Ann. I. H. Poincaré – AN 20, 5 (2003) 759–777

10.1016/S0294-1449(02)00030-6/FLA

MULTIPLE POSITIVE SOLUTIONS FOR SINGULARLY PERTURBED ELLIPTIC PROBLEMS IN EXTERIOR DOMAINS ☆

Giovanna CERAMI [a,*], **Riccardo MOLLE** [b]

[a] *Dipartimento di Matematica ed Applicazioni, Università di Palermo, Via Archirafi n. 34, 90123 Palermo, Italy*

[b] *Dipartimento di Matematica, Università di Roma "Tor Vergata", Via della Ricerca Scientifica n. 1, 00133 Roma, Italy*

Received 3 March 2002, revised 20 June 2002

ABSTRACT. – The equation $-\varepsilon^2\Delta u + a_\varepsilon(x)u = u^{p-1}$ with boundary Dirichlet zero data is considered in an exterior domain $\Omega = \mathbb{R}^N \setminus \bar\omega$ (ω bounded and $N \geqslant 2$). Under the assumption that $a_\varepsilon \geqslant a_0 > 0$ concentrates round a point of Ω as $\varepsilon \to 0$, that $p > 2$ and $p < 2N/(N-2)$ when $N \geqslant 3$, the existence of at least three positive distinct solutions is proved.

MSC: 35J20; 35J60

Keywords: Exterior domains; Lack of compactness; Multiplicity of solutions

RÉSUMÉ. – Dans cet article on étude l'équation $-\varepsilon^2\Delta u + a_\varepsilon(x)u = u^{p-1}$ dans l'ouvert extérieur $\Omega = \mathbb{R}^N \setminus \bar\omega$ (ω borné et $N \geqslant 2$), avec la condition de Dirichlet $u = 0$ sur $\partial\Omega$. En supposant que $a_\varepsilon \geqslant a_0 > 0$ se concentre autour d'un point du domaine Ω quand $\varepsilon \to 0$, que $p > 2$ et que $p < 2N/(N-2)$ quand $N \geqslant 3$, on démontre que le problème possède au moins trois solutions distinctes.

1. Introduction

In this paper we consider the problem

$$(P_\varepsilon) \qquad \begin{cases} -\varepsilon^2\Delta u + a_\varepsilon(x)u = u^{p-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

☆ Work supported by Italian M.I.U.R., national research project "Metodi variazionali e topologici nello studio di fenomeni non lineari".

* Corresponding author.

E-mail addresses: cerami@math.unipa.it (G. Cerami), molle@mat.uniroma2.it (R. Molle).

where $\Omega=\mathbb{R}^N\setminus\bar{\omega}$, ω being a nonempty, bounded domain having smooth boundary $\partial\omega=\partial\Omega$, $N\geqslant 2$, $\varepsilon\in\mathbb{R}^+\setminus\{0\}$, $p>2$ and $p<2N/(N-2)$ when $N\geqslant 3$. a_ε is a given nonnegative function that, as $\varepsilon\to 0$, concentrates round a point $x_0\in\Omega$, namely a_ε has the form

$$a_\varepsilon(x)=a_0+\alpha\left(\frac{x-x_0}{\varepsilon}\right) \tag{1.1}$$

and satisfies

(A_1) $\quad a_0\in\mathbb{R}^+\setminus\{0\},\ x_0\in\Omega,\ \alpha(x)\geqslant 0,\ \alpha\in L^{N/2}(\mathbb{R}^N),\ |\alpha|_{L^{N/2}(\mathbb{R}^N)}\neq 0,$

(A_2) $\quad \int_{\mathbb{R}^N}\alpha(x)\,\mathrm{e}^{2|x|}\big(1+|x|^{\frac{N-1}{2}\sigma}\big)\,dx<\infty$ for some $\sigma\in(1,2]$.

Problem (P_ε) has a variational structure: the solutions of (P_ε) can be characterized as the nonnegative functions that are critical points of the functional $\mathcal{I}_\varepsilon : H_0^1(\Omega)\to\mathbb{R}$

$$\mathcal{I}_\varepsilon(u)=\int_\Omega\big(\varepsilon^2|\nabla u|^2+a_\varepsilon(x)u^2\big)\,dx$$

constrained to lie on the manifold

$$\mathcal{M}=\big\{u\in H_0^1(\Omega)\mid |u|_{L^p(\Omega)}=1\big\}.$$

However, it is well known that the unboundedness of the domain gives rise to a lack of compactness, not allowing a straight application of the usual variational techniques. In particular (P_ε) cannot be solved by minimization, in fact (see Section 2), the infimum of $\mathcal{I}_\varepsilon$ on $\mathcal{M}$ is not achieved, moreover the functional $\mathcal{I}_\varepsilon$ does not satisfy the Palais-Smale condition in every energy level (see [1] and [3] for a careful analysis of the compactness question). The study of (P_ε) needs subtle tools as the minimax theory together with topological arguments.

In recent years problems like (P_ε) have been object of several researches, here we only recall that, without any symmetry assumption on ω, the existence of one solution for (P_ε) has been proved, first, in [3], in the case $a_\varepsilon(x)\equiv a_0$, then in [1], under more general assumptions; multiplicity results have been obtained, when $a_\varepsilon(x)\equiv a_0$, in domains having several holes [7,8,11,15] relating the number of solutions of (P_ε) to the metric and/or topological properties of Ω. We also remark that, for equations in $\mathbb{R}^N$ having nonconstant, nonsymmetric coefficients, the existence of one positive solution has been stated in [2,4], while multiple solutions have been found in [13].

In this work, motivated by former results, [6,9], that emphasize the role that a concentrating potential a_ε can play in obtaining multiplicity of solutions for problems like (P_ε) in bounded domains, we investigate the effect of such a potential when Ω is an unbounded exterior domain.

The result we obtain is stated in the following

THEOREM 1.1. – *Let a_ε be as in* (1.1) *and let the assumptions* (A_1) *and* (A_2) *be satisfied. Then there exists $\bar{\varepsilon}>0$ such that for every $\varepsilon\in(0,\bar{\varepsilon})$ Problem (P_ε) has at least three distinct solutions $u_{1,\varepsilon}$, $u_{2,\varepsilon}$, $u_{3,\varepsilon}$. Moreover*

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^{N(1-2/p)}}\mathcal{I}_\varepsilon\left(\frac{u_{1,\varepsilon}}{|u_{1,\varepsilon}|_{L^p(\Omega)}}\right)=m, \tag{1.2}$$

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^{N(1-2/p)}}\mathcal{I}_\varepsilon\left(\frac{u_{2,\varepsilon}}{|u_{2,\varepsilon}|_{L^p(\Omega)}}\right)\in\left(m,2^{1-2/p}m\right), \tag{1.3}$$

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^{N(1-2/p)}}\mathcal{I}_\varepsilon\left(\frac{u_{3,\varepsilon}}{|u_{3,\varepsilon}|_{L^p(\Omega)}}\right)=2^{1-2/p}m, \tag{1.4}$$

where

$$m=\inf\left\{\int_{\mathbb{R}^N}\left[|\nabla u|^2+a_0u^2\right]dx \;\middle|\; u\in H^1\left(\mathbb{R}^N\right),\ |u|_{L^p(\mathbb{R}^N)}=1\right\}.$$

We remark that the above theorem gives the existence of at least three solutions whatever Ω is, even the complement of a convex domain.

It is worth observing, also, that the asymptotic energy estimates give some information about the shape of the solutions. Indeed $u_{1,\varepsilon}$ is a "single peak" solution, that is a function that, suitably translated and scaled, tends, as $\varepsilon\to 0$, to a solution of the limit problem

$$(P_\infty)\qquad \begin{cases}-\Delta u+a_0u=u^{p-1} & \text{in } \mathbb{R}^N,\\ u>0 & \text{in } \mathbb{R}^N,\\ u(x)\to 0 & \text{as } |x|\to\infty,\end{cases}$$

and, on the other hand, $u_{3,\varepsilon}$ must be a "two-peaks" solution, in fact its energy, suitably scaled, tends to the energy of a pairs of not interacting solutions of (P_∞). About the last solution, $u_{2,\varepsilon}$, we can guess (but we have not a rigorous proof) that it, suitably scaled in x_0, as $\varepsilon\to 0$, tends to a solution of

$$(P_\alpha)\qquad \begin{cases}-\Delta u+(a_0+\alpha(x))u=u^{p-1} & \text{in } \mathbb{R}^N,\\ u>0 & \text{in } \mathbb{R}^N,\\ u(x)\to 0 & \text{as } |x|\to\infty\end{cases}$$

whose shape depends on α (see [13]).

Finally, we point out that we can look at problem (P_ε) in a "dual" way: an equation not depending on ε, considered in an exterior domain whose complement, as $\varepsilon\to 0$, widens and becomes far and far from the relevant part (in the sense of $L^{N/2}(\mathbb{R}^N)$) of α.

Actually, considering, for instance $\Omega_{\varepsilon,x_0}=\{x\in\mathbb{R}^N \mid \varepsilon x+x_0\in\Omega\}$ an easy scale change shows that to any solution of (P_ε) there corresponds, in a one to one way, a solution of

$$\begin{cases}-\Delta u+(a_0+\alpha(x))u=u^{p-1} & \text{in } \Omega_{\varepsilon,x_0},\\ u>0 & \text{in } \Omega_{\varepsilon,x_0},\\ u=0 & \text{on } \partial\Omega_{\varepsilon,x_0}.\end{cases}$$

Thus the conclusion of Theorem 1.1 can be expressed equivalently as follows:

THEOREM 1.2. – *Let a_0 and α satisfy (A_1) and (A_2). Let $\Omega_n\subset\mathbb{R}^N$ be a sequence of exterior domains such that for some $y_n\in\mathbb{R}^N$ and $r_n\to\infty$*

$$B(y_n,r_n)\subset\mathbb{R}^N\setminus\Omega_n,\qquad B(x_0,r_n)\subset\Omega_n.$$

Then there exists $\bar n \in \mathbb{N}$ such that for all $n > \bar n$ the equation $-\Delta u + (a_0 + \alpha(x))u = u^{p-1}$ with zero Dirichlet boundary data in Ω_n has at least three positive solutions, $\bar u_{1,n}$, $\bar u_{2,n}$, $\bar u_{3,n}$. Moreover

$$\lim_{n\to+\infty} \frac{\int_{\Omega_n} (|\nabla \bar u_{1,n}(x)|^2 + (a_0+\alpha(x))\bar u_{1,n}^2(x))\,dx}{|\bar u_{1,n}|^2_{L^p(\Omega_n)}} = m,$$

$$\lim_{n\to+\infty} \frac{\int_{\Omega_n} (|\nabla \bar u_{2,n}(x)|^2 + (a_0+\alpha(x))\bar u_{2,n}^2(x))\,dx}{|\bar u_{2,n}|^2_{L^p(\Omega_n)}} \in \left(m, 2^{1-2/p}m\right),$$

$$\lim_{n\to+\infty} \frac{\int_{\Omega_n} (|\nabla \bar u_{3,n}(x)|^2 + (a_0+\alpha(x))\bar u_{3,n}^2(x))\,dx}{|\bar u_{3,n}|^2_{L^p(\Omega_n)}} = 2^{1-2/p}m.$$

The paper is organized as follows: Section 2 is devoted to introducing some notations and recalling some known results and useful relations; in Section 3 some useful tools are introduced and some basic asymptotic estimates are proved, Section 4 contains the proof of Theorem 1.1. Arguing as in proving Theorem 1.1, it is a simple matter to get the proof of Theorem 1.2.

2. Notations, known facts and useful remarks

Throughout the paper we make use of the following notations.

- $L^p(\mathcal{D})$, $1 \leqslant p < +\infty$, $\mathcal{D} \subseteq \mathbb{R}^N$, denotes a Lebesgue space; the norm in $L^p(\mathcal{D})$ is denoted by $|\cdot|_{p,\mathcal{D}}$.
- $H_0^1(\mathcal{D})$, $\mathcal{D} \subset \mathbb{R}^N$ and $H^1(\mathbb{R}^N)$ denote the Sobolev spaces obtained, respectively, as closure of $C_0^\infty(\mathcal{D})$ and $C_0^\infty(\mathbb{R}^N)$ with respect to the norms

$$\|u\|_{\mathcal{D}} = \left[\int_{\mathcal{D}} \left(|\nabla u|^2 + a_0 u^2\right) dx\right]^{1/2}, \qquad \|u\|_{\mathbb{R}^N} = \left[\int_{\mathbb{R}^N} \left(|\nabla u|^2 + a_0 u^2\right) dx\right]^{1/2}.$$

- If $\mathcal{D}_1 \subset \mathcal{D}_2 \subseteq \mathbb{R}^N$ and $u \in H_0^1(\mathcal{D}_1)$, we denote also by u its extension to $\mathcal{D}_2$ obtained setting $u \equiv 0$ outside $\mathcal{D}_1$.
- $\mathcal{D}_\varepsilon$ denotes the subset of $\mathbb{R}^N$ $\{y \in \mathbb{R}^N \mid \varepsilon y \in \mathcal{D}\}$, $\mathcal{D} \subset \mathbb{R}^N$.
- $B(y,\rho)$ denotes the open ball, of $\mathbb{R}^N$, having radius ρ and centered at y.

In what follows, without any loss of generality, *we assume $a_0 = 1$ and $x_0 = 0$.*

Setting

$$u_\varepsilon(x) = \varepsilon^{N/p} u(\varepsilon x)$$

an easy computation shows that for every $u \in H_0^1(\Omega)$ $u_\varepsilon \in H_0^1(\Omega_\varepsilon)$, $u \in \mathcal{M}$ if and only if $|u_\varepsilon|_{p,\Omega_\varepsilon} = 1$ and

$$\begin{aligned}\mathcal{I}_\varepsilon(u) &= \int_\Omega \left[\varepsilon^2|\nabla u|^2 + \left(1+\alpha\left(\frac{x}{\varepsilon}\right)\right)u^2\right] dx \\ &= \varepsilon^{(1-2/p)N} \int_{\Omega_\varepsilon} \left[|\nabla u_\varepsilon|^2 + \left(1+\alpha(x)\right)u_\varepsilon^2\right] dx. \qquad (2.1)\end{aligned}$$

Thus looking for critical points of $\mathcal{I}_\varepsilon$ on $\mathcal{M}$ is equivalent to searching for critical points of the “rescaled” energy functional

$$E_\varepsilon(u) = \int_{\Omega_\varepsilon} \left[|\nabla u|^2 + \big(1+\alpha(x)\big)u^2\right] dx$$

on the manifold

$$M_\varepsilon = \left\{u \in H_0^1(\Omega_\varepsilon) \mid |u|_{p,\Omega_\varepsilon} = 1\right\}.$$

Let us set

$$m_\varepsilon = \inf\left\{E_\varepsilon(u) \mid u \in M_\varepsilon\right\} \tag{2.2}$$

and

$$m = \inf\left\{\|u\|^2_{\mathbb{R}^N} \mid u \in H^1\big(\mathbb{R}^N\big),\ |u|_{p,\mathbb{R}^N} = 1\right\}. \tag{2.3}$$

The infimum in (2.3) *is achieved* (*see* [16] *or* [5]) *by a positive function w, that is unique modulo translations* (*see* [12]) *and radially symmetric about the origin, decreasing when the radial co-ordinate increases and such that*

$$\lim_{|x|\to+\infty} \left|D^j w(x)\right| |x|^{\frac{N-1}{2}} \mathrm{e}^{|x|} = d_j > 0, \quad d_j \in \mathbb{R},\ j = 0, 1 \tag{2.4}$$

(*see* [5] *and* [10]).

On the contrary we have

PROPOSITION 2.1. – *Let α satisfy (A_1). Then*

$$m_\varepsilon = m \tag{2.5}$$

and the minimization problem (2.2) *has no solution.*

Proof. – Since we may consider $H_0^1(\Omega_\varepsilon)$ as a subspace of $H^1(\mathbb{R}^N)$,

$$m_\varepsilon \geqslant m.$$

To prove that the equality holds, we consider the sequence

$$w_{\varepsilon,y_n}(x) := \frac{\phi_\varepsilon(x) w(x - y_n)}{|\phi_\varepsilon(x) w(x - y_n)|_{p,\Omega_\varepsilon}} \tag{2.6}$$

where $y_n \in \Omega_\varepsilon$, $\lim_{n\to+\infty} |y_n| = +\infty$, w is the function realizing (2.3) and $\phi_\varepsilon(x) = \phi(\varepsilon x)$ with $\phi : \mathbb{R}^N \to [0,1]$ a C^∞-function such that: $\phi(x) = 0$ if $x \in \omega$, $0 \leqslant \phi(x) \leqslant 1$, $\operatorname{supp}(1-\phi)$ is compact, and we show that

$$\lim_{n\to+\infty} E_\varepsilon(w_{\varepsilon,y_n}) = m. \tag{2.7}$$

Indeed, using (2.4) it is not difficult to show that

$$\left|\phi_\varepsilon(x)w(x-y_n)-w(x-y_n)\right|_{p,\mathbb{R}^N}=\mathrm{o}\big(1/|y_n|\big), \tag{2.8}$$

$$\left\|\phi_\varepsilon(x)w(x-y_n)-w(x-y_n)\right\|_{\mathbb{R}^N}=\mathrm{o}\big(1/|y_n|\big). \tag{2.9}$$

On the other hand, for every fixed $\eta>0$, we can find $\rho=\rho(\eta)>0$ so that

$$\left|\phi_\varepsilon(x)w(x-y_n)\right|_{\frac{2N}{N-2},\Omega_\varepsilon\setminus B(y_n,\rho)}<\eta$$

and

$$|\alpha|_{N/2,B(y_n,\rho)}<\eta,$$

if n is large enough; hence

$$\begin{aligned}
&\int_{\Omega_\varepsilon}\alpha(x)\big[\phi_\varepsilon(x)w(x-y_n)\big]^2\,dx\\
&\quad=\int_{B(y_n,\rho)}\alpha(x)\big[\phi_\varepsilon(x)w(x-y_n)\big]^2\,dx+\int_{\Omega_\varepsilon\setminus B(y_n,\rho)}\alpha(x)\big[\phi_\varepsilon(x)w(x-y_n)\big]^2\,dx\\
&\quad\leqslant\eta\left|\phi_\varepsilon(x)w(x-y_n)\right|_{\frac{2N}{N-2},\mathbb{R}^N}+\eta|\alpha|_{N/2,\mathbb{R}^N}
\end{aligned}$$

from which

$$\lim_{n\to+\infty}\int_{\Omega_\varepsilon}\alpha(x)\big[\phi_\varepsilon(x)w(x-y_n)\big]^2\,dx=0 \tag{2.10}$$

follows.

Hence (2.8), (2.9) and (2.10) give (2.7).

Let us now assume that the minimization problem (2.2) has a solution $u^*\geqslant 0$. Then

$$m\leqslant\|u^*\|^2_{\mathbb{R}^N}=\|u^*\|^2_{\Omega_\varepsilon}\leqslant\|u^*\|^2_{\Omega_\varepsilon}+\int_{\Omega_\varepsilon}\alpha(x)\big(u^*(x)\big)^2\,dx=m.$$

Thus we deduce

$$u^*(x)=w(x-y^*)\quad\text{for some } y^*\in\mathbb{R}^N$$

and, by (A_1) and $w(x)>0\ \forall x\in\mathbb{R}^N$,

$$0=\int_{\Omega_\varepsilon}\alpha(x)\big(u^*(x)\big)^2\,dx=\int_{\Omega_\varepsilon}\alpha(x)w^2(x-y^*)\,dx>0,$$

a contradiction. □

The functional E_ε constrained on M_ε does not verify globally the Palais-Smale condition, however, as proved in [3], the compactness is preserved in some energy range.

LEMMA 2.2. – *Let $(u_n)_n$ be a Palais-Smale sequence for E_ε constrained on M_ε, i.e. $u_n\in M_\varepsilon$*

$$\begin{cases}\lim\limits_{n\to\infty}E_\varepsilon(u_n)=c,\\ \lim\limits_{n\to\infty}\nabla E_{\varepsilon|M_\varepsilon}(u_n)=0.\end{cases}$$

If $c \in (m, 2^{1-2/p}m)$ then $(u_n)_n$ is relatively compact.

The following lemma states a lower bound for the energy of a critical point u of E_ε on M_ε that changes sign; the proof, that can be easily deduced using the definition of m, can be found in [7].

LEMMA 2.3. – *Let $u \in H_0^1(\Omega_\varepsilon)$ be such that*

$$|u|_{p,\Omega_\varepsilon} = 1, \qquad E_\varepsilon(u) = c, \qquad \nabla E_{\varepsilon|M_\varepsilon}(u) = 0.$$

Then $u^+ \not\equiv 0$ and $u^- \not\equiv 0$ implies $c > 2^{1-2/p}m$.

This lemma and the maximum principle ensure that critical points of E_ε on M_ε in the range $(m, 2^{1-2/p}m)$ give rise to positive solutions of problem (P_ε).

3. Tools, preliminary remarks, basic estimates

For what follows we need to introduce some barycenter type function.

For $u \in L^p(\mathbb{R}^N)$ we set

$$\tilde{u}(x) = \frac{1}{|B(x,1)|} \int_{B(x,1)} |u(y)|\, dy$$

$|B(x,1)|$ being the Lebesgue measure of $B(x,1)$, and

$$\hat{u}(x) = \left[\tilde{u}(x) - \frac{1}{2} \max_{\mathbb{R}^N} \tilde{u}(x)\right]^+;$$

we then define $\beta : L^p(\mathbb{R}^N) \setminus \{0\} \to \mathbb{R}^N$ by

$$\beta(u) = \frac{1}{|\hat{u}|^p_{p,\mathbb{R}^N}} \int_{\mathbb{R}^N} x\left[\hat{u}(x)\right]^p dx. \tag{3.1}$$

We remark that β is well defined for all $u \in L^p(\mathbb{R}^N) \setminus \{0\}$, because $\hat{u} \not\equiv 0$ and has compact support, moreover β is continuous.

We define also, for every $\varepsilon > 0$, another map $\beta_\varepsilon : L^p(\mathbb{R}^N) \setminus \{0\} \to \mathbb{R}^N$ by

$$\beta_\varepsilon(u) = \frac{1}{|u|^p_{p,\mathbb{R}^N}} \int_{\mathbb{R}^N} \chi(x - \bar{x}_\varepsilon)\left|u(x)\right|^p dx \tag{3.2}$$

where $\bar{x}_\varepsilon = \bar{x}/\varepsilon$, *$\bar{x}$ being a fixed point in $\omega = \mathbb{R}^N \setminus \overline{\Omega}$* and χ is the function

$$\chi(x) = \frac{x}{1 + |x|}.$$

We remark that β_ε is a continuous map in $L^p(\mathbb{R}^N) \setminus \{0\}$; we observe also that $\beta_\varepsilon(w(x - \bar{x}_\varepsilon)) = 0$.

We put

$$\mathcal{B}_0 := \inf\left\{ \int_{\mathbb{R}^N} \left[|\nabla u|^2 + \big(1+\alpha(x)\big)u^2\right] dx \;\Big|\; u \in H^1(\mathbb{R}^N),\ |u|_{p,\mathbb{R}^N} = 1,\ \beta(u) = 0 \right\} \tag{3.3}$$

and, for all $\varepsilon > 0$, we set

$$\mathcal{B}_{0,\varepsilon} := \inf\{E_\varepsilon(u) \mid u \in M_\varepsilon,\ \beta(u) = 0\}, \tag{3.4}$$

$$\mathcal{B}_{\bar{x}_\varepsilon} := \inf\{E_\varepsilon(u) \mid u \in M_\varepsilon,\ \beta(u) = \bar{x}_\varepsilon\}, \tag{3.5}$$

$$\mathcal{B}_{0,\beta_\varepsilon} := \inf\{E_\varepsilon(u) \mid u \in M_\varepsilon,\ \beta_\varepsilon(u) = 0\}. \tag{3.6}$$

We denote by L_ε the segment joining 0 and $\bar{x}_\varepsilon$, i.e.

$$L_\varepsilon = \{t\bar{x}_\varepsilon \mid t \in [0,1]\}$$

and by

$$\mathcal{A}_\varepsilon := \inf\{E_\varepsilon(u) \mid u \in M_\varepsilon,\ \beta(u) \in L_\varepsilon\}. \tag{3.7}$$

Fixed a point $\zeta \in \partial B(0,1)$ we denote by $\Sigma = \partial B(\zeta, 2)$ i.e.

$$\Sigma = \{z \in \mathbb{R}^N \mid |z - \zeta| = 2\}. \tag{3.8}$$

For every $\varepsilon > 0$ and $\rho > 0$ we define the operator

$$\psi_{\varepsilon,\rho} : \Sigma \times [0,1] \to M_\varepsilon$$

by

$$\psi_{\varepsilon,\rho}[z,t](x) = \frac{\phi_\varepsilon(x)[(1-t)w(x-\rho z) + tw(x-\rho\zeta)]}{|\phi_\varepsilon(x)[(1-t)w(x-\rho z) + tw(x-\rho\zeta)]|_{p,\Omega_\varepsilon}} \tag{3.9}$$

where ϕ_ε is the cut-off function introduced in Proposition 2.1 to define the sequence (2.6).

We put for all $z \in \mathbb{R}^N$

$$w_{\varepsilon,z}(x) = \frac{\phi_\varepsilon(x)w(x-z)}{|\phi_\varepsilon(x)w(x-z)|_{p,\Omega_\varepsilon}} \tag{3.10}$$

and we remark that $\forall z \in \Sigma$

$$\psi_{\varepsilon,\rho}[z,0](x) = w_{\varepsilon,\rho z}(x), \qquad \psi_{\varepsilon,\rho}[z,1](x) = w_{\varepsilon,\rho\zeta}(x).$$

We consider, also, for every $\rho > 0$, the operator

$$\psi_\rho : \Sigma \times [0,1] \to \{u \in H^1(\mathbb{R}^N) \mid |u|_{p,\mathbb{R}^N} = 1\}$$

defined by

$$\psi_\rho[z,t](x) = \frac{(1-t)w(x-\rho z) + tw(x-\rho\zeta)}{|(1-t)w(x-\rho z) + tw(x-\rho\zeta)|_{p,\mathbb{R}^N}}. \tag{3.11}$$

PROPOSITION 3.1. – *Let α satisfy (A_1). Let $\mathcal{B}_0$, $\mathcal{B}_{0,\varepsilon}$ and m as defined, respectively, in* (3.3), (3.4), (2.3). *Then the relation*

$$\mathcal{B}_{0,\varepsilon} \geqslant \mathcal{B}_0 > m \tag{3.12}$$

holds for all $\varepsilon > 0$.

Proof. – Clearly, $\forall \varepsilon > 0$, $\mathcal{B}_{0,\varepsilon} \geqslant \mathcal{B}_0$ and $\mathcal{B}_0 \geqslant m$, so, in order to prove (3.12), we have to show that the equality $\mathcal{B}_0 = m$ cannot be true.

Arguing by contradiction, we assume $\mathcal{B}_0 = m$. Hence a sequence of nonnegative functions $(u_n)_n$ in $H^1(\mathbb{R}^N)$ must exist so that

$$\left.\begin{array}{lr} \beta(u_n) = 0 & \text{(a)} \\ |u_n|_{p,\mathbb{R}^N} = 1, \displaystyle\int_{\mathbb{R}^N} \left[|\nabla u_n|^2 + \left(1+\alpha(x)\right)u_n^2\right] dx \to m & \text{(b)} \end{array}\right\}. \tag{3.13}$$

Moreover (A_1), (2.3) and (3.13)(b) imply $\lim_{n\to+\infty} \|u_n\|^2_{\mathbb{R}^N} = m$.

Then, by the uniqueness of the solution of (2.3), a sequence of points $(z_n)_n$ in $\mathbb{R}^N$ and a sequence of functions $(\varphi_n)_n$ in $H^1(\mathbb{R}^N)$ exist so that, up to a subsequence still denoted by $(u_n)_n$,

$$u_n(x) = w(x - z_n) + \varphi_n(x), \quad x \in \mathbb{R}^N,$$
$$\lim_{n\to+\infty} \varphi_n(x) = 0 \quad \text{in } H^1\left(\mathbb{R}^N\right) \text{ and in } L^p\left(\mathbb{R}^N\right)$$

and, by the same arguments of Proposition 2.1, $\lim_{n\to+\infty} |z_n| = +\infty$.

On the other hand

$$\lim_{n\to+\infty} \sup_{x\in\mathbb{R}^N} \left|\tilde{u}_n(x+z_n) - \widetilde{w}(x)\right| = 0,$$

and, as a consequence,

$$\left|\beta\left(u_n(x)\right) - \beta\left(w(x-z_n)\right)\right| \to 0 \quad \text{as } n \to +\infty,$$

that is

$$\left|\beta\left(u_n(x)\right) - z_n\right| \to 0 \quad \text{as } n \to +\infty,$$

contradicting (3.13)(a). □

LEMMA 3.2. – *Let Σ, $\psi_{\varepsilon,\rho}$, $\mathcal{B}_{0,\varepsilon}$ be as defined, respectively, in* (3.8), (3.9), (3.4). *Then for every $\rho > 0$ there exists $\varepsilon_\rho > 0$ such that for all $\varepsilon \in (0, \varepsilon_\rho)$*

$$\mathcal{B}_{0,\varepsilon} \leqslant \max_{\Sigma\times[0,1]} E_\varepsilon\left(\psi_{\varepsilon,\rho}[z,t]\right). \tag{3.14}$$

Proof. – In view of (2.4), of the radial symmetry round 0 of $w(x)$ and of the fact that $\mathrm{dist}(\bar{\omega}_\varepsilon, 0) \to +\infty$ as $\varepsilon \to 0$, it is not difficult to verify that, for every fixed $\rho > 0$,

$$\lim_{\varepsilon\to 0} \max_{\Sigma} \left|\beta \circ \psi_{\varepsilon,\rho}[z,0] - \rho z\right| = 0.$$

Thus, for all $\varepsilon > 0$ small enough, $\beta \circ \psi_{\varepsilon,\rho}(\Sigma \times \{0\})$ is homotopically equivalent in $\mathbb{R}^N \setminus \{0\}$ to $\rho\Sigma$ and, then, there exists $(\hat{z}_\varepsilon, \hat{t}_\varepsilon) \in \Sigma \times [0,1]$ such that $\beta \circ \psi_{\varepsilon,\rho}[\hat{z}_\varepsilon, \hat{t}_\varepsilon] = 0$, hence

$$\mathcal{B}_{0,\varepsilon} \leqslant E_\varepsilon\bigl(\psi_{\varepsilon,\rho}[\hat{z}_\varepsilon, \hat{t}_\varepsilon]\bigr) \leqslant \max_{\Sigma\times[0,1]} E_\varepsilon\bigl(\psi_{\varepsilon,\rho}[z,t]\bigr). \qquad \square$$

PROPOSITION 3.3. – *Let α satisfy (A_1), (A_2) then there exist constants $\rho_\alpha > 0$, $\mu_\alpha > 0$ and $\varepsilon_1 > 0$, such that for all $\varepsilon \in (0, \varepsilon_1)$*

$$\max_{\Sigma\times[0,1]} E_\varepsilon\bigl(\psi_{\varepsilon,\rho_\alpha}[z,t]\bigr) < \mu_\alpha < 2^{1-2/p} m, \tag{3.15}$$

$$\max_{\Sigma} E_\varepsilon\bigl(\psi_{\varepsilon,\rho_\alpha}[z,0]\bigr) < \mathcal{B}_0. \tag{3.16}$$

Proof. – The proof is carried out in three steps.

Step 1. *There exists $\rho_1 > 0$ such that $\forall \rho > \rho_1$*

$$\max_{\Sigma\times[0,1]} \int_{\mathbb{R}^N} \left[\left|\nabla\psi_\rho[z,t]\right|^2 + \bigl(1+\alpha(x)\bigr)\bigl(\psi_\rho[z,t]\bigr)^2\right] dx := \hat{\mu}_\rho < 2^{1-2/p} m. \tag{3.17}$$

The argument is very similar to that of Lemma 3.5 in [8] so we only sketch it for the reader's convenience.

We define

$$N_\rho[z,t] = \int_{\mathbb{R}^N} \Bigl[\bigl|\nabla\bigl((1-t)w(x-\rho z) + t w(x-\rho\zeta)\bigr)\bigr|^2 + \bigl(1+\alpha(x)\bigr)\bigl((1-t)w(x-\rho z) + t w(x-\rho\zeta)\bigr)^2\Bigr] dx,$$

$$D_\rho[z,t] = \bigl|(1-t)w(x-\rho z) + t w(x-\rho\zeta)\bigr|^p_{p,\mathbb{R}^N}.$$

To verify (3.17) we must prove that if ρ is large enough

$$\max_{\Sigma\times[0,1]} \frac{N_\rho[z,t]}{(D_\rho[z,t])^{2/p}} < 2^{1-2/p} m. \tag{3.18}$$

Taking into account that $-\Delta w + w = m w^{p-1}$ in $\mathbb{R}^N$ we obtain

$$N_\rho[z,t] = \bigl[(1-t)^2 + t^2\bigr] m + 2t(1-t) m \eta_\rho + 2t^2\theta_\rho + 2(1-t)^2\delta_\rho$$

where

$$\eta_\rho = \int_{\mathbb{R}^N} w(x-\rho z)^{p-1} w(x-\rho\zeta)\, dx = \int_{\mathbb{R}^N} w(x-\rho z) w(x-\rho\zeta)^{p-1}\, dx,$$

$$\theta_\rho = \int_{\mathbb{R}^N} \alpha(x)\big|w(x-\rho\zeta)\big|^2\,dx,$$

$$\delta_\rho = \int_{\mathbb{R}^N} \alpha(x)\big|w(x-\rho z)\big|^2\,dx.$$

Using Lemma 2.2 of [1], (2.4) and condition (A_2) we then deduce

$$\lim_{\rho\to+\infty} \eta_\rho\big[2\rho^{\frac{N-1}{2}}\,\mathrm{e}^{2\rho}\big] = C_1 > 0,$$

$$\lim_{\rho\to+\infty} \theta_\rho\big[\rho^{\frac{N-1}{2}\sigma}\,\mathrm{e}^{2\rho}\big] = C_2 \geqslant 0,$$

$$\lim_{\rho\to+\infty} \delta_\rho\big[\rho^{\frac{N-1}{2}\sigma}\,\mathrm{e}^{2\rho}\big] = C_3 \geqslant 0,$$

that allow to obtain

$$N_\rho[z,t] = \big[(1-t)^2+t^2\big]m + 2t(1-t)m\eta_\rho + g(\rho)$$

with $g(\rho) = \mathrm{o}(\eta_\rho)$, because $\sigma \in (1,2]$.

On the other hand, using Lemma 2.7 of [8] we get

$$D_\rho[z,t] \geqslant \big[(1-t)^p + t^p\big] + (p-1)\big[(1-t)^{p-1}t + t^{p-1}(1-t)\big]\eta_\rho.$$

Hence

$$\frac{N_\rho[z,t]}{(D_\rho[z,t])^{2/p}} \leqslant \frac{[(1-t)^2+t^2]}{[(1-t)^p+t^p]^{2/p}}m + 2\gamma(t)m\eta_\rho + \mathrm{o}(\eta_\rho)$$

where

$$\gamma(t) = \frac{(1-t)t}{[(1-t)^p+t^p]^{2/p}}\left\{1 - \frac{p-1}{p}\,\frac{(1-t)^2+t^2}{(1-t)^p+t^p}\big[(1-t)^{p-2}+t^{p-2}\big]\right\}.$$

Now $\gamma(1/2) < 0$, so there exists a neighbourhood $I(1/2)$ such that $\gamma(t) < c < 0$ $\forall t \in I(1/2)$ and

$$\max\left\{\frac{N_\rho[z,t]}{(D_\rho[z,t])^{2/p}} \;\Big|\; z\in\Sigma,\ t\in I\left(\frac{1}{2}\right)\right\} \leqslant 2^{1-2/p}m + 2cm\eta_\rho + \mathrm{o}(\eta_\rho) < 2^{1-2/p}m \tag{3.19}$$

for ρ large enough. Moreover the relation

$$\lim_{\rho\to+\infty}\max\left\{\frac{N_\rho[z,t]}{(D_\rho[z,t])^{2/p}} \;\Big|\; z\in\Sigma,\ t\in[0,1]\setminus I(1/2)\right\} = m\max\left\{\frac{[(1-t)^2+t^2]}{[(1-t)^p+t^p]^{2/p}} \;\Big|\; t\in[0,1]\setminus I(1/2)\right\} < 2^{1-2/p}m$$

holds and together with (3.19) gives (3.18) as desired.

Step 2. *There exists* $\hat\rho \geqslant \rho_1$ *such that* $\forall \rho \geqslant \hat\rho$

$$\max_{\Sigma} \int_{\mathbb{R}^N} \big[\big|\nabla \psi_\rho[z,0]\big|^2 + \big(1+\alpha(x)\big)\big(\psi_\rho[z,0]\big)^2\big]\,dx < \mathcal{B}_0. \tag{3.20}$$

Since (3.12) holds and

$$\begin{aligned}
&\int_{\mathbb{R}^N} \big[\big|\nabla \psi_\rho[z,0]\big|^2 + \big(1+\alpha(x)\big)\big(\psi_\rho[z,0]\big)^2\big]\,dx \\
&\quad= \int_{\mathbb{R}^N} \big[\big|\nabla w(x-\rho z)\big|^2 + \big(1+\alpha(x)\big) w(x-\rho z)^2\big]\,dx \\
&\quad= m + \int_{\mathbb{R}^N} \alpha(x) w(x-\rho z)^2\,dx,
\end{aligned}$$

to prove (3.20) we only need the relation

$$\lim_{|\xi|\to+\infty} \int_{\mathbb{R}^N} \alpha(x) w(x-\xi)^2\,dx = 0$$

that follows, easily, arguing as in Proposition 2.1 to prove relation (2.10).

Step 3. *Let* $\rho_\alpha \geqslant \hat\rho$ *and* $\mu_\alpha \in (\hat\mu_{\rho_\alpha}, 2^{1-2/p}m)$ *be fixed, then there exists* $\varepsilon_1 > 0$ *such that* (3.15) *and* (3.16) *hold for all* $\varepsilon \in (0, \varepsilon_1)$.

Because of the choice of ρ_α, the inequalities (3.17) and (3.20) hold true when $\rho = \rho_\alpha$. Then in order to obtain (3.15) and (3.16) it is enough to observe that for all compact set $K \subset \Sigma \times [0,1]$

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \max_{(z,t)\in K} E_\varepsilon\big(\psi_{\varepsilon,\rho_\alpha}[z,t]\big) \\
&\quad= \max_{(z,t)\in K} \int_{\mathbb{R}^N} \big(\big|\nabla \psi_{\rho_\alpha}[z,t]\big|^2 + \big(1+\alpha(x)\big)\big(\psi_{\rho_\alpha}[z,t]\big)^2\big)\,dx.
\end{aligned} \tag{3.21}$$

In fact, let ε_n and $(z_n,t_n) \in K$ be such that $\lim_{n\to+\infty} \varepsilon_n = 0$ and $\lim_{n\to+\infty}(z_n,t_n) = (z_0,t_0) \in K$, then in view of (2.4) and of the fact that $\operatorname{dist}(\omega_{\varepsilon_n}, 0) \to +\infty$ it is not difficult to see that

$$\lim_{n\to+\infty} \psi_{\varepsilon_n,\rho_\alpha}[z_n,t_n] = \psi_{\rho_\alpha}[z_0,t_0] \quad \text{in } H^1\big(\mathbb{R}^N\big)$$

hence

$$\lim_{n\to+\infty} E_{\varepsilon_n}\big(\psi_{\varepsilon_n,\rho_\alpha}[z_n,t_n]\big) = \int_{\mathbb{R}^N} \big(\big|\nabla \psi_{\rho_\alpha}[z_0,t_0]\big|^2 + \big(1+\alpha(x)\big)\big(\psi_{\rho_\alpha}[z_0,t_0]\big)^2\big)\,dx$$

so (3.21) and the claim easily follow. □

PROPOSITION 3.4. – *Let $\mathcal{B}_{\bar{x}_\varepsilon}$ be as defined in* (3.5). *Let α satisfy (A_1). Then there exists a constant $\mathcal{C}_{\bar{x}} > m$ such that the relation*

$$\mathcal{B}_{\bar{x}_\varepsilon} \geqslant \mathcal{C}_{\bar{x}} > m \tag{3.22}$$

holds for all $\varepsilon > 0$.

Proof. – To prove the claim, we argue by contradiction; so, we assume that a sequence $(\varepsilon_n)_n$ exists such that $\mathcal{B}_{\bar{x}_{\varepsilon_n}} \to m$, as $n \to +\infty$. We can also assume $\varepsilon_n \to 0$, as $n \to +\infty$, otherwise we get a contradiction at once, observing that $\varepsilon_n \geqslant \lambda > 0$ for some $\lambda \in \mathbb{R}$ implies $\bar{x}_{\varepsilon_n} \in \widetilde{\omega}_\lambda := \cup_{\varepsilon \geqslant \lambda} \omega_\varepsilon$ and

$$\mathcal{B}_{\bar{x}_{\varepsilon_n}} \geqslant \mathcal{C}_\lambda := \inf\left\{ \int_{\mathbb{R}^N} \left(|\nabla u|^2 + \big(1+\alpha(x)\big)u^2\right) dx \;\middle|\; u \in H^1\big(\mathbb{R}^N\big),\ |u|_{p,\mathbb{R}^N} = 1, \quad \beta(u) \in \widetilde{\omega}_\lambda \right\},$$

and that, in view of the boundedness of $\widetilde{\omega}_\lambda$, arguing as in Proposition 3.1, it is not difficult to conclude $\mathcal{C}_\lambda > m$.

So a sequence of nonnegative functions $(u_n)_n$, $u_n \in H_0^1(\Omega_{\varepsilon_n})$, must exist, such that $E_{\varepsilon_n}(u_n) \to m$, $\varepsilon_n \to 0$ as $n \to +\infty$, $|u_n|_{p,\Omega_{\varepsilon_n}} = 1$ and $\beta(u_n) = \bar{x}/\varepsilon_n$. Hence there exist sequences $(z_n)_n$ in $\mathbb{R}^N$ and $(\varphi_n)_n$ in $H^1(\mathbb{R}^N)$ such that, up to a subsequence,

$$u_n(x) = w(x - z_n) + \varphi_n(x) \quad \forall x \in \mathbb{R}^N, \tag{3.23}$$

and

$$\lim_{n\to+\infty} \varphi_n(x) = 0 \quad \text{strongly in } H^1\big(\mathbb{R}^N\big) \text{ and in } L^p\big(\mathbb{R}^N\big).$$

So by the continuity of β, we infer

$$\left|\frac{\bar{x}}{\varepsilon_n} - z_n\right| = \left|\beta(u_n) - z_n\right| \to 0 \quad \text{as } n \to +\infty$$

from which the relation

$$\lim_{n\to+\infty} \operatorname{dist}(\Omega_{\varepsilon_n}, z_n) = +\infty$$

follows. Thus, for any $R > 0$ and for n large enough, $B(z_n, R) \cap \Omega_{\varepsilon_n} = \emptyset$ that implies

$$\int_{B(z_n,R)} \left|u_n(x)\right| dx = 0.$$

The above relation contradicts the relation

$$\lim_{n\to+\infty} \int_{B(z_n,R)} \left|u_n(x)\right| dx = \int_{B(0,R)} w(x)\, dx > 0$$

that follows from the properties of w and (3.23). $\square$

PROPOSITION 3.5. – *Let* α *satisfy* (A_1). *Let* $\mathcal{A}_\varepsilon$, $\mathcal{B}_0$, $w_{\varepsilon,z}$, $\mathcal{C}_{\bar{x}}$ *be as defined respectively in* (3.7), (3.3), (3.10) *and in Proposition* 3.4. *Let* $R \in \mathbb{R}$, $R > 0$ *be chosen so that* $\overline{B(0,R)} \subset \Omega$. *Then there exists* $\varepsilon_2 > 0$ *such that*

$$m < \mathcal{A}_\varepsilon \leqslant \max_{|z|=R/2\varepsilon} E_\varepsilon(w_{\varepsilon,z}) < \min(\mathcal{B}_0, \mathcal{C}_{\bar{x}}) \tag{3.24}$$

for all $\varepsilon \in (0, \varepsilon_2)$.

Proof. – Clearly, for every fixed ε, by the same arguments of Proposition 2.1, $m < \mathcal{A}_\varepsilon$.

Let us, now, observe that, in view of (2.4), of the radial symmetry of w and of the fact that $\operatorname{dist}(\partial B(0, R/2\varepsilon), \bar{\omega}_\varepsilon) \to +\infty$ as $\varepsilon \to 0$, we have

$$\lim_{\varepsilon\to 0} \max_{|z|=R/2\varepsilon} \left\| w_{\varepsilon,z}(x) - w(x-z) \right\|_{\mathbb{R}^N} = 0 \tag{3.25}$$

and

$$\lim_{\varepsilon\to 0} \max_{|z|=R/2\varepsilon} \left| \beta(w_{\varepsilon,z}) - z \right| = 0. \tag{3.26}$$

(3.25) implies $\lim_{\varepsilon\to 0} \max_{|z|=R/2\varepsilon} E_\varepsilon(w_{\varepsilon,z}) = m$ and this relation, with (3.12) and (3.22), gives the third inequality for small ε.

As a consequence of (3.26), for small ε, the map

$$z \to \beta(w_{\varepsilon,z})$$

is homotopic to the identity map i on $\partial B(0, R/2\varepsilon)$ by the homotopy

$$\mathcal{K}(\theta, z) = \theta\beta(w_{\varepsilon,z}) + (1-\theta)z, \quad 0 \leqslant \theta \leqslant 1, \tag{3.27}$$

and $\mathcal{K}(\theta, z) \notin \{0, \bar{x}_\varepsilon\}$, $\forall\theta \in [0,1]$ $\forall z \in \partial B(0, R/2\varepsilon)$.

Then there exists $\tilde{z} \in \partial B(0, R/2\varepsilon)$ such that $\beta(w_{\varepsilon,\tilde{z}}) \in L_\varepsilon$, hence the relation

$$\mathcal{A}_\varepsilon \leqslant E_\varepsilon(w_{\varepsilon,\tilde{z}}) \leqslant \max_{|z|=R/2\varepsilon} E_\varepsilon(w_{\varepsilon,z})$$

gives the second inequality. □

PROPOSITION 3.6. – *Let* α *satisfy* (A_1). *Let* $\mathcal{B}_{0,\beta_\varepsilon}$ *be as defined in* (3.6). *Let* μ *be a constant such that* $\mu \in (m, 2^{1-2/p}m)$ *then there exists* $\varepsilon_\mu > 0$ *such that*

$$\mathcal{B}_{0,\beta_\varepsilon} > \mu \tag{3.28}$$

for all $\varepsilon \in (0, \varepsilon_\mu)$.

Proof. – The claim follows from the asymptotic estimate

$$\lim_{\varepsilon\to 0} \mathcal{B}_{0,\beta_\varepsilon} = 2^{1-2/p} m$$

that can be obtained arguing exactly as in Lemma 3.3 and Remark 3.4 of [15]. □

LEMMA 3.7. – *Let* Σ, $\psi_{\varepsilon,\rho}$, $\mathcal{B}_{0,\beta_\varepsilon}$ *be as defined respectively in* (3.8), (3.9), (3.6). *Then for every* $\varepsilon > 0$ *there exists* $\hat{\rho}_\varepsilon > 0$ *such that for all* $\rho > \hat{\rho}_\varepsilon$

$$\mathcal{B}_{0,\beta_\varepsilon} \leqslant \max_{\Sigma\times[0,1]} E_\varepsilon\big(\psi_{\varepsilon,\rho}[z,t]\big). \tag{3.29}$$

Proof. – In view of (2.4), of the radial symmetry of w and by the definition (3.2) of β_ε, it is not difficult to verify that, for every fixed $\varepsilon > 0$,

$$\lim_{\rho\to+\infty} \max_{z\in\Sigma} \big|\beta_\varepsilon \circ \psi_{\varepsilon,\rho}[z,0] - \chi(\rho z - \bar{x}_\varepsilon)\big| = 0.$$

Hence, for all ρ large enough, the set $\beta_\varepsilon \circ \psi_{\varepsilon,\rho}(\Sigma \times \{0\})$ is homotopically equivalent in $\mathbb{R}^N \setminus \{0\}$ to $\rho\Sigma$ and, then, there exists $(\bar{z}_\rho, \bar{t}_\rho) \in \Sigma \times [0,1]$ such that $\beta_\varepsilon \circ \psi_{\varepsilon,\rho}(\bar{z}_\rho, \bar{t}_\rho) = 0$, thus

$$\mathcal{B}_{0,\beta_\varepsilon} \leqslant E_\varepsilon\big(\psi_{\varepsilon,\rho}(\bar{z}_\rho, \bar{t}_\rho)\big) \leqslant \max_{\Sigma\times[0,1]} E_\varepsilon\big(\psi_{\varepsilon,\rho}[z,t]\big). \qquad \square$$

PROPOSITION 3.8. – *Let* α *satisfy* (A_1) *and let* μ *be so that* $\mu \in (m, 2^{1-2/p}m)$. *For every* $\varepsilon > 0$ *there exists* $\bar{\rho}_{\varepsilon,\mu} > 0$ *such that for all* $\rho \geqslant \bar{\rho}_{\varepsilon,\mu}$

$$\max_{\Sigma\times[0,1]} E_\varepsilon\big(\psi_{\varepsilon,\rho}[z,t]\big) < 2^{1-2/p}m, \tag{3.30}$$

$$\max_{\Sigma} E_\varepsilon\big(\psi_{\varepsilon,\rho}[z,0]\big) < \mu. \tag{3.31}$$

Proof. – The proof is carried out in three steps.

Step 1. *For every* $\varepsilon > 0$ *there exists* $\bar{\rho}_{\varepsilon,1} > 0$ *such that for all* $\rho > \bar{\rho}_{\varepsilon,1}$

$$\max_{\Sigma\times[0,1]} \int_{\Omega_\varepsilon} \big[\big|\nabla\psi_{\varepsilon,\rho}[z,t]\big|^2 + \big(\psi_{\varepsilon,\rho}[z,t]\big)^2\big]\,dx \leqslant 2^{1-2/p}m. \tag{3.32}$$

The proof of this step is just Lemma 3.5 in [8].

Step 2. *For every* $\varepsilon > 0$ *there exists* $\bar{\rho}_{\varepsilon,2} > \bar{\rho}_{\varepsilon,1}$ *such that*

$$\max_{\Sigma} \int_{\Omega_\varepsilon} \big[\big|\nabla\psi_{\varepsilon,\rho}[z,0]\big|^2 + \big(\psi_{\varepsilon,\rho}[z,0]\big)^2\big]\,dx \leqslant \mu \tag{3.33}$$

holds for all $\rho > \bar{\rho}_{\varepsilon,2}$.

By (2.4), the shape of w and the choice of ϕ_ε we have

$$\lim_{|z|\to+\infty} \big\|\phi_\varepsilon(x)w(x-z) - w(x-z)\big\|_{\mathbb{R}^N} = 0$$

from which

$$\lim_{\rho\to\infty} \max_{\Sigma} \big[\big\|\psi_{\varepsilon,\rho}[z,0]\big\|^2_{\mathbb{R}^N} - \big\|w(x-\rho z)\big\|^2_{\mathbb{R}^N}\big] = 0$$

that implies

$$\lim_{\rho\to+\infty}\max_{\Sigma}\int_{\Omega_\varepsilon}\left[\left|\nabla\psi_{\varepsilon,\rho}[z,0]\right|^2+\left(\psi_{\varepsilon,\rho}[z,0]\right)^2\right]dx=m.$$

Step 3. *For every* $\varepsilon>0$ *there exists* $\bar\rho_\varepsilon>\bar\rho_{\varepsilon,2}$ *such that* (3.30) *and* (3.31) *hold for all* $\rho>\bar\rho_\varepsilon$.

Taking into account that $|\phi_\varepsilon(x)[(1-t)w(x-\rho z)+tw(x-\rho\zeta)]|_{p,\Omega_\varepsilon}\geqslant c>0$, arguing as in Proposition 2.1 to prove (2.10) it is not difficult to see that

$$\lim_{\rho\to+\infty}\max_{\Sigma\times[0,1]}\int_{\Omega_\varepsilon}\alpha(x)\left(\psi_{\varepsilon,\rho}[z,t](x)\right)^2dx=0.$$

Hence

$$\lim_{\rho\to+\infty}\max_{\Sigma\times[0,1]}\left[E_\varepsilon\left(\psi_{\varepsilon,\rho}[z,t]\right)-\int_{\Omega_\varepsilon}\left[\left|\nabla\psi_{\varepsilon,\rho}[z,t]\right|^2+\left(\psi_{\varepsilon,\rho}[z,t]\right)^2\right]dx\right]=0$$

that, with (3.32) and (3.33), gives (3.30) and (3.31). □

4. Proof of Theorem 1.1

To prove the theorem we show that, for small ε, E_ε has on M_ε three distinct critical values, lying in the energy range $(m,2^{1-2/p}m)$, to which there correspond at least three distinct solutions of (P_ε), positive by Lemma 2.3.

In what follows ρ_α, μ_α are the constants whose existence is stated in Proposition 3.3, moreover we choose $\bar\varepsilon=\min(\varepsilon_{\rho_\alpha},\varepsilon_1,\varepsilon_2,\varepsilon_{\mu_\alpha})$ where ε_1, ε_2 are, respectively, the numbers found in Propositions 3.3 and 3.5 and $\varepsilon_{\rho_\alpha}$, ε_{μ_α} are as stated in Lemma 3.2 and Proposition 3.6.

We remark that, by the results of Section 3, for all $\varepsilon\in(0,\bar\varepsilon)$ the following inequalities hold

$$\begin{aligned}m<\mathcal{A}_\varepsilon\leqslant\max_{|z|=R/2\varepsilon}E_\varepsilon(w_{\varepsilon,z})&<\min(\mathcal{B}_0,\mathcal{C}_{\bar x}),\\ \max_{\Sigma}E_\varepsilon\left(\psi_{\varepsilon,\rho_\alpha}[z,0]\right)<\mathcal{B}_0\leqslant\mathcal{B}_{0,\varepsilon}&\leqslant\max_{\Sigma\times[0,1]}E_\varepsilon\left(\psi_{\varepsilon,\rho_\alpha}[z,t]\right)\\ &<\mu_\alpha<\mathcal{B}_{0,\beta_\varepsilon}<2^{1-2/p}m.\end{aligned}\tag{4.1}$$

and, fixed $\varepsilon\in(0,\bar\varepsilon)$, for all $\rho>\max(\hat\rho_\varepsilon,\bar\rho_{\varepsilon,\mu_\alpha},\rho_\alpha)$ ($\hat\rho_\alpha$ and $\bar\rho_{\varepsilon,\mu_\alpha}$ being the numbers whose existence is stated in Lemma 3.7 and Proposition 3.8, respectively)

$$\max_{\Sigma}E_\varepsilon\left(\psi_{\varepsilon,\rho}[z,0]\right)<\mu_\alpha<\mathcal{B}_{0,\beta_\varepsilon}\leqslant\max_{\Sigma\times[0,1]}E_\varepsilon\left(\psi_{\varepsilon,\rho}[z,t]\right)<2^{1-2/p}m.\tag{4.2}$$

We consider a fixed $\varepsilon\in(0,\bar\varepsilon)$ and we carry out the proof in three steps: first we prove, in Step 1, the existence of a critical value $c_{1,\varepsilon}$ satisfying

$$\mathcal{A}_\varepsilon\leqslant c_{1,\varepsilon}\leqslant\max_{|z|=R/2\varepsilon}E_\varepsilon(w_{\varepsilon,z}),$$

then, in Step 2, we show that another critical level $c_{2,\varepsilon}$ exists so that

$$\mathcal{B}_{0,\varepsilon} \leqslant c_{2,\varepsilon} \leqslant \max_{\Sigma\times[0,1]} E_\varepsilon\big(\psi_{\varepsilon,\rho_\alpha}[z,t]\big),$$

finally, in Step 3, we state the existence of a third critical level $c_{3,\varepsilon} \geqslant \mathcal{B}_{0,\beta_\varepsilon}$.

The above levels are distinct because, by (4.1), (4.2),

$$m < c_{1,\varepsilon} < \mathcal{B}_0 \leqslant c_{2,\varepsilon} < \mu_\alpha < c_{3,\varepsilon} < 2^{1-2/p}m.$$

Moreover, since, by (3.25), $\lim_{\varepsilon\to 0}\max_{|z|=R/2\varepsilon} E_\varepsilon(w_{\varepsilon,z}) = m$, and, by Proposition 3.6, the asymptotic estimate $\lim_{\varepsilon\to 0}\mathcal{B}_{0,\beta_\varepsilon} = 2^{1-2/p}m$ holds, using again (4.1), we deduce

$$\lim_{\varepsilon\to 0} c_{1,\varepsilon} = m, \qquad \lim_{\varepsilon\to 0} c_{2,\varepsilon} \in [\mathcal{B}_0, \mu_\alpha] \subset \big(m, 2^{1-2/p}m\big), \qquad \lim_{\varepsilon\to 0} c_{3,\varepsilon} = 2^{1-2/p}m,$$

that, with (2.1), imply (1.2)–(1.4).

In what follows, for a given $\gamma \in \mathbb{R}$, we set $E_\varepsilon^\gamma = \{u \in M_\varepsilon \mid E_\varepsilon(u) \leqslant \gamma\}$.

Step 1. Let us denote by $\mathcal{S}_{R,\varepsilon} = \max_{|z|=R/2\varepsilon} E_\varepsilon(w_{\varepsilon,z})$. We assume, by contradiction, that

$$\big\{u \in M_\varepsilon \mid \mathcal{A}_\varepsilon \leqslant E_\varepsilon(u) \leqslant \mathcal{S}_{R,\varepsilon},\ \nabla E_{\varepsilon|M_\varepsilon}(u) = 0\big\} = \emptyset.$$

Since the pair $(E_\varepsilon, M_\varepsilon)$ satisfies the Palais-Smale condition, using a well known deformation lemma (see f.i. [17]), we find a positive number $\delta_1 > 0$ and a continuous map $\eta : [0,1] \times E_\varepsilon^{\mathcal{S}_{R,\varepsilon}} \to E_\varepsilon^{\mathcal{S}_{R,\varepsilon}}$ such that

$$\begin{aligned} &\eta(0,u) = u, \quad \forall u \in E_\varepsilon^{\mathcal{S}_{R,\varepsilon}},\\ &\eta\big(1, E_\varepsilon^{\mathcal{S}_{R,\varepsilon}}\big) \subseteq E_\varepsilon^{\mathcal{A}_\varepsilon - \delta_1}. \end{aligned} \tag{4.3}$$

Then we define $\forall\theta \in [0,1]$ and $\forall z \in \partial B(0, R/2\varepsilon)$ the continuous map

$$\mathcal{G}(\theta, z) = \begin{cases} \mathcal{K}(2\theta, z) & 0 \leqslant \theta \leqslant 1/2,\\ \beta(\eta(2\theta - 1, w_{\varepsilon,z})) & 1/2 \leqslant \theta \leqslant 1, \end{cases}$$

$\mathcal{K}$ being the map defined in (3.27). By the definition of $\mathcal{K}$, $\mathcal{G}(\theta, z) \notin \{0, \bar{x}_\varepsilon\}$ $\forall\theta \in [0, 1/2]$ $\forall z \in \partial B(0, R/2\varepsilon)$, moreover, by the relations (4.1) $\mathcal{S}_{R,\varepsilon} < \min(\mathcal{B}_0, \mathcal{C}_{\bar{x}}) \leqslant \min(\mathcal{B}_{0,\varepsilon}, \mathcal{B}_{\bar{x}_\varepsilon})$, $\mathcal{G}(\theta, z) \notin \{0, \bar{x}_\varepsilon\}$ $\forall\theta \in [1/2, 1]$, $\forall z \in \partial B(0, R/2\varepsilon)$. Hence, taking into account that $\mathcal{K}(0, z) = z$ $\forall z \in \partial B(0, R/2\varepsilon)$, we deduce the existence of $\hat{z} \in \partial B(0, R/2\varepsilon)$ such that

$$\mathcal{G}(1, \hat{z}) = \beta \circ \eta(1, w_{\varepsilon,\hat{z}}) \in L_\varepsilon. \tag{4.4}$$

On the other hand by (4.3) and (3.7)

$$\mathcal{G}\big(1, \partial B(0, R/2\varepsilon)\big) \cap L_\varepsilon = \emptyset,$$

that contradicts (4.4).

Step 2. Set $\mathcal{Q}_{\rho_\alpha,\varepsilon} = \max_{\Sigma\times[0,1]} E_\varepsilon(\psi_{\varepsilon,\rho_\alpha}[z,t])$. We assume, by contradiction, that

$$\big\{u \in M_\varepsilon \mid \mathcal{B}_{0,\varepsilon} \leqslant E_\varepsilon(u) \leqslant \mathcal{Q}_{\rho_\alpha,\varepsilon},\ \nabla E_{\varepsilon|M_\varepsilon}(u) = 0\big\} = \emptyset,$$

then, arguing as in the previous step, we find a number $\delta_2 > 0$ and a continuous function $\sigma : E_\varepsilon^{\mathcal{Q}_{\rho_\alpha,\varepsilon}} \to E_\varepsilon^{\mathcal{B}_{0,\varepsilon}-\delta_2}$ such that

$$\sigma(u) = u \quad \forall u \in E_\varepsilon^{\mathcal{B}_{0,\varepsilon}-\delta_2}, \tag{4.5}$$

furthermore, by (3.12) and (3.16), δ_2 can be chosen in such a way that

$$\max_\Sigma E_\varepsilon\big(\psi_{\varepsilon,\rho_\alpha}[z,0]\big) < \mathcal{B}_{0,\varepsilon} - \delta_2. \tag{4.6}$$

Setting

$$\widetilde{\Sigma} = \frac{\Sigma \times [0,1]}{\sim}$$

where $\sim$ identifies the points $(z,1)$, we define a map $\mathcal{H}$ on $\widetilde{\Sigma}$ by

$$\mathcal{H}[z,t] = \beta\big(\sigma\big(\psi_{\varepsilon,\rho_\alpha}[z,t]\big)\big).$$

Since $\varepsilon < \varepsilon_{\rho_\alpha}$, by Lemma 3.2, (4.5) and (4.6), $\mathcal{H}$ maps $\partial\widetilde{\Sigma}$ in a set homotopically equivalent to $\rho_\alpha\Sigma$ (and then to Σ) in $\mathbb{R}^N \setminus \{0\}$. Moreover $\mathcal{H}$ is continuous, so a point $(\tilde{z},\tilde{t}) \in \widetilde{\Sigma}$ must exist, for which

$$0 = \mathcal{H}(\tilde{z},\tilde{t}) = \beta\big(\sigma\big(\psi_{\varepsilon,\rho_\alpha}[\tilde{z},\tilde{t}]\big)\big).$$

This is impossible because $\sigma(\widetilde{\Sigma}) \subset \sigma(E_\varepsilon^{\mathcal{Q}_{\rho_\alpha,\varepsilon}}) \subset E_\varepsilon^{\mathcal{B}_{0,\varepsilon}-\delta_2}$ and by (3.4), so we are in contradiction.

Step 3. Considering a fixed $\rho > \max(\hat{\rho}_\varepsilon, \bar{\rho}_{\varepsilon,\mu_\alpha}, \rho_\alpha)$, taking into account (4.2) and using the same argument displayed in Step 2, we deduce, as desired, that

$$\Big\{u \in M_\varepsilon \mid \mathcal{B}_{0,\beta_\varepsilon} \leqslant E_\varepsilon(u) \leqslant \max_{\Sigma\times[0,1]} E_\varepsilon\big(\psi_{\varepsilon,\rho}[z,t]\big),\ \nabla E_{\varepsilon|M_\varepsilon}(u) = 0\Big\} \neq \emptyset.$$

REFERENCES

[1] A. Bahri, P.L. Lions, On the existence of a positive solution of semilinear elliptic equations in unbounded domains, Ann. Inst. H. Poincaré Anal. Non Linéaire 14 (3) (1997) 365–413.

[2] A. Bahri, Y.Y. Li, On a min-max procedure for the existence of a positive solution for certain scalar field equations in $\mathbb{R}^N$, Rev. Mat. Iberoamericana 6 (1–2) (1990) 1–15.

[3] V. Benci, G. Cerami, Positive solutions of some nonlinear elliptic problems in exterior domains, Arch. Rational Mech. Anal. 99 (1987) 283–300.

[4] V. Benci, G. Cerami, Existence of positive solutions of the equation $-\Delta u + a(x)u = u^{(N+2)/(N-2)}$ in $\mathbb{R}^N$, J. Funct. Anal. 88 (1) (1990) 90–117.

[5] H. Berestycki, P.L. Lions, Nonlinear scalar fields equations – I. Existence of a ground-state, Arch. Rational Mech. Anal. 82 (1983) 313–346.

[6] G. Cerami, C. Maniscalco, Multiple positive solutions for a singularly perturbed Dirichlet problem in "geometrically trivial" domains, Topol. Methods Nonlin. Anal. 19 (1) (2002) 63–76.

[7] G. Cerami, D. Passaseo, Existence and multiplicity of positive solutions for nonlinear elliptic problems in exterior domains with “rich” topology, Nonlinear Analysis TMA 18 (2) (1992) 109–119.
[8] G. Cerami, D. Passaseo, Existence and multiplicity results for semilinear elliptic Dirichlet problems in exterior domains, Nonlinear Analysis TMA 24 (11) (1995) 1533–1547.
[9] G. Cerami, D. Passaseo, Effect of concentrating potentials in some singularly perturbed problems, Calculus of Variations and PDE, to appear.
[10] B. Gidas, W.M. Ni, L. Nirenberg, Symmetry of positive solutions of nonlinear elliptic equations in $\mathbb{R}^N$, in: Mathematical Analysis and Applications, Part A, in: Advances in Mathematics Supplementary Studies, Vol. 7-A, Academic Press, 1981, pp. 369–402.
[11] M. Grossi, D. Passaseo, Nonlinear elliptic Dirichlet problems in exterior domains: the role of geometry and topology of the domain, Comm. Appl. Nonlinear Anal. 2 (2) (1995) 1–31.
[12] M.K. Kwong, Uniqueness of positive solutions of $\Delta u - u + u^p = 0$, Arch. Rational Mech. Anal. 105 (1989) 243–266.
[13] R. Molle, M. Musso, D. Passaseo, Positive solutions for a class of nonlinear elliptic problems in $\mathbb{R}^N$, Proc. Roy. Soc. Edinburgh Sect. A 130 (1) (2000) 141–166.
[14] R. Molle, D. Passaseo, On the behaviour of the solutions for a class of nonlinear elliptic problems in exterior domains, Discrete Contin. Dynam. Systems 4 (3) (1998) 445–454.
[15] R. Molle, D. Passaseo, Multiple solutions of nonlinear elliptic Dirichlet problems in exterior domains, Nonlinear Anal. Ser. A: Theory Methods 39 (4) (2000) 447–462.
[16] W.A. Strauss, Existence of solitary waves in higher dimensions, Comm. Math. Phys. 55 (1977) 149–162.
[17] M. Struwe, Variational Methods – Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems, Springer-Verlag, Berlin, 1990.