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New Inequalities for the Jacobian

by

L. GRECO [1]

Dipartimento di Matematica e
Applicazioni "R. Caccioppoli",
Complesso Monte S. Angelo - Edificio T,
Via Cintia - 80126 Napoli,
Italy

and

T. IWANIEC [2]

Dept. of Mathematics,
Syracuse University,
Syracuse, New York 13244,
U.S.A.

ABSTRACT. – We derive integral estimates for the Jacobian by using the Rochberg-Weiss theory of nonlinear commutators. A refinement of Müller's result is obtained. We demonstrate how local estimates can be used to improve the degree of integrability of the Jacobian.

Key words : Jacobian, Singular Integrals, Nonlinear Commutators.

RÉSUMÉ. – On établit des estimations intégrales pour le jacobien au moyen de la théorie des commutateurs non linéaires par Rochberg-Weiss. On obtient ainsi un raffinement du résultat de Müller. On démontre comment l'emploi des estimations locales permit d'améliorer le degré d'intégrabilité du jacobien.

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1. INTRODUCTION

Let Ω be a domain in $\mathbb{R}^n$ and $f=(f_1, f_2, \ldots, f_n):\Omega \to \mathbb{R}^n$ be a mapping of the Sobolev class $W^{1,\mathbf{p}}(\Omega, \mathbb{R}^n)$, where $\mathbf{p}=(p_1, p_2, \ldots, p_n)$ is an n-tuple of exponents $p_1, \ldots, p_n \in (1, \infty)$ such that $1/p_1 + \ldots + 1/p_n = 1$. Thus the gradient of each co-ordinate f_k, $k=1, 2, \ldots, n$, belongs to $L^{p_k}(\Omega)$:

$$\|\nabla f_k\|_{p_k} = \left(\int_\Omega |\nabla f_k(x)|^{p_k} dx \right)^{1/p_k} < \infty.$$

We denote the differential of f by $Df(x): \mathbb{R}^n \to \mathbb{R}^n$. The operator norm of $Df(x)$ is then defined by $|Df(x)| = \sup \{ |Df(x)\xi| : \xi \in S^{n-1} \}$.

We say that f is an orientation preserving mapping if its Jacobian determinant $J=J(x, f)=\det Df(x)$ is non-negative almost everywhere. Clearly, the Jacobian is an integrable function. Because of Hadamard's inequality we have $J(x, f) \leqq |\nabla f_1| \ldots |\nabla f_n|$. Hence Hölder's inequality implies the following estimate

$$\int_\Omega J(x, f)\, dx \leqq \|\nabla f_1\|_{p_1} \ldots \|\nabla f_n\|_{p_n}.$$

In case of $p_1 = \ldots = p_n = n$, S. Müller [M] discovered, using maximal theorems, that the Jacobian actually belongs to the Zygmund class L Log L. This remarkable result inspired a new study of the Jacobian function and related non-linear quantities (null Lagrangians), *see* [BFS], [CLMS], [G], [IS1], [IL]. For later use we record a more general result corresponding to arbitrary Hölder conjugate exponents $1 < p_1, \ldots, p_n < \infty$. If $\Omega = Q = Q(a, R)$ is a cube in $\mathbb{R}^n$ we then have a rather precise estimate [IL],

$$\int_{\sigma Q} J(x, f) \log \left(e + \frac{J(x, f)}{J_{\sigma Q}} \right) dx \leqq \frac{c(n, \mathbf{p})}{1-\sigma} \|\nabla f_1\|_{p_1} \ldots \|\nabla f_n\|_{p_n}, \quad (1.1)$$

where $\sigma Q = Q(a, \sigma R)$, $0 < \sigma < 1$, and $J_{\sigma Q}$ is the integral mean of J over the cube σQ. S. Müller also showed that for $f \in W^{1,n}(\Omega, \mathbb{R}^n)$ the above degree of summability of the Jacobian is the best possible.

In this paper we avoid maximal theorems by using the technique of the Hodge decomposition. This technique was developed in [I] to treat non-linear problems concerning quasiregular mappings, *see* also [IL] and [IS2]. Our approach is similar to that of [IS1] and leads to the following estimate.

THEOREM 1. – *Let* $f=(f_1, \ldots, f_n) \in W^{1,\mathbf{p}}(B, \mathbb{R}^n)$ *be an orientation preserving mapping,* $\mathbf{p}=(p_1, \ldots, p_n)$, $1 < p_1, \ldots, p_n < \infty$, $1/p_1 + \ldots + 1/p_n = 1$, *on a ball* $B \subset \mathbb{R}^n$. *Then* $(\det Df) \log |Df|$ *belongs to* $L^1_{loc}(B)$ *and for each*

$0<\sigma<1$ *we have*

$$\int_{\sigma B} \log\left(e+\frac{|Df(x)|}{|Df|_B}\right) J(x,f)\,dx \leqq \frac{c(n,\mathbf{p})}{1-\sigma}\|\nabla f_1\|_{p_1}\ldots\|\nabla f_n\|_{p_n} \quad (1.2)$$

Here, as usually, $|Df|_B$ denotes the integral mean over the ball B. For $p_1=\ldots=p_n=n$ this result is sharp, in the sense that the logarithm in inequality (1.2) cannot be replaced by any function $L:\mathbb{R}_+\to\mathbb{R}_+$ such that $\lim_{t\to\infty}\dfrac{\log t}{L(t)}=0$.

REMARK. – In case $p_1=\ldots=p_n=n$ the local integrability of $(\det Df)\log|Df|$ follows almost immediately from Müller's result. We thank P. L. Lions for pointing out this new argument. Actually, we shall close this paper by showing how his argument can be used to improve inequality (1.7). Estimate (1.2) can be derived from (1.1) similarly by using inequality (6.3) with $s=1$.

It is illuminating and rewarding to discuss first a smooth mapping $f=(f_1,\ldots,f_n)\in C_0^\infty(\mathbb{R}^n,\mathbb{R}^n)$. Using the linear theory of Hodge decomposition, we split the matrix-field $(Df)\log|Df|$ as

$$Df(x)\log|Df(x)|=Dg(x)+H(x), \quad (1.3)$$

where $g=(g_1,\ldots,g_n)\in W^{1,s}(\mathbb{R}^n,\mathbb{R}^n)$ and $H\in L^s(\mathbb{R}^n,GL(n))$ is a divergence free matrix-field, for all $1<s<\infty$. The matrix H can be expressed by an integral formula; $H=T(Df\log|Df|)$, where

$$T:L^s(\mathbb{R}^n,GL(n))\to L^s(\mathbb{R}^n,GL(n))$$

is a singular integral operator of the Calderòn-Zygmund type. From the uniqueness of the Hodge decomposition it follows that $T(Df)=0$. We shall now refer to the known result of R. Rochberg and G. Weiss [RW] on non-linear commutators. Accordingly

$$\|T(F\log|F|)-(TF)\log|TF|\|_s\leqq c(n,s)\|F\|_s$$

for all $F\in L^s(\mathbb{R}^n,GL(n))$, $1<s<\infty$. Applying this to $F=Df$ and $s=n$, we arrive at the following homogeneous estimate for H

$$\left(\int_{\mathbb{R}^n}|H(x)|^n\,dx\right)^{1/n}\leqq c(n)\left(\int_{\mathbb{R}^n}|Df(x)|^n\,dx\right)^{1/n}. \quad (1.4)$$

From now on it will be easier to use the calculus of differential forms. Thus $J(x,f)\,dx=df_1\wedge df_2\wedge\ldots\wedge df_n$ is an n-form, while (1.3) can be rewritten as

$$(\log|Df|)\,df_k=dg_k+h_k,\qquad k=1,2,\ldots,n,$$

where h_k are the differential forms of degree 1 whose coefficients coincide with the entries of the k-th column of H. Letting k be equal to 1, we

compute that

$$\int_{\mathbb{R}^n} J(x,f)\log|Df(x)|\,dx = \int_{\mathbb{R}^n} dg_1\wedge df_2\wedge\ldots\wedge df_n + \int_{\mathbb{R}^n} h_1\wedge df_2\wedge\ldots\wedge df_n.$$

By Stokes' theorem we notice that the integral $\int dg_1\wedge df_2\wedge\ldots\wedge df_n$ vanishes, because $dg_1\in L^{p_1}(\mathbb{R}^n)$, $df_2\in L^{p_2}(\mathbb{R}^n)$, ..., $df_n\in L^{p_n}(\mathbb{R}^n)$ with some Hölder conjugate exponents $1<p_1,\ldots,p_n<\infty$. With the aid of Hadamard's inequality, we estimate the second integral by

$$\int_{\mathbb{R}^n} h_1\wedge df_2\wedge\ldots\wedge df_n \leqq \int_{\mathbb{R}^n} |H|\,|Df|^{n-1}\,dx \leqq \|H\|_n\,\|Df\|_n^{n-1} \leqq c(n)\,\|Df\|_n^n.$$

The latter being an immediate consequence of (1.4). We then conclude with the following estimate

THEOREM 2. — *Let $f:\mathbb{R}^n\to\mathbb{R}^n$ be a mapping of class $C_0^\infty(\mathbb{R}^n,\mathbb{R}^n)$. Then*

$$\int_{\mathbb{R}^n} J(x,f)\log|Df(x)|\,dx \leqq c(n)\int_{\mathbb{R}^n}|Df(x)|^n\,dx \qquad (1.5)$$

This result seems to be of some interest for the future study of the Jacobian function.

The sign condition on the Jacobian determinant is crucial for absolute convergence of the integrals in (1.1) and (1.2).

Having inequality (1.5) with constant $c(n)$ independent of f, it is now possible to extend it to mappings f with $Df\in L^n(\mathbb{R}^n, GL(n))$. By using an approximation argument one can give a meaning to the integral in the left hand side, for instance as $\limsup \int J(x,f_j)\log|Df_j(x)|\,dx$, where $\{f_j\}$ is any sequence of C_0^∞-mappings such that $Df_j\to Df$ in L^n.

Another way to interpret (1.5) for such mappings is by using the well known duality between BMO and the Hardy space $\mathscr{H}^1$, due to Fefferman and Stein [FS]. As pointed out by P. L. Lions, we can decompose

$$\int J(x,f)\log|Df| = \int J(x,f)\log M|Df| + \int J(x,f)\log\frac{|Df|}{M|Df|}.$$

Here the Jacobian determinant J belongs to $\mathscr{H}^1$, *see* [CLMS]. On the other hand $\log M|Df|\in BMO(\mathbb{R}^n)$, *see* Coifman and Rochberg [CR], so can be regarded as a bounded functional on $\mathscr{H}^1$. The first intergral has, therefore, a meaning. The last one is obviously converging.

In an entirely routine manner one can deduce local variants of (1.5). Unfortunately, inequality (1.5) and their local variants cannot be derived for mappings $f \in W^{1,n}(\mathbb{R}^n, \mathbb{R}^n)$. This is due to the fact that, for $Df \in L^n(\mathbb{R}^n, GL(n))$, the term Dg in the Hodge decomposition (1.3) need not belong to $L^n(\mathbb{R}^n, GL(n))$, which is required for the cancellation of the integral $\int dg_1 \wedge df_2 \wedge \ldots \wedge df_n$. It is worth noting, however, that the term H belongs to $L^n(\mathbb{R}^n, GL(n))$ and satisfies (1.4).

The limit theorems for integrals also fail. For $f \in W^{1,n}(\Omega, \mathbb{R}^n)$ an orientation preserving mapping (as in theorem 1), one might try local variants of (1.5) and then use Fatou's lemma. This is still a nontrivial task. The reason being that it is not always possible to approximate f by smooth mappings with non-negative Jacobian. We overcome these difficulties by proving first certain L^s-estimates with exponents $s = n - \varepsilon$ below the dimension n. Then letting ε go to zero establishes the result.

As it might be expected, the local estimates between the integral averages such as (1.1) imply higher degree of integrability of $\det Df$, provided the differential Df belongs to an appropriate integrability class. To illustrate this fact, in section 6 we examine an orientation preserving mapping $f = (f_1, \ldots, f_n) : \Omega \to \mathbb{R}^n$, where we assume that each of the gradients ∇f_i belongs to the Zygmund class $L^{p_i} \log L(\Omega)$ $i = 1, 2, \ldots, n$, $1 < p_1, \ldots, p_n < \infty$, $1/p_1 + \ldots + 1/p_n = 1$, that is

$$[\![\nabla f_i]\!]_{p_i} := \left\{ \int_\Omega |\nabla f_i(x)|^{p_i} \log\left(e + \frac{|\nabla f_i(x)|}{|\nabla f_i|_\Omega} \right) dx \right\}^{1/p_i} < \infty.$$

Under this assumption we obtain

THEOREM 3. — *The Jacobian determinant* $J(x) = \det Df(x)$ *belongs to the Orlicz space* $L \log^2 L(\Omega)$ *and*

$$\int_{\sigma\Omega} J(x) \log^2\left(e + \frac{J(x)}{J_{\sigma\Omega}} \right) dx \leqq \frac{c(n, \mathbf{p})}{(1-\sigma)\sigma^n} [\![\nabla f_1]\!]_{p_1} \ldots [\![\nabla f_n]\!]_{p_n}, \quad (1.6)$$

where Ω *is a cube in* $\mathbb{R}^n$ *and* $0 < \sigma < 1$.

In the case $|Df| \in L^n \log L(\Omega)$, *we have*

$$\int_{\sigma\Omega} J(x) \log^2\left(e + \frac{J(x)}{J_{\sigma\Omega}} \right) dx \leqq \frac{c(n)}{(1-\sigma)\sigma^n} \int_\Omega |Df|^n \log\left(e + \frac{|Df|}{|Df|_\Omega} \right) dx. \quad (1.7)$$

Let us stress that inequality (1.6) follows directly from (1.1), that is, no specific properties of the Jacobian function are invoked. To this effect we introduce a new maximal operator of the functions of the Zygmund

class $L \log L$. This little innovation seems to be interesting on its own accord.

It is possible to show that if $\nabla f_i \in L^{p_i} \log^s L$, $i=1, \ldots, n$, then $J = \det Df$ belongs to the Orlicz space $L \log^{s+1} L$ for $s=0$ [M], $s=-1$ [IS1], arbitrary $s \in (-1, 0)$ [BFS], $s>0$ [Mo] and $s<-1$ [G2]. Some new approachs to these questions are discussed by Milman [Mi]. We shall report on estimates of the Jacobian in more general Orlicz spaces in [GIM].

2. HODGE DECOMPOSITION

Arguments proving theorem 1 are based on the following

LEMMA 2.1. – *Let* $B = B(a, R)$ *be a ball in* $\mathbb{R}^n$ *and let* u *be a function of the Sobolev class* $W^{1,r}(B)$, $r>1$. *Then for each* $\varepsilon \in (1-r, 1)$ *the vector field* $|\nabla u|^{-\varepsilon} \nabla u \in L^{r/(1-\varepsilon)}(B, \mathbb{R}^n)$ *can be decomposed as*

$$|\nabla u(x)|^{-\varepsilon} \nabla u(x) = \nabla \varphi(x) + H(x) \tag{2.1}$$

for almost every $x \in B$, *where* $\varphi \in W^{1, r/(1-\varepsilon)}(\mathbb{R}^n)$ *and* $H \in L^{r/(1-\varepsilon)}(\mathbb{R}^n, \mathbb{R}^n)$ *is a (divergence free) vector field such that*

$$\left(\int_{\mathbb{R}^n} |H(x)|^{r/(1-\varepsilon)} dx \right)^{(1-\varepsilon)/r} \leqq A(n, r) |\varepsilon| \left(\int_B |\nabla u(x)|^r dx \right)^{(1-\varepsilon)/r} \tag{2.2}$$

The constant $A(n, r)$ *depends only on* r *and the dimension.*

This result is a refinement of theorem 3 in [IS2], where we have considered $u \in W_0^{1,r}(B)$.

Proof. – It may be assumed that B is centered at the origin. With the aid of the inversion about the sphere ∂B, we extend u to the whole space $\mathbb{R}^n$ by

$$u(x) = u\left(\frac{R^2 x}{|x|^2} \right) \qquad \text{for} \quad x \in \mathbb{R}^n - B.$$

It is easy to see that

$$\int_{R<|x|<2R} |\nabla u(x)|^r dx = \int_{R/2<|x|<R} |\nabla u(x)|^r \left| \frac{x}{R} \right|^{2r-2n} dx.$$

Hence

$$\int_{2B} |\nabla u(x)|^r dx \leqq 4^n \int_B |\nabla u(x)|^r dx.$$

Next, let $\eta \in C_0^\infty(2B)$ be a cut-off function such that $0 \leqq \eta \leqq 1$, $\eta = 1$ on B and $|\nabla \eta| \leqq c(n)/R$. We define an auxiliary function

$$\omega \in W_0^{1,r}(2B) \subset W^{1,r}(\mathbb{R}^n)$$

by

$$\omega(x)=\eta(x)[u(x)-u_{2B}].$$

Then Poincaré's inequality yields

$$\int_{\mathbb{R}^n}|\nabla\omega|^r\,dx\leqq A(n,r)\int_{2B}|\nabla u|^r\,dx\leqq 4^n A(n,r)\int_B|\nabla u|^r\,dx. \quad (2.3)$$

In this way the problem reduces to theorem 3 from [IS2]. To apply this theorem, we consider the following Hodge decomposition in $\mathbb{R}^n$.

$$|\nabla\omega|^{-\varepsilon}\nabla\omega=\nabla\varphi+H, \quad (2.4)$$

with $\varphi\in W^{1,r/(1-\varepsilon)}(\mathbb{R}^n)$ and $H\in L^{r/(1-\varepsilon)}(\mathbb{R}^n,\mathbb{R}^n)$ a vector field such that

$$\|H\|_{r/(1-\varepsilon)}\leqq c(n,r)\,|\varepsilon|\,\|\,|\nabla\omega|\,\|_r^{1-\varepsilon};$$

this, in view of (2.3), implies (2.2). What remains is to observe that the Hodge decomposition (2.4) coincides with (2.1) on the ball B.

3. TWO INEQUALITIES

We shall need to prove two elementary inequalities that clear away quite long computations in section 4.

LEMMA 3.1. – *Let* $0<\varepsilon<1/2$ *and* $\delta>0$. *Then for each* $a\geqq 0$ *we have*

$$|a-a^{1-\varepsilon}|\leqq\varepsilon(2+\delta^{-1}a^{1+\delta}), \quad (3.1)$$

$$\varepsilon a^{1-\varepsilon}\log(2+a)\leqq(a-a^{1-\varepsilon})+4\varepsilon. \quad (3.2)$$

Proof. – We begin with the familiar inequality

$$1-1/x\leqq\log x\leqq x-1,\qquad x>0.$$

For $x=a^\varepsilon$ this reads as

$$1-a^{-\varepsilon}\leqq\varepsilon\log a\leqq a^\varepsilon-1$$

where ε can be any number. Hence, for $\varepsilon>0$,

$$a-a^{1-\varepsilon}\leqq\varepsilon a\log a\leqq\varepsilon a\delta^{-1}(a^\delta-1)\leqq\varepsilon\delta^{-1}a^{1+\delta}.$$

Similarly, for $0<\varepsilon<1/2$

$$a-a^{1-\varepsilon}\geqq\varepsilon a^{1-\varepsilon}\log a>\varepsilon a^{1-\varepsilon}(1-\varepsilon)^{-1}(1-a^{\varepsilon-1})\geqq-2\varepsilon.$$

These two estimates obviously imply (3.1).

To prove (3.2), we write

$$\log\left(1+\frac{2}{a}\right)\leqq\log[(1+2^{1-\varepsilon}a^{\varepsilon-1})^{1/(1-\varepsilon)}]\leqq(1-\varepsilon)^{-1}\log(1+2a^{\varepsilon-1})$$

$$\leqq 2(1-\varepsilon)^{-1}a^{\varepsilon-1}\leqq 4a^{\varepsilon-1}.$$

Hence

$$\varepsilon a^{1-\varepsilon}\log(2+a)\leqq 4\varepsilon+\varepsilon a^{1-\varepsilon}\log a\leqq 4\varepsilon+(a-a^{1-\varepsilon})$$

as desired.

4. THE PROOF OF THEOREM

Our theorem will follow almost immediately from the following estimate

$$\frac{1}{|\mathrm{B}|}\int_{\sigma\mathrm{B}}\log\left(2+\frac{|df_1|}{|df_1|_{\mathrm{B}}}\right)|df_1|^{-\varepsilon}\,df_1\wedge df_2\wedge\ldots\wedge df_n$$
$$\leqq\frac{c(n,\mathbf{p})}{1-\sigma}[df_1]_{p_1}^{1-\varepsilon}[df_2]_{p_2}\ldots[df_n]_{p_n}\quad(4.1)$$

where ε is any sufficiently small positive number, for instance $0<\varepsilon<\dfrac{p_1}{2n(p_1-1)p_2}$. Hereafter the symbol $[g]_p=\left(\dfrac{1}{|\mathrm{B}|}\displaystyle\int_{\mathrm{B}}|g|^p\right)^{1/p}$ stands for the p-average of g over the ball $\mathrm{B}=\mathrm{B}(a,\mathrm{R})\subset\mathbb{R}^n$ and $\sigma\mathrm{B}=\mathrm{B}(a,\sigma\mathrm{R})$, $0<\sigma<1$.

Notice that the integral in the left-hand side of (4.1) is converging. Because of the same degree of homogeneity of both sides of (4.1), we may assume that

$$|df_1|_{\mathrm{B}}=\frac{1}{|\mathrm{B}|}\int_{\mathrm{B}}|df_1|\,dx=1.$$

This yields

$$1=[df_1]_1\leqq[df_1]_{p_1}^{1-\varepsilon}.\quad(4.2)$$

Then, we may interpolate to obtain

$$[|df_1|^{1+\delta}]_{np_1p_2/(p_1+np_2)}\leqq[df_1]_1^{\varepsilon+\delta}[df_1]_{p_1}^{1-\varepsilon}=[df_1]_{p_1}^{1-\varepsilon}\quad(4.3)$$

for $\delta=\dfrac{1}{np_2}+\dfrac{\varepsilon}{p_1}-\varepsilon>\dfrac{1}{2np_2}$.

Next observe that adding a constant to the function f_2 does not affect its differential df_2, so we may assume that the integral of f_2 over B is equal to zero. With this assumption we can write the Poincaré-Sobolev inequality

$$[f_2]_{p_2}\leqq[f_2]_{np_2/(n-1)}\leqq c(n,p_2)\,\mathrm{R}\,[df_2]_{p_2}.\quad(4.4)$$

Fix a cut-off function $\lambda \in C_0^\infty(B)$ such that $0 \leqq \lambda \leqq 1$, $\lambda \equiv 1$ on σB and $|\nabla \lambda| \leqq c(n)/(1-\sigma) R$. Thus (4.4) implies that

$$[f_2 \, d\lambda]_{np_2/(n-1)} \leqq \frac{c(n, p_2)}{1-\sigma} [df_2]_{p_2} \tag{4.5}$$

and

$$[d(\lambda f_2)]_{p_2} \leqq \frac{c(n, p_2)}{1-\sigma} [df_2]_{p_2}. \tag{4.6}$$

We shall examine the following integral

$$I = \varepsilon \fint_B \lambda \log(2 + |df_1|) \, |df_1|^{-\varepsilon} \, df_1 \wedge \ldots \wedge df_n. \tag{4.7}$$

Since the n-form $df_1 \wedge \ldots \wedge df_n = J(x, f) \, dx$ was assumed to be non-negative, we may apply inequality (3.2) to get

$$I \leqq \fint_B \lambda (1 - |df_1|^{-\varepsilon}) \, df_1 \wedge \ldots \wedge df_n + 4\varepsilon \fint_B \lambda \, |df_1|^{-1} \, df_1 \wedge df_2 \wedge \ldots \wedge df_n.$$

We then break the first integral in accordance with the formula $\lambda \, df_2 = d(\lambda f_2) - f_2 \, d\lambda$ to write that

$$I \leqq I_1 + I_2 + I_3, \tag{4.8}$$

where

$$I_1 = \fint_B (1 - |df_1|^{-\varepsilon}) \, df_1 \wedge d(\lambda f_2) \wedge df_3 \wedge \ldots \wedge df_n;$$

$$I_2 = \fint_B (1 - |df_1|^{-\varepsilon}) \, df_1 \wedge f_2 \, d\lambda \wedge df_3 \wedge \ldots \wedge df_n;$$

$$I_3 = 4\varepsilon \fint_B \lambda \, |df_1|^{-1} \, df_1 \wedge df_2 \wedge df_3 \wedge \ldots \wedge df_n.$$

It follows by Hölder's inequality and (4.2) that

$$I_3 \leqq 4\varepsilon [df_1]_{p_1}^{1-\varepsilon} [df_2]_{p_2} \ldots [df_n]_{p_n}. \tag{4.9}$$

The second term I_2 is estimated with the aid of inequality (3.1) and Hölder's inequality:

$$I_2 \leqq \varepsilon \int_B (2 + \delta^{-1} |df_1|^{1+\delta}) \, |f_2 \, d\lambda| \, |df_3| \ldots |df_n| \, dx$$

$$\leqq \varepsilon [2 + \delta^{-1} |df_1|^{1+\delta}]_{np_1 p_2/(p_1 + np_2)} [f_2 \, d\lambda]_{np_2/(n-1)} [df_3]_{p_3} \ldots [df_n]_{p_n}.$$

Here, in view of (4.2) and (4.3), the first factor is controlled by $(2+\delta^{-1})[df_1]_{p_1}^{1-\varepsilon}$. This together with the inequality (4.5) yields

$$I_2 \leqq \frac{c(n, \mathbf{p})}{1-\sigma} \varepsilon [df_1]_{p_1}^{1-\varepsilon} [df_2]_{p_2} \dots [df_n]_{p_n}. \tag{4.10}$$

It remains to estimate the integral I_1. To this effect we decompose

$$|df_1|^{-\varepsilon} df_1 = d\varphi + h$$

as in lemma 2.1. Inequality (2.2), applied with

$$r = (1-\varepsilon) p_1 > \frac{1}{2}(1+p_1) > 1,$$

shows that the p_1-average of $|h|$ is small compared to $[df_1]_{p_1}$. Indeed, we have

$$[h]_{p_1} \leqq c(n, p_1) \varepsilon [|df_1|^{1-\varepsilon}]_{p_1} \leqq c(n, p_1) \varepsilon [df_1]_{p_1}^{1-\varepsilon}. \tag{4.11}$$

Now, because of the decomposition $(1-|df_1|^{-\varepsilon}) df_1 = df_1 - d\varphi - h$, we split I_1 into two integrals

$$I_1 = \fint_B d(f_1 - \varphi) \wedge d(\lambda f_2) \wedge df_3 \wedge \dots \wedge df_n - \fint_B h \wedge d(\lambda f_2) \wedge df_3 \wedge \dots \wedge df_n.$$

By Stockes' theorem the first integral vanishes. We then estimate the second one by Hölder's inequality to obtain

$$I_1 \leqq [h]_{p_1} [d(\lambda f_2)]_{p_2} [df_3]_{p_3} \dots [df_n]_{p_n}.$$

Finally, using (4.11) and (4.6) we infer that

$$I_1 \leqq \frac{c(n, \mathbf{p})}{1-\sigma} \varepsilon [df_1]_{p_1}^{1-\varepsilon} [df_2]_{p_2} \dots [df_n]_{p_n}. \tag{4.12}$$

This together with (4.10), (4.9) and (4.8) implies

$$I \leqq \frac{c(n, \mathbf{p})}{1-\sigma} \varepsilon [df_1]_{p_1}^{1-\varepsilon} [df_2]_{p_2} \dots [df_n]_{p_n}.$$

Recalling the definition of I, *see* (4.7), completes the proof of (4.1).

The final step is to pass, by means of a limiting argument, from (4.1) to (1.2). First, letting ε go to zero, by Fatou's lemma we deduce that $J(., f) \log\left(2 + \frac{|df_1|}{|df_1|_B}\right)$ is integrable on σB. We also obtain the estimate

$$\int_{\sigma B} \log\left(2 + \frac{|df_1(x)|}{|df_1|_B}\right) J(x, f)\, dx \leqq \frac{c(n, \mathbf{p})}{1-\sigma} \|df_1\|_{p_1} \dots \|df_n\|_{p_n}.$$

Clearly, one can replace f_1 in the left-hand side by any other co-ordinate function f_k, $k=1,2,\ldots,n$. Taking into account that

$$\log\left(e+\frac{|Df(x)|}{|Df|_B}\right)\leqq\sum_{k=1}^{n}\log\left(2+\frac{|df_k(x)|}{|df_k|_B}\right),$$

inequality (1.2) is immediate, proving theorem 1. For $p_1=\ldots=p_n=n$, this inequality reads as:

$$\int_{\sigma B}\log\left(e+\frac{|Df(x)|}{|Df|_B}\right)J(x,f)\,dx\leqq\frac{c(n)}{1-\sigma}\int_B|Df(x)|^n\,dx,$$

which is an extension of Müller's result [M].

5. A MAXIMAL OPERATOR IN L log L

Let Ω be a cube in $\mathbb{R}^n$ and f a measurable function on Ω. We denote the integral mean of $|f|$ over a subcube $Q\subset\Omega$ by

$$|f|_Q=\int_Q|f(x)|\,dx=\frac{1}{|Q|}\int_Q|f(x)|\,dx.$$

Accordingly, the Hardy-Littlewood maximal operator is defined by

$$Mf(x)=\sup\{|f|_Q;\ x\in Q\subset\Omega\}.$$

Although this maximal operator plays a primary role in the theory of Lebesgue spaces $L^p(\Omega)$, there are larger classes of Orlicz spaces in which specific variants of the maximal operator appear to be appropriate tools. In this section we introduce one of such operators acting on the Zygmund class $L\log L(\Omega)$. We mimic the lines of the classical theory of maximal operators. There are, however, new interesting details which may be useful for further development of the differentiation of integrals.

The Zygmund space consists of the functions $f:\Omega\to\mathbb{R}$ such that

$$\langle f\rangle_\Omega:=\fint_\Omega|f(x)|\log\left(e+\frac{|f(x)|}{|f|_\Omega}\right)dx<\infty \tag{5.1}$$

Notice that $L\log L(\Omega)$ is a Banach space and $\langle\ \rangle_\Omega$ is an order preserving norm, that is $\langle f\rangle_\Omega\leqq\langle g\rangle_\Omega$ whenever $|f|\leqq|g|$ pointwise, *see* [IK].

The well known theorem of E. Stein [S1] asserts that $f\in L\log L$ if and only if $Mf\in L^1(\Omega)$. The following estimates establish the equivalence

between these two norms in $L \log L(\Omega)$

$$3^{-n-2} \fint_\Omega M f(x)\, dx \leqq \fint_\Omega |f(x)| \log\left(e + \frac{|f(x)|}{|f|_\Omega}\right) dx \leqq 2^{n+1} \fint_\Omega M f(x)\, dx. \quad (5.2)$$

It should be notified in advance that the maximal function Mf depends on the cube Ω. In particular, the Lebesgue differentiation theorem does not apply, so no point-wise estimates follow from (5.2).

For $f \in L \log L(\Omega)$ we shall consider a maximal function

$$Zf(x) = \sup \{ \langle f \rangle_Q;\ x \in Q \subset \Omega \}, \quad (5.3)$$

where, as usually, the supremum is taken over all cubes $Q \subset \Omega$ containing the given point $x \in \Omega$. Thus $Z : L \log L(\Omega) \to \mathcal{M}_0(\Omega)$ (here $\mathcal{M}_0(\Omega)$ stands for the space of all measurable functions which are a. e. finite) is an order preserving sublinear operator. This latter means that $Z(f+g) \leqq Zf + Zg$. In honor of Professor Antoni Zygmund we call Z the Zygmund maximal operator. We shall prove the following analogue of Stein's result

THEOREM 4. — *The maximal function Zf is integrable on Ω if and only if f belongs to the Orlicz class $L \log^2 L(\Omega)$. Moreover, we have the following inequalities*

$$\fint_\Omega Zf(x)\, dx \leqq 3^{n+4} \fint_\Omega |f(x)| \log^2\left(e + \frac{|f(x)|}{|f|_\Omega}\right) dx \quad (5.4)$$

and conversely

$$\fint_\Omega |f(x)| \log^2\left(e + \frac{|f(x)|}{|f|_\Omega}\right) dx \leqq 2^{n+4} \fint_\Omega Zf(x)\, dx. \quad (5.5)$$

Before the proof, we recall the Luxemburg norm of a function $h \in L \log L(Q)$, with Q a cube in Ω:

$$\langle\langle h \rangle\rangle_Q := \inf \left\{ \lambda > 0;\ \fint_Q \frac{|h|}{\lambda} \log\left(e + \frac{|h|}{\lambda}\right) < 1 \right\}.$$

In other words, for $h \not\equiv 0$ the number $\langle\langle h \rangle\rangle_Q$ is the unique solution of the equation

$$\langle\langle h \rangle\rangle_Q = \fint_Q |h| \log\left(e + \frac{|h|}{\langle\langle h \rangle\rangle_Q}\right). \quad (5.6)$$

These two norms in the space $L \log L(Q)$ compares to each other by

$$\langle\langle h \rangle\rangle_Q \leqq \langle h \rangle_Q \leqq 2 \langle\langle h \rangle\rangle_Q \quad (5.7)$$

To see this, we first observe that $\langle\langle h\rangle\rangle_Q \geqq |h|_Q$. Hence

$$\langle\langle h\rangle\rangle_Q \leqq \fint_Q |h| \log\left(e + \frac{|h|}{|h|_Q}\right) = \langle h\rangle_Q$$

On the other hand

$$\begin{aligned}\langle h\rangle_Q &= \fint_Q |h| \log\left(e + \frac{h}{\langle\langle h\rangle\rangle_Q} \cdot \frac{\langle\langle h\rangle\rangle_Q}{|h|_Q}\right) \\ &\leqq \fint_Q |h| \log\left(e + \frac{h}{\langle\langle h\rangle\rangle_Q}\right) + \fint_Q |h| \log\left(1 + \frac{\langle\langle h\rangle\rangle_Q}{|h|_Q}\right) \\ &\leqq \langle\langle h\rangle\rangle_Q + \fint_Q |h| \frac{\langle\langle h\rangle\rangle_Q}{|h|_Q} = 2\langle\langle h\rangle\rangle_Q\end{aligned}$$

as claimed.

Proof of inequality (5.4). – For $t>0$ we shall consider the distribution set E_t of the Zygmund maximal function $Z : L \log L(\Omega) \to \mathcal{M}_0$

$$E_t = \{x \in \Omega;\ Zf(x) > t\}. \tag{5.8}$$

We need to estimate the measure of E_t by

$$|E_t| \leqq \frac{3^n 6}{t} \int_{|f|>t/3} |f| \log\left(e + \frac{6|f|}{t}\right) \tag{5.9}$$

The proof of this inequality is based on Vitali's lemma. First we split f as $f = h + (f-h)$, where

$$h(x) = \begin{cases} f(x), & \text{if } |f(x)| > t/3, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly $|f(x)| \leqq |h(x)| + t/3$. Hence we obtain the pointwise inequality $Zf(x) \leqq Zh(x) + \log(1+e)\, t/3 \leqq Zh(x) + 2t/3$. Therefore

$$|E_t| \leqq |\{x \in \Omega;\ Zh(x) > t/3\}| \tag{5.10}$$

Next, as a consequence of the definition of the maximal function Zh, it follows that for each $x \in E_t$ there exists a cube $Q_x \subset \Omega$ containing x such that $\langle h\rangle_{Q_x} > t/3$. This combined with (5.7) yields

$$\langle\langle h\rangle\rangle_{Q_x} = \fint_{Q_x} |h| \log\left(e + \frac{|h|}{\langle\langle h\rangle\rangle_{Q_x}}\right) > \frac{t}{6}.$$

Hence

$$\fint_{Q_x} |h| \log\left(e + \frac{6|h|}{t}\right) > \frac{t}{6}$$

or, equivalently

$$|Q_x| \leqq \frac{6}{t} \int_{Q_x} |h| \log\left(e + \frac{6|h|}{t}\right).$$

Finally, with the aid of Vitali's lemma [S] we select mutually disjoint cubes Q_x, $x \in E$, to conclude with the estimate

$$|E_t| \leqq 3^n \sum |Q_x| \leqq \frac{3^n 6}{t} \int_\Omega |h| \log\left(e + \frac{6|h|}{t}\right).$$

Here the summation is taken over selected cubes. This in view of the definition of h is the same as (5.9).

Now we compute the integral of Zf. Let $a = |f|_Q$, then

$$\begin{aligned}
\int_\Omega Zf(x)\,dx &= \int_0^\infty |E_t|\,dt \leqq a|\Omega| + \int_a^\infty |E_t|\,dt \\
&\leqq a|\Omega| + 3^n 6 \int_a^\infty \frac{1}{t} \int_{|f|>t/3} |f| \log\left(e + \frac{6|f|}{t}\,dx\,dt\right) \\
&= a|\Omega| + 3^n 6 \int_{|f|\geqq a/3} |f| \left(\int_a^{3|f|} \log\left(e + \frac{6|f|}{t}\right) \frac{dt}{t} \right) dx.
\end{aligned}$$

Here we have changed the order of integration. Finally we make the substitution $s = 6|f|/t$ to obtain

$$\begin{aligned}
\fint_\Omega Zf(x)\,dx &\leqq a + \frac{3^n 6}{|\Omega|} \int_{|f|\geqq a/3} |f| \int_2^{6|f|/a} \log(e+s) \frac{ds}{s}\,dx \\
&\leqq a + \frac{3^n 6}{|\Omega|} 4 \int_{|f|\geqq a/3} |f| \int_2^{6|f|/a} \frac{\log(e+s)}{e+s}\,ds\,dx \\
&= a + \frac{3^n 12}{|\Omega|} \int_{|f|>a/3} |f| \left[\log^2\left(e + \frac{6|f|}{a}\right) - \log^2(e+2) \right] dx \\
&\leqq a + 3^{n+1} \cdot 12 \fint_B |f| \log^2\left(e + \frac{|f|}{a}\right) dx \\
&\leqq (1 + 3^{n+1} \cdot 12) \fint_B |f| \log^2\left(e + \frac{|f|}{|f|_q}\right) dx.
\end{aligned}$$

This completes the proof of inequality (5.4).

Proof of inequality (5.5). – Because of homogeneity we may wish f to be normalized so that $\langle f \rangle_\Omega = 1$. Our first objective is to reverse inequality (5.9) for the distribution function of Zf. More precisely, we shall need

to show that

$$|E_t| \geqq \frac{1}{2^n t} \int_{|f|>t} |f| \log\left(e + \frac{|f|}{2^n t}\right) \tag{5.11}$$

for all $t \geqq 1$, which requires a slight modification of the Calderón-Zygmund lemma. This is fairly easy to achieve by deploying the familiar dyadic partition of Ω, as in the celebrated construction by Calderón-Zygmund. The reader is referred to [S] for details. First notice that, if P is one of the 2^n congruent subcubes of a cube Q (obtained by bisecting the sides of Q), then the corresponding L log L-averages compare as:

$$\langle f \rangle_P \leqq 2^n \langle f \rangle_Q.$$

Indeed, using the order preserving property of the norm $\langle \ \rangle_Q$ yields

$$\langle f \rangle_P = 2^n \fint_Q |\chi_P f| \log\left(e + \frac{|\chi_P f|}{2^n |\chi_P f|_Q}\right) \leqq 2^n \langle \chi_P f \rangle_Q \leqq 2^n \langle f \rangle_Q.$$

Here χ_P stands for the characteristic function of $P \subset Q \subset \Omega$. Notice also that by Lebesgue's differentiation theorem, for almost every $x \in \Omega$ there exists the limit

$$\lim \langle f \rangle_{Q_x} = |f(x)| \log(e+1),$$

where Q_x are those dyadic subcubes of Ω which contain x. These observations allow us to establish the following decomposition, analogous to that of Calderón-Zygmund.

For each $t \geqq \langle f \rangle_\Omega = 1$, *there exists a disjoint family* $\mathcal{F}$ *of dyadic subcubes* $Q \subset \Omega$ *such that*

$$t < \fint_Q |f| \log\left(e + \frac{|f|}{|f|_Q}\right) \leqq 2^n t \tag{5.12}$$

for each cube $Q \in \mathcal{F}$. *Moreover*

$$|f(x)| \leqq \frac{t}{\log(1+e)} < t \tag{5.13}$$

for almost every $x \in \Omega - \bigcup \mathcal{F}$.

In particular $|f|_Q \leqq 2^n t$, which combined with the right hand side of (5.12) yields

$$|Q| \geqq \frac{1}{2^n t} \int_Q |f| \log\left(e + \frac{|f|}{2^n t}\right).$$

On the other hand, inequalities (5.12) and (5.13) imply that $Zf(x) > t$ for $x \in \bigcup \mathcal{F}$ and $\bigcup \mathcal{F} \supset \{x; |f(x)| > t\}$. Hence estimate (5.11) follows

immediately

$$|\mathrm{E}_t| \geqq \sum_{Q \in \mathcal{F}} |Q| \geqq \sum_{Q \in \mathcal{F}} \frac{1}{2^n t} \int_Q |f| \log\left(e + \frac{|f|}{2^n t}\right) \geqq \frac{1}{2^n t} \int_{|f|>t} |f| \log\left(e + \frac{|f|}{2^n t}\right).$$

The rest of the proof of theorem 4 proceeds in much the same way as the proof of inequality (5.4). Namely, changing the order of integration yields

$$\begin{aligned} \int_\Omega \mathrm{Z} f(x)\, dx &\geqq \int_1^\infty |\mathrm{E}_t|\, dt \\ &\geqq \frac{1}{2^n} \int_1^\infty \frac{1}{t} \int_{|f|>t} |f| \log\left(e + \frac{|f|}{2^n t}\right) dx\, dt \\ &= \frac{1}{2^n} \int_{|f|>1} |f| \int_1^{|f|} \log\left(e + \frac{|f|}{2^n t}\right) \frac{dt}{t} dx. \end{aligned}$$

We make the substitution $s = \dfrac{|f|}{2^n t}$ to obtain

$$\begin{aligned} \int_\Omega \mathrm{Z} f &\geqq \frac{1}{2^n} \int_{|f|>1} |f| \int_{2^{-n}}^{2^{-n}|f|} \log(e+s)\, ds\, dx \\ &\geqq \frac{1}{2^n} \int_{|f|>1} |f| \int_{2^{-n}}^{2^{-n}|f|} \frac{\log(e+s)}{e+s} ds\, dx \\ &= \frac{1}{2^{n+1}} \int_{|f|>1} |f| \log^2(e+s) \Big|_{2^{-n}}^{2^{-n}|f|} dx \\ &\geqq \frac{1}{2^{n+1}} \int_{|f|>1} |f| \log^2\left(e + \frac{|f|}{2^n}\right) dx - \frac{1}{2^n} \int_{|f|>1} |f|\, dx. \end{aligned}$$

On the other hand we have a trivial estimate

$$\int_{|f|<1} |f| \log^2\left(e + \frac{|f|}{2^n}\right) dx \leqq 2 \int_{|f|<1} |f|\, dx$$

which combined with the previous one yields

$$\int_\Omega |f| \log^2\left(e + \frac{|f|}{2^n}\right) dx \leqq 2^{n+1} \int_\Omega |\mathrm{Z} f|\, dx + 2 \int_\Omega |f|\, dx.$$

Finally, using an elementary inequality $\log^2(e+ab) \leqq 2\log^2(e+a)+8+8b$, for $a, b \geqq 0$, we complete our estimation as follows

$$\begin{aligned}
&\fint_\Omega |f| \log^2\left(e+\frac{|f|}{|f|_\Omega}\right) \\
&\quad \leqq 2\fint_\Omega |f| \log^2\left(e+\frac{|f|}{2^n}\right)+8\fint_\Omega |f|\left(1+\frac{2^n}{|f|_\Omega}\right) \\
&\quad \leqq 2^{n+2}(Zf)_\Omega+4|f|_\Omega+8|f|_\Omega+8.2^n \\
&\quad = 2^{n+1}(Zf)_\Omega+12|f|_\Omega+2^{n+3}\langle f\rangle_\Omega \\
&\quad \leqq (12+2^{n+1})(Zf)_\Omega+2^{n+3}\inf_\Omega Zf \\
&\quad \leqq (12+2^{n+1}+2^{n+3})(Zf)_\Omega \\
&\quad \leqq 2^{n+4}(Zf)_\Omega.
\end{aligned}$$

This ends the proof of theorem 4.

6. THE PROOF OF THEOREM 3

First we rephrase inequality (1.1) as

$$\begin{aligned}
&\fint_{\sigma Q} J(x)\log\left(e+\frac{J(x)}{J_{\sigma Q}}\right)dx \\
&\qquad \leqq \frac{c(n,\mathbf{p})}{\sigma^n(1-\sigma)}\left(\fint_Q |\nabla f_1|^{p_1}\right)^{1/p_1}\cdots\left(\fint_Q |\nabla f_n|^{p_n}\right)^{1/p_n} \qquad (6.1)
\end{aligned}$$

for each cube $Q \subset \Omega$.

Let Z denote the Zygmund type maximal operator associated with the cube $\sigma\Omega$, that is

$$Zh(x)=\sup\left\{\fint_Q h\log\left(e+\frac{h}{h_Q}\right);\ x\in Q\subset\sigma\Omega\right\}.$$

We denote by M_p, $1 \leqq p < \infty$, the Hardy-Littlewood type maximal operator associated with the cube Ω,

$$M_p h(x)=\sup\left\{\left(\fint_Q |h|^p\right)^{1/p};\ x\in Q\subset\Omega\right\}.$$

Then applying (6.1) yields a pointwise inequality between the maximal functions

$$(Zf)(x) \leqq \frac{c(n,\mathbf{p})}{\sigma^n(1-\sigma)} M_{p_1}(\nabla f_1)\ldots M_{p_n}(\nabla f_n)$$

for all $x\in\sigma\Omega$.

Observe that $M_{p_i}(\nabla f_i) \in L^{p_i}(\Omega)$. Indeed, by (5.2) we have

$$\begin{aligned} \| M_{p_i}(\nabla f_i) \|_{p_i} &= \| M(|\nabla f_i|^{p_i}) \|_1^{1/p_i} \\ &\leqq \left[3^{n+2} \int_B |\nabla f_i|^{p_i} \log\left(e + \frac{|\nabla f_i|^{p_i}}{(|\nabla f_i|^{p_i})_\Omega} \right) \right]^{1/p_i} \\ &\leqq \left[3^{n+2} p_i \int_\Omega |\nabla f_i|^{p_i} \log\left(e + \frac{|\nabla f_i|}{|\nabla f_i|_\Omega} \right) \right]^{1/p_i}. \end{aligned}$$

In case $p_1 = \ldots = p_n = n$ this estimate takes the form

$$\| M_n(\nabla f_i) \|_n \leqq \| M_n(Df) \|_n \leqq \left[3^{n+2} n \int_\Omega |Df|^n \log\left(e + \frac{|Df|}{|Df|_\Omega} \right) \right]^{1/n}. \quad (6.2)$$

Therefore, by Hölder's inequality we find that ZJ is integrable on $\sigma\Omega$ and its L^1-norm is controlled by

$$\int_{\sigma\Omega} (ZJ)(x)\, dx \leqq \frac{c(n, \mathbf{p})}{\sigma^n (1-\sigma)} 3^{n+2} n [\![\nabla f_1]\!]_{p_1} \ldots [\![\nabla f_n]\!]_{p_n}.$$

Here we used the elementary inequality $p_1^{1/p_1} \ldots p_n^{1/p_n} \leqq n$.

Finally, by theorem 4 we deduce that $J \in L \log^2 L(\Omega)$ and by (5.5) we conclude with inequality (1.6). Similarly, in case of $p_1 = \ldots = p_n = n$ estimate (6.2) yields inequality (1.7), completing the proof of theorem 3.

It is possible to improve slightly inequality (1.7) by replacing the Jacobian under the logarithm in (1.7) by $|Df(x)|^n$, as mentioned in the Introduction. Indeed, by using an elementary inequality

$$A \log^s\left(e + \frac{B}{t} \right) \leqq c_s A \log^s\left(e + \frac{A}{t} \right) + c_s B \quad (6.3)$$

for A, B, s, $t > 0$, we obtain

$$\begin{aligned} &\int_{\sigma\Omega} J(x) \log^2\left(e + \frac{|Df(x)|^n}{J_{\sigma\Omega}} \right) dx \\ &\leqq c_2 \int_{\sigma\Omega} J(x) \log^2\left(e + \frac{J(x)}{J_{\sigma\Omega}} \right) dx + c_2 \int_{\sigma\Omega} |Df(x)|^n\, dx \\ &\leqq \frac{c(n)}{(1-\sigma)\sigma^n} \int_\Omega |Df(x)|^n \log\left(e + \frac{|Df(x)|}{|Df|_\Omega} \right) dx. \quad (6.4) \end{aligned}$$

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