v, v\rangle_P=\langle \Psi'_0(P)\,v, \Psi'_0(P)\,v\rangle \tag{3.2}$$

for all $P\in\mathfrak{M}_0$ and $v\in T_P\mathfrak{M}_0$. Here $\langle\ .,.\ \rangle$ denotes the Euclidean inner product in $\mathbb{R}^N$.

Our aim is to get a more explicit form of the functional (2.1) using the embedding Ψ_0.

In the following we set

$$M_0=\Psi_0(\mathfrak{M}_O),\qquad x=\Psi_0(P),\qquad \xi=\Psi'_0(P)\,v, \tag{3.3}$$

where $P\in\mathfrak{M}_0$ and $v\in T_P\mathfrak{M}_0$.

We improperly set

$$\beta(x)=\beta(P), \qquad \text{where} \quad x=\Psi_0(P).$$

Moreover we shall identify $\mathbb{R}^N$ with its dual space. Thus we have that $T_x M_0 \subset \mathbb{R}^N$.

Then the study of the geodesics on $\mathfrak{M}$ for the metric g is equivalent to the study of the geodesics on $M_0 \times \mathbb{R}$ for the metric $\tilde{g}$, defined by

$$\tilde{g}(x)\left[\begin{pmatrix}\xi_1\\ \tau_1\end{pmatrix}, \begin{pmatrix}\xi_2\\ \tau_2\end{pmatrix}\right]=\langle \xi_1, \xi_2\rangle-\beta(x)\tau_1\tau_2, \tag{3.4}$$

where $x\in M_0$, ξ_1, $\xi_2\in T_x M_0$ and τ_1, $\tau_2\in\mathbb{R}$.

We now introduce our functional setting.

If $I=[0, 1]$, $W^1(I, \mathbb{R}^N)$ denotes the ordinary Sobolev space of $\mathbb{R}^N$-valued functions defined on I.

We denote by

$$\|x\|_1=\left(\int_0^1 (|\dot{x}(s)|^2+|x(s)|^2)\,ds\right)^{1/2}$$

its norm. Here the dot " . " denotes the derivative with respect to s and $|\ |$ the Euclidean norm in $\mathbb{R}^N$.

Let x_1, $x_2\in M_0$ and introduce the space of the W^1-curves on M joining x_1 and x_2.

$$\begin{aligned}\Omega^1 &:=\Omega^1(M_0, x_1, x_2)\\ &=\{x\in W^1(I, \mathbb{R}^N): \forall s\in I,\ x(s)\in M_0, \quad \text{and} \quad x(0)=x_1,\ x(1)=x_2\}.\end{aligned} \tag{3.5}$$

Ω^1 is a closed submanifold of $W^1(I, \mathbb{R}^N)$ (*cf.* e. g. [23] and its references) and its tangent space at $x\in\Omega^1$ is given by

$$T_x\Omega^1=\{\xi\in W_0^1(I, \mathbb{R}^N): \xi(s)\in T_{x(s)}M_0, s\in I\} \tag{3.6}$$

where

$$W_0^1(I, \mathbb{R}^N)=\{\xi\in W^1(I, \mathbb{R}^N): \xi(0)=\xi(1)=0\}.$$

Analogously we define the space of the W^1-closed curves on M_0

$$\Lambda^1:=\Lambda^1 M_0=\{x\in W^1(S^1, \mathbb{R}^N): \forall s\in S^1,\ x(s)\in M_0\}, \tag{3.7}$$

($S^1=\mathbb{R}/\mathbb{Z}=I/\{0, 1\}$), whose tangent space at $x\in\Lambda^1$ in given by

$$T_x\Lambda^1=\{\xi\in W^1(S^1, \mathbb{R}^N): \xi(s)\in T_{x(s)}M_0, s\in S^1\}. \tag{3.8}$$

On the following we shall use the notation W^1 to denote $W^1(I, \mathbb{R}^N)$ or $W^1(S^1, \mathbb{R}^N)$ depending on the case.

If M_0 is complete, Ω^1 and Λ^1 are complete Riemannian manifolds with the Riemannian structure inherited from W^1, *see* e. g. [1], [20].

Now we recall a technical Lemma which will be useful to prove that the functional J [defined by (2.3)] on Ω^1 and Λ^1 satisfies the Palais-Smale condition (*cf.* [8], lemma 2.1].

LEMMA 3.2. — *Let* $\{x_k\}$ *be a sequence in* Ω^1 (*or* Λ^1) *such that*

$$x_k \text{ converges to } x_0 \text{ weakly in } W^1.$$

Then there exist two sequences

$$\{\xi_k\} \subset T_{x_k}\Omega^1 \qquad (or \quad T_{x_k}\Lambda^1)$$

and

$$\{\nu_k\} \subset W_0^1(I, \mathbb{R}^N) \qquad [or\, W^1(S^1, \mathbb{R}^N)]$$

such that

$$x_k = \xi_k + \nu_k + x_0,$$

ξ_k *converges to* 0 *weakly in* W^1, *and* ν_k *converges to* 0 *strongly in* W^1.

4. PROOF OF THEOREMS 1.1, 1.3, 1.5

By Theorems 2.1 and 2.2 the study of the geodesics for g joining two given events $Q_1, Q_2 \in \mathfrak{M}$, or the study of the T-periodic trajectories on $\mathfrak{M}$, is equivalent to the study of the critical points of the functional J [*cf*. (2.3)] on Ω^1 or on Λ^1.

Now we introduce the well known condition (C) of Palais-Smale.

DEFINITION 4.1. — *Let* X *be a Riemannian manifold modelled on a Hilbert space and let* $F \in C^1(X, \mathbb{R})$. *We say that* F *satisfies* (P.S.) *if any sequence* $\{z_n\} \subset X$ *such that* $F(z_n)$ *is bounded and* $F'(z_n) \to 0$, *possesses a convergent subsequence*.

We shall prove that the functional J defined in (2.3) satisfies (P.S.). More precisely the following Lemma holds

LEMMA 4.2. — *Let* β *satisfy assumption* (1.5). *Then* J *satisfies* (P.S.) *on* Ω^1. *If we assume also that* $\mathfrak{M}_0$ (*and therefore* M_0) *is compact, then* J *satisfies* (P.S.) *on* Λ^1.

Proof. — It will be useful to use $\mathcal{O}^1$ as joint notation for the space Ω^1 and Λ^1.

Let $\{x_n\} \subset \mathcal{O}^1$ such that

$$\exists K \in \mathbb{R} \quad \text{s.t.}\, \forall n \in \mathbb{N}, \qquad J(x_n) \leqq K, \tag{4.1}$$

$$\frac{J'(x_n)[\xi]}{\|\xi\|_1} \to 0 \quad \text{uniformely for } \xi \in T_{x_n}\mathcal{O}^1. \tag{4.2}$$

From (4.1) we deduce that

$$\forall n \in \mathbb{N}, \quad \int_0^1 \langle \dot{x}_n, \dot{x}_n \rangle \leqq \Delta^2 \left[\int_0^1 \sigma(x_n)\, ds \right]^{-1} + K;$$

and, since σ is bounded, we get that

$$\int_0^1 \langle \dot{x}_n, \dot{x}_n \rangle \, ds \quad \text{is bounded independently of } n. \tag{4.3}$$

If $\mathcal{O}^1 = \Omega^1$ or if $\mathcal{O}^1 = \Lambda^1$ and M_0 is compact, (4.3) implies that

$$\{x_n\} \text{ is bounded in } W^1 \text{ independently of } n.$$

Then, passing eventually to a subsequence, we get that

$$x_n \text{ converges to } x \text{ weakly in } W^1. \tag{4.4}$$

Now, by (4.2), we have

$$2\int_0^1 \langle \dot{x}_n, \dot{\xi} \rangle \, ds + \Delta^2 \left[\int_0^1 \sigma(x_n) \, ds \right]^{-2} \int_0^1 \langle \operatorname{grad} \sigma(x_n), \xi \rangle \, ds = o(1) \tag{4.5}$$

uniformly for $\xi \in T_{x_n} \mathcal{O}^1$ and $\|\xi\|_1 \leqq 1$, where $o(1)$ denotes an infinitesimal sequence and $\sigma(x) = \dfrac{1}{\beta(x)}$.

By Lemma 3.2 we can take, in (4.5),

$$\xi = \xi_n = x_n - x - v_n$$

where

$$v_n \to 0 \quad \text{in } W^1. \tag{4.6}$$

Then we have

$$\int_0^1 \langle \dot{x}_n, \dot{x}_n - \dot{x} - \dot{v}_n \rangle \, ds + \Delta^2 \left[\int_0^1 \sigma(x_n) \, ds \right]^{-2} \int_0^1 \langle \operatorname{grad} \sigma(x_n), x_n - x - v_n \rangle \, ds = o(1). \tag{4.7}$$

Since W^1 is compactly embedded into L^∞, by (4.4) we deduce that

$$x_n \to x \quad \text{in } L^\infty. \tag{4.8}$$

Moreover by (4.4) and (4.6) we have

$$x_n - x - v_n \text{ converges to 0 weakly in } W^1. \tag{4.9}$$

Now, by (4.7), (4.8) and (4.9) we get

$$\int_0^1 \langle \dot{x}_n, \dot{x}_n - \dot{x} - \dot{v}_n \rangle \, ds = o(1) \tag{4.10}$$

so (4.6) and (4.10) imply that

$$\int_0^1 \langle \dot{x}_n, \dot{x}_n - \dot{x} \rangle \, ds = o(1). \tag{4.11}$$

Moreover, since $\{\dot{x}_n\}$ converges weakly in L^2 to $\dot{x}$, we have

$$\int_0^1 \langle \dot{x}, \dot{x}_n - \dot{x} \rangle ds = o(1). \tag{4.12}$$

(4.11) and (4.12) imply that

$$\int_0^1 |\dot{x}_n - \dot{x}_0|^2 = o(1) \tag{4.13}$$

and finally from (4.8) and (4.13) we deduce that

$$x_n \to x \quad \text{in } W^1. \quad \blacksquare$$

Proof of Theorem (1.1). – Since M_0 is connected, (P.S.) holds, and J is bounded from below, by virtue of Theorem 2.1, we easily get that M is geodesically connected by a minimum argument.

Now if $P_1 \neq P_2$ we have $\inf_{M_0} \int_0^1 \langle \dot{x}(s), \dot{x}(s) \rangle_{x(s)} ds > 0$, so, if Δ is small enough, by (1.5) we have $\inf_{\Omega^1} J > 0$ and we get (i).

Moreover if all the geodesics joining Q_1 and Q_2 are non degenerate, then all the critical points of J on Ω^1 are isolated. Then if we assume that the are infinitely many time-like geodesics we get a contradiction, because by Lemma 4.2 and Theorem 2.1 we get a sequence of critical points of J, converging to a critical point of J.

Now assume that M_0 is not contractible in itself and there exist a retraction of M_0 on a compact subset. Under this assumptions, it is possible to apply a well known theorem of Serre (*see* [25]), in order to prove that

$$\text{cat}_{\Omega^1}(\Omega^1) = +\infty, \quad (^4)$$

(*see* e.g. [10]). Notice that we need the assumption about the retraction of M_0 on a compact subset, because, whenever $\pi_1(M_0)$ is finite, in order to apply the Serre Theorem, the Betti numbers of M_0 and of its universal covering space must be finite (and not only eventually zero).

For every $c \in \mathbb{R}$, we set $J^c = \{x \in \Omega^1 : J(x) \leqq c\}$. Since J satisfies (P.S.) and $\inf_{\Omega^1} J > -\infty$, we have that

$$\text{cat}_{\Omega_1}(J^c) < +\infty \tag{4.14}$$

(4) Here $\text{cat}_X A$ denotes the Liusternik-Schnirelman category of $A \subset X$ in the topological space X.

Indeed, if there exist $c \in \mathbb{R}$ such that $\mathrm{cat}_{\Omega_1}(J^c) = +\infty$, we can consider

$$\bar{c} = \inf\{c \in \mathbb{R} : \mathrm{cat}_{\Omega_1}(J^c) = +\infty\}.$$

Moreover, by virtue of (P.S.), the set Z of the critical points of J is compact, hence there exists a neighbourhood U_Z of Z, such that

$$\mathrm{cat}_{\Omega_1}(U_Z) < +\infty.$$

Now, by a well known deformation Lemma (*see* e.g. [19]), there exists $\varepsilon > 0$ such that $J^{\bar{c}-\varepsilon}$ includes a strong deformation retract of $J^{\bar{c}+\varepsilon} \setminus U_Z$. Then

$$\mathrm{cat}_{\Omega_1}(J^{\bar{c}+\varepsilon}) \leqq \mathrm{cat}_{\Omega_1}(J^{\bar{c}+\varepsilon} \setminus U_Z) + \mathrm{cat}_{\Omega_1}(U_Z) \leqq \mathrm{cat}_{\Omega_1}(J^{\bar{c}-\varepsilon}) + \mathrm{cat}_{\Omega_1}(U_Z) < +\infty,$$

and this contradicts the definition of $\bar{c}$. Then (4.14) is proved.

Now, by (4.14), we get immediately that there exists a sequence x_n of critical points for J on Ω^1 such that

$$\lim_{n \to +\infty} J(x_n) = +\infty,$$

so our assertion on γ_n followss by Theorem 2.1.

Let us prove (iii). Under our assumptions on the topology of M_0, we have the existence of a sequence K_m of compact subsets of Ω^1 such that

$$\lim_{m \to +\infty} \mathrm{cat}_{\Omega^1}(K_m) = +\infty, \tag{4.15}$$

(*see* e.g. [10]). Moreover, by the minimax characterization of the critical level $J(x_n)$ (*see* [23]), if

$$\Gamma_n = \{A \subset \Omega^1 : \mathrm{cat}_{\Omega^1}(A) \geqq n\}$$

we have

$$J(x_n) = \inf_{A \in \Gamma_n} \sup_{x \in A} J(A).$$

If two of these critical levels are equal, there exist infinitely many critical points having such a critical level (*see* [23]). Then by (4.15) we get easily (iii). ■

Proof of Theorem 1.3. – By Theorem 1.1 the set $G(Q_1, Q_2)$ of geodesics joining $Q_1, Q_2 \in \mathfrak{M}$ is not empty.

Let $\{\gamma\}_n \subset G(Q_1, Q_2)$ such that

$$l(\gamma_n) \to d(a_1, a_2) \quad \text{for } n \to \infty \quad [cf. [1.7)].$$

Now

$$E_{\gamma_n} = c_n, \quad \sqrt{|c_n|} = l(\gamma_n), \tag{4.16}$$

and from (4.16) and by Theorem 2.1 we get

$$c_n = f(\gamma_n) = J(x_n) \quad \text{for all } n \in \mathbb{N}.$$

Since $|c_n|=(l(\gamma_n))^2$, we deduce that $\{J(x_n)\}$ is bounded. Then, by lemma 4.2, we deduce that

$$x_n \to x \quad \text{in } W^1, \qquad J'(x)=0.$$

So, using once again Theorem 2.1, we get that there exists $\gamma \in G(Q_1, Q_2)$ with $l(\gamma)=d(a_1, a_2)$. ■

In order to prove Theorem 1.5 we cannot use the Liusternik-Schnirelman category since in general $\text{cat}_{\Lambda^1}(\Lambda^1)$ is not known. However we can exploit some of the results on the cohomology of Λ^1 proved in [28] to obtain our result.

First of all we recall one of those results.

PROPOSITION 4.3. – *If* M_0 *is compact and* $\pi_1(M_0)=0$ *then there exists an infinite set of positive integers* $Q \subset \mathbb{N}$ *such that*

$$H^q(\Lambda^1) \neq 0 \quad \textit{for every } q \in Q,$$

where $H^q(\Lambda^1)$ *is the q-th group of cohomology of* Λ^1.

Now consider $\alpha \in H^*(\Lambda^1)$, $\alpha \neq 0$, *and set*

$$\Gamma_\alpha = \{ B \subset \Lambda^1 \mid i_B^*(\alpha) \neq 0 \} \tag{4.17}$$

where $i_B^*: H^*(\Lambda^1) \to H^*(B)$ *is the homomorphism induced by the inclusion* $i_B: B \to \Lambda^1$.

Observe that Γ_α defined in (4.17) is not empty and contains compact sets (in fact it contains the support of k-chain, $k=\deg \alpha$, which are not homologous to a constant).

For every $c>0$ we set

$$\begin{aligned} E^c &= \left\{ x \in \Lambda^1 \,\middle|\, \int_0^1 \langle \dot{x}(s), \dot{x}(s) \rangle_{x(s)}\, ds < c \right\} \\ E_c &= \left\{ x \in \Lambda^1 \,\middle|\, \int_0^1 \langle \dot{x}(s), \dot{x}(s) \rangle_{x(s)}\, ds \geq c \right\}, \end{aligned} \tag{4.18}$$

and the following Proposition holds

PROPOSITION 4.4. – *For every* $c>0$ *there exists* $\bar{q}=\bar{q}(c) \in \mathbb{N}$ *such that*

$$H^q(E^c)=0 \qquad \textit{for} \quad q>\bar{q}.$$

Proof. – By a well known result (*cf*. e. g. [16]) E^c is a strong deformation retract of a finite dimensional manifold, whose dimension depends on c (*cf*. also [5]).

Then, if we take $\bar{q}(c)=n$ we get the conclusion. ■

LEMMA 4.5. – *Let the assumption of lemma* 4.2 *be satisfied and let* $\alpha \in H^*(\Lambda^1)$, $\alpha \neq 0$. *Then the number*

$$c_\alpha = \inf_{B \in \Gamma_\alpha} \sup J(B) \tag{4.19}$$

is a critical value of J *on* Λ^1. *Moreover, if we assume that* $H^q(\Lambda^1) \neq 0$ *for infinitely many* q, *then there exists a sequence* $\{c_\alpha\}$ *of critical values of* J *defined as in* (4.10) *and such that*

$$c_\alpha \to +\infty \quad as \ \deg \alpha \to +\infty. \qquad (4.20)$$

Proof. – First of all we prove that c_α in (4.19) is well defined. Since Γ_α contains compact subsets of Λ^1, we have $c_\alpha < +\infty$. Moreover, since $\beta(x) \leqq M$, J is bounded from below and $c_\alpha > -\infty$.

To prove that c_α is a critical value of J we follow a standard argument. Arguing indirectly we assume that c_α is not a critical value of J. Since J satisfies the (P.S.) condition on Λ^1 (*see* Lemma 4.2), by a well known deformation Lemma (*see* e. g. [19]), there exists $\varepsilon > 0$ and an homeomorphism η on Λ^1 s. t.

$$\eta(J^{-1}(]-\infty, c_\alpha+\varepsilon]) \subset J^{-1}(]-\infty, c_\alpha-\varepsilon]). \qquad (4.21)$$

Now we claim that

$$\forall B \in \Gamma_\alpha, \quad \eta(B) \in \Gamma_\alpha. \qquad (4.22)$$

In fact, let $\eta^* : H^q(\eta(\Lambda^1)) \to H^q(\Lambda^1)$ be the isomorphism induced from η, and $B \in \Gamma_\alpha$. Then

$$i^*_{\eta(B)}(\alpha) = (\eta^*)^{-1} \circ i^*_B(\alpha) \neq 0,$$

so we conclude that $\eta(B) \in \Gamma_\alpha$.

Now, by the definition of c_α there exists $B \in \Gamma_\alpha$ such that $\sup J(B) < c_\alpha + \varepsilon$; then by (4.21), we have

$$\sup J(\eta(B)) \leqq c_\alpha - \varepsilon, \qquad (4.23)$$

while (4.22) and (4.23) contradict the definition of c_α.

Finally we prove the last part of lemma 4.5 and assume that $H^q(\Lambda^1) \neq 0$ for $q \in Q \subset \mathbb{N}$, Q infinite.

If $n \in \mathbb{N}$, by Proposition 4.4, there exists $\bar{q} = \bar{q}(n) \in \mathbb{N}$ such that

$$H^q(E^n) = 0 \qquad \text{for} \quad q > \bar{q}. \qquad (4.24)$$

Now let $q_n \in Q$, with $q_n > \bar{q}(n)$ and consider $\alpha \in H^{q_n}(\Lambda^1)$, $\alpha \neq 0$. We claim that

$$\forall B \in \Gamma_\alpha, \quad B \cap E_n \neq \varnothing \qquad (4.25)$$

Arguing by contradiction we assume that there exists $B \in \Gamma_\alpha$ such that

$$B \subset \Lambda^1 \setminus E_n := E^n,$$

then

$$H^{q_n}(\Lambda^1) \overset{i_2^*}{\to} H^{q_n}(E^n) \overset{i_1^*}{\to} H^{q_n}(B) \qquad (4.26)$$

where i_2^*, i_1^* are the homomorphisms induced by the inclusion maps

$$i_2 : E^n \to \Lambda^1, \qquad i_1 : \quad B \to E^n.$$

Then, since $B \in \Gamma_\alpha$, we have

$$i_1^* \circ i_2(\alpha) = i_B^*(\alpha) \neq 0. \qquad (4.27)$$

From (4.26) and (4.27) we deduce that $H^{q_n}(E_n) \neq 0$ and this contradicts (4.24). Then the intersection property (4.25) holds.

By (4.25) and the definition (4.19) of c_α we easily get

$$c_\alpha \geqq n - \Delta^2 M, \qquad (4.28)$$

where M is un upper bound for β. Clearly from (4.28) we deduce (4.20). ■

Proof of Theorem 1.5. – In order to prove that J on Λ^1 possesses infinitely many critical points $\{x_n\}$ such that

$$\lim_{n \to +\infty} J(x_n) = +\infty, \qquad (4.29)$$

we distinguish two cases.

First case. – Assume that M_0 is simply connected, then (4.29) immediately follows from Proposition 4.3 and Lemma 4.5.

Second case. – Assume $\pi_1(M_0) \neq 0$ and finite. Denote by $(\tilde{M}_0, p)$ the universal covering of M_0. $\pi_1(\tilde{M}_0) = 0$ and $\tilde{M}_0$ is compact since $\pi_1(M_0)$ is finite.

Then arguing as in the first case, we prove the existence of infinitely many critical points $\{\tilde{x}_n\}$ of J, such that

$$J(\tilde{x}_n) \to +\infty \quad \text{as } n \to \infty.$$

Therefore, if we set $x_n = p(\tilde{x}_n)$, we get the existence of infinitely many critical points.

Then by Theorem (2.2) we get that there exists a sequence γ_n of T-periodic trajectories (x_n, t_n), where t_n defined by (2.11), such that $E_{\gamma_n} \underset{n}{\to} +\infty$. Notice that by (4.29) the sequence $\{x_n\}$ of critical points of J consists at most of finitely many constant curves, because on the constant curves J is bounded from above. Then, up to consider a subsequence, we can assume that all the T-periodic trajectories γ_n are non trivial.

In order to prove that curves $\gamma_n (n \in \mathbb{N})$ are geometrically distinct, assume that two (non trivial) T-periodic trajectories $\gamma_i = (x_i, t_i)$ and $\gamma_j = (x_j, t_j)$ $(i \neq j)$ are not geometrically distinct. Then there exists $\varphi(s)$ such that

$$\gamma_j(s) = \gamma_i(\varphi(s)).$$

Then we have, for all s,

$$0 = \nabla_s(\dot{\gamma}_j)(s) = (\varphi'(s))^2 \nabla_s(\dot{\gamma}_i)(\varphi(s)) + \varphi''(s)\dot{\gamma}_i(\varphi(s)) = \varphi''(s)\dot{\gamma}_i(\varphi(s)).$$

Now, since γ_i is non trivial, by the uniqueness on the Cauchy problem

$$\left.\begin{array}{l}\nabla_s(\dot{\gamma})(s)=0\\ \gamma(s_0)=\gamma_0\\ \dot{\gamma}(s_0)=0\end{array}\right\}$$

we have $\dot{\gamma}_i(\varphi(s))\neq 0$ for all s, hence

$$\varphi''(s)=0 \quad \text{for all } s. \tag{4.30}$$

Moreover $t_j(s)=t_i(\varphi(s))$ for all s, t_i is strictly increasing and [*see* (1.8)] $t_j(0)=t_i(0)=0$, $t_j(1)=t_i(1)=T$. Therefore $\varphi(0)=0$ and $\varphi(1)=1$, hence, by (4.30), $\varphi(s)=s$ for all s.

The proof of (i) is the same of (ii) of (1.1). ■

5. PROOF OF THEOREM 1.6

By virtue of Theorem 2.2, we are reduced to studying the functional

$$J(x)=\int_0^1 \langle \alpha(x)\dot{x}, \dot{x}\rangle\, ds - T^2\left[\int_0^1 \sigma(x)\, ds\right]^{-1} \tag{5.1}$$

where $x\in W^1(S^1, \mathbb{R}^n)$, $\sigma(x)=\dfrac{1}{\beta(x)}$, $\langle\ ,\ \rangle$ is the usual scalar product of $\mathbb{R}^n$ and $\alpha(x)$ is a symmetric matrix having minimum eigenvalue far from zero.

For technical reasons we modify J as follows

$$I(x)=-\frac{1}{T^2}J(x)=\left[\int_0^1 \sigma(x)\, ds\right]^{-1}-\frac{1}{T^2}\int_0^1 \langle \alpha(x)\dot{x}, \dot{x}\rangle\, ds. \tag{5.2}$$

Clearly it is enough to show that the multiplicity result contained in Theorem 1.6 holds for the functional I.

Standard calculations show that I is of class C^2 on W^1. Moreover I is invariant under the unitary representation of the group $S^1=\mathbb{R}/\mathbb{Z}$ on W^1 given by the time translations [namely $J(x(s+\vartheta))=J(x(s))$ for all $\vartheta\in\mathbb{R}$ and $x\in W^1$].

Let us now recall the following abstract critical point Theorem:

THEOREM 5.1. — *Let* E *be a real Hilbert space on which a unitary representation* G *of the group* $S^1=\mathbb{R}/\mathbb{Z}$ *acts.*

Let I *be a* C^2 *functional on* E *satisfying the following properties:*

(I_1) *is* G-*invariant* [i.e. $\forall u\in E$, $\forall g\in G$, $I(u)=I(g(u))$].

(I_2) I *satisfies the Palais-Smale condition* (P.S.) *in* $]0, \mu[$, $\mu>0$ [i.e. *for all* $c\in]0, \mu[$ *any sequence* $\{u_k\}\subset E$ *s. t.* $I'(u_k)\to 0$ *as* $k\to\infty$ *and* $I(u_k)\to c$ *contains a convergent subsequence*].

(I_3) *There exist two closed* G-*invariant subspaces* E_1 *and* E_2 of E *with* $\operatorname{codim} E_2 < +\infty$ *and a bounded* G-*invariant neighborhood* B *of* 0 *such that:*

(*a*) $I(u) \geqq \lambda > I(0)$, $\forall u \in E_1 \cap \partial B$ (*∂B being the boundary of* B);

(*b*) $\sup I(E_2) < \mu$;

(*c*) $\operatorname{Fix} G \subset E_1$ *or* $\operatorname{Fix} G \subset E_2$, *where* $\operatorname{Fix} G = \{ u \in E : g(u) = u, \forall g \in G \}$,

Suppose moreover that

(I_4) $\forall u \in \operatorname{Fix} G$, *with* $I'(u) = 0$, *we have* $I(u) < \lambda$.

Then I *possesses at least*

$$\frac{1}{2}[\dim(E_1 \cap E_2) - \operatorname{codim}(E_1 + E_2)]$$

distinct critical points ([5]) *whose critical values belong to the interval* $[\lambda, \sup I(E_2)]$.

Variants of Theorem 5.1 have been proved in [3] and therefore we omit its proof.

In order to verify that I satisfy the assumptions of Theorem 5.1 we need some Lemmas.

Notice that under our assumptions $\sup_{W^1} I = d_0$ and $I(x) < d_0$ for all $x \in W^1$.

LEMMA 5.2. – *Under the assumptions* (1.9), (1.10), I *satisfies the* (P.S.) *in* $]-\infty, d_0[$, i.e. *any sequence* $\{x_n\} \subset W^1$ *such that*

$$I'(x_n) \to 0 \quad \text{as } n \to +\infty, \tag{5.3}$$

$$I(x_n) \to c < d_0 \quad \text{as } n \to +\infty, \tag{5.4}$$

contains a convergent subsequence.

Proof. – Let $\{x_n\}$ be a sequence satisfying (5.3), (5.4). Since β is bounded and (1.9) holds, (5.4) implies that

$$\|\dot{x}_n\|_{L^2} \text{ is bounded.} \tag{5.5}$$

We shall prove that

$$\|x_n\|_{L^\infty} \text{ is bounded.} \tag{5.6}$$

Arguing by contradiction assume that there exists a subsequence (which we continue to call by $\{x_n\}$) such that

$$\|x_n\|_{L^\infty} \to +\infty. \tag{5.7}$$

By (5.5) and (5.7) we have

$$\inf_{s \in [0,1]} |x_n(s)| \to +\infty \quad \text{as } n \to +\infty, \tag{5.8}$$

([5]) We say that two critical points u_1, u_2 are distinct if $u_1 \neq g(u_2)$ for all $g \in G$.

then by (1.10) we get

$$\begin{gathered} \|\operatorname{grad}\sigma(x_n)\|_{L^\infty}\to 0, \\ \left\|\sigma(x_n)-\frac{1}{d_0}\right\|_{L^\infty}\to 0 \quad \text{as } n\to+\infty. \end{gathered} \tag{5.9}$$

Now set

$$\tilde{x}_n = x_n - \int_0^1 x_n\, ds.$$

By (5.5) we deduce that $\left\|\frac{d}{ds}\tilde{x}_n\right\|_{L^2}$ is bounded, then by Wirtinger inequality,

$$\|\tilde{x}_n\|_{W^1} \quad \text{is bounded.} \tag{5.10}$$

Since W^1 is continuously embedded into L^∞, from (5.10) we deduce that

$$\|\tilde{x}_n\|_{L^\infty} \quad \text{is bounded.} \tag{5.11}$$

Now (5.3) implies that

$$I'(x_n)[\tilde{x}_n]=\varepsilon_n\|\tilde{x}_n\|_{W^1}, \qquad \varepsilon_n\to 0, \quad \text{as } n\to+\infty \tag{5.12}$$

Then

$$\left[\int_0^1 \sigma(x_n)\,ds\right]^{-2}\int_0^1 \langle \operatorname{grad}\sigma(x_n), \tilde{x}_n\rangle\, ds + \frac{1}{T^2}\int_0^1\left[\langle \alpha'(x_n)[\tilde{x}_n]\dot{x}_n, \dot{x}_n\rangle + 2\left\langle \alpha(x_n)\dot{x}_n, \frac{d}{ds}\tilde{x}_n\right\rangle\right]ds=\varepsilon_n\|\tilde{x}_n\|_{W^1}. \tag{5.13}$$

By (1.9) and (5.8) we have

$$\|\alpha'(x_n)[\tilde{x}_n]\|_{L^\infty}\to 0, \quad \text{as } n\to+\infty \tag{5.14}$$

Moreover (5.9) and (5.11) imply that

$$\left[\int_0^1 \sigma(x_n)\,ds\right]^{-2}\int_0^1 \langle \operatorname{grad}\sigma(x_n), \tilde{x}_n\rangle\, ds = o(1) \tag{5.15}$$

and (5.14) and (5.5) imply that

$$\int_0^1 [\langle \alpha'(x_n)[\tilde{x}_n]\dot{x}_n, \dot{x}_n\rangle]\, ds = o(1). \tag{5.16}$$

Then from (5.13), (5.15) and (5.16) we deduce that

$$\int_0^1 \left\langle \alpha(x_n)\dot{x}_n, \frac{d}{dt}\tilde{x}_n \right\rangle ds = o(1). \tag{5.17}$$

Since $\frac{d}{ds}\tilde{x}_n = \dot{x}_n$, (5.17) and (5.9) imply that

$$\mathrm{I}(x_n) = d_0 + o(1)$$

and this contradicts (5.4).

Then we conclude that $\{x_n\}$ is bounded in W^1 and therefore it contains a subsequence $\{x_n^1\}$ weakly convergent to x in W^1.

Now the same argument used in the proof of Lemma 4.2 (obviously here we take $v_n \equiv 0$) permit to prove that x_n^1 converges to x strongly in W^1. ■

Let $k \in \mathbb{N}$ and set

$$W_k = \mathrm{span}\{ e_i \sin(2\pi js), e_i \cos(2\pi js) : i=1, \dots, n; j=0, \dots, k \} \tag{5.18}$$

where e_i, $i=1, \dots, n$ is the standard basis in $\mathbb{R}^n$. Moreover we set

$$\tilde{W} = \left\{ x \in W^1(S^1, \mathbb{R}^n) : \int_0^1 x\, ds = 0 \right\}, \tag{5.19}$$

$$S_k^r = \{ x \in W_k : \|x\|_{W^1} = r \}, \qquad r > 0. \tag{5.20}$$

Notice that W_k includes Fix G which is the set of the constant curves.

The following Lemma holds:

LEMMA 5.3. – *Suppose that* β *satisfies assumptions* (1.9), (1.10) *and* (1.11).

Then for any $k \in \mathbb{N}$ *there exist* $\bar{T} = \bar{T}(k)$, $\delta = \delta(k)$, $r = r(k)$ *such that for any* $T \geqq \bar{T}$

$$\mathrm{I}(x) \geqq c_0 + \delta > \mathrm{I}(0), \qquad \forall x \in S_k^r, \tag{5.21}$$

where $c_0 = \sup_{\mathbb{R}^n} \{ \beta(x) : \mathrm{grad}\, \beta(x) = 0 \}$.

Proof. – Since W_k is finite dimensional, S_k^r is compact for every $r > 0$. Then there exists $x_r \in S_k^r$ such that

$$\mu(r) := \left[\int_0^1 \sigma(x_r)\, ds \right]^{-1} = \min\left\{ \left[\int_0^1 \sigma(x)\, ds \right]^{-1} : x \in S_k^r \right\}. \tag{5.22}$$

We claim that

$$\lim_{r \to +\infty} \mu(r) = d_0 > c_0, \tag{5.23}$$

where $d_0 = \sup_{\mathbb{R}^n} \beta$.

Obviously we have

$$\mu(r) \leq d_0, \qquad \forall r > 0.$$

Then, if by contradiction (5.23) does not hold, there exists $r_n \to +\infty$ as $n \to +\infty$, such that

$$\lim_{n \to +\infty} \mu(r_n) < d_0. \tag{5.24}$$

For every $n \in \mathbb{N}$ consider $\lambda_n \in \mathbb{R}^+$ and $x_n \in S_k^r$ (i.e. $x_n \in W_k$ and $\| x_n \|_{W^1} = 1$) such that

$$\lambda_n x_n = x_{r_n},$$

where x_{r_n} is defined in (5.22) and

$$\lambda_n = \| x_{r_n} \|_{W^1} = r_n \underset{n}{\to} +\infty.$$

By (5.24) we get

$$\lim_{n \to +\infty} \int_0^1 \frac{1}{\beta(\lambda_n x_n(s))} ds > \frac{1}{d_0}.$$

Then there exists $\varepsilon_0 > 0$ such that for every $n \in \mathbb{N}$

$$\text{meas}\left\{ s \in [0, 1] : \frac{1}{\beta(\lambda_n x_n(s))} > \frac{1}{d_0 - \varepsilon_0} \right\} > \varepsilon_0, \tag{5.25}$$

where meas denote the Lebesgue measure.

Let

$$A_n = \left\{ s \in [0, 1] : \frac{1}{\beta(\lambda_n x_n(s))} > \frac{1}{d_0 - \varepsilon_0} \right\}, \tag{5.26}$$

and $R > 0$ such that

$$\beta(x) > d_0 - \varepsilon_0 \quad \text{for every } x \text{ such that } |x| \geq R.$$

Then by virtue of (5.26) we have that

$$\forall s \in A_n, \quad \lambda_n |x_n(s)| < R. \tag{5.27}$$

Now x_n is a bounded sequence of curves included in W_k, hence, up to consider a subsequence, x_n is uniformly convergent to a continuous curve $\bar{x}$ such that

$$\| \bar{x} \|_{W^1} = 1. \tag{5.28}$$

Moreover by (5.25), (5.26) and (5.27) the function

$$s \mapsto | \bar{x}(s) |^2 \tag{5.29}$$

vanishes in a subset of [0, 1] having Lebesgue measure greater than ε_0.

On the other hand the space W_k consists of analytic curves, so the function (5.29) is an analytic function. Now the locus of zeros of an analytic functions which is not identically zero is a discrete set. Then $\bar{x}(s)=0$ for every $s\in[0, 1]$ and this contradicts (5.28).

(5.23) is so proved.

Now we have, for any $x\in S_k^r$,

$$I(x)\geqq \mu(r)-\frac{1}{T^2}\int_0^1 \langle \alpha(x)\dot{x}, \dot{x}\rangle ds \geqq \mu(r)-\frac{K r^2}{T^2}, \tag{5.30}$$

where $K=\sup_{\mathbb{R}^n}\{\|\alpha(x)\|_{L^\infty}: x\in W^1\}<+\infty$ because of (1.9).

At this point we choose $r=r(k)$ and $\delta=\delta(k)>0$ such that

$$\mu(r)=c_0+\delta> I(0),$$

and, because of (5.30), we can choose $\bar{T}$ sufficiently large in order to get (5.21). ■

LEMMA 5.4. – *Under the assumptions* (1.9), (1.10) *and* (1.11)

$$\sup I(\tilde{W})<d_0$$

where $\tilde{W}$ *is defined in* (5.19) *and* $d_0=\sup_{\mathbb{R}^n}\beta$.

Proof. – Let $x\in\tilde{W}$. Obviously $I(x)\leqq d_0$. Arguing by contradiction assume that

$$\sup I(\tilde{W})=d_0. \tag{5.31}$$

Then there exists a sequence $\{x_n\}\subset\tilde{W}$ such that

$$\int_0^1 \langle \alpha(x_n)\dot{x}_n, \dot{x}_n\rangle ds\to 0, \qquad \text{and} \qquad \left[\int_0^1 \sigma(x_n)\, ds\right]^{-1}\to d_0 \tag{5.32}$$

as $n\to+\infty$. Then, by (1.9), $\|\dot{x}_n\|_{L^2}\to 0$ as $n\to+\infty$ and, since $x_n\in\tilde{W}$, $\forall n$, we deduce that

$$\|x_n\|_{W^1}\to 0 \quad \text{as } n\to+\infty.$$

Consequently $\|x_n\|_{L^\infty}\to 0$ as $n\to+\infty$. Then we have

$$\left[\int_0^1 \sigma(x_n)\, ds\right]^{-1}\to\beta(0)<d_0$$

and this contradicts (5.32). ■

Now are ready to prove Theorem 1.6.

Proof of Theorem 1.6. – We have already observed that it is enough to study the critical point of I. To this aim we show that I satisfies assumptions (I_1), (I_2), (I_3), (I_4) in Theorem 5.1.

Clearly I satisfies I_1. By virtue of lemma 5.2 also (I_2) is satisfied with $\mu=d_0$.

Now let $m \in \mathbb{N}$ and set

$$k=\left[\frac{m}{n}\right]+1, \tag{5.33}$$

where $\left[\frac{m}{n}\right]$ denotes the greatest integer less or equal than $\frac{m}{n}$.

By Lemmas 5.3 and 5.4 we obtain that if $T \geqq \bar{T}$ (cf. Lemma 5.3), I satisfies assumption (I_3) with $E_1 = W_k$, $E_2 = \tilde{W}$, $\lambda = c_0 + \delta$, $\partial B = S_k^r$.

Moreover by (1.11) also I_4 is satisfied, because a constant curve x is a critical point for I iff grad $\beta(x)=0$, so $I(x) \leqq c_0$.

Since codim $(W_k + \tilde{W}) = 0$ and $\dim (W_k \cap \tilde{W}) = 2nk$, we obtain, by using theorem 5.1, that, if $T \geqq \bar{T}$, I possesses at least

$$nk=\left(\left[\frac{m}{n}\right]+1\right)n \geqq m$$

critical points x_j ($j=1, \ldots, nk$).

By Theorem 1.5 we get also

$$c_0 < I(x_j) < d_0, \tag{5.34}$$

so every x_j is not constant. Moreover, if we denote by $\gamma_j = (x_j, t_j)$ the T-periodic trajectory such that t_j satisfies (2.11), and by E_{γ_j} its energy [*cf.* (1.4)], we have [by (2.6)],

$$E_{\gamma_j} = -T^2 I(x_j); \tag{5.35}$$

and from (5.34) and (5.35) we get

$$-d_0 T^2 < E_{\gamma_j} < -c_0 T^2, \qquad \forall j=1, \ldots, nk.$$

Finally, as in Theorem (1.5) we see that the T-periodic trajectories γ_j are geometrically distinct. ■

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