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# On the vanishing viscosity approximation to the Cauchy problem for a $2 \times 2$ system of conservation laws

by

**Bruno RUBINO**

Scuola Normale Superiore,
Piazza dei Cavalieri, 7, 56126 Pisa, Italy

ABSTRACT. – This paper is concerned with a non-strictly hyperbolic system of conservation laws. We study the existence of weak solutions to the associated Cauchy problem, in the framework of $L^\infty$, using the methods of compensated compactness. Some previous results for quadratic systems have been generalized.

*Key words :* Conservation laws, entropy, Goursat problem, invariant region, quasi-convexity, rarefaction wave, Riemann function, Riemann invariant, umbilical point.

RÉSUMÉ. – Étant donné un certain système de lois de conservation faiblement hyperbolique, on étudie l'existence d'une solution faible dans $L^\infty$ du problème de Cauchy associé en utilisant les méthodes de la compacité par compensation. De cette façon, on généralise quelques-uns des résultats connus pour les systèmes quadratiques.

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## 1. INTRODUCTION

This paper is devoted to investigate the convergence of vanishing viscosity approximation to weak solutions, satisfying the *entropy inequality*, of the following, quasilinear $2 \times 2$ hyperbolic system of conservation laws

$$\begin{cases} u_t + \left(\left(\dfrac{1}{2} + a\right) u^2 + f(v)\right)_x = 0 \\ v_t + (uv)_x = 0 \\ (u, v)_{|t=0} = (u_0, v_0) \end{cases} \tag{1.1}$$

for real valued functions $u$, $v$ on $\mathbb{R}_x \times \mathbb{R}_t^+$. We assume that $a > \frac{1}{2}$, $f \in C^2(\mathbb{R})$ is even and superlinear and for all $v$ satisfying

$$\begin{cases} f'(v)\, v > 0 & \text{for} \quad v \neq 0 \\ f''(v) > 0 & \text{for} \quad v \neq 0. \end{cases}$$

The system (1.1) fails to be strictly hyperbolic under these assumptions since there is an umbilical point in $(u, v) \equiv (0, 0)$. Moreover the system is not genuinely non-linear (*see* below for the definition) along the $u$ axis.

A system of this type, when $f(v) = \frac{1}{2} v^2$ has been investigated by Kan [14] in his doctoral thesis, and we also employ some of his ideas to extend his results to our case. Systems of this type have been widely investigated in connection with oil reservoir engineering models ([12], [13], [24]) and also in some situations arising in problems of mathematical physics ([1], [2], [7], [15]).

A complete account of the literature on these topics goes beyond the scope of this paper, but among many other important contributions we wish to mention ([5] to [8], [10], [16] to [21], [30]).

In this framework the method of compensated compactness is the main tool developed to analyze the vanishing viscosity method and we refer to the classical papers of Tartar [29], Murat [22] and DiPerna ([5] to [8]) for the crucial ideas in the theory. Further some important contributions are also due to Chen, Ding and Luo [1], Chen [2] and Serre [26].

In order to use the compensated compactness methods we will need *a priori* estimates in $L^\infty$, independent of the viscosity. These bounds will be obtained using the theory of invariant domains due to Chueh, Conley and Smoller [3] (*see* also [27]) and the classical maximum principle.

We extend the result of [14], from the case $f(v) = \frac{1}{2} v^2$ to a general even and superlinear function, showing that, in order to overcome the difficulty of explicit computation, it is only necessary to provide the asymptotic behaviour of the coefficients of the differential equation concerning the entropies, getting over the problem of difficulty of the explicit computation. We hope to develop in future papers the theory for more general non-linearities. In particular, the convergence of some numerical schemes will be discussed in a forthcoming paper.

Let us recall the system (1.1) can be written in the vector form as a single conservation law,

$$\begin{cases} \mathcal{U}_t + \mathrm{F}(\mathcal{U})_x = 0 \\ \mathcal{U}_{|t=0} = \mathcal{U}_0 \end{cases} \tag{1.2}$$

by denoting $\mathcal{U} = (u, v)^{\mathrm{T}}$, $\mathrm{F}(\mathcal{U}) = \left(\left(\frac{1}{2} + a\right) u^2 + f(v),\ uv\right)^{\mathrm{T}}$.

The system (1.1) is said to be strictly hyperbolic if the two characteristics speeds, namely the eigenvalues $\lambda_{\mp} = \lambda_{\mp}(u, v)$, satisfy

$$\lambda_- < \lambda_+.$$

In our case, let us denote by

$$\mathrm{G}(u, v) = a^2 u^2 + v f'(v)$$

and

$$g_{\mp}(u, v) = -au \mp \sqrt{\mathrm{G}(u, v)},$$

then the characteristic speed of the problem are given by

$$\lambda_{\mp}(u, v) = (1 + a) u \mp \sqrt{\mathrm{G}(u, v)}$$

and the corrisponding right and left eigenvectors by

$$r_{\mp}(u, v) = \begin{pmatrix} f'(v) \\ g_{\mp}(u, v) \end{pmatrix}, \qquad l_{\mp}(u, v) = \begin{pmatrix} \pm \dfrac{g_x}{\sqrt{\mathrm{G}(u, v)}} \\ \mp \dfrac{1}{\sqrt{\mathrm{G}(u, v)}} \end{pmatrix}$$

Since $v f'(v) > 0$ for all $v \neq 0$, it follows

$$\lambda_-(u, v) < \lambda_+(u, v)$$

for all $(u, v) \neq (0, 0)$. So the origin is an isolated umbilical point for (1.1). The system (1.1) is said to be genuinly non-linear if $\nabla \lambda_{\mp} . r_{\mp} \neq 0$ for all $(u, v)$; in this case we have

$$\nabla \lambda_{\mp} . r_{\mp} = \frac{1}{2\sqrt{\mathrm{G}(u, v)}} \{ [(2a + 3)\sqrt{\mathrm{G}(u, v)} \mp a(2a - 1) u] f'(v) + [\sqrt{\mathrm{G}(u, v)} \mp au] v f''(v) \}$$

so the genuine non-linearity fails on the $u$ axis: $\{ (u, v) \in \mathbb{R}^2 : v = 0 \}$. A weak solution to (1.2) is a bounded measurable function $\mathcal{U} = \mathcal{U}(x, t)$ such that for all $\Phi \in C_0^1$ one has

$$\int_0^{+\infty}\int_{-\infty}^{+\infty} (\mathcal{U}(x, t)\Phi_t + F(\mathcal{U}(x, t))\Phi_x)\,dx\,dt + \int_{-\infty}^{+\infty} \mathcal{U}_0(x)\Phi(x, 0)\,dx = 0.$$

An entropy – entropy flux pair of the system (1.1) is defined to be a

$$\nabla q = \nabla \eta \, \nabla F. \tag{1.3}$$

In order to select the appropriate weak solution, the essential idea of the vanishing viscosity method consists in first introducing a small viscous perturbation of the system and then considering limits of solutions $\mathcal{U}^\varepsilon$ of such system which further satisfy the *entropy inequality* in the sense of Kružkov [16] and Lax [19] (*see* also Hörmander [11]):

$$\eta(\mathcal{U}^\varepsilon)_t + q(\mathcal{U}^\varepsilon)_x \leqq 0$$

in $\mathcal{D}'$ for any convex $\eta$. However the uniqueness under this condition remains still an open problem.

The plan of paper is the following: in section 2 we study the existence of the invariant domain and the *a priori* estimates in $L^\infty$; in section 3 we consider the viscous system. Section 4 is devoted to show the existence of infinitely many entropies of different types, needed to apply the arguments of Tartar and DiPerna. We shall prove existence of product type, sum type and weak entropies. When $f$ is polynomial, we also prove the existence of polynomial entropy. We then prove the energy estimates and apply them to investigate the properties of the entropy rate. The final section is concerned with strong convergence in $L^p$ required for the vanishing viscosity method.

## 2. INVARIANT REGIONS

Here we consider the vanishing viscosity approximation to the system (1.1):

$$\left\{\begin{array}{l} u_t + \left(\left(\dfrac{1}{2} + a\right)u^2 + f(v)\right)_x = \varepsilon\, u_{x_x} \\ v_t + (uv)_x = \varepsilon\, v_{xx} \\ (u, v)_{|t=0} = (u_0, v_0) \end{array}\right. \tag{2.1$_\varepsilon$}$$

for all $\varepsilon > 0$. In order to establish *a priori* estimates, independent of $\varepsilon$, we will use the theory of invariant regions due to Chueh, Conley and Smoller [3]. The results of [3] can be summarized in the following

THEOREM. – *Let $g_\mp$ be two smooth functions, $g_\mp : \mathbb{R}^2 \to \mathbb{R}$ and $\Sigma = \{(u, v) : g_\mp(u, v) \leqq 0\}$. Assume that, for any $t > 0$ and $(\bar{u}, \bar{v}) \in \partial\Sigma$, the* *following conditions hold:*

*a)* $\nabla g_{\mp}$ *is a left eigenvector of* $\nabla F(\bar{u}, \bar{v})$;

*b)* $g_{\mp}$ *is quasi-convex in* $(\bar{u}, \bar{v})$, i.e. *for all* $\xi \in \mathbb{R}^2$:

$$\xi . \nabla g_{\mp} = 0 \quad \Rightarrow \quad \nabla^2 g_{\mp}(\xi, \xi) \geqq 0.$$

*Then* $\Sigma$ *is an invariant region for* $(2.1)_\varepsilon$ *for all* $\varepsilon > 0$, *namely, if* $(u_0, v_0) \in \Sigma$ *for all* $x$, *then* $(u^\varepsilon(x, t), v^\varepsilon(x, t)) \in \Sigma$, *for all* $(x, t)$.

We say that $\omega_- \in C^1$ ($\omega_+$ respectively) is a first (second) Riemann invariant for (1.1) if for all $(u, v)$

$$(\nabla \omega_-(u, v))^T . r_+(u, v) = 0$$

$((\nabla \omega_+(u, v))^T . r_-(u, v) = 0)$. Therefore $\nabla \omega_{\mp}$ and $l_{\pm}$ are parallel.

Further, it is well known that any classical solution $(u, v)$ of (1.1) satisfies

$$\begin{cases} \dfrac{\partial \omega_-}{\partial t} + \lambda_+ \dfrac{\partial \omega_-}{\partial x} = 0 \\ \dfrac{\partial \omega_+}{\partial t} + \lambda_- + \dfrac{\partial \omega_+}{\partial x} = 0. \end{cases}$$

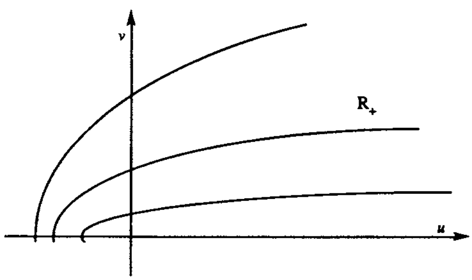


Fig. 1. – Integral curves $R_+$ for system (1.1).

Finally let us recall that the integral curves $R_{\mp}$ of $r_{\mp}$ in the state space are respectively called the first and second rarefaction wave curves. Thus, in our case, a first (second) rarefaction curve satisfies the ordinary first order differential equation

$$\frac{dv}{du} = \frac{g_{\mp}}{f'(v)}. \tag{2.2}$$

Since $f$ is even, the curves $R_-$ are obtained by a mirror reflection of the curves $R_+$ about the $v$-axis; we can hence restrict our attention to the $R_+$ family (*Fig.* 1). Further, we notice that the right hand side of $(2.2)_+$ is odd in $v$, so that the $R_+$ curves have an up-down symmetry; in the upper half plane they have a positive sign and so the $R_+$ curves cannot be closed.

LEMMA 2.1. — *Under the above hypothesis the following properties hold:*

*a) the* $R_+$ *curves do not intersect the positive u-axis;*

*b) the* $R_+$ *curves are in one to one corrispondence with the points of the negative u-semiaxis;*

*c) each one of the* $R_+$ *curves is confined to the upper half plane and tends to* $(0, 0)$ *as* $v \to 0$;

*d) the half line* $\{(u, v): u>0, v=0\}$ *is itself an* $R_+$ *curve;*

*e) every* $R_+$ *curve which does not stay on the u-axis tends to* $\infty$ *as* $u \to +\infty$.

*Hence we can conclude that an* $R_+$ *curve either starts from the origin and stays on the u-axis going to infinity to the right or it intersects the negative u-axis and goes to infinity to the right.*

*Finally, for* (1.1) *we can construct Riemann invariants* $\omega_-$, $\omega_+$ *so that*

$$\omega_-(u, v) \leqq 0 \leqq \omega_+(u, v).$$

*Proof.* — We limit ourselves to prove the last statement. In fact, if we suppose that there exists $M>0$ so that, for an $R_+$ curve that stays in the upper half plane $\frac{M}{2}<v<M$ for all $u>\bar{u}$, we obtain $\lim_{u \to +\infty} \frac{dv}{du}=0$. On the other hand, one has

$$\frac{dv}{du}=\frac{v}{au+\sqrt{G(u, v)}} \geqq \frac{M}{5au}$$

for all $u>\bar{\bar{u}}$, then we arrived at a contradiction.

In order to study the convexity of the $R_+$ curves, we compute the second derivative

$$\frac{d^2 v}{du^2}=\frac{g_+}{2f'(v)\sqrt{G(u, v)}}\left[(1-2a)-\frac{v}{f'(v)}f''(v)+2a\frac{f''(v)}{f'(v)^2}ug_+(u, v)\right].$$

Since

$$\frac{\partial}{\partial u}ug_+(u, v)=\frac{g_+^2}{\sqrt{G(u, v)}} \geqq 0,$$

then

$$\sup_{u \in \mathbb{R}} ug_+(u, v)=\lim_{u \to +\infty} ug_+(u, v)=\frac{vf'(v)}{2a},$$

so that we have

$$\frac{d^2 v}{du^2} \leqq \frac{(1-2a)}{2}\,\frac{g_+(u, v)}{f'(v)\sqrt{G(u, v)}} \leqq 0.$$

Therefore the $R_+$ curves are concave in the positive half-plane and convex in the negative half. Similarly $R_-$ curves are convex in the positive half plane and concave in the negative half.

Since $\omega_\mp$ is constant along every $R_\pm$ curve, if we prescribe $\omega_\mp(u, 0)$ as follows:

$$\omega_-(u, 0)=\begin{cases} u & u<0 \\ 0 & u\geqq 0, \end{cases} \qquad \omega_+(u, 0)=\begin{cases} 0 & u\leqq 0 \\ u & u>0, \end{cases}$$

then

$$\omega_-(u, v)\leqq 0\leqq \omega_+(u, v). \quad \blacksquare$$

From the geometry of the wave curves we also have the following

COROLLARY 2.2. – *The Riemann invariants constructed as before satisfy*

$$\begin{cases} \dfrac{1}{f'(v)}\dfrac{\partial\omega_-}{\partial v}<0 \\ \dfrac{1}{f'(v)}\dfrac{\partial\omega_+}{\partial v}>0. \end{cases}$$

*From now on, we shall use the Riemann invariants $\omega_-$, $\omega_+$ provided by the previous construction.*

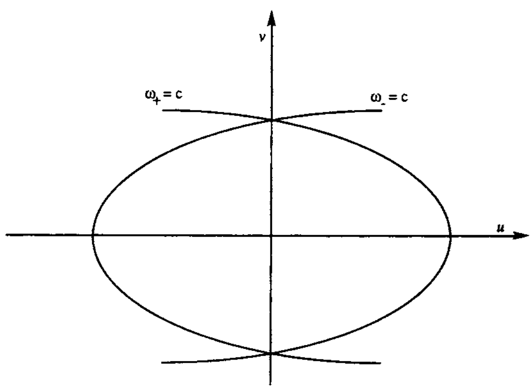


Fig. 2. – Invariant region for the system (1.1).

Since by definition $\nabla\omega_\mp$ are left eigenvectors for $\nabla F$, the Riemann invariants are the functions $\omega_\mp$ used to define the invariant regions following [3]. Therefore we will find in this way a family of invariant regions (*Fig.* 2) given by

$$\Sigma_c=\{\omega_-+c\geqq 0\}\cap\{\omega_+-c\leqq 0\}, \qquad c>0.$$

These regions are bounded by the wave curves of the two families. On the other hand, $\Sigma_c$ is strictly increasing in $c$ and it spans the whole state space as $c \to +\infty$.

We have by the above construction

LEMMA 2.3. – *Away from the origin,*

$$\left\{\begin{array}{l}\nabla^2 \omega_+ (r_-, r_-) \geqq 0 \\ \nabla^2 \omega_- (r_+, r_+) \leqq 0\end{array}\right.$$

*namely* $\omega_+$ *are quasi-convex (respectively quasi-concave) in the sense of* [3].

*Proof.* – In our case $\omega_\mp$ satisfy

$$f'(v) \frac{\partial \omega_\mp}{\partial u} + g_\mp \frac{\partial \omega_\mp}{\partial v} = 0.$$

Differentiating with respect to $u$

$$f'(v) \frac{\partial^2 \omega_\mp}{\partial u^2} + g_\pm \frac{\partial^2 \omega_\mp}{\partial u \, \partial v} + \frac{\partial \omega_\mp}{\partial v} \frac{\partial g_\pm}{\partial u} = 0$$

and multiplying by $\dfrac{1}{f'(v)} \left( \dfrac{\partial \omega_\mp}{\partial v} \right)^2$, then

$$\left( \frac{\partial \omega_\mp}{\partial v} \right)^2 \frac{\partial^2 \omega_\mp}{\partial u^2} - \frac{\partial \omega_\mp}{\partial u} \frac{\partial \omega_\mp}{\partial v} \frac{\partial^2 \omega_\mp}{\partial u \, \partial v} = - \left( \frac{\partial \omega_\mp}{\partial v} \right)^3 \frac{1}{f'(v)} \frac{\partial g_\mp}{\partial u} \tag{2.3}$$

On the other hand, if we differentiate (2.3) respect to $v$, it follows

$$f'(v) \frac{\partial^2 \omega_\mp}{\partial u \, \partial v} + f''(v) \frac{\partial \omega_\mp}{\partial u} + g_\pm \frac{\partial^2 \omega_\mp}{\partial v^2} + \frac{\partial \omega_\mp}{\partial v} \frac{\partial g_\pm}{\partial v} = 0.$$

Then we multiply by $\dfrac{1}{f'(v)} \dfrac{\partial \omega_\mp}{\partial u} \dfrac{\partial \omega_\mp}{\partial v}$ and therefore

$$\begin{aligned} \left( \frac{\partial \omega_\mp}{\partial u} \right)^2 \frac{\partial^2 \omega_\mp}{\partial v^2} - \frac{\partial^2 \omega_\mp}{\partial u \, \partial v} \frac{\partial \omega_\mp}{\partial u} \frac{\partial \omega_\mp}{\partial v} &= \frac{f''(v)}{f'(v)} \frac{\partial \omega_\mp}{\partial v} \left( \frac{\partial \omega_\mp}{\partial u} \right)^2 \\ &\quad + \left( \frac{\partial \omega_\mp}{\partial v} \right)^2 \frac{\partial \omega_\mp}{\partial u} \frac{\partial g_\pm}{\partial v} \frac{1}{f'(v)} \end{aligned} \tag{2.4}$$

Since one has

$$\nabla^2 \omega_\mp (r_\pm, r_\pm) = \frac{\partial^2 \omega_\mp}{\partial u^2} \left( \frac{\partial \omega_\mp}{\partial v} \right)^2 - 2 \frac{\partial^2 \omega_\mp}{\partial u \, \partial v} \frac{\partial \omega_\mp}{\partial u} \frac{\partial \omega_\mp}{\partial v} + \frac{\partial^2 \omega_\mp}{\partial v^2} \left( \frac{\partial \omega_\mp}{\partial u} \right)^2,$$

adding (2.3) and (2.4), we get

$$\nabla^2 \omega_{\mp}(r_{\pm}, r_{\pm}) = \frac{f''(v)}{f'(v)} \frac{\partial \omega_{\mp}}{\partial v} \left(\frac{\partial \omega_{\mp}}{\partial u}\right)^2 + \left(\frac{\partial \omega_{\mp}}{\partial v}\right)^2 \frac{\partial \omega_{\mp}}{\partial u} \frac{\partial g_{\pm}}{\partial v} \frac{1}{f'(v)} - \left(\frac{\partial \omega_{\mp}}{\partial v}\right)^3 \frac{\partial g_{\pm}}{\partial u} \frac{1}{f'(v)}$$

and by using (2.3) we obtain

$$\nabla^2 \omega_{\mp}(r_{\pm}, r_{\pm}) = \frac{1}{f'(v)^3} \left(\frac{\partial \omega_{\mp}}{\partial v}\right)^3 \left[f''(v)\, g_{\pm}^2 - f'(v)^2 \frac{\partial g_{\pm}}{\partial u} - f'(v)\, g_{\pm} \frac{\partial g_{\pm}}{\partial v}\right].$$

Because of the corollary 2.2, we have only to prove that

$$\mathrm{L}_{\mp}(u, v) \stackrel{\text{def}}{=} f''(v)\, g_{\pm}^2 - f'(v)^2 \frac{\partial g_{\pm}}{\partial u} - f'(v)\, g_{\pm} \frac{\partial g_{\pm}}{\partial v} \tag{2.5}$$

is non negative. We can express (2.5) also as

$$\mathrm{L}_{\mp}(u, v) = g_{\pm}\left[g_{\pm} f''(v) - f'(v)^2 \left(\pm \frac{1-a}{\sqrt{\mathrm{G}(u, v)}}\right) - f'(v) \left(\pm \frac{f'(v) + v f''(v)}{2\sqrt{\mathrm{G}(u, v)}}\right)\right].$$

Since $\pm g_{\pm}(u, v) \geqq 0$, we need only to show that

$$\begin{aligned} \mathrm{Q}_{\mp}(u, v) &\stackrel{\text{def}}{=} \pm \left[g_{\pm} f''(v) \pm \frac{a f'(v)^2}{\sqrt{\mathrm{G}(u, v)}} \mp \frac{f' v)}{\sqrt{\mathrm{G}(u, v)}} \frac{(f'(v) + v f''(v))}{2}\right] \\ &= \pm \left[f''(v) \left(g_{\pm} \mp \frac{v f'(v)}{2\sqrt{\mathrm{G}(u, v)}}\right) \pm \frac{f'(v)^2}{\sqrt{\mathrm{G}(u, v)}} \frac{(2a-1)}{2}\right] \end{aligned}$$

is non negative. But one has

$$\pm \left(g_{\mp} \mp \frac{v f'(v)}{2\sqrt{\mathrm{G}(u, v)}}\right) = \left(\frac{2 a^2 u^2 + v f'(v) \pm 2 a u \sqrt{\mathrm{G}(u, v)}}{2\sqrt{\mathrm{G}(u, v)}}\right).$$

Then the lemma holds when $\mp u \geqq 0$. In the case $\mp u \leqq 0$, one has

$$4 a^4 u^4 + 4 a^2 u^2 v f'(v) + v^2 f'(v)^2 \geqq 4 a^2 u^2 [a^2 u^2 + v f'(v)],$$

then

$$2 a^2 u^2 + v f'(v) \geqq 2 a u \sqrt{\mathrm{G}(u, v)}.$$

Hence, in any case, $\mathrm{Q}_{\mp}(u, v) \geqq 0$. ■

Because of the quasi-convexity property, we can conclude with the following

THEOREM 2.4. – *Assume that* $(u_0, v_0) \in L^\infty(\mathbb{R}) \times L^\infty(\mathbb{R})$ *and consider the viscous problem* $(2.1)_\varepsilon$, *then the solution* $(u^\varepsilon, v^\varepsilon)$ *is bounded* a priori *in* $L^\infty$ *independent of* $\varepsilon$.

## 3. STUDY OF THE VISCOUS SYSTEM

We now show that $(2.1)_\varepsilon$ has a global solution in time. The proof will be carried out using the result of section 2 and the contraction mapping theorem in a standard way.

Let $\mathscr{C}_T = C(\mathbb{R})$ be provided with the norm

$$\|(u, v)\| \overset{\text{def}}{=} \sup_x |u| + \sup_x |v|$$

and $C([0, T] \times \mathbb{R})$

$$\|(u, v)\|_T \overset{\text{def}}{=} \sup_{0 \leq \tau \leq T} \|(u, v)(\tau)\|.$$

Let

$$\Gamma \overset{\text{def}}{=} \{(u, v) \in C((0, T) \times \mathbb{R}) : \|(u, v)(t) - Z * (u_0, v_0)\| \leq \|(u_0, v_0)\|\}$$

where $Z(x, t) \overset{\text{def}}{=} \dfrac{1}{\sqrt{4\pi t}} \exp\left(-\dfrac{x^2}{4t}\right)$ is the heat kernel and $*$ denotes convolution in $x$.

By using the formula of the method of variation of constants, we have

$$\mathscr{U}(x, t) = \int_{-\infty}^{+\infty} Z(x-y, t)\, \mathscr{U}_0(y)\, dy + \int_0^t \int_{-\infty}^{+\infty} Z_x(x-y, t-\tau)\, F(\mathscr{U}(y, \tau))\, dy\, d\tau.$$

Let

$$\mathscr{F}(\mathscr{U}(x, t)) = \int_{-\infty}^{+\infty} Z(x-y, t)\, \mathscr{U}_0(y)\, dy + \int_0^t \int_{-\infty}^{+\infty} Z_x(x-y, t-\tau)\, F(\mathscr{U}(y, \tau))\, dy\, d\tau;$$

than $\mathscr{F}$ maps $\Gamma_T$ into itself if $T > 0$ is sufficiently small. Indeed we have the well known estimate

$$\int_{-\infty}^{+\infty} |Z_x(x-y, t-\tau)|\, dy \leq \frac{c}{\sqrt{(t-\tau)}}.$$

On the other hand, if $\mathcal{U} \in \Gamma_T$ then

$$|\mathcal{U}(y, \tau)| \leqq 2 \|\mathcal{U}_0\|.$$

Finally, denoting by

$$f_{\mathcal{U}_0} \overset{\text{def}}{=} \max_{\mathcal{U} \in \Gamma_T} |f(v)| \frac{1}{\|\mathcal{U}_0\|^2},$$

we have

$$\sup_{y \in \mathbb{R}} |F(\mathcal{U}(y, \tau))| \leqq (4a + 6 + f_{\mathcal{U}_0}) \|\mathcal{U}_0\|^2.$$

Thus it follows that

$$|\mathcal{F}(\mathcal{U}) - Z * \mathcal{U}_0| \leqq c(4a + 6 + f_{\mathcal{U}_0}) \|\mathcal{U}_0\|^2 \int_0^t \frac{1}{\sqrt{t-\tau}} d\tau \leqq c(4a + 6 + f_{\mathcal{U}_0}) \|\mathcal{U}_0\|^2 \sqrt{t}.$$

The right hand side is bounded by $\|\mathcal{U}_0\|$, provided that

$$t \leqq T_1 \overset{\text{def}}{=} \frac{1}{[c(4a + 6 + f_{\mathcal{U}_0}) \|\mathcal{U}_0\|]^2}.$$

Now as T is small enough,

$$\mathcal{F}: \quad \Gamma_T \to \Gamma_T$$

is a contraction mapping on $\Gamma_T$; indeed if $\mathcal{U}_1, \mathcal{U}_2 \in \Gamma_T$, we easily see that there exists a constant $c_{a, \mathcal{U}_0} > 0$ such that

$$|F(\mathcal{U}_1) - F(\mathcal{U}_2)| \leqq c_{a, \mathcal{U}_0} \|\mathcal{U}_1 - \mathcal{U}_2\|,$$

so that we can conclude

$$\|\mathcal{F}(\mathcal{U}_1) - \mathcal{F}(\mathcal{U}_2)\|_t \leqq \int_0^t c c_{a, f, \mathcal{U}_0} \frac{\|\mathcal{U}_1 - \mathcal{U}_2\|}{\sqrt{t-\tau}} d\tau \leqq c c_{a, f, \mathcal{U}_0} \|\mathcal{U}_1 - \mathcal{U}_2\| \sqrt{t}.$$

Now the right hand side is bounded by $\|\mathcal{U}_1 - \mathcal{U}_2\|$, provided that

$$t \leqq T_2 \overset{\text{def}}{=} \frac{1}{(c c_{a, \mathcal{U}_0})^2}.$$

Therefore, we conclude that $(2.1)_\varepsilon$ has a unique solution in $C((0, T^*), \mathcal{C}(T^*))$, where we have set

$$T^* \overset{\text{def}}{=} \frac{1}{2} \frac{1}{[\max(c(4a + 6 + f_{\mathcal{U}_0}) \|\mathcal{U}_0\|, c c_{a, \mathcal{U}_0})]^2}$$

Because of the existence of an invariant region, $T^*$ depends only on $\|\mathcal{U}_0\|$. Then we can iterate the previous argument by taking $(u, v)_{|t = T^*}$ as initial datum. Thus we have proved

THEOREM 3.1. — *Under the above hypotheses the approximation* $(2.1)_\varepsilon$ *for the problem* (1.1) *has a unique smooth solution inside a suitable bounded invariant region.*

We conclude this section showing that, when the data lies in the upper half plane, the solution remains in it for all $t>0$. To establish this result we apply a positive solution theorem for parabolic equations of second order (*see* [9]).

Consider the linear operator

$$L\psi=\varepsilon\frac{\partial^2\psi}{\partial x^2}-u\frac{\partial\psi}{\partial x}-\frac{\partial\psi}{\partial t}-u_x\psi$$

for any smooth function $\psi$, then $Lv=0$ because of the equation $(2.1)_\varepsilon$ and by the theorem 2.4 there exists $M>0$ such that $|u|$, $|v|<M$. Therefore, for all $(x, t)$, $\beta>0$, we have

$$v\geqq -M\geqq -M\exp(\beta|x|^2).$$

We now deduce an *a priori* bound on $|\mathscr{U}_x|_\infty$ in terms of $|(u_0)_x|_\infty$ and $|(v_0)_x|_\infty$. Indeed, if we differentiate $(2.1)_\varepsilon$ with respect to $x$, one has

$$\mathscr{U}_x(x, t)=\int_{-\infty}^{+\infty} Z(x-y, t)(\mathscr{U}_0)_x(y)\,dy + \int_0^t\int_{-\infty}^{+\infty} Z_x(x-y, t-\tau)\,F(\mathscr{U}(\tau, y))_x\,dy\,d\tau.$$

Let us denote by

$$\Phi(\tau)=\max\{\sup_x|u_x(\tau, x)|, \sup_x|v_x(\tau, x)|\},$$

then

$$|F(\mathscr{U})_x|\overset{\text{def}}{=}|\nabla F(\mathscr{U})\,\mathscr{U}_x|\leqq c_{a, M, f}\,\Phi(\tau),$$

where $c_{a, M, f}$ depends on $a$, M and $f$; therefore

$$\begin{aligned}\Phi(t)-\Phi(0)&=\max\{\sup_x|u_x(t, x)|, \sup_x|v_x(t, x)|\}\\ &\quad-\max\{\sup_x|(u_0)_x|, \sup_x|(v_0)_x|\}\\ &\leqq\sup_x|\mathscr{U}_x(x, t)-\mathscr{U}_x(x, 0)|\leqq c\int_0^t\frac{c_{a, M, f}\,\Phi(\tau)\,d\tau}{\sqrt{t-\tau}}d\tau.\end{aligned}$$

Assume that $T>0$; then define $A: C([0, T])\to C([0, T])$ by

$$(A\Psi)(t)=\int_0^t\frac{\Psi(\tau)}{\sqrt{t-\tau}}d\tau.$$

We can easily check that this definition is well posed and

$$|A\Psi|_\infty \leqq 2\sqrt{T}\,|\Psi|_\infty,$$

namely

$$\|A\| \leqq 2\sqrt{T}.$$

Then $(I - cc_{a,M,f}A)$ is an invertible operator when $T < \dfrac{1}{(2cc_{a,M,f})^2}$.

Denote by

$$\Psi(t) = [(I - cc_{a,M,f}A)^{-1}\Phi(0)](t).$$

Then

$$\Psi(t) = \Phi(0) + cc_{a,M,f}\int_0^t \frac{\Psi(\tau)}{\sqrt{t-\tau}}\,d\tau.$$

On the other hand, for all $t \in [0, T]$

$$\Phi(t) \leqq \Psi(t).$$

Since $g \overset{\text{def}}{=} \max\{0, \Phi - \Psi\}$ satisfies

$$\begin{cases} g(t) \leqq cc_{a,M,f}\displaystyle\int_0^t \frac{g(\tau)}{\sqrt{t-\tau}}\,d\tau \\ g(0) = 0, \end{cases}$$

we then get

$$\sup_{0 \leqq t \leqq T} g(t) \leqq cc_{a,M,f}\int_0^T \frac{1}{\sqrt{T-\tau}} \sup_{0 \leqq t \leqq T} g(t)\,d\tau = 2cc_{a,M,f}\sqrt{T}\sup_{0 \leqq t \leqq T} g(t).$$

However, $2cc_{a,M,f}\sqrt{T} < 1$, which is impossible, unless $g(t) = 0$ for all $t \in [0, T]$. Therefore

$$\Phi(t) \leqq \Psi(t) \quad \text{for all } t \in [0, T].$$

Thus we conclude that $\Phi$ is bounded on $[0, T]$ and, since T depends only on M, we can iterate the previous arguments.

Therefore, for all T, we obtain the following *a priori* bound

$$\sup_x |u_x|(t) \leqq c_{T, |(u_0)_x|_\infty, |(v_0)_x|_\infty}$$

for all $t \in [0, T]$.

We summarize our conclusion in the following result

PROPOSITION 3.2. – *Assume that* $u_0, v_0, \dfrac{\partial u_0}{\partial x}, \dfrac{\partial v_0}{\partial x} \in L^\infty(\mathbb{R})$, $v_0(x) \geqq 0$ *for all* $x$, *then the solution* $(u^\varepsilon, v^\varepsilon)$ *to* $(2.1)_\varepsilon$ *satisfies* $v^\varepsilon(x, t) \geqq 0$ *for all* $(x, t)$.

COROLLARY 3.3. — *Under the previous assumptions we have, for all* $(x, t)$,

$$w^* - \lim_{\varepsilon \to 0^+} (u^\varepsilon, v^\varepsilon) \in \{ (u, v) : v \geqq 0 \}.$$

## 4. ENTROPIES

In this section we discuss several properties of the entropy pair associated to our problem which we shall use later.

The inner product of (1.3) with the right eigenvectors $r_\mp$ of $\nabla F$ produces the characteristic form

$$(\lambda_\mp \nabla \eta - \nabla q) . r_\mp = 0 \tag{4.1}$$

which is equivalent to (1.3). Hence the Riemann invariants $(\omega_-, \omega_+)$ are well defined coordinates because

$$T: \quad (u, v) \mapsto (\omega_-, \omega_+)$$

is a one to one map which defines a change of coordinates in the half plane $\{ (u, v) : v \geqq 0 \}$.

Indeed we have

$$\left\{ \begin{aligned} &\frac{\partial}{\partial \omega_-} = r_+ . \nabla \\ &\frac{\partial}{\partial \omega_+} = r_- . \nabla, \end{aligned} \right.$$

which transforms the system (4.1) into

$$\left\{ \begin{aligned} &\lambda_- \eta_{\omega_-} = q_{\omega_-} \\ &\lambda_+ \eta_{\omega_+} = q_{\omega_+}. \end{aligned} \right.$$

The equations (1.3) can be written explicitely as

$$\left\{ \begin{aligned} &\frac{\partial q}{\partial u} = (2a+1) u \frac{\partial \eta}{\partial u} + v \frac{\partial \eta}{\partial v} \\ &\frac{\partial q}{\partial v} = f'(v) \frac{\partial \eta}{\partial u} + u \frac{\partial \eta}{\partial v}. \end{aligned} \right.$$

Eliminating $q$, we get a second order partial differential equation in $\eta$:

$$v \frac{\partial^2 \eta}{\partial v^2} - f'(v) \frac{\partial^2 \eta}{\partial u^2} + 2 a u \frac{\partial^2 \eta}{\partial u \, \partial v} = 0. \tag{4.2}$$

The same equation, in coordinates $\omega=(\omega_-, \omega_+)$, becomes

$$\frac{\partial^2 \eta}{\partial\omega_- \partial\omega_+}+\frac{1}{\lambda_+(\omega)-\lambda_-(\omega)}\left(\frac{\partial\lambda_+(\omega)}{\partial\omega_-}\frac{\partial\eta}{\partial\omega_+}-\frac{\partial\lambda_-(\omega)}{\partial\omega_+}\frac{\partial\eta}{\partial\omega_-}\right)=0 \quad (4.3)$$

and we shall only consider (4.3) when $\omega_-\leqq 0\leqq\omega_+$.

We shall study, in the following, some particular types of entropies.

### 4.1. Entropies of product type

We now look for solutions of the form

$$\eta(u, v)=\alpha(u)\beta(v).$$

For such a function $\eta(u, v)$, (4.2) assumes the form

$$v\,\alpha(u)\beta''(v)-f'(v)\alpha''(u)\beta(v)+2au\alpha'(u)\beta'(v)=0. \quad (4.4)$$

Dividing by $\alpha(u)\beta(v)f'(v)$ and differentiating with respect to $u$, we obtain

$$\left(\frac{\alpha''(u)}{\alpha(u)}\right)'=2a\left(u\frac{\alpha'(u)}{\alpha(u)}\right)'\frac{\beta'(v)}{\beta(v)f'(v)},$$

which allows us to conclude that there exists a $k\in\mathbb{R}$ such that

$$\left\{\begin{array}{l}\alpha(u)=\displaystyle\int_0^u \exp\left(\frac{k}{2}t^2\right)dt\\ \beta(v)=\exp\left(\dfrac{k}{2a}f(v)\right).\end{array}\right.$$

Thus we have found a one parameter family of entropies of the following type

$$\eta_k(u, v)=\int_0^u \exp\left(\frac{k}{2}\left(\frac{f(v)}{a}+t^2\right)\right)dt.$$

On the other hand, dividing (4.4) by $\alpha(u)\beta(v)f'(v)$ and differentiating with respect to $v$, we get

$$\left(\frac{v\,\beta''(v)}{f'(v)\beta(v)}\right)'=-2au\frac{\alpha'(u)}{\alpha(u)}\left(\frac{\beta'(v)}{f'(v)\beta(v)}\right)'$$

then there exists an $h\in\mathbb{R}$ such that

$$\left\{\begin{array}{l}\alpha(u)=\exp(-hu^2)\\ \beta(v)=v^{ah+1}.\end{array}\right.$$

Thus we have found an other one parameter family of entropies of the type

$$\eta_h(u, v)=v^{h+1}\exp(-hu^2). \quad (4.5)$$

Observe that (4.5) is an entropy for our problem which does not depend on $f$, and $\eta_h$ is convex provided that $h \geqq 0$.

### 4.2. Sum type entropies

We now look for solutions of the form

$$\eta(u, v) = \alpha(u) + \beta(v).$$

In this case (4.2) reduces to

$$v\,\beta''(v) - f'(v)\,\alpha''(u) = 0,$$

which admits the following solutions

$$\begin{cases} \beta(v) = \displaystyle\int_0^v \int_0^y \frac{f'(z)}{z}\,dz\,dy \\ \alpha(u) = \dfrac{1}{2} u^2, \end{cases}$$

that is,

$$\eta(u, v) = \frac{1}{2} u^2 + \int_0^v \int_0^y \frac{f'(z)}{z}\,dz\,dy.$$

It is a well defined convex entropy. The corrisponding flux is given by

$$q(u, v) = \frac{2a+1}{3} u^3 + uv \int_0^v \frac{f'(z)}{z}\,dz.$$

### 4.3. Polynomial entropies

In this paragraph we assume $f$ is polynomial of degree $n$. Under this hypothesis we will prove the existence of polynomial entropies by an iterative process. Suppose first $f(v) = \frac{1}{2} v^2$, which is not restrictive since if $f(v) = \frac{c^2}{2} v^2$, by means of the change of variable $v \mapsto \frac{v}{c}$, we obtain

$$\begin{cases} u_t + \left(\left(\dfrac{1}{2} + a\right) u^2 + \dfrac{1}{2} v^2\right)_x = 0 \\ \dfrac{1}{c} v_t + \dfrac{1}{c} (uv)_x = 0, \end{cases}$$

namely the problem (1.1), with $f(v) = \frac{1}{2} v^2$.

In this particular case the problem of finding polynomial entropies was treated by [14]. Indeed (4.2) reduces to

$$v\frac{\partial^2 \eta}{\partial v^2}-v\frac{\partial^2 \eta}{\partial u^2}+2\,au\frac{\partial^2 \eta}{\partial u\,\partial v}=0, \tag{4.6}$$

which is invariant under the dilatation

$$(u,\ v,\ \eta)\mapsto(cu,\ cv,\ c\,\eta).$$

Therefore we can look for particular solutions by reducing (4.6) to an ordinary differential equation in the new variables $\xi=\frac{u}{v}$, $\tilde{\Phi}=\frac{\Phi}{v^\alpha}$, where $\alpha>0$ is an arbitrary integer. Then [13] obtains a polynomial solution of degree $\alpha$ in $u$ and $v$ of the form

$$\Phi(u,\ v)=v^\alpha\,\tilde{\Phi}\left(\frac{u}{v}\right),$$

where $\tilde{\Phi}$ is a polynomial of degree $\alpha$.

Now, we suppose that $f$ is a polynomial of degree $n$ which can be written as

$$f(v)=\frac{1}{2}v^2+g(v),$$

where $g$ is a polynomial of degree greater than two. Let

$$\mathscr{L}(\eta)=v\left(\frac{\partial^2 \eta}{\partial v^2}-\frac{\partial^2 \eta}{\partial u^2}\right)+2\,au\frac{\partial^2 \eta}{\partial u\,\partial v}-g'(v)\frac{\partial^2 \eta}{\partial u^2},$$

then (4.2) can be written as

$$\mathscr{L}(\eta)=0. \tag{4.7}$$

Let us denote by

$$\mathscr{L}_0(\eta)=v\left(\frac{\partial^2 \eta}{\partial v^2}-\frac{\partial^2 \eta}{\partial u^2}\right)+2\,au\frac{\partial^2 \eta}{\partial u\,\partial v},$$

then we can easily find polynomial solutions to

$$\mathscr{L}_0(\eta)=0 \tag{4.8}$$

and we want to find such solutions for (4.7). Let $\eta_0$ a solution to (4.8), then we can write the solution to (4.7) in the following way

$$\eta=\eta_0+H_1^0+H_1^R,$$

where $H_1^0$ is a solution to the inhomogeneous equation

$$\mathscr{L}_0(\eta)=g'(v)\frac{\partial^2 \eta_0}{\partial u^2}. \tag{4.9}$$

Therefore $H_1^R$ solves also

$$\mathscr{L}(\eta)=g'(v)\frac{\partial^2 H_1^0}{\partial u^2}.$$

Let us now split $H_1^R$ in the following way

$$H_1^R=H_2^0+H_2^R,$$

with $H_2^0$ solution to

$$\mathscr{L}_0(\eta)=g'(v)\frac{\partial^2 H_1^0}{\partial u^2}.$$

We find that $H_2^R$ is a solution to

$$\mathscr{L}(\eta)=g'(v)\frac{\partial^2 H_2^0}{\partial u^2}.$$

Then our solution $\eta$ is given by

$$\eta=\eta_0+H_1^0+H_2^0+H_2^R.$$

Then, iterating this procedure, after $k$ steps we obtain

$$\eta=\eta_0+\sum_{j=1}^{k} H_j^0+H_k^R,$$

where

$$\left\{\begin{array}{l} \mathscr{L}_0(\eta_0)=0 \\ \mathscr{L}_0(H_j^0)=g'(v)\dfrac{\partial^2 H_{j-1}^0}{\partial u^2} \qquad \text{for} \quad j\geqq 1 \end{array}\right. \tag{4.10}$$

Finally we have

PROPOSITION 4.1. — *Under the above hypothesis the following properties hold:*

*a) if* $H_{j-1}^0$ *is polynomial, then there exists* $H_j^0$ *which is a polynomial;*

*b) there exists* $k>0$ *such that* $H_k^R\equiv 0$.

*Proof.* — We suppose that $H_{j-1}^0(u, v)$ has degree $m_1$ in $u$, $m_2$ in $v$, *i.e.*

$$H_{j-1}(u, v)=\sum_{\substack{k=0 \\ h=0}}^{\substack{m_1 \\ m_2}} d_{k,h}u^k v^h;$$

then

$$\frac{\partial^2 H_{j-1}^0}{\partial u^2}=\sum k(k-1)\,d_{k,h}u^{k-2}v^h$$

and there exist $\tilde{d}_{k,h}$ such that

$$g'(v)\frac{\partial^2 H^0_{j-1}}{\partial u^2}=\sum_{\substack{k=0\\h=2}}^{\substack{m_1-2\\m_2+n-1}} \tilde{d}_{k,h}u^k v^h;$$

and so, if we assume that

$$H^0_j(u, v)=\sum_{\substack{k_1\geqq 0\\h_1\geqq 0}} c_{k_1,h_1}u^{k_1}v^{h_1},$$

then $(4.10)_j$ becomes

$$-\sum_{\substack{k_1\geqq 2\\h_1\geqq 0}} k_1 c_{k_1,h_1}(k_1-1)u^{k_1-2}v^{h_1+1} + \sum_{\substack{k_1\geqq 0\\h_1\geqq 2}} h_1 c_{k_1,h_1}(h_1-1)u^{k_1}v^{h_1-1}=\sum_{\substack{k=0\\h=0}} \tilde{d}_{k,h}u^k v^h$$

or, equivalently,

$$\sum_{\substack{k_1\geqq 0\\h_1\geqq 1}} h_1(h_1+1)c_{k_1,h_1+1}u^{k_1}v^{h_1} - \sum_{\substack{k_1\geqq 0\\h_1\geqq 1}} (k_1+2)(k_1+1)c_{k_1+2,h_1-1}u^{k_1}v^{h_1} + \sum_{\substack{k_1\geqq 1\\h_1\geqq 0}} 2ac_{k_1,h_1+1}(h_1+1)k_1 u^{k_1}v_{h_1}=\sum_{\substack{k=0\\h=2}} \tilde{d}_{k,h}u^k v^h,$$

from which we obtain

$$\begin{cases} c_{k,1}=0 & \text{if} \quad k\in\mathbf{N}\\ c_{k,2}=0 & \text{if} \quad k\in\mathbf{N}\\ (h+1)(h+(a-1)k)c_{k,h+1}-(k+1)(k+2)c_{k+2,h-1}=\tilde{d}_{k,h} & \\ \qquad\qquad \text{if} \quad k\leqq m_1-2, \quad h\leqq m_2+n-1 & \\ c_{k,h}=0 & \text{otherwise}\end{cases}$$

Here we can choose some of the coefficients $c_{k,h}$ arbitrarily, for example,

$$\begin{cases} c_{m_1-1,h}=0\\ c_{m_1,h}=0.\end{cases}$$

With this choice we obtain $H^0_j(u, v)$ which is of degree $m_1-2$ in $u$, indeed if $\eta_0$ has degree $\gamma$ in $u$, after $\left[\frac{\gamma}{2}\right]+1$ steps we obtain a function which is independent of $u$, and our iteration procedure is complete. ■

Thus we have proved that

THEOREM 4.2. — *If we consider* (1.1) *with a polynomial $f$ of degree $n$, there exist polynomial entropies in* $(u, v)$.

### 4.4. Weak entropy

We now consider the Riemann invariant coordinate system $(\omega_-, \omega_+)$. In this case we find that the characteristic curves for the entropy equation are the straight lines parallel to the coordinates axes. The Goursat problem consists in finding a solution $\eta$ of (4.3) when its value on two incident characteristics are known.

Let us take two constant $\omega_-^*$, $\omega_+^*$ with $\omega_-^* \leqq \omega_+^*$: we shall study the Goursat problem

$$\left\{\begin{array}{l} \dfrac{\partial^2 \eta}{\partial\omega_-\,\partial\omega_+}+\dfrac{1}{\lambda_+(\omega)-\lambda_-(\omega)}\left(\dfrac{\partial\lambda_+(\omega)}{\partial\omega_-}\,\dfrac{\partial\eta}{\partial\omega_+}-\dfrac{\partial\lambda_-(\omega)}{\partial\omega_+}\,\dfrac{\partial\eta}{\partial\omega_-}\right)=0 \\ \eta(\omega_-, \omega_+^*)=\theta_-(\omega_-) \\ \eta(\omega_-^*, \omega_+)=\theta_+(\omega_+) \end{array}\right. \tag{4.11}$$

where $\theta_-$ and $\theta_+$ are given smooth functions.

We recall the following standard result due to Sobolev [28]:

THEOREM. — *The Goursat problem* (4.11) *admits a solution as regular as the initial data $\theta_-$ and $\theta_+$ on any bounded domain away from the umbilical point and from the $\omega_+$ axis.*

This solution will be, in general, singular along the $\omega_+$ axis. This is due to the fact that the domain of dependence of any point on the $\omega_+$ axis contains the umbilical point, where the coefficients of (4.3) become singular.

We shall only consider special classes of Goursat data, namely:

$$\left\{\begin{array}{ll} \theta_-(\omega_-)=0 & \text{in} \quad \omega_- \leqq \omega_-^* \\ \theta_+(\omega_+)\equiv 0 & \end{array}\right.$$

and we shall consider special constants, namely:

$$\left\{\begin{array}{l} \omega_-^* < 0 \\ \omega_+^* = 0. \end{array}\right.$$

In this case the solution possesses some convenient vanishing properties and, in particular,

$$\eta(\omega)=0 \qquad \text{for} \quad \omega_- < \omega_-^*.$$

We shall see later that the same condition has to imposed on $\theta_-$ to make the entropy function regular everywhere.

The technical complication that one finds with this method is becouse of our incapacity to write the change of coordinates T explicitely.

However, as we can easily see, there exist Riemann invariants $(\omega_-, \omega_+)$ with the above properties for which we have

$$\left\{\begin{array}{l} \dfrac{\partial u}{\partial \omega_-}=\dfrac{1}{2} \\ \dfrac{\partial v}{\partial \omega_-}=\dfrac{-au-\sqrt{a^2 u^2+vf'(v)}}{2f'(v)} \\ \dfrac{\partial u}{\partial \omega_+}=\dfrac{1}{2} \\ \dfrac{\partial v}{\partial \omega_+}=\dfrac{-au+\sqrt{a^2 u^2+vf'(v)}}{2f'(v)}, \end{array}\right.$$

In this way we can estimate the coefficients of the first order terms of the equation (4.3): we have

$$\begin{aligned} \frac{\partial \lambda_+}{\partial \omega_-}=&\frac{1}{2}\left(a+1+\frac{a^2 u}{\sqrt{a^2 u^2+vf'(v)}}\right) \\ &+\left(\frac{-\sqrt{a^2 u^2+vf'(v)}-au}{2f'(v)}\right)\left(\frac{vf''(v)+f'(v)}{2\sqrt{a^2 u^2+vf'(v)}}\right) \\ =&\frac{1}{2}\left(1+a-\left(\frac{vf''(v)+f'(v)}{2f'(v)}\right)\right) \\ &+\frac{1}{2}\frac{au}{\sqrt{a^2 u^2+vf'(v)}}\left(a-\frac{vf''(v)+f'(v)}{2f'(v)}\right) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial \lambda_-}{\partial \omega_+}=&\frac{1}{2}\left(a+1-\frac{a^2 u}{\sqrt{a^2 u^2+vf'(v)}}\right) \\ &+\left(\frac{\sqrt{a^2 u^2+vf'(v)}-au}{2f'(v)}\right)\left(-\frac{vf''(v)+f'(v)}{2\sqrt{a^2 u^2+vf'(v)}}\right) \\ =&\frac{1}{2}\left(1+a-\left(\frac{vf''(v)+f'(v)}{2f'(v)}\right)\right) \\ &-\frac{1}{2}\frac{au}{\sqrt{a^2 u^2+vf'(v)}}\left(a-\frac{vf''(v)+f'(v)}{2f'(v)}\right). \end{aligned}$$

We are interested in the order of the infinitesimal $\lambda_+(\omega)-\lambda_-(\omega)$ with respect to $(\omega_-, \omega_+)$.

Now, we notice that $\lambda_+(u,v)-\lambda_-(u,v)=2\sqrt{a^2u^2+vf'(v)}$; hence we have

$$\frac{\partial}{\partial\omega_-}\sqrt{a^2u^2+vf'(v)}=-\frac{vf''(v)+f'(v)}{4f'(v)} +\frac{1}{2}\frac{au}{\sqrt{a^2u^2+vf'(v)}}\left(a-\frac{vf''(v)+f'(v)}{2f'(v)}\right)$$

and

$$\frac{\partial}{\partial\omega_+}\sqrt{a^2u^2+vf'(v)}=\frac{vf''(v)+f'(v)}{4f'(v)} +\frac{1}{2}\frac{au}{\sqrt{a^2u^2+vf'(v)}}\left(a-\frac{vf''(v)+f'(v)}{2f'(v)}\right)$$

LEMMA 4.3. – *Let $f$ be a function such that*

$$f(v)=kv^{2n}+O(|v|^{2n+1}) \tag{4.12}$$

*for some $k\in\mathbb{R}^+$ and $n\in\mathbb{N}$. Then, the quantities*

$$\frac{\partial\lambda_-}{\partial\omega_+},\quad \frac{\partial\lambda_+}{\partial\omega_-},\quad \frac{\partial}{\partial\omega_-}(\lambda_+-\lambda_-),\quad \frac{\partial}{\partial\omega_+}(\lambda_+-\lambda_-)$$

*just computed are continuous functions in a neighbourhood of the umbilical point if and only if $a=n$.*

*In the other case, $\omega=0$ is an essential singularity.*

*Proof.* – In fact, with the hypothesis (4.12), the limit, as $v\to 0$, for the special term

$$\frac{vf''(v)+f'(v)}{2f'(v)}$$

is $n$. On the other hand, as it is easy to see, the term

$$\frac{au}{\sqrt{a^2u^2+vf'(v)}}$$

is uniformly bounded but it does not admit a limit as $|(u,v)|\to 0$. So, it is necessary that the associated term

$$a-\frac{vf''(v)+f'(v)}{2f'(v)}$$

tends to 0 and the lemma is proved. ■

Therefore, to continue our method, it is necessary to take the special coefficient $a=n$. From now on, we make this hypothesis and hence we obtain in the previous calculation

$$\left\{\begin{aligned} &\lim_{|\omega|\to 0} \frac{\partial\lambda_+(\omega)}{\partial\omega_-}=\frac{1}{2}\\ &\lim_{|\omega|\to 0} \frac{\partial\lambda_-(\omega)}{\partial\omega_+}=\frac{1}{2}\\ &\lim_{|\omega|\to 0} \frac{\partial}{\partial\omega_-}(\lambda_+(\omega)-\lambda_-(\omega))=-n\\ &\lim_{|\omega|\to 0} \frac{\partial}{\partial\omega_+}(\lambda_+(\omega)-\lambda_-(\omega))=+n \end{aligned}\right.$$

and, in particular we have, in a neighbourhood of the umbilical point, it turns out that

$$\lambda_+(\omega)-\lambda_-(\omega)=n(\omega_+-\omega_-)+O(|\omega|)$$

and that the singularity of the equation (4.3) at the umbilical point depends only on

$$\frac{1}{\lambda_+(\omega)-\lambda_-(\omega)}.$$

*Remark* 4.4. – The case studied in [14], where $a=1$ and $f(v)=\frac{1}{2}v^2$, falls within the scope of our result.

The result just found means that our equation behaves in a neighbourhood of the umbilical point as in the case of [14]. We finally remark that this generalization is possible for all $f\in C^2(\mathbb{R})$, even and superlinear, not only for a perturbation of the quadratic term $\frac{1}{2}v^2$.

On the other hand, we recall that for our problem (4.11), the following Riemann representation formula holds (*see* [4], pp. 449-461).

$$\begin{aligned}\eta(\omega)=&\int_{\omega_-^*}^{\omega_-} \mathscr{R}(t,0,\omega_-,\omega_+)\\ &\times\left(\frac{\partial\eta}{\partial\omega_-}(t,\omega_+)+\frac{\eta(t,\omega_+)}{\lambda_+(t,\omega_+)-\lambda_-(t,\omega_+)}\frac{\partial\lambda_+}{\partial\omega_-}(t,\omega_+)\right)_{|\omega_+=0} dt\\ =&\int_{\omega_-^*}^{\omega_-} \mathscr{R}(t,0,\omega_-,\omega_+)\left(\theta'_-(t)+\frac{\theta(t)}{\lambda_+(t,0)-\lambda_-(t,0)}\frac{\partial\lambda_+}{\partial\omega_-}(t,0)\right)dt\end{aligned}$$

where $\mathscr{R}=\mathscr{R}(t,s,\omega_-,\omega_+)$ is the Riemann function associated to the equations.

The Riemann function $\mathcal{R}$ is subject to the following condition:
(i) $\mathcal{R}(x, y, \alpha, \beta)$ satisfies the equation (4.3), namely

$$\frac{\partial^2 \mathcal{R}}{\partial\alpha\,\partial\beta}+\frac{1}{\lambda_+(\alpha, \beta)-\lambda_-(\alpha, \beta)}\left(\frac{\partial\lambda_+}{\partial\alpha}(\alpha, \beta)\frac{\partial\mathcal{R}}{\partial\beta}-\frac{\partial\lambda_-}{\partial\beta}(\alpha, \beta)\frac{\partial\mathcal{R}}{\partial\alpha}\right)=0;$$

(ii) we have, on the axis $y=\beta$

$$\frac{\partial\mathcal{R}}{\partial x}(x, y, \alpha, \beta)=\frac{\mathcal{R}(x, y, \alpha, \beta)}{\lambda_+(x, \beta)-\lambda_-(x, \beta)}\,\frac{\partial\lambda_+}{\partial\omega_-}(x, \beta)$$

and on the axis $x=\alpha$

$$\frac{\partial\mathcal{R}}{\partial y}(x, y, \alpha, \beta)=-\frac{\mathcal{R}(x, y, \alpha, y)}{\lambda_+(\alpha, y)-\lambda_-(\alpha, y)}\,\frac{\partial\lambda_-}{\partial\omega_+}(\alpha, y);$$

(iii)

$$\mathcal{R}(\alpha, \beta, \alpha, \beta)=1.$$

So, the singularity of the Riemann function $\mathcal{R}$ depends only on the singularity of the first order terms of the equation. Therefore, by using the method of [26] as in [14] one has the following result

THEOREM 4.5. – *Consider the Goursat problem* (4.3). *There exists a datum* $\theta_-$ *vanishes when* $-\delta\leqq\omega_-\leqq 0$ *for some* $\delta>0$ *that balances the singularity of the Riemann function* $\mathcal{R}$ *in the neighbourhood of the umbilical point so that the solution* $\eta$ *and its derivatives with respect to* $(u, v)$ *up to the second order are bounded on bounded sets in the state space* $(u, v)$.

The solution just found of the Goursat problem (4.3) with the previous conditions is said, following Serre [26], to be an east type entropy with limit $\omega_-^*$. It is easy to show that $\eta$ satisfies the condition

$$\eta(\omega)\equiv 0 \qquad \text{for} \quad \omega_-\leqq\omega_-^*.$$

A similar theorem holds for entropies of west types with limit $\omega_-^*$ and likewise for entropy of south and nord type with limit $\omega_+^*=0$.

These four canonical types of entropies each vanishing in a half plane will be used later on to reduce the Young measure.

### 4.5. Entropy rate

In order to apply the theory of compensated compactness (*see* [5], [11], [22], [29]) we need to show the following fact; for any entropy pair $(\eta, q)$, $\eta(\mathcal{U}^\varepsilon)_t+q(\mathcal{U}^\varepsilon)_x$ lies in a relative compact subset of $H_{loc}^{-1}$. The first step in this direction will be an energy-type estimate in $L^2$, in order to control both the $L^2$-norm of the solutions and the $L^2$-rate of explosion of $u_x^\varepsilon$ and $v_x^\varepsilon$, along shock waves.

We have the following result

LEMMA 4.6. — *Assume that* $(u^\varepsilon, v^\varepsilon)$ *is an* $L^2$*-solution that satisfies* $(2.1)_\varepsilon$. *Then there exists a constant* $M>0$ *such that we have the following estimate*

$$\varepsilon \int_0^T \int_{-\infty}^{+\infty} (u_x^\varepsilon)^2 + \frac{f'(v)}{v} (v_x^\varepsilon)^2 \, dx \, dt \leqq M \tag{4.13}$$

*for all* $\varepsilon>0$.

*Proof.* — We have determined a strictly convex entropy in section 4.2 above. Multiplying $(2.1)_\varepsilon$ by $\nabla \eta$ and using (1.3), when $q$ is the entropy flux corresponding to $\eta$, we get

$$\frac{\partial}{\partial t}\eta^\varepsilon + \frac{\partial}{\partial x}\psi^\varepsilon = \varepsilon \nabla \eta^\varepsilon \mathscr{U}_{xx}^\varepsilon = -\varepsilon (\nabla \eta^\varepsilon \mathscr{U}_x^\varepsilon)_x - \varepsilon \nabla^2 \eta^\varepsilon (\mathscr{U}_x^\varepsilon, \mathscr{U}_x^\varepsilon) \tag{4.14}$$

where $\eta^\varepsilon = \eta(\mathscr{U}^\varepsilon)$ and $q^\varepsilon = q(\mathscr{U}^\varepsilon)$. Integrating (4.14) over the domain

$$\{(x, t) \,|\, 0 \leqq t \leqq T\},$$

we get

$$\int_{-\infty}^{+\infty} \eta(\tilde{\mathscr{U}}^\varepsilon(x, 0))\, dx \geqq \varepsilon \int_0^T \int_{-\infty}^{+\infty} \nabla^2 \eta (\tilde{\mathscr{U}}_x^\varepsilon, \tilde{\mathscr{U}}_x^\varepsilon)\, dx\, dt$$

provided that $\tilde{\mathscr{U}}_x^\varepsilon(., t)$ decays sufficiently rapidly at infinity. But

$$\left\{ \begin{aligned} &\frac{\partial^2}{\partial u^2}\eta = 1 \\ &\frac{\partial^2}{\partial v^2}\eta = \frac{f'(v)}{v} \end{aligned} \right.$$

and, since $(u^\varepsilon, v^\varepsilon) \in \Sigma_c \cap \{(u, v) \,|\, v \geqq 0\}$, we obtain for some constant $M>0$ the estimate (4.13)

LEMMA 4.7. — *For any entropy−entropy flux pair* $\eta_t^\varepsilon + q_x^\varepsilon$ *lies in a relatively compact set of* $H_{\text{loc}}^{-1}$, *where* $\eta^\varepsilon = \eta(u^\varepsilon, v^\varepsilon)$, $q^\varepsilon = q(u^\varepsilon, v^\varepsilon)$.

*Proof.* — It follows immediately from a lemma due to Murat [22] and the method of [29] and [5]. ■

If we now define the Young measure $\nu_{(x,t)}$ as the limit, in the sense of weak topology, of the Dirac measure sequence $\delta_{(u^\varepsilon, v^\varepsilon)(x,t)}$ [determined by $(u^\varepsilon, v^\varepsilon)(x, t)$], we have

COROLLARY 4.8. — *For any entropy−entropy flux pair* $(\eta_j, q_j)$, $j=1, 2$ *the commutation relation of Tartar, namely*

$$\begin{aligned} &\langle \nu_{(x,t)}, \eta_1 q_2 - \eta_2 q_1 \rangle \\ &\qquad = \langle \nu_{(x,t)}, \eta_1 \rangle \langle \nu_{(x,t)}, q_2 \rangle - \langle \nu_{(x,t)}, \eta_2 \rangle \langle \nu_{(x,t)}, q_1 \rangle \end{aligned} \tag{4.15}$$

*holds.*

*Proof.* — It follows by applying the *div-curl* lemma (*see* [29]). ■

## 5. STRONG CONVERGENCE

Now we establish a strong convergence theorem of the vanishing viscosity approximation to the hyperbolic system (1.1). To accomplish this, we make use of the entropies constructed in section 4.4 and apply the theory of compensated compactness.

THEOREM 5.1. – *Under the hypothesis of Proposition* 3.2, *the approximate solutions* $\{(u^\varepsilon, v^\varepsilon)\}$ *converge (taking eventually a subsequence) strongly in* $L^p_{loc}$, $p<+\infty$, *to a weak solution* $(u, v)$ *to the system* (1.1).

The proof of this result will take several steps. First, we reduce the Young measure to a point mass in the Riemann invariant space. We then conclude that the Young measure is also a point mass in the state space $(u, v)$.

Let $\nu$ denote the Young measure. Suppose that $\nu$ is not a Dirac mass and $\mathscr{R}$ is the minimal rectangle in $(\omega_-, \omega_+)$ space containing the support of $\nu$. We assume that $\mathscr{R}$ is not a line segment parallel to any axis and contains the umbilical point, namely

$$\mathscr{R}=[\omega_-^-, 0]\times[0, \omega_+^+],$$

for some $\omega_-^-<0<\omega_+^+$ (the other cases are simpler and can be dealt with using a similar method).

Now we can use the results in [14] and generalize to our case the theory developped by [26] for strictly hyperbolic systems. It is easy to show that the following results are true:

LEMMA 5.2. – *Let* $\alpha$ *be any number satisfying* $\omega_-^-<\alpha<-\delta$, $\varepsilon$ *be such that*

$$\omega_-^-<\omega_-^*=\alpha-\varepsilon<\alpha<\omega_-^\#=\alpha+\varepsilon<-\delta,$$

*let* $(\eta, q)$ *be of east type with limit* $\omega_-^*$ *and* $(\bar\eta, \bar q)$ *be of west type with limit* $\omega_-^*$. *Then the following conditions hold:*

*a) If, for all east type entropy* $\hat\eta$ *with limit* $\omega_-^*$, *we have* $\langle \nu, \eta\rangle=0$, *then*

$$\operatorname{supp}\nu\cap\{(\omega_-, \omega_+):\omega_-^\#\leqq\omega_-\leqq-\delta,\ \omega_+\geqq 0\}=\varnothing.$$

*b) Let* $\theta_-$ *and* $\bar\theta_-$ *be the respective Goursat data for the entropies* $\eta$ *and* $\bar\eta$ *and suppose that their derivatives up to the second order valued at* $\alpha$ *are nonzero constants independent of* $\varepsilon$. *If we suppose that, for all* $\varepsilon$,

$$\langle \nu, \eta\bar q-\bar\eta q\rangle=0 \tag{5.1}$$

*then*

$$\operatorname{supp}\nu\cap\{(\omega_-, \omega_+):\omega_-=\alpha,\ \omega_+\geqq 0\}=\varnothing.$$

*c) There exists a constant c independent of* $\omega_-^*$ *and of the entropy such that*

$$\langle \nu, q\rangle=c\langle \nu, \eta\rangle.$$

*d) If there exists an east type entropy* $\hat{\eta}$ *with limit* $\omega_-^{\#}$ *so that* $\langle \nu, \hat{\eta} \rangle \neq 0$, *then for all west type entropy pairs of limit* $\omega_-^{\#}$ *we have*

$$\langle \nu, \bar{q} \rangle = c \langle \nu, \bar{\eta} \rangle$$

*with the same constant c corresponding to the entropies of east type with limit* $\omega_-^{*}$ *and satisfy the hypothesis* (5.1).

The reduction process will take two steps. First, we show that the support of $\nu$ must be concentrated only at the four corners of $\mathcal{R}$, *i.e.* $\nu$ is the sum of four delta functions. Then, with a standard method ([1], [2], [8], [26]), we reduce these delta-functions to one.

PROPOSITION 5.3. — *The support of* $\nu$ *is concentrated at most at the points* $(0, 0)$, $(\omega_-^-, 0)$, $(0, \omega_+^+)$, $(\omega_-^-, \omega_+^+)$.

*Proof.* — Let $\delta > 0$ be the constant used in the construction of the entropies in section 4.4 and assume that it is chosen *a priori* to satisfy $\omega_-^- < -\delta$. Define

$$\omega_-(\delta) = \inf\{y : \omega_-^- \leqq y \leqq -\delta,\ \operatorname{supp}\nu \cap \{\omega : y \leqq \omega_- < -\delta\} = \varnothing\}.$$

We claim that $\omega_-(\delta) = \omega_-^-$ and so the support of $\nu$ is concentrated on the line $\omega_- = \omega_-^-$ and the strip $-\delta \leqq \omega_- \leqq 0$.

If we suppose that $\omega_-(\delta) = -\delta$, we can use the lemma 5.2 and conclude that $\operatorname{supp}\nu$ does not intersect the line $\omega_- = \alpha$ for any $\alpha \in (\omega_-^-, -\delta)$. Therefore we again conclude that the support of $\nu$ is concentrated on the line $\omega_- = \omega_-^-$ and the strip defined by $-\delta \leqq \omega_- \leqq 0$.

On the other hand, if we suppose that $\omega_-(\delta) \in (\omega_-^-, -\delta)$, by an argument similar to the previous one, we can show that for any $\alpha \in (\omega_-^-, \omega_-(\delta))$, $\operatorname{supp}\nu$ does not intersect the line $\omega_- = \alpha$. Therefore $\nu$ is concentrated on the lines $\omega_- = \omega_-(\delta)$ and $\omega_- = \omega_-^-$ and the strip defined by $-\delta \leqq \omega_- \leqq 0$.

We shall now show, using lemma 5.2, that this contradicts the minimality of $\mathcal{R}$. Let $\varepsilon > 0$ such that $\omega_-^- + \varepsilon < \omega_-(\delta)$. Let $(\eta, q)$ be of east type with limit $\omega_-^-$ and $(\bar{\eta}, \bar{q})$ be of west type with limit $\omega_-^- + \varepsilon$. Let $\theta_-$ and $\bar{\theta}_-$ be their respective Goursat data which we assume are smooth enough. By lemma 5.2 we have

$$\langle \nu, \eta \bar{q} - \bar{\eta} q \rangle = 0$$

for all $\varepsilon > 0$ small enough. But, as our system (1.1) is genuinly non-linear away from $\omega_- = 0$, we have

$$\frac{\partial \lambda_-}{\partial \omega_-}(\omega) > 0$$

and as in [26], we can obtain a contradiction as $\varepsilon \to 0$. So

$$\operatorname{supp}\nu \subseteqq \mathcal{Q}_- \overset{\text{def}}{=} \{(\omega_-, \omega_+) \mid 0 \leqq \omega_+ \leqq \omega_+^+,\ \omega_- = \omega_-^- \text{ or } -\delta \leqq \omega_- \leqq 0\}.$$

By a similar argument, by using entropies of north and south type, we have

$$\operatorname{supp} \nu \subseteqq \mathcal{Q}_+ \stackrel{\text{def}}{=} \{(\omega_-, \omega_+) \mid \omega_-^- \leqq \omega_- \leqq 0,\ \omega_+ = \omega_+^+ \text{ or } 0 \leqq \omega_+ \leqq \delta\}.$$

So we obtain

$$\operatorname{supp} \nu \subseteqq \mathcal{Q}_\delta \stackrel{\text{def}}{=} \mathcal{Q}_- \cap \mathcal{Q}_+.$$

Because of $\delta > 0$ is arbitrarily fixed, letting $\delta \to 0$, the result is proved. ■

PROPOSITION 5.4. — *The Young measure* $\nu$ *is a point mass in the* $(\omega_-, \omega_+)$ *plane.*

*Proof.* — By the genuine nonlinearity of (1.1) in the interior of $\mathcal{R}$, using theorem 6.1 in Serre [24], we can exclude one of the four corners of $\mathcal{R}$. First, suppose that

$$\nu = \beta_0 \delta_0 + \beta_- \delta_- + \beta_+ \delta_+.$$

where $\delta_0 = \delta_{(0,0)}$, $\delta_- = \delta_{(\omega_-^-, 0)}$, $\delta_+ = \delta_{(0, \omega_+^+)}$ and $\beta_0 + \beta_- + \beta_+ = 1$, $\beta_0$, $\beta_-$, $\beta_+ \geqq 0$.

Let $(\eta, q)$ be of west type with limit $\frac{1}{2}\omega_-^-$. Since entropies and entropy fluxes are defined only up to additive constants, we pick $(\bar{\eta}, \bar{q})$ as entropy pair as constructed in section 4.2 but shifted such that

$$\beta_+ \bar{q}_+ + \beta_0 \bar{q}_0 = \beta_+ \bar{\eta}_+ + \beta_0 \bar{\eta}_0.$$

Substituting $\nu$ into the commutation relation of Tartar (4.15), we obtain

$$\beta_- - \beta_-^2 = 0.$$

Thus $\beta_- = 0$ or $\beta_- = 1$ and $\nu$ is concentrated at, at most, two points.

With a suitable choice of the entropies we can similarly conclude in the other cases. So we conclude in generale that $\nu$ is the sum of at most two delta functions.

Let $\nu = \gamma_P \delta_P + \gamma_Q \delta_Q$, where $\gamma_P + \gamma_Q = 1$, $\gamma_P, \gamma_Q \geqq 0$. Let $(\eta_1, q_1)$, $(\eta_2, q_2)$ be any two pairs of entropies as constructed in section 4.2. We assume that $P \neq (0, 0)$ and let

$$\eta_i(Q) = q_i(Q) = 0.$$

Then the commutation relation (4.15) gives

$$\gamma_P - \gamma_P^2 = 0$$

for the arbitrariety of the entropies and we conclude that $\gamma_P = 0$ or $\gamma_Q = 1$.

This completes the proof of the proposition. ■

*Proof of the theorem* 5.1. — By the up-down symmetry of (1.1), $\omega_-$ and $\omega_+$ are even functions of $\nu$. So, as a consequence of proposition 5.4, ν is a sum of two delta functions in the $(u, v)$ plane concentrated at two points symmetric about the line $v=0$. If we now assume as in proposition 3.2 that the data to (1.1) satisfies the condition $v_0(x) \geqq 0$ for all $x$, we have the conclusion of corollary 3.3 and ν is a point mass in the $(u, v)$ plane. ■

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