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On a quasi-periodic Hopf bifurcation

by

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ABSTRACT. — In this paper we study quasi-periodic Hopf bifurcations for the model problem of a quasi-periodically forced oscillator, where the frequencies remain fixed. For this purpose we first consider Stoker's problem for small damping.

Key-words: Hopf Bifurcation, Quasi-periodic solution, Invariant tori, Stoker's problem, Normal form, Small divisors, Center manifold.

RÉSUMÉ. — Dans cet article, nous étudions les bifurcations de Hopf quasi-périodiques pour le problème d'un oscillateur quasi-périodique forcé où les fréquences restent fixes. Dans ce but, nous commençons par analyser le problème de Stoker pour un amortissement faible.

§ 1. INTRODUCTION

a. Setting of the problem.

Consider the forced oscillator

$$\ddot{x} + c\dot{x} + ax = f(t, x, \dot{x}), \tag{1.1}$$

where f is quasi periodic in t with the fixed, rationally independent frequencies $\omega_1, \omega_2, \ldots, \omega_n$. This means that $f(t, x, \dot{x}) = F(\omega_1 t, \omega_2 t, \ldots, \omega_n t, x, \dot{x})$ for a function $F = F(\theta_1, \theta_2, \ldots, \theta_n, x, \dot{x})$ which is periodic with period 2π in all θ_i $(1 \leq i \leq n)$. Special examples are the Duffing and the Van der Pol equation with quasi-periodic forcing.

Our first problem comes from [Sto], appendix 2 who refers to Friedrichs, although its origin seems to lie in the days of Planck. We shall refer to it as Stoker's problem. If in (1.1) the coefficients c and $a\,(a>0)$ are considered as parameters, then, for which values of these parameters does the equation (1.1) have quasi-periodic solutions? These solutions are required to have the same frequencies $\omega_1, \omega_2, \ldots, \omega_n$ as the forcing term f and to be « close to $x = \dot{x} = 0$, measured in terms of the size of f ».

Before we state our second problem we rephrase the above in terms of invariant n-dimensional tori of an autonomous system. Since our main interest is in the case of small damping, we restrict ourselves to the parameter region $c^2 < 4a$, where we can introduce a convenient complex notation. Here the characteristic equation $\lambda^2 + c\lambda + a = 0$ has complex roots. Writing $\lambda := -\frac{c}{2} + i\sqrt{a - \frac{c^2}{4}}$ we have $\ddot{x} + c\dot{x} + ax = \left(\frac{d}{dt} - \lambda\right)\left(\frac{d}{dt} - \bar{\lambda}\right)x$. So putting $z := \dot{x} - \lambda x$ we transform the equation (1.1) into the following autonomous system, defined on $\mathbf{T}^n \times \mathbf{C}$ (where $\mathbf{T}^n = \mathbf{R}^n/(2\pi\mathbf{Z})^n$ denotes the n-dimensional torus):

$$(1.2) \qquad \begin{cases} \dot{\theta} = \omega \\ \dot{z} = \lambda z + g(\theta, z, \lambda). \end{cases}$$

Here $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$, $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ and

$$g(\theta, z, \lambda) = \mathrm{F}\left(\theta, \frac{z - \bar{z}}{\lambda - \bar{\lambda}}, \frac{\lambda z - \bar{\lambda}\bar{z}}{\lambda - \bar{\lambda}}\right).$$

The parameter λ varies over the complex upper half plane $\mathbf{C}^+$. Stoker's problem now asks for invariant n-tori of the system (1.2), carrying ω-quasi-periodic flow. In fact, if $z = z(\theta, \lambda)$ is an embedded n-torus, which is invariant under the vector field, then it satisfies the following non-linear partial differential equation

$$\left(\frac{\partial z}{\partial \theta}, \omega\right) = z + g(\theta, z, \lambda).$$

Since $\dot{\theta} = \omega$ it then follows that $z(t) := z(\theta_0 + t\omega, \lambda)$ is the required family of quasi-periodic solutions. The function g is considered as a perturbation and the invariant n-tori have to be perturbations of $\mathbf{T}^n \times \{0\}$ in $\mathbf{T}^n \times \mathbf{C}$.

The normal behavior of (1.2) on these Stoker n-tori, provided that they exist, roughly speaking is dominated by the linear term $\dot{z} = \lambda z$. So the n-torus is normally hyperbolic for $\operatorname{Re}\lambda \neq 0$: asymptotically stable for $\operatorname{Re}\lambda < 0\,(c > 0)$ and unstable for $\operatorname{Re}\lambda > 0\,(c < 0)$.

Now our second and main problem is to analyze the dynamics close to the Stoker n-tori when λ is near the imaginary axis. In particular we look for quasi-periodic Hopf bifurcations, where invariant $(n + 1)$-tori branch off at the moment the Stoker n-torus loses stability.

b. Some literature on Stoker's problem.

We present a brief overview of some literature on Stoker's problem. First in [Sto], appendix 2, in the case of a model, the question is answered affirmatively for large damping, using a straight forward contraction argument. Below we shall briefly touch this classical method. For small damping this approach cannot be applied, due to the presence of small divisors.

Secondly [Mo1] considers a model problem without damping. It is Duffing's equation

$$\ddot{x} + a(\mu)x - bx^3 = \mu h(t), \tag{1.3}$$

where the ω-quasi-periodic function h satisfies the reversibility condition $h(-t) = h(t)$. Also the choice of the frequencies $\omega_1, \omega_2, \ldots, \omega_n$ is restricted by a strong non resonance condition of the following form. For given constants $\tau > n$ and $\gamma > 0$ and for all integer vectors $k \in \mathbf{Z}^n \backslash \{ 0 \}$ we have the estimate

$$|(\omega, k)| \geq \gamma |k|^{-\tau}, \tag{1.4}$$

where $(\omega, k) = \Sigma_{j=1}^n \omega_j k_j$ and $|k| = \Sigma_{j=1}^n |k_j|$.

This condition serves to overcome the small divisor problem mentioned above. Indeed it appears that now the equation (1.3) has ω-quasi-periodic solutions, provided that the coefficient a varies appropriately in dependence of the perturbation parameter μ.

Thirdly, [Fr 1,2] attempts a more general approach of Stoker's problem, integrating the methods used in the two cases mentioned above. The (c, a)-regime where the damping $|c|$ is sufficiently large to have the classical contraction method work, we shall refer to as the Stoker domain. The parameter regime with ω-quasi-periodic solutions now is extended by an uncountable number of cusps, which connect the two components of this Stoker domain, cf. fig. 1. These cusps are almost horizontal and in

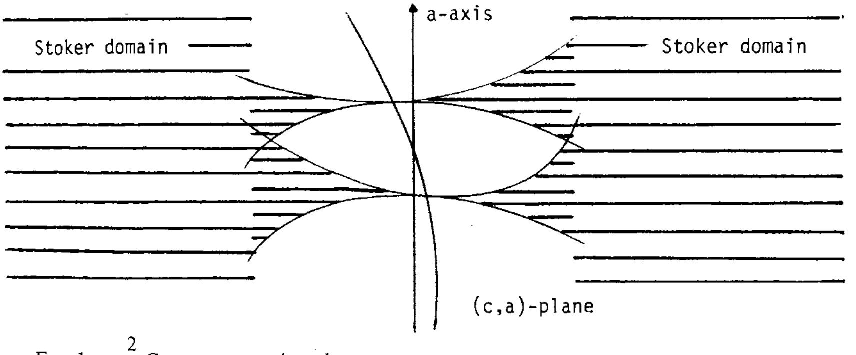

FIG. 1. — $\frac{2}{1}$-Cusps connecting the components of the Stoker domain, cf. [Fr1, 2].

the cusp points the order of tangency is quadratic. In these cusp points the quasi-periodic solution changes from asymptotically stable to unstable if one moves from left to right. Because of the small divisors the choice of the frequencies again is restricted by the condition (1.4).

c. Organization of this paper.

In § 2 we formulate a normal form theorem for systems (1,2) for a λ-domain that contains a part of the imaginary axis. This normal form is

$$(1.5)\qquad \begin{cases} \dot{\theta} = \omega \\ \dot{\zeta} = \mu\zeta + \Sigma_{l=1}^{N}\alpha_l(\mu)\zeta\,|\,\zeta\,|^2 + \text{higher order terms in } |\,\zeta\,|, \end{cases}$$

provided that certain small divisor conditions on ω and μ are fulfilled, compare (1.4). The conjugacy to the normal form is of the type

$$\Phi : (\theta, \zeta, \mu) \;\rightarrow\; (\theta, z, \lambda) = (\theta, \zeta + V(\theta, \zeta, \mu), \mu + U(\mu))$$

where V is analytic in θ, polynomial in ζ and $\bar{\zeta}$ and where both V and U are Whitney-smooth in μ. (Note that the parameters are transformed too !). Also the coefficients α_l ($1 \leqq l \leqq N$) are Whitney-smooth. Compare [Be], where a linear normal form is obtained in the hyperbolic case. Our proof of the normal form theorem, which is written down in § 5, uses the KAM-theory as developed by [Ze, Pö].

In § 3 we consider the problems mentioned in *a*. Our analysis heavily rests on the normal from technique form § 2. We start looking for Stoker n-tori. Here we use the normal form (1.5) with $N = 0$, so the linear case. Then in the new coordinates $\zeta = 0$ is the required n-toruş. This approach only works for a certain subset of the μ-plane containing no interior points. This subset of the μ-plane can be fattened to a family of cusps, where also Stoker n-tori exist. So we retrace the steps of [Fr, 1,2], improving these results as follows. In the first place we find a family of cusps which at the cusp points are almost vertical instead of horizontal, the tangency being of infinite order. Secondly the whole family of cusps is a Whitney-smooth bundle over a Cantor set of large measure. This implies that the « hole » in the ω-quasi-periodic parameter regime, relative to the region we study, has small measure (cf. fig. 2).

Next we come to the quasi-periodic Hopf bifurcation. As in the usual theory of Hopf bifurcations, see e. g. [MM], we now need a non-linear normal form. We shall work with the case $N = 1$ and $\alpha_1 < 0$ (the case $\alpha_1 > 0$ is similar). Then we show that in a right hand neighborhood of the « hole » there exist invariant $(n + 1)$-tori, again cf. fig. 2.

The invariant n- and $(n + 1)$-tori are found by applying center manifold theory to our normal form. In the case of the Stoker n-tori this is very similar to the classical contraction method mentioned before. For the

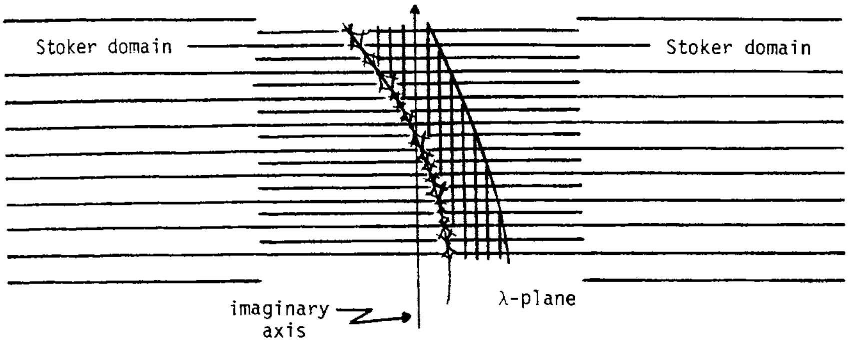

FIG. 2. — Family of flat cusps with Stoker n-tori (shaded) and $(n + 1)$-tori (doubly shaded) in the case $\alpha_1 < 0$.

$(n + 1)$-tori we use a generalization of this, compare [Ha1, CH]. A similar analysis can be found in [CI, Se, F1], also see e. g. [Io (Ch. VI)]. Our normal form technique, however, is of great help, both facilitating the analysis and sharpening the results. In a future paper we shall explore this technique further in the case of autonomous systems, see § 2 below. Related results concerning a saddle node bifurcation can be found in [Ch].

We end § 3 with some remarks on the invariant $(n + 1)$-tori which carry quasi-periodic flow. A result is announced that can be proved using [Mo2, Ze, Pö]; a full proof will be given in forthcoming work of Huitema.

Finally in § 4 we present some examples and applications. First we apply the theory on the Stoker problem to the case of Duffing's equation.

Secondly we consider a forced equation of the special type

$$\ddot{x} + c\dot{x} + ax + h(x, \dot{x}, c, a) = f(t), \tag{1.6}$$

where $h(x, \dot{x}, c, a) = O(x^2 + \dot{x}^2)$, so containing the non-linearities, and where f is ω-quasi-periodic in t, as before. We compare Hopf bifurcations of the free oscillator to quasi-periodic Hopf bifurcations of the forced oscillator. Applications to the Van der Pol and the Duffing equation will be discussed.

Thirdly we conclude this section by giving an example which shows that also in our special case of fixed frequencies $\omega_1, \omega_2, \ldots, \omega_n$ the invariant $(n + 1)$-tori are in general not as smooth as the system one starts with (i. e. (1.2)), but that one has to deal with losses of differentiability. Our example is similar to [Str, Sij].

In this paper, for simplicity, we restrict ourselves to the case where the system (1.2) is real analytic in all variables, but obviously one could extend the results further to the cases of finite or infinite differentiability. A natural topology on the space of real analytic functions is that of locally uniform convergence on complexified domains. That is why we shall measure the size of the perturbation in (1.2) by its supremum norm on a complexified domain. For more details see § 2 below.

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d. Some « linear » remarks.

In this final part of the introduction we briefly discuss the linear equation

$$\ddot{x} + c\dot{x} + ax = f(t) \tag{1.7}$$

or, equivalently, the system

$$\begin{cases} \dot{\theta} = \omega \\ \dot{z} = \lambda z + g(\theta, \lambda), \end{cases} \tag{1.8}$$

compare (1.1) and (1.2). We do this in order to indicate some of the difficulties as well as some techniques for later use.

First we formally solve Stoker's problem, which in this case is the following linear inhomogeneous partial differential equation for $z = z(\theta, \lambda)$:

$$\left(\frac{dz}{\partial\theta}, \omega\right) - \lambda z = g(\theta, \lambda), \quad \operatorname{Im} \lambda > 0 .$$

Formally solving by Fourier series

$$g(\theta, \lambda) = \sum_{k \in Z^n} g_k(\lambda) e^{i(\theta, k)}$$

and

$$z(\theta, \lambda) = \sum_{k \in Z^n} z_k(\lambda) e^{i(\theta, k)} .$$

then it follows that for all $k \in \mathbf{Z}^n$ we have

$$z_k(\lambda) = \frac{g_k(\lambda)}{i(\omega, k) - \lambda}. \tag{1.9}$$

Here one sees the small divisors entering: the denominator of (1.9) vanishes at the resonance points $\lambda = i(\omega, k)$, $k \in \mathbf{Z}^n$, so in a dense subset of the imaginary λ-axis. Recall that $\operatorname{Re} \lambda = -\frac{c}{2}$ and compare the remarks on the small divisor problem made in part *b* above.

Secondly we expose two different ways to materialize the formal solution (1.9). The first is rather familiar, but for the sake of completeness we here briefly touch it. It assumes that $\operatorname{Re}\lambda \neq 0$, for simplicity we restrict to the case $\operatorname{Re}\lambda < 0$. In § 3 below, it will serve us to give the classical contraction argument which produces Stoker's n-torus in the case of large damping, compare e. g. [Ha2]. Also compare [Sto], appendix 2, where the setting is slightly different. In [Ma, BN, Ha2] similar methods are used in the more general case of almost periodic f. Consider the Banach space $\mathscr{B}$ of continuous, complex valued functions defined on the n-torus $\mathbf{T}^n$, endowed with the supremum norm. Then the formal solution (1.9) extends to the linear operator $\mathrm{T}: \mathscr{B} \to \mathscr{B}$ given by

$$(1.10) \qquad (\mathrm{T}g)(\theta) = \int_0^\infty e^{\lambda s} g(\theta - s\omega)ds\,.$$

One easily sees that $|\mathrm{T}| = |\operatorname{Re}\lambda|^{-1}$, so indeed the operator is unbounded on the imaginary λ-axis.

The second materialization points in the direction of KAM-theory. Let $\tau > n$ and $\gamma > 0$ be given constants and define as a subset of the upper half λ-plane $\mathbf{C}^+$

$$(1.11) \qquad \mathrm{C}_\gamma := \{ \lambda \in \mathbf{C}^+ \mid \forall k \in \mathbf{Z}^n \backslash \{0\} : |\lambda - i(\omega, k)| \geq \gamma |k|^{-\tau} \}$$

(compare (1.4)). For $\lambda \in \mathrm{C}_\gamma$ the growth of the coefficient z_k in (1.9) for $|k| \to \infty$ is controlled. In fact for real analytic g the coefficients g_k decay exponentially as $|k| \to \infty$ (Paley-Wiener). Therefore the formal solution (1.9) converges for $\lambda \in \mathrm{C}_\gamma$. Moreover, by the same argument this solution is again analytic: the z_k also exhibit exponential decay as $|k| \to \infty$.

Now let us examine the set C_γ. Observe that C_γ excludes from the complex upper half plane a countable number of open discs, numbered by $k \in \mathbf{Z}^n \backslash \{0\}$. The centers of these discs are the points $i(\omega, k)$, the corresponding radii $\gamma |k|^{-\tau}$. Therefore, by the Cantor-Bendixson theorem, cf. [Hau (p. 159)], C_γ intersects the imaginary λ-axis in the union of a countable set and a Cantor set. This intersection is in the complement of the set of resonance points $\{ i(\omega, k) \mid k \in \mathbf{Z}^n \backslash \{0\} \}$, which densely fills up the imaginary axis, see above. Next observe that for $l \in \mathbf{N}$ the number of multi indices $k \in \mathbf{Z}^n$ with $|k| = l$ can be estimated by $2^n l^{n-1}$. From this it directly follows that the measure of the set $\{ \operatorname{Re}\lambda = 0 \} \backslash \mathrm{C}_\gamma$ in the imaginary λ-axis, is of order γ as $\gamma \downarrow 0$. Similarly we see that the « bead string » $\mathrm{C} \backslash C_\gamma$ has a measure of order γ^2 as $\gamma \downarrow 0$.

Remark. — If for a moment we consider the linear equation for its own sake, we may conclude that a real analytic system (1.8) has a real analytic Stoker n-torus as soon as the parameter λ is an element of $\cup_{\gamma > 0} \mathrm{C}_\gamma$. Notice that this union in the imaginary axis leaves out a residual set of measure zero, containing the resonance points $i(\omega, k)$, $k \in \mathbf{Z}^n \backslash \{0\}$. In § 2 we shall see that in the general non linear case such a conclusion is impossible, since then γ enters in the smallness condition on the perturbation g.

§ 2. THE NORMAL FORM THEOREM

a. Introduction.

Consider the family of differential equations

$$(2.1) \qquad \begin{cases} \dot{\theta} = \omega \\ \dot{z} = \lambda z + g(\theta, z, \lambda), \end{cases}$$

where $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$ varies over the n-torus $\mathbf{T}^n = \mathbf{R}^n/(2\pi\mathbf{Z})^n$, meaning that the function g is 2π-periodic in all θ_j $(1 \leq j \leq n)$, and where the variable z and the parameter λ both have open complex domains. We assume that the λ-domain contains an interval of the imaginary axis.

In the following we shall put (2.1) on a normal form

$$(2.2) \qquad \begin{cases} \dot{\theta} = \omega \\ \dot{\zeta} = \mu\zeta + \Sigma_{l=1}^{\mathrm{N}} \alpha_l(\mu)\zeta \mid \zeta \mid^{2l} + \mathrm{O}(\mid \zeta \mid^{\mathrm{M}}) \end{cases}$$

near the invariant n-torus with equation $\zeta = 0$. Here N is a prefixed non-negative integer and $\mathrm{M} = 2\mathrm{N} + 2$ or $2\mathrm{N} + 3$. The transformation to this form is of the type

$$(2.3) \qquad \Phi : (\theta, \zeta, \mu) \rightarrow (\theta, z, \lambda) = (\theta, \zeta + \mathrm{V}(\theta, \zeta, \mu), \mu + \mathrm{U}(\mu)),$$

where
$$\mathrm{V}(\theta, \zeta, \mu) = \sum_{k+l=0}^{\mathrm{M}-1} \mathrm{V}_{kl}(\theta, \mu)\zeta^k \bar{\zeta}^l$$

is polynomial in ζ and $\bar{\zeta}$ of the same degree as the normalized part of (2.2). Note that this normalized part is symmetric for the rotations generated by the linear part $\dot{\zeta} = \mu\zeta$, where μ is purely imaginary. This is similar to the usual normal form theory near a singular point or a closed orbit, compare e. g. [Poi, Ta, Br1, VdM]. In our present case however, due to the presence of small divisors, we shall have to restrict the choice of the frequency vector $\omega \in \mathbf{R}^n$. For the same reason also the μ-domain of the conjugacy Φ has to be restricted, compare § 1. Below we shall be more precise.

[Fr 1,2] obtains a similar normal form (2.2) for $\mathrm{M} = 2$, though with less regularity than ours. Such a linear normal form provides a kind of Floquet theory for the linearized system (2.1) near the Stoker n-torus given by the equation $\zeta = 0$.

Our consideration of the normal form follows the ideas of [Ze, Pö].

This means that the conjugacy Φ and the normal form coefficients α_l $(1 \leq l \leq \mathrm{N})$ will be Whitney-differentiable in the parameter μ, which varies over a Cantor-like set. At this point the regularity of our result on the Stoker problem exceeds [Fr 1,2].

In [Be], also see [Ar2], a similar normal form is given in the hyperbolic case, so where the λ-domain, and hence also the μ-domain, are bounded away from the imaginary axis. In that case the entire normal form is linear. On the other hand the setting of [Be] is somewhat more general since (2.1) is replaced by

$$\left\{\begin{array}{l}\dot{\theta} = \omega + f(\theta, z, \omega, \lambda) \\ \dot{z} = \lambda z + g(\theta, z, \omega, \lambda).\end{array}\right. \tag{2.1 a}$$

where ω also is a parameter of the problem. As said before, in a forthcoming paper we shall integrate this set up with ours and derive for (2.1 *a*) a normal form similar to (2.2) in the non-hyperbolic case. Such an approach is of interest e. g. for the analysis of subordinate quasi-periodic Hopf bifurcations in autonomous system.

b. Formulation of the theorem.

In order to be able to formulate the normal form theorem more precisely we first introduce the z- and λ-domains of the function g in (2.1). The z-domain of g may be any neighborhood of 0 in $\mathbf{C}$, which does not have to be specified any further since in z all manipulations will be polynomial. The λ-domain of g is an open rectangle

$$\Lambda = \Lambda_1 \times \Lambda_2$$

in the complex upper half plane (so $\lambda_1 = \operatorname{Re} \lambda \in \Lambda_1$ and $\lambda_2 = \operatorname{Im} \lambda \in \Lambda_2$).

We recall the assumption that g be real analytic on its domain of definition. This means that g can be extended as a complex analytic function of the independent complex variables $\theta_1, \theta_2, \ldots, \theta_n, z, \bar{z}, \lambda$ and $\bar{\lambda}$. We have to specify the extended domain of g in somewhat more detail. For this purpose we introduce the following notation. If $\mathrm{I} \subseteq \mathbf{R}^m$ and positive constants $\rho_1, \rho_2, \ldots, \rho_m$ are given, then

$$\mathrm{I} + (\rho_1, \rho_2, \ldots, \rho_m) := \cup_{x \in \mathrm{I}} \{ z \in \mathbf{C}^m \mid \text{for } 1 \leq j \leq m : | z_j - x_j | < \rho_j \},$$

and similarly for $\mathbf{T}^n$ and $\sigma > 0$

$$\mathbf{T}^n + \sigma := \cup_{x \in \mathbf{T}^n} \{ z \in \mathbf{C}^n / 2\pi \mathbf{Z}^n \mid \text{for } 1 \leq j \leq m : | z_j - x_j | < \sigma \}.$$

Now assume that there exist positive constants σ, ρ_1 and ρ_2, such that the θ-, the λ_1- and the λ_2-domains can be extended to

$$\mathbf{T}^n + \sigma, \Lambda_1 + \rho_1 \quad \text{and} \quad \Lambda_2 + \rho_2 \tag{2.4}$$

respectively. The $(z, \bar{z})$-complexification does not have to be specified, see above. In order to avoid heavy notation we often shall suppress the dependence on $\bar{z}$ and λ in our formulae. Also we usually shall omit a formula with $\partial/\partial\bar{\lambda}$ when a similar one with $\partial/\partial\lambda$ is already present.

In view of the « linear » situation, where g does not explicitly depend on z, it may not be expected that the conjugacy Φ can be defined on the full μ-domain: at least one seems to have to stay away from the resonance points $\mu = i(\omega, h)$, $h \in \mathbf{Z}^n$. See § 1 *e*. In fact we introduce a constant $\tau > n$ and a parameter $\gamma > 0$ and first restrict the choice of the frequency vector $\omega \in \mathbf{R}^n$ by the « small divisor condition » that for all $h \in \mathbf{Z}^n \backslash \{ 0 \}$

$$|(\omega, h)| \geq \gamma |h|^{-\tau}. \tag{2.5a}$$

Here $(\omega, h) = \Sigma_{j=1}^n \omega_j h_j$ and $|h| = \sum_{j=1}^n |h_j|$.

Note that the set of $\omega \in \mathbf{R}^n$ with (2.5 *a*) has a large Lebesgue measure. Modulo measure zero it has the structure of Cantor set cross interval. Here again we used the Cantor-Bendixson theorem, see [Hau]. Note that the components $\omega_1, \omega_2, \ldots, \omega_n$ of such a vector ω certainly will be rationally independent. Now secondly we define closed subsets $\Lambda_{1,\gamma} \subseteq \Lambda_1$ and $\Lambda_{2,\gamma}^{\mathrm{M}} \subseteq \Lambda_2$ as follows. To begin with $\Lambda_{1,\gamma}$ consists of all $\mu_1 \in \Lambda_1$ with a distance not less than γ to the boundary $\partial\Lambda_1$. This same condition also holds for $\Lambda_{2,\gamma}^{\mathrm{M}} \subseteq \Lambda_2$, but here we moreover require that for all $h \in \mathbf{Z}^n \backslash \{ 0 \}$ and for $l = 1, 2, \ldots, \mathrm{M}$

$$|(\omega, h) - l\mu_2| \geq \gamma |h|^{-\tau}. \tag{2.5b}$$

This is again a « small divisor condition » which, for any ω as in (2.5 *a*), up to a countable set, determines $\Lambda_{2,\gamma}^{\mathrm{M}}$ as a Cantor set, such that the measure of $\Lambda_2 \backslash \Lambda_{2,\gamma}^{\mathrm{M}}$ is of order γ as $\gamma \downarrow 0$. See above and § 1 *e*.

Finally we define

$$\Lambda_\gamma^{\mathrm{M}} := \Lambda_{1,\gamma} \times \Lambda_{2,\gamma}^{\mathrm{M}}.$$

We are now able to formulate the normal form theorem.

NORMAL FORM THEOREM. — *Let* $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ *satisfy (2.5a) and assume g to be « real » analytic in all its arguments, with* θ-, λ_1- *and* λ_2-*domains (2.4), for positive constants* σ, ρ_1 *and* ρ_2. *Also assume that* $0 \in \mathrm{Int}\,(\Lambda_{1,\gamma})$.

Then there exists a constant $\delta > 0$, *depending on* n, τ, M *and* σ, *but not on* γ, ρ_1, ρ_2 *and* Λ, *such that the following holds. First let* $0 < \gamma \leq \min \{ \rho_1, \rho_2 \}$. *Next let* $\Omega = (\mathbf{T}^n + \sigma) \times \mathrm{O} \times (\Lambda_1 + \rho_1) \times (\Lambda_2 + \rho_2)$, *where* O *is any neighborhood of* 0 *in* $\mathbf{C}^2$, *such that g can be extended as a complex analytic function to the domain* Ω *with*

$$|g|_\Omega \leq \gamma\delta.$$

Then a map $\Phi : \mathbf{T}^n \times \mathbf{C} \times \Lambda_\gamma^{\mathrm{M}} \to \mathbf{T}^n \times \mathbf{C} \times \Lambda$ *exists as in (2.3), with*

i) Φ *is of class* C^∞ *in the sense of Whitney, even real analytic in the variables* θ *and* μ_1 *and polynomial of degree* $\mathrm{M} - 1$ *in* ζ *and* $\bar{\zeta}$;

ii) Φ *takes (2.1) into the normal form (2.2), where the functions* α_l, $1 \leq l \leq N$, *are of class* C^∞ *in the sense of Whitney and even real analytic in* μ_1.

Remarks. — *i)* According to the Whitney extension theorem (see the appendix) the map Φ can be extended as a C^∞ map on $\mathbf{T}^n \times \mathbf{C} \times \Lambda$, analytic in θ, ζ and μ_1. Also we can choose this extension as a diffeomorphism in μ and as a diffeomorphism in ζ on a neighborhood of $\zeta = 0$. We shall ensure that the extended Φ is close to the identity map in the C^∞-topology. In fact we shall prove the following. Let $\sigma_\gamma : \mathbf{T}^n \times \mathbf{C} \times \mathbf{C} \to \mathbf{T}^n \times \mathbf{C} \times \mathbf{C}$ be the stretching operator defined by $(\theta, z, \lambda) \to (\theta, z, \gamma\lambda)$. Then for all $j \geq 1$ one has that in the C^j-norm $\| - \|_j$ on $\sigma_\gamma^{-1}(\Omega)$

$$\| \sigma_\gamma^{-1} \circ (\Phi - id) \circ \sigma_\gamma \|_j \to 0 \tag{2.6}$$

as $| g |_\Omega \to 0$. We emphasize the fact that the extension of Φ is not unique and that for $\mu \notin \Lambda_\gamma^M$ it looses its conjugation property;

ii) Similarly the coefficients α_l, $1 \leq l \leq N$, can be extended as C^∞-functions on Λ, remaining analytic in μ_1, such that for all j in the C^j-norm on « $\sigma_\gamma^{-1}(\Lambda)$ », also to be denoted by $\| - \|_j$, we have that

$$\| \sigma_\gamma^{-1} \circ \alpha_l \circ \sigma_\gamma \|_j \to 0 \tag{2.7}$$

as $| g |_\Omega \to 0$. It will appear that since g is real valued also the α_l, $1 \leq l \leq N$, will be real valued;

iii) Consider the component function $\mu \to \lambda(\mu)$ of the map Φ, cf. (2.3). This component function maps the product $\Lambda_\gamma^M = \Lambda_{1,\gamma} \times \Lambda_{2,\gamma}^M$ diffeomorphically into $\Lambda_1 \times \Lambda_2$. Since Φ is C^1-close to the identity map, see (2.6), the measure of Λ_γ^M and that of its image under $\mu \to \lambda(\mu)$ have the same asymptotic behavior as $\gamma \to 0$;

iv) Unlike the linear case, cf. § 1 *e*, we cannot allow the parameter γ to vary independent of $| g |_\Omega$, since it enters into the smallness condition on $| g |_\Omega$.

A proof of the normal form theorem will be given in § 5, below.

We now conclude this section with a corollary to the normal form theorem, which is of convenience for our applications in the following.

COROLLARY. — *Let* Φ *be extended to* $\mathbf{T}^n \times \mathbf{C} \times \Lambda$ *as a* C^∞*-map, which is a local diffeomorphism in* ζ *near* 0 *in* $\mathbf{C}$ *and a diffeomorphism in* μ *on* Λ. *This extension conjugates (2.1) on* $\mathbf{T}^n \times \mathbf{C} \times \Lambda$ *to the following normal form which is analytic in* θ *and* ζ *and* μ_1 *and of class* C^∞ *in* μ_2:

$$\begin{cases} \dot\theta = \omega \\ \dot\zeta = \mu\zeta + \Sigma_{l=1}^N \alpha_l(\mu)\zeta \, | \zeta |^{2l} + r(\theta, \zeta, \mu) + p(\theta, \zeta, \mu), \end{cases} \tag{2.8}$$

where $r = O(| \zeta |^M)$ *and* p *is infinitely flat on* Λ_γ^M.

Proof. — Extension of Φ directly yields from (2.2) the C^∞-form

$$\begin{cases} \dot{\theta} = \omega \\ \dot{\zeta} = \mu\zeta + \Sigma_{l=1}^{N}\alpha_l(\mu)\zeta\,|\,\zeta\,|^{2l} + G(\theta, \zeta, \mu) \end{cases}$$

for (2.1), where the $\alpha_l(\mu)$, $1 \leq l \leq N$, also are Whitney-C^∞ extensions from (2.2). Next put $G = p + r$, where p is polynomial of degree $M - 1$ in ζ and $\bar{\zeta}$ and where $r = O(|\,\zeta\,|^M)$. By the normal form theorem p vanishes identically on Λ_γ^M.

Now $\Lambda_\gamma^M = \Lambda_{1,\gamma} \times \Lambda_{2,\gamma}^M$, where $\Lambda_{2,\gamma}^M$ is perfect, being a Cantor set. From this the flatness of p on Λ_γ^N immediately follows. As a typical example we consider $\partial^2 p/\partial\mu_1\partial\mu_2$. Clearly $\partial p/\partial\mu_1 \equiv 0$ on Λ_γ^M. It then follows that also $\partial^2 p/\partial\mu_1\partial\mu_2 \equiv 0$ on Λ_γ^M, since any point $\mu_1 + i\mu_2 \in \Lambda_\gamma^M$ is the limit of a sequence $\{ \mu_1 + i\mu_{2j} \}_{j=0}^{\infty}$ in $\Lambda_\gamma^M \backslash \{ \mu_1 + i\mu_2 \}$. etc. Q. E. D.

Remark. — In general it will be impossible to extend Φ analytically in the μ_2-direction, so in this sense our result is optimal, compare [Pö, 5*a*], who obtains similar regularity results in the Hamiltonian context. In fact an analytical continuation of Φ would give formula (2.8), where p vanishes identically on the whole of Λ, implying that Stoker n-tori would exist for all $\mu \in \Lambda$, so particularly in the resonance points $\mu = i(\omega, k)$, $k \in Z^n$. In view of § 1 *e* this generally would lead to a contradiction.

§ 3. STOKER'S PROBLEM AND THE QUASI-PERIODIC HOPF BIFURCATION

We consider a real analytic system

$$(3.1) \qquad \begin{cases} \dot{\theta} = \omega \\ \dot{z} = \lambda z + g(\theta, z, \lambda), \end{cases}$$

where θ varies over the n-torus $\mathbf{T}^n$ and where z and λ are complex variables, compare (1.2) and (2.1). This system, for $|\operatorname{Re}\lambda|$ small, will be explored for ω-quasi-periodic invariant n-tori (Stoker n-tori) and for quasi-periodic Hopf bifurcation where invariant $(n + 1)$-tori branch off. Here we mention valuable private communications we had with Gerard Iooss and with Floris Takens.

a. The Stoker problem.

*a*1. For completeness sake we present a brief exposition of he classical contraction lemma, which serves to find the normally hyperbolic Stoker n-tori, see [Ha2 (§ IV, 2)]. Also compare e. g. [Sto, Bo, Fr1].

3.1. CONTRACTION LEMMA. — *Consider a system*

$$(*)\qquad \begin{cases} \dot{\theta} = \omega \\ \dot{z} = \lambda z + g(\theta, z, \lambda), \end{cases}$$

where g is of class C^1. *Assume that* λ *varies over an open complex domain on which* $\operatorname{Re}\lambda \neq 0$ *and that a constant* $\alpha > 0$ *exists such that for all* $|z| \leq \alpha$ *and all* λ *under consideration one has*

$$|g| \leq \alpha |\operatorname{Re}\lambda| \quad \text{and} \quad \left|\frac{\partial g}{\partial z}\right| + \left|\frac{\partial g}{\partial \bar{z}}\right| < |\operatorname{Re}\lambda|.$$

Then $(*)$ *has a Stoker n-torus* $z = v(\theta, \lambda)$, *which is of class* C^1 *and unique in the region* $|z| \leq \alpha$.

Moreover, if g is real analytic in θ *and z, then v is real analytic in* θ *and of class* C^∞ *in* λ.

Indication of a Proof. — As in § 1 *e* for simplicity take $\operatorname{Re}\lambda < 0$. Again consider the Banach space $\mathscr{B}$ of continuous functions $\mathbf{T}^n \to \mathbf{C}$ with the supremum norm. Also consider the linear operator T on $\mathscr{B}$, cf. (1.10), which yields the unique Stoker *n*-torus in the linear case. Recall that $|T| = |\operatorname{Re}\lambda|^{-1}$. Then define

$$(Sv)(\theta, \lambda) := Tg(\theta, v(\theta, \lambda), \lambda)$$

and prove in a straight forward way that S is a contraction and so has a unique fixed point, which corresponds to the desired *n*-torus. The regularity of this torus follows from that of *g* by applying [CH (thm. 2.2)]. Q. E. D.

Remark. — If *g* is real analytic in λ as well and we extend into the complex with respect to λ and $\bar{\lambda}$, adapting the conditions of the lemma suitably, then the *n*-tori $z = v(\theta, \lambda)$ also depend analytically on λ. See e. g. [CH (thm. 2.2)].

Before applying the contraction lemma to our small damping problem we observe the following. A direct application of the lemma to system (3.1) solves Stoker's problem for all parameter values λ with $|\operatorname{Re}\lambda|$ large compared to $|g|$, so in the case of large damping. Compare § 1 *b*, also see the first example of the next section.

*a*2. In our present case of small damping we take as a starting point the normal form (2.8) with M = 2. We consider a fixed $\omega_0 \in \Delta^2_{2,\gamma}$ and let the parameter μ vary in a neighborhood of the horizontal line with equation $\mu_2 = \omega_0$. If we write $\mu = i\omega_0 + \nu$ then the form (2.8) may be rewritten as

$$(3.2)\qquad \begin{cases} \dot{\theta} = \omega \\ \dot{\zeta} = (i\omega_0 + \nu)\zeta + O(|\zeta|^2 + |\nu_2|^L). \end{cases}$$

Here $L \in \mathbf{N}$ is arbitrarily prefixed. Recall that (3.2) is of class C^∞ in ν_2

and analytic in all other variables. We shall apply the contraction lemma 3.1 to a system (*) where

$$\lambda = i\omega_0 + \nu, \quad \text{so} \quad \operatorname{Re}\lambda = \nu_1, \quad \text{and}$$
$$g = O(|\zeta|^2 + |\nu_2|^L).$$

Now we have *a priori* estimates

$$|g| \leq K(|\zeta|^2 + |\nu_2|^L),$$
$$\left|\frac{\partial g}{\partial \zeta}\right| + \left|\frac{\partial g}{\partial \bar{\zeta}}\right| \leq K(|\zeta| + |\nu_2|^L).$$

where $K = K(L)$ is a positive constant. If we start with a bounded parameter domain Λ, cf. § 2, and take K sufficiently large, then these *a priori* estimates are uniform in θ, ν_1 and ω_0. Note that the second inequality follows from the first by application of the Cauchy integral formula.

So the contraction lemma applies as soon as for some $\alpha > 0$ one has

$$K(\alpha^2 + |\nu_2|^L) \leq \alpha |\nu_1|,$$
$$K(\alpha + |\nu_2|^L) < |\nu_1|.$$

Next take $\alpha := \dfrac{1}{2K}|\nu_1|$, then it clearly is sufficient to have

$$K|\nu_2|^L \leq \frac{1}{4K}|\nu_1|^2 \qquad \text{and} \qquad K|\nu_2|^L < \frac{1}{2}|\nu_1|.$$

Therefore, if K is chosen sufficiently large, then the only remaining condition is that

$$|\nu_2| \leq c(L)|\nu_1|^{2/L}.$$

Here we abbreviated $c(L) := (4K^2(L))^{-1/L}$.

Summarizing the above, returning to the μ-variable, we define the $\dfrac{2}{L}$-cusp

$$\tilde{C}(\omega_0, L) := \{ \mu \in \Lambda \mid |\mu_2 - \omega_0| \leq c(L)|\mu_1|^{2/L} \}$$

and conclude that system (2.8) has a C^∞-family of real analytic Stoker n-tori, parametrized by the cusp $\tilde{C}(\omega_0, L)$. Finally let $C(\omega_0, L)$ be the image in the λ-plane of $\tilde{C}(\omega_0, L)$ under the λ-component function of the extended conjugacy Φ, cf. § 2, and define the Cantor bundle

$$C(L) := \cup_{\omega_0 \in \Lambda^2_{2,\gamma}} C(\omega_0, L).$$

We now have proved (cf. § 1 *c*, fig. 2.)

3.2. Theorem. — *Assume that in (3.1) the function g satisfies the conditions of the normal form theorem for* $M = 2$. *Then for all* $L \in \mathbb{N}$ *there exists a bundle* $C(L)$ *as above, such that for all* $\lambda \in C(L)$ *system (3.1) has a Stoker n-torus, which is normally hyperbolic except in the cusp points. These tori are real analytic and depend in a* C^∞*-manner on* λ.

Remarks. — *i*) From the above proof it follows that the Stoker n-torus is unique in the region $|\zeta| \leq \frac{1}{2K}|\nu_1|$;

ii) Consider the whole ω-quasi-periodic regime. Part of it consists of the Stoker domain, corresponding to the case of large damping, see above. The components of this Stoker domain apparently are connected by the union $C := \cup_{L \in N} C(L)$. Note that the order of contact with the λ-image of the imaginary axis is infinite;

iii) The above approach yields Stoker n-tori with C^∞-dependence on the parameters. First observe that as a direct consequence of our normal form theory we know that, restricted to the horizontal lines of Λ_γ^2, these n-tori depend analytically on μ_1.

This regularity could be further improved as follows. Take any of the horizontal lines $\mu_2 = \omega_0$ in Λ_γ^2. Instead of the global C^∞ extension Φ, cf. § 2, we consider the Taylor approximation of Φ to the order $L - 1$ in the local variable $\nu_2 = \mu_2 - \omega_0$. This leads to a real analytic system of the form (3.2). Now proceed as above, keeping in mind the remark to the contraction lemma 3.1. Thus one could establish analytic parameter dependence of the n-tori in a neighborhood of the punctured horizontal line $\mu_2 = \omega_0$, $\mu_1 \neq 0$. Taking the union with respect to $\omega_0 \in \Lambda_{2,\gamma}^2$ we so would cover a neighborhood of the Cantor line bundle Λ_γ^2. Note that since Φ-id is small in the C^∞-topology, cf. § 2, this neighborhood is not much smaller than the parameter domain covered by theorem 3.2. In fact at most finitely many « strips » will be missed, corresponding to the larger holes in the Cantor set $\Lambda_{2,\gamma}^2$.

b. The quasi-periodic Hopf bifurcation.

In conformity with the local theory of the Hopf bifurcation, see e. g. [MM], we here take our normal form for $M = 4$ and assume that the coefficient α_1 does not vanish on $\{0\} \times \Lambda_2 \subseteq \Lambda_1 \times \Lambda_2$. See § 2. For simplicity we shall take $\alpha_1(i\mu_2) < 0$. It will appear that in this case we have a quasi-periodic Hopf bifurcation of supercritical type: speaking in terms of the normal form (2.8), asymptotically stable invariant $(n + 1)$-tori emanate from the points $\omega_0 \in \{0\} \times \Lambda_{2,\gamma}^4$ for $\mu_1 > 0$.

To be more precise we have

3.3. THEOREM. — *Assume that in (3.1) the function g satisfies the conditions of the normal form theorem for* $M = 4$. *Also assume that* $\omega_0 \in \Lambda_{2,\gamma}^4$ *and that* $\alpha_1(i\omega_0) < 0$.

Then for all $L \in N$ *a* $\frac{2}{L}$*-cusp* $\tilde{C}(\omega_0, L)$ *exists, as before, such that for*

$\mu \in \tilde{C}(\omega_0, L)$ and $0 < \mu_1$ sufficiently small, there exists a normally hyperbolic (asymptotically stable) invariant $(n + 1)$-torus.

Moreover this 2-parameter family of $(n + 1)$-tori constitutes a codimension 1 submanifold M of $T^n \times C \times C$, which is of class C^r for any given $r \in N$, provided that one starts with $0 < \mu_1$ sufficiently small.

Proof. — Similar to (3.2) we now obtain as a « local » version of the normal form (2.8) for $M = 4$:

$$(3.4) \qquad \begin{cases} \dot{\theta} = \omega \\ \dot{\theta}_0 = \omega_0 + \nu_2 + \rho^{-1}O(\rho^4 + |\nu_2|^L) \\ \dot{\rho} = \nu_1\rho - \beta(\nu)\rho^3 + O(\rho^4 + |\nu_2|^L). \end{cases}$$

Here $\beta(\nu) := -\alpha_1(i\omega_0 + \nu)$. Clearly the normalized, symmetric part of system (3.4) has an invariant $(n + 1)$-torus given by the equation

$$\rho = \sqrt{\frac{\nu_1}{\beta(\nu)}},$$

provided that $\beta(\nu) > 0$.

Restricting $0 < \nu_1$ sufficiently to have this, we introduce the local variable y by

$$\rho := (1 + y)\sqrt{\frac{\nu_1}{\beta(\nu)}}.$$

Thus we obtain from (3.4) the form

$$(3.5) \qquad \begin{cases} \dot{\theta} = \omega \\ \dot{\theta}_0 = \omega_0 + \nu_2 + \Theta(\theta, \theta_0, y, \nu) \\ \dot{y} = -2\nu_1 y + Y(\theta, \theta_0, y, \nu), \end{cases}$$

where

$$\Theta(\theta, \theta_0, y, \nu) = \nu_1^{-1/2}O(\nu_1^2 + |\nu_2|^L) \quad \text{and}$$
$$Y(\theta, \theta_0, y, \nu) = O(\nu_1 y^2 + \nu_1^{3/2} + \nu_1^{-1/2}|\nu_2|^L).$$

So if we restrict to the cusp $\tilde{C}(\omega_0, L)$ defined by $|\nu_2| \leq c(L)\nu_1^{2/L}$, as before, we get

$$(3.6) \qquad \Theta(\theta, \theta_0, y, \nu) = O(\nu_1^{3/2}) \quad \text{and} \quad Y(\theta, \theta_0, y, \nu) = O(\nu_1 y^2 + \nu_1^{3/2}).$$

Our analysis of the form (3.5) with the perturbations (3.6) now follows [CH (§ 12.5)], which is a generalization of the above method used to find the Stoker n-tori. Also compare [CH (§§ 12.6, 12.7)].

We start with a brief digression on the « linear » system, compare § 1 *e*. In fact we consider

$$(3.7) \qquad \begin{cases} \dot{\theta} = \omega \\ \dot{\theta}_0 = \omega_0 + \nu_2 + \tilde{\Theta}(\theta, \theta_0, \nu) \\ \dot{y} = -2\nu_1 y + \tilde{Y}(\theta, \theta_0, \nu) \end{cases}$$

and look for invariant $(n + 1)$-tori of the form $y = v(\theta, \theta_0, \nu)$.

First we solve the first two equations of system (3.7), which are decoupled from the third. Let $\theta(t; \phi)$ and $\theta_0(t; \phi, \phi_0)$ denote the respective solutions with initial conditions respectively ϕ and ϕ_0 for $t = 0$. Note that $\theta(t; \phi) = \phi + t\omega$, see above.

Substituting these solutions in the third equation of (3.7) yields

$$\dot{y}(t) = -2v_1 y + \tilde{Y}(\theta(t; \phi), \theta_0(t; \phi, \phi_0), v)$$

with a general solution

$$y_{\phi,\phi_0,v}(t) = ce^{-2v_1 t} + \int_{-\infty}^{t} e^{-2v_1(t-s)}\tilde{Y}(\theta(s; \phi), \theta_0(s; \phi, \phi_0), v)ds .$$

Since we are looking for solutions on an $(n + 1)$-torus, which are bounded for all time, we have to take $c = 0$. Defining

$$v(\phi, \phi_0, v) := y_{\phi,\phi_0,v}(0)$$

we claim that $y = v(\phi, \phi_0, v)$ is an invariant $(n + 1)$-torus, the flow on which is given by $\theta(t; \phi)$ and $\theta_0(t; \phi, \phi_0)$. This invariance follows from the group property of the θ- and θ_0-flows.

Let us denote this invariant $(n+1)$-torus in the system (3.7) by $v = T(v, \tilde{\Theta}, \tilde{Y})$, then we have

$$(3.8) \quad T(v, \tilde{\Theta}, \tilde{Y})(\phi, \phi_0) = \int_{0}^{\infty} e^{-2v_1 s}\tilde{Y}(\theta(-s; \phi), \theta_0(-s; \phi, \phi_0), v)ds .$$

Observe that T is linear in $\tilde{Y}$ and that for $\tilde{\Theta} = 0$ it reduces to formula (1.10) with $n + 1$ frequencies.

This ends our digression.

Returning to the general non linear form (3.5) we are going to construct a contraction S similar to that from the proof of lemma 3.1.

To this end we first introduce some spaces. Let $C^r(\mathbf{T}^{n+1}, \mathbf{R})$ with the C^r-norm $|\,.\,|_r$ $(r \geq 0)$ be the Banach space of C^r-functions on $\mathbf{T}^{n+1}$. For $0 < \alpha < 1$ and $\varepsilon > 0$ define

$$U_\alpha := \{ v \in C^r(\mathbf{T}^{n+1}, \mathbf{R}) \mid |v|_r \leq \alpha \} \quad \text{and}$$
$$V_\varepsilon := \{ v \in \tilde{C}(\omega_0, L) \mid 0 < v_1 < \varepsilon \} .$$

Next for $v \in U_\alpha$, $v \in V_\varepsilon$ consider (3.7) with

$$\tilde{\Theta}(\theta, \theta_0, v) := \Theta(\theta, \theta_0, v(\theta, \theta_0), v)$$
$$\tilde{Y}(\theta, \theta_0, v) := Y(\theta, \theta_0, v(\theta, \theta_0)v) .$$

Then define

$$S(v, v) := T(v, \tilde{\Theta}, \tilde{Y}) .$$

First observe that for ε sufficiently small, this defines a map

$$S : V_\varepsilon \times U_\alpha \to C^r(T^{n+1}, \mathbf{R}) ,$$

compare [CH (§ 12.5)]. Here the upper bound on ε decreases with r.

To this map we want to apply the uniform contraction principle, cf. [CH (§ 2.2)]. For this purpose one has to verify that for ε sufficiently small S is of class C^r in both ν and v. Also one has to check that for ε and α sufficiently small

$$S : V_\varepsilon \times U_\alpha \to U_\alpha$$

is a uniform contraction on U_α. Provided all this one obtains as a conclusion that there is a unique fixed point $v = v(\theta, \theta_0, \nu)$ of S, which is of class C^r in θ, θ_0 and ν. Then $y = v(\theta, \theta_0, \nu)$ is the desired invariant $(n + 1)$-torus of our theorem.

In order to check the contraction property, for $v_1, v_2 \in U_\alpha$ introduce the following notation

θ_0^j being the solution θ_0 of (3.7) corresponding to

$$\tilde{\Theta}_j(\theta, \theta_0, \nu) := \Theta(\theta, \theta_0, v_j(\theta, \theta_0, \nu), \nu);$$
$$\tilde{Y}_j(\theta, \theta_0, \nu) := Y(\theta, \theta_0, v_j(\theta, \theta_0, \nu), \nu);$$
$$\tilde{\tilde{Y}}_j(t, \phi, \phi_0, \nu) := \tilde{Y}_j(\phi + t\omega, \theta_0^j(t; \phi, \phi_0), \nu).$$

We then have $S(\nu, v_1)(\phi, \phi_0) = \int_0^\infty e^{-2\nu_1 s}\tilde{\tilde{Y}}_1(-s, \phi, \phi_0, \nu)ds$ and

$$(S(\nu, v_1) - S(\nu, v_2))(\phi, \phi_0) = \int_0^\infty e^{-2\nu_1 s}\{\tilde{\tilde{Y}}_1(-s, \phi, \phi_0, \nu) - \tilde{\tilde{Y}}_2(-s, \phi, \phi_0, \nu)\}\, ds,$$

which we have to estimate in the C^r-norm.

First we consider the C^0-norms of $S(\nu, v_1)$ and $S(\nu, v_1) - S(\nu, v_2)$. It directly follows from (3.6) that $| S(\nu, v_1) |_0 = O(\alpha^2 + \nu_1^{1/2})$.

Next from Gronwall's inequality we obtain

$$(3.10) \qquad |(\theta_0^1 - \theta_0^2)(t; \phi, \phi_0)| = \frac{1}{\kappa}(e^{\kappa|t|} - 1)\,| v_1 - v_2 |\, O(\nu_1^{3/2}).$$

Here κ is the Lipschitz constant of the function Θ with respect to the variables θ_0 and y, so we can estimate $\kappa = O(\nu_1^{3/2})$. Here we use the Cauchy integral formula and the normal form theorem.

Similarly we have

$$(3.11) \qquad \begin{aligned} | D_{\theta_0}Y(\theta, \theta_0, y, \nu) | &= O(\nu_1 y^2 + \nu_1^{3/2}) \quad \text{and} \\ | D_y Y(\theta, \theta_0, y, \nu) | &= O(\nu_1 | y | + \nu_1^{3/2}). \end{aligned}$$

Now for t fixed we write

$$| \tilde{\tilde{Y}}_1 - \tilde{\tilde{Y}}_2 |_0 \leq (| D_{\theta_0} Y |_0 + | D_y Y |_0\, D_{\theta_0} v_1 |_0)\, | \theta_0^1 - \theta_0^2 |_0 + | D_y Y |_0\, | v_1 - v_2 |_0$$

and substituting (3.10) and (3.11) we conclude that since $|v_j|_r \leq \alpha < 1$ we have in the C^0-norm, for $\varepsilon > 0$ sufficiently small

$$|S(\nu, v_1) - S(\nu, v_2)|_0 = O(1)\,|v_1 - v_2|_0 \left\{ (\nu_1^{5/2}\alpha^2 + \nu_1^3) \int_0^\infty \frac{1}{\kappa} e^{-2\nu_1 s}(e^{\kappa s} - 1)\,ds + (\nu_1\alpha + \nu_1^{3/2}) \int_0^\infty e^{-2\nu_1 s}\,ds \right\} = O(\alpha + \nu_1^{1/2})\,|v_1 - v_2|_0 .$$

Similarly, after some calculations, for $\varepsilon > 0$ sufficiently small we obtain in the C^r-norm

$$|S(\nu, v_1)|_r = O(\alpha^2 + \nu_1^{1/2}) \quad \text{and}$$

$$|S(\nu, v_1) - S(\nu, v_2)|_r = O(\alpha + \nu_1^{1/2})\,|v_1 - v_2|_r,$$

which confirms the claim that S is a contraction in this norm for $\varepsilon > 0$ sufficiently small and suitable $0 < \alpha < 1$. Q. E. D.

Remarks. — *i*) The estimates in the above theorem by the Whitney smoothness are uniform with respect to ω_0. So we find constants $d = d(L, r)$ and $\alpha = \alpha(L, r)$, not depending on ω_0, such that in the parameter domain $\{ \mu \in \tilde{C}(\omega_0, L) \mid 0 < \mu_1 < d \}$ there exists an invariant $(n + 1)$-torus which is of class C^r, also depending C^r on μ, cf. § 1 *c*, fig. 2, which is unique in the region $|y| \leq \alpha$.

Also if a parameter point μ lies in two or more overlapping cusps, we still find a unique $(n + 1)$-torus in the region

$$\left| \sqrt{\frac{-\alpha_1(\mu)}{\mu_1}}\,|\zeta| - 1 \right| \leq \alpha .$$

This uniqueness result moreover can be extended to a full neighbourhood of $z = 0$ in $\mathbf{T}^n \times \mathbf{C} \times C(L)$.

In order to see this we take $0 < \nu_1$ sufficiently small and consider both the divergence $2\nu_1 - 4\beta(\nu)\rho^2 + 0(\rho^3 + |\nu_2|^L)$ of the system and its directional derivative of the Lyapunov function ρ^2, i. e.

$$2\rho(\nu_1\rho - \beta(\nu)\rho^3 + 0(\rho^4 + |\nu_2|^L)).$$

First one verifies that the divergence is positive for $\rho \leqq \frac{1}{2}\sqrt{\nu_1/\beta(\nu)}$ and negative for $(1 + \alpha)\sqrt{\nu_1/\beta(\nu)} \leqq \rho \leqq C$, where C is a suitable bound. We conclude that in these regions there can be no invariant $(n + 1)$-torus. It remains to show that there is no invariant $(n + 1)$-torus for

$$\frac{1}{2}\sqrt{\nu_1/\beta(\nu)} \leqq \rho \leqq (1 - \alpha)\sqrt{\nu_1/\beta(\nu)}.$$

Here the above directional derivative is easily shown to be positive, which gives the desired result.

ii) The theorems (3.2) and (3.3) also can be proven by the implicit function theorem, cf. [CH (§ 2.2, § 12.5, ff.)], also compare [Ca]. We indicate how to do this in the case of theorem (3.3), see above. It is convenient to rescale the parameters (ν_1, ν_2) to (η_1, η_2) by $\nu_1 := \eta_1^{2L}$, $\nu_2 := \eta_1^4\eta_2$ and also to work with the rescaled variable y. It is our aim to solve the equation

$$v - S(\eta_1, \eta_2, v) = 0$$

near $(\eta_1, \eta_2, v) = (0, 0, 0)$. From (3.5) it immediately follows that for $\eta_1 = \eta_2 = 0$ one has $v = 0$ as a solution, while from the estimate $| S(\nu, v_1) - S(\nu, v_2) |_r = O(\alpha + \nu^{1/2}) | v_1 - v_2 |_r$ (see above) one directly obtains that $D_v S(0, 0, 0) = 0$. Now an application of the implicit function theorem again gives the desired result;

iii) A similar analysis as in *a* and *b* holds for the countably many ω_0 satisfying (2.5*b*), but which are isolated from the Cantor sets $\Lambda^2_{2,\gamma}$, resp. $\Lambda^4_{2,\gamma}$. Then the normal form is less powerful and in general we obtain oblique cusps instead of vertical ones;

iv) The invariant $(n + 1)$-tori apparently are C^r-dependent on the parameters. Another way to see that the parameter dependence is at least as differentiable as the $(n + 1)$-torus itself, is the following. In terms of [HPS] our conditions imply r-hyperbolicity. Now if the invariant $(n + 1)$-torus is r-hyperbolic near some parameter point ν_0, then for ν near ν_0 the « vertical » system

$$\begin{cases} \dot\theta = \omega \\ \dot\theta_0 = \omega_0 + \nu_2 + \Theta(\theta, \theta_0, y, \nu) \\ \dot y = -2\nu_1 y + Y(\theta, \theta_0, y, \nu) \\ \dot\nu = 0 \end{cases}$$

(cf. (3.5)) has a 2-parameter family of invariant $(n + 1)$-tori, which as a submanifold of $\mathbf{T}^n \times \mathbf{C} \times \mathbf{C}$ clearly also is r-hyperbolic. Hence this family is of class C^r in ν.

c. On some $(n + 1)$-tori with quasi-periodic flow.

Again consider the system (3.5)

$$\begin{cases} \dot\theta = \omega \\ \dot\theta_0 = \omega_0 + \nu_2 + \Theta(\theta, \theta_0, y, \nu) \\ \dot y = -2\nu_1 y + Y(\theta, \theta_0, y, \nu) \end{cases}$$

for $\nu \in \tilde{C}(\omega_0, L)$, implying that by (3.6) we have

$$\Theta(\theta, \theta_0, y, \nu) = O(| \nu_1 |^{3/2}) \quad \text{and}$$
$$Y(\theta, \theta_0, y, \nu) = O(| \nu_1 y^2 | + | \nu_1 |^{3/2}).$$

Recall that $\omega_0 \in \Lambda^4_{2,\gamma}$ and that $v \in \tilde{C}(\omega_0, L)$ if and only if $|v_2| \leq c(L)|v_1|^{2/L}$. If we consider the normalized part

$$(3.13) \qquad \begin{cases} \dot{\theta} = \omega \\ \dot{\theta}_0 = \omega_0 + v_2 \\ \dot{y} = -2v_1 y \end{cases}$$

of (3.5), with its invariant $(n+1)$-tori given by the equation $y = 0$, then in part *b* of this section we proved persistence of these tori for small $0 < v_1$. For this family of tori we obtained the usual center manifold regularity: it becomes smoother if one starts out from sufficiently restricted $0 < v_1$. It is well known that analytic systems may have invariant manifolds that are only of finite differentiability, see e. g. [Str, Sij]. In the next section we shall give an example showing that even in our specific setting this can be the case. Although the Stoker n-tori are all analytic, cf. part *a* of this section, the regularity of the invariant $(n+1)$-tori in general cannot be improved further. In our example we shall take a small perturbation of a system (3.13) where $\omega_0 + v_2$ is rationally dependent on the fixed numbers $\omega_1, \omega_2, \ldots, \omega_n$.

Here we want to discuss the opposite case where $\omega_0 + v_2$ is badly commensurable with $\omega_1, \omega_2, \ldots, \omega_n$. Then, in the normal form (3.13), the invariant $(n+1)$-torus is quasi-periodic with $n+1$ independent frequencies. To be more precise we assume that for constants $\tau > n+1$ and $\gamma > 0$ and for all integer vectors $(h_0, h_1, \ldots, h_n) \in \mathbf{Z}^{n+1} \setminus \{0\}$ we have

$$(3.14) \qquad |(\omega_0 + v_2)h_0 + (\omega, h)| \geq \gamma(|h_0| + |h|)^{-\tau}.$$

Here, as in (2.5), we abbreviated $(\omega, h) = \sum_{j=1}^{n} \omega_j h_j$ and $|h| = \sum_{j=1}^{n} |h_j|$.

In the same spirit as in § 2 we now can use KAM-theory, particularly [Mo2, Ze, Pö], to prove persistence of these $(n+1)$-tori, at the same time showing that they are all analytic.

In fact condition (3.14) in the cusp $\tilde{C}(\omega_0, L)$ defines a horizontal Cantor line bundle Λ_γ, as in § 2. On each of these horizontal lines the extra frequency $\omega_0 + v_2$ is constant. For $\delta > 0$ let $\Lambda_{\gamma,\delta}$ denote the intersection of Λ_γ with the vertical strip $0 < v_1 < \delta$. With help of these notions we are able to announce.

3.4. THEOREM. — *For $0 < \delta$ sufficiently small there exists a map*

$$\Phi : \mathbf{T}^{n+1} \times \mathbf{R} \times \Lambda_{\gamma,\delta} \to \mathbf{T}^{n+1} \times \mathbf{R} \times \tilde{C}(\omega_0, L)$$

which is real analytic in θ, θ_0 and y and Whitney-C^∞ in v, and which, near $y = 0$, conjugates the normal form (3.13) to its perturbation (3.5).

This theorem can be proven along the same lines as the normal form theorem, cf. § 2, also compare [BB, Br2]. For more details we refer to forthcoming work of G. B. Huitema.

Remarks. — The map Φ can be chosen affine in the y-variable. Also Φ-id in the C^∞-topology is close to zero in terms of δ. This last remark means that the horizontal lines in $\Lambda_{\gamma,\delta}$ remain almost horizontal when mapped under the third component function of Φ. So in the system (3.5), or (3.1), the corresponding parameter lines are close to parameter lines given by the normal form theorem, cf. (2.5).

§ 4. SOME APPLICATIONS AND EXAMPLES

In this section we study three different items. The first is Stoker's problem for Duffing's equation $\ddot{x} + c\dot{x} + ax - bx^3 = f(t)$. The second subject consists of the quasi-periodic Hopf bifurcation for a specific class of forced oscillators, containing e. g. Duffing's and Van der Pol's equation. They are of the form $\ddot{x} + c\dot{x} + ax + h(x, \dot{x}, c, a) = f(t)$, where the function h contains only non linear terms in x and $\dot{x}$. Finally our third item consists of an example showing that the invariant $(n + 1)$-tori which branch off after a quasi-periodic Hopf bifurcation are in general not C^∞.

a. Stoker's problem for Duffing's equation.

The Duffing equation reads

$$\ddot{x} + c\dot{x} + ax = bx^3 + f(t). \tag{4.1}$$

where, as above, $a > 0$ and f is a real analytic, ω-quasi-periodic function. We shall consider Stoker's problem in the whole (c, a)-plane.

a1. *The large damping case.*

We cannot use our complex notation now, but nevertheless we have a straightforward analogue of Lemma 3.1, and so of [Sto], for this solution. To this end we first consider the linear equation $\ddot{x} + c\dot{x} + ax = f(t)$. For $c \neq 0$ we give the Stoker solution of this equation by a linear operator, defined on the Banach space $\mathscr{B}$ of continuous functions $\mathbf{T}^n \to \mathbf{C}$, endowed with the supremum norm, compare § 1 *e* and the proof of Lemma 3.1. Now, for $c > 0$ and $c^2 \neq 4$, this operator is given by

$$(\mathrm{TF})(\theta) := \int_0^\infty \frac{e^{\lambda_+ s} - e^{\lambda_- s}}{\lambda_+ - \lambda_-} \mathrm{F}(\theta - s\omega)ds.$$

where $\lambda_\pm = -\frac{c}{2} \pm \sqrt{\frac{c^2}{4} - a}$ are the roots of the characteristic equation $\lambda^2 + c\lambda + a = 0$ and where $f(t) = \mathrm{F}(t\omega)$. So the Stoker solution of the linear equation then is $x(t) = (\mathrm{TF})(t\omega)$.

Next we calculate the norm of T.

4.1. PROPOSITION. — *The norm of the operator* $T : \mathscr{B} \to \mathscr{B}$ *is given by*

$$|\mathrm{T}| = \begin{cases} \frac{1}{a} & \text{if} \quad c^2 \geq 4a, \\ \frac{1}{a} \operatorname{cotanh}\left(\frac{c\pi}{2\sqrt{4a - c^2}}\right) & \text{if} \quad c^2 \leq 4a. \end{cases}$$

Proof. — If $c^2 > 4a$, then $\lambda_\pm \in \mathbf{R}$ and we easily see that $|\mathrm{T}| = 1/a$. Now, if $c^2 < 4a$, then $\lambda_\pm = -\frac{c}{2} \pm i\sigma$, where $\sigma = \sqrt{a - \frac{c^2}{4}}$. Hence, if $c > 0$ and $c^2 < 4a$

$$(\mathrm{TF})(\theta) = \frac{1}{\sigma} \int_0^\infty e^{-\frac{c}{2}s} \sin(\sigma s) \mathrm{F}(\theta - s\omega) ds$$

and therefore

$$|\mathrm{T}| = \frac{1}{\sigma} \int_0^\infty e^{-\frac{c}{2}s} |\sin(\sigma s)| \, ds = \frac{1}{a} \operatorname{cotanh} \frac{c\pi}{4\sigma}.$$

The cases $c^2 = 4a$ and $c < 0$ can be treated analogously. Q. E. D.

Then we have

4.2. THEOREM. — *Define*

$$\text{(4.3)} \qquad \mathrm{R}^3 := \frac{27}{4} |b| |f|^2 \qquad \text{and} \qquad \rho(a) := \frac{16a\left(\operatorname{artanh}\frac{\mathrm{R}}{a}\right)^2}{\pi^2 + 4\left(\operatorname{artanh}\frac{\mathrm{R}}{a}\right)^2}.$$

Then, on the open parameter domain given by

$$\text{(4.4)} \qquad a < \mathrm{R} \qquad \text{and} \qquad c^2 > \rho(a),$$

system (4.1) has an analytic Stoker solution, depending analytically on c and a. This solution remains for all time in the region $|x| < \frac{3}{2} |\mathrm{T}| \, |f|$.

Remark. — The region (4.4) is what in § 1 was called the Stoker domain of equation (4.1). The function ρ is strictly decreasing on (R, ∞), with $\lim_{a \to \infty} \rho(a) = 0$ and $\lim_{a \downarrow \mathrm{R}} \rho(a) = -\infty$. Cf. fig. 3.

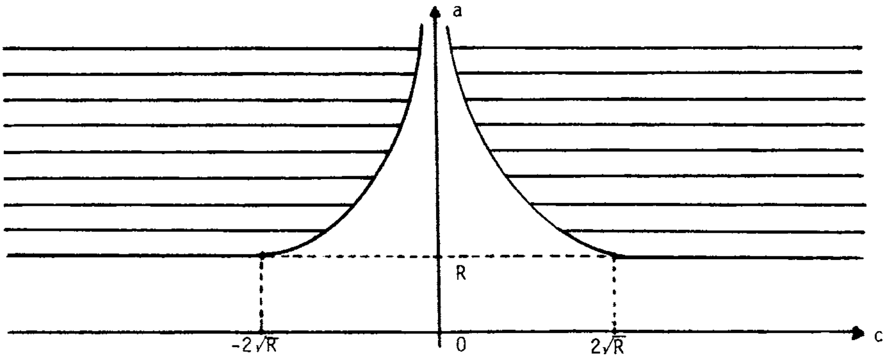


FIG. 3. — Stoker domain for Duffing's equation.

Proof. — Similar to lemma 3.1 we have for $\alpha > 0$ and $g(x, t) = bx^3 + f(t)$, that if

$$(4.5) \qquad |\mathrm{T}|\,|g| \leq \alpha \quad \text{and} \quad |\mathrm{T}\left|\frac{\partial g}{\partial x}\right| < 1,$$

then the equation (4.1) has a unique Stoker solution x with $|x| \leq \alpha$. We now indicate how (4.4) implies (4.5). For this purpose let $p := |T|\,|b|$ and $q := |T|\,|f|$. Then (4.5) is satisfied if

$$(4.6) \qquad 3p\alpha^2 < 1 \quad \text{and} \quad q \leq \alpha - p\alpha^3 .$$

Next, let $\alpha_0 := 1/\sqrt{3p}$. Then, if $q < \frac{2}{3}\alpha_0$, there exists $\alpha \in (0, \alpha_0)$ with $q = \alpha - p\alpha^3$, satisfying (4.6). This is due to the fact that

$$\max\{\alpha - p\alpha^3 \mid 0 \leq \alpha \leq \alpha_0\} = \alpha_0 - p\alpha_0^3 = \frac{2}{3}\alpha_0 .$$

Hence we find our solution for $q < \frac{2}{3}\alpha_0$, or equivalently $\mathrm{R}\,|\mathrm{T}| < 1$, see (4.3).

According to ptoposition (4.1), this amounts to

$$\text{(I)} \qquad a > \mathrm{R} \quad \text{if} \quad c^2 \geq 4a ,$$

$$\text{(II)} \qquad a \tanh\left(\frac{c^2}{2\sqrt{4a - c^2}}\right) > \mathrm{R} \quad \text{if} \quad c^2 \leq 4a .$$

Now (II) is equivalent to (4.4) and $c^2 \leq 4a$. Also (I) is equivalent to (4.4) and $c^2 \geq 4a$, since $\rho(a) \leq 4a$ (cf. (4.3)). Hence (I) and (II) are equivalent to (4.4).

Finally we have $|x| \leq \alpha = \dfrac{q}{1 - p\alpha^2} < \dfrac{q}{1 - p\alpha_0^2} = \dfrac{3}{2} q = \dfrac{3}{2}|\mathrm{T}|\,|f|$.

We conclude the unique existence of analytic Stoker solutions depending in a C^∞-manner on the parameters c and a. We can even conclude analytic dependence on c and a, using continuity of the norm $|T|$ in these parameters. Compare the remark to lemma 3.1. Q. E. D.

a2. *The small damping case.*

Now we use our complex form $\dot{\theta} = \omega$, $\dot{z} = \lambda z + g(\theta, z, \lambda)$, with

$$g(\theta, z, \lambda) = F(\theta) + b\left(\frac{\operatorname{Im} z}{\operatorname{Im} \lambda}\right)^3,$$

compare § 1. The condition of the normal form theorem, see § 2, is that $|g|_\Omega < \gamma\delta$, so necessarily we require that $|f|$ be sufficiently small. In that case the remaining condition is of the form $|\operatorname{Im} z| \leq cst.\, |\operatorname{Im} \lambda|$. From this it follows that we may take for the parameter domain $\Lambda = \Lambda_1 \times \Lambda_2$ in § 2 : $\Lambda_1 = \mathbf{R}$ and $\Lambda_2 = (\lambda_2^*, \infty)$, for some $\lambda_2^* > 0$. Cf. fig. 4.

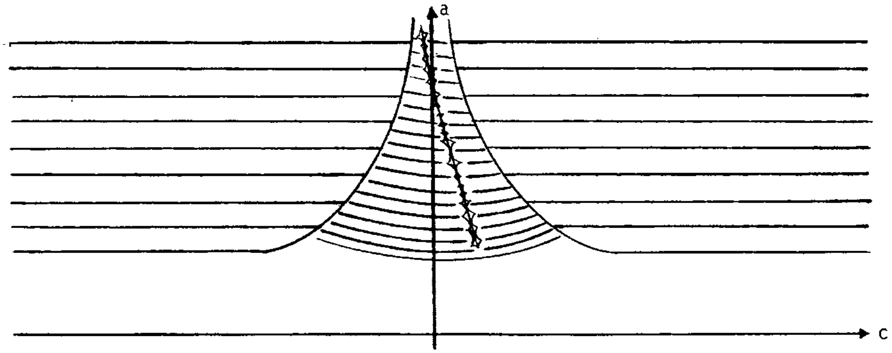


FIG. 4. — ω-quasi-periodic parameter region for Duffing's equation.

b. On free and forced oscillators.

Unlike the usual normal form theory near singular points, see e. g. [Poi, Ta, Br1, VdM], the coefficients α_l $(1 \leq l \leq N)$ in the normal form (2.2) are not determined by a finite algorithm, but in an infinite process. Compare § 5 below.

Here we shall present a simple special case where this difficulty is not felt, namely we consider analytic systems like (1.1) and (3.1) of the form

$$(4.7) \qquad \begin{cases} \dot{\theta} = \omega \\ \dot{z} = \lambda z + H(z, \lambda) + F(\theta), \end{cases}$$

where $H(z, \lambda) = O(|z|^2)$. So here $g(\theta, z, \lambda) = H(z, \lambda) + F(\theta)$. Observe that (4.7) corresponds to a forced equation of type

$$\ddot{x} + c\dot{x} + ax + h(x, \dot{x}, c, a) = f(t). \tag{4.8}$$

Recall that the forcing term $f(t) = F(t\omega + \theta_0)$ is quasi-periodic in t with the frequencies $\omega_1, \omega_2, \ldots, \omega_n$, see § 1. In (4.8) the term h contains the non linearities. Note that important examples like the Duffing and the Van der Pol equation are of this type.

The corresponding free oscillator is

$$\begin{aligned} &\ddot{x} + c\dot{x} + ax + h(x, \dot{x}, c, a) = 0 \text{ or, equivalently,} \\ &\dot{z} = \lambda z + H(z, \lambda) \end{aligned} \tag{4.9}$$

and the usual normal form theory directly decides whether for Re $\lambda = 0$ one has an ordinary Hopf bifurcation or not, compare e. g. [MM].

We now have (for the terminology see the beginning of § 3 *b*, above).

4.3. THEOREM. — *Assume that the free oscillator (4.9) has a Hopf bifurcation for* Re $\lambda = 0$. *Then the forced oscillator (4.7) has a quasi-periodic Hopf bifurcation for* Re λ *near 0, provided that* $|F|$ *is sufficiently small on the considered domain. The criticality type of both bifurcations is the same.*

Sketch of a Proof. — Our argument is straightforward. The normal form from § 2, for small $|F|$, is close to the decomposed system

$$\begin{cases} \dot{\theta} = \omega \\ \dot{\zeta} = \lambda\zeta + \alpha(\lambda)\zeta|\zeta|^2 + O(|\zeta|^4), \end{cases}$$

where the second line is « the » ordinary normal form of the free oscillator (4.9). We may assume that the coefficient $\alpha(\lambda)$ is real. Now the sign of this coefficient is preserved under sufficiently small perturbation. Q.E.D.

EXAMPLES. — *i*) The Van der Pol equation $\ddot{x} + (c + x^2)\dot{x} + ax = f(t)$. It is well known that the free oscillator has a supercritical Hopf bifurcation in the parameter $\lambda_1 = -\frac{c}{2}$. So for a quasi-periodic forcing term f, with $|f|$ sufficiently small, we also have a quasi-periodic Hopf bifurcation, again supercritical, in $\lambda_1 = -\frac{c}{2}$.

ii) The Duffing equation $\ddot{x} + c\dot{x} + ax - bx^3 = f(t)$. Now the free oscillator for $c = 0$ near $x = \dot{x} = 0$ is completely integrable (Hamiltonian), therefore all normal form coefficients vanish and our method does not apply. Nevertheless we do expect a family of invariant $(n + 1)$-tori although it probably does not collapse quadratically, but perhaps has an infinite order of contact with the (θ, z, λ_2)-directions. Also from the viewpoint of KAM-theory, compare § 3 *b*2, one might expect some persistence of quasi-periodic $(n + 1)$-tori.

c. The invariant $(n + 1)$-tori in general are not C^∞.

In order to illustrate this point consider the following parametrized system

$$(4.10) \qquad \begin{cases} \dot\theta_0 = \omega_0(\mu) \\ \dot\theta \ = \omega \\ \dot r \ = r(\mu_1 - r^2). \end{cases}$$

Here, as before, $\mu = \mu_1 + i\mu_2$ is a complex parameter. Moreover r and θ_0 are polar coordinates in the z-plane. Note that (4.10), for $\mu_1 > 0$, has an invariant $(n + 1)$-torus with equations $r_1 = \sqrt{\mu_1}$. The restriction of (4.10) to this $(n + 1)$-torus is of the form (3.4), see above. As is said before we shall consider the case where $\omega_0(\mu)$ is rationally dependent on the fixed components $\omega_1, \omega_2, \ldots, \omega_n$ of the vector ω. Below we shall give a suitable analytic perturbation of (4.10) and follow the invariant $(n + 1)$-torus during this perturbation, showing that it is only of a finite differentiability. Our example is similar to [Str, Sij].

First we construct some more convenient angular coordinates. By the rational dependence of ω_0 on $\omega_1, \omega_2, \ldots, \omega_n$ we have $\omega_0 = \sum_{j=1}^{n} a_j\omega_j$ for rational numbers $a_1, a_2, \ldots, a_n$.

Let k be the least common multiple of the denominators of $a_1, a_2, \ldots, a_n$. If we now define

$$\phi := k\left(\theta_0 - \sum_{j=1}^{n} a_j\theta_j\right),$$

then the map $(\theta_0, \theta) \to (\phi, \theta)$ is an endomorphism of $\mathbf{T}^{n+1}$, in fact it is a k-fold covering map and in particular a local diffeomorphism.

In the variables $\phi, \theta_1, \theta_2, \ldots, \theta_n$ and r the system (4.10) takes the form

$$(4.11) \qquad \begin{cases} \dot\phi = 0 \\ \dot\theta = \omega \\ \dot r = r(\mu_1 - r^2). \end{cases}$$

Now we perturb to

$$(4.12) \qquad \begin{cases} \dot\phi = \varepsilon \cos \phi \\ \dot\theta = \omega \\ \dot r = r(\mu_1 - r^2) + \varepsilon\delta r \cos \phi, \end{cases}$$

where $\varepsilon > 0$ and δ are real parameters, varying near 0. Note that by returning to the θ_0-variable (4.12) transforms to a small perturbation of (4.10), which has the right form (1.2), (2.1) or (3.1). Moreover an invariant $(n + 1)$-torus $r = r(\phi, \theta)$ of (4.12) corresponds to an invariant $(n + 1)$-torus of this perturbation of (4.10), which covers the first torus k times. Hence it is sufficient to study invariant tori of (4.12).

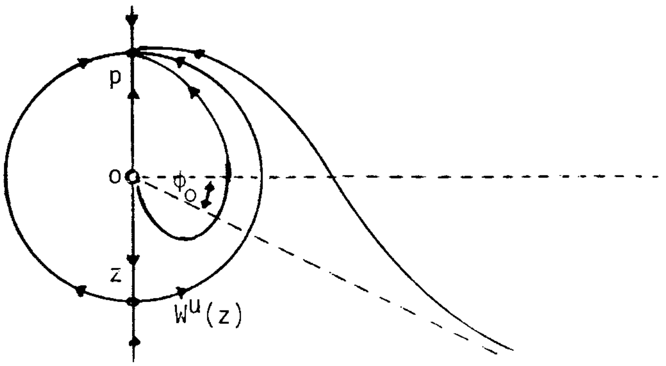


FIG. 5. — A reduction of system (4.12) to the (r, ϕ)-plane.

In fig. 5 we depicted a phase portrait of the reduction of (4.12) to the (r, ϕ)-plane. Observe that for $(r, \phi) = \left(\mu_1, \pm \frac{\pi}{2}\right)$ this reduction has an equilibrium point: a saddle z for $\phi = -\frac{\pi}{2}$, and a sink p for $\phi = \frac{\pi}{2}$. Also observe that all solutions for $t \to \infty$ tend to the sink p, except for the ones that start on the half line $\phi = -\frac{\pi}{2}$. The eigenvalues in p are $-2\mu_1$ and $-\varepsilon$ and we let $\beta := \frac{2\mu_1}{\varepsilon}$ denote their ratio.

The equilibria z and p correspond to invariant n-tori in (4.12), while all integral curves correspond to invariant $(n + 1)$-manifolds. The unstable manifold $W^u(z)$ of the saddle corresponds to our invariant $(n + 1)$-torus. We have

4.4. THEOREM. — *For $m = 2, 3, 4, \ldots$ and $m - 1 < \beta \leqq m$ and for δ in a sufficiently small punctured neighborhood of 0, the invariant $(n + 1)$-torus in (4.12) that corresponds to the unstable manifold $W^u(z)$ is of class C^{m-1}, but not of class C^m.*

Remarks. — *i*) System (4.12) can be solved in a straightforward manner, see the proof below. It appears that for $0 < \mu_1$ small, for the non vertical invariant manifolds through p we have the following possibilities, cf. fig. 5.

For $\beta = m \in \mathrm{N}$ all these non vertical invariant manifolds have a logarithmic singularity in p, making them of class C^{m-1}, but not of class C^m.

For $\beta \notin \mathrm{N}$ and $m-1=[\beta]$ all but one of these manifolds are of class C^{m-1} and not of class C^m, this remaining one however is analytic. So in this case the problem is to show that this analytic manifold does not coincide with our $(n+1)$-torus, i. e. with the unstable manifold $W^u(z)$;

ii) The « loss of differentiability » in the sink p is maximal according to [HPS]: the vector field (4.12) on the n-torus corresponding to p is $(m-1)$-hyperbolic;

iii) [Str, Sij] study the differentiability of a center manifold in a planar system with a parameter. [Str] considers a polynomial vector field corresponding to our case $\beta \in \mathbf{N}$, see above. It is shown that there is a maximal « loss of differentiability » in the sense of [HPS]. For a quadratic system [Sij] obtains results corresponding to our cases $\beta \in \mathbf{N}$ and $\beta \notin \mathbf{N}$,

iv) A different approach in the non resonant case where $\beta \notin \mathbf{Q}$, uses [Ster] to obtain a C^∞-linearization of the reduced vector field near the sink p. It now is easy to produce a C^∞-example with a non-C^∞ $(n+1)$-torus.

Proof of Theorem 4.4. — In our system reduced to the (r, φ)-plane we consider solutions of the form $r = r(\phi)$. In order to study these we eliminate time so obtaining

$$\frac{dr}{d\phi} = \frac{r(\mu_1 - r^2)}{\varepsilon \cos \phi} + \delta r . \tag{4.13}$$

This equation is of Bernoulli-type. By substituting $y = 1/r^2$ we get the linear equation

$$\frac{dy}{d\phi} = -2\left(\frac{\mu_1}{\varepsilon \cos \phi} + \delta\right)y + \frac{2}{\varepsilon \cos \phi},$$

compare [Sij], § II.1.

The general solution of the last equation is

$$y(\phi) = \frac{2}{\varepsilon}\left(\tan\left(\frac{\pi}{4} - \frac{\phi}{2}\right)\right)^{\beta}\left\{c + \int_{\phi_0}^{\phi}\left(\tan\left(\frac{\pi}{4} - \frac{\psi}{2}\right)\right)^{-\beta} e^{2\delta(\psi-\phi)}\frac{d\psi}{\cos\psi}\right\},$$

where c is an arbitrary constant and ϕ_0 may be chosen suitably, compare fig. 5. If we take $c = 0$ and $\phi_0 = -\frac{\pi}{2}$ this solution exactly is the unstable manifold $W^u(z)$. We shall analyze this solution near $\phi = \frac{\pi}{2}$, where it passes through the sink p.

To this end substitute $t := \tan\left(\frac{\pi}{4} - \frac{\phi}{2}\right)$, $\tau := \tan\left(\frac{\pi}{4} - \frac{\psi}{2}\right)$ and write $y = y(t)$ for our solution. Integration by parts yields

$$y(t) = \frac{1}{\mu_1} - \frac{2\delta}{\mu_1} e^{4\delta \arctan t} \mathrm{I}(t, \delta, \beta), \tag{4.14}$$

where $$\mathrm{I}(t, \delta, \beta) = t^\beta \int_t^\infty \tau^{-\beta} e^{-4\delta \arctan \tau} \frac{d\tau}{1 + \tau^2}.$$

So we have to determine the behavior of I near $t = 0$. In fact we shall show that

$$\mathrm{I}(t, \delta, \beta) = \begin{cases} c_1(\delta, m) t^m \log t + f_1(t, \delta, m), & \text{for } \beta = m, m = 2, 3, \ldots \\ c_2(\delta, \beta) t^\beta + f_2(t, \delta, \beta), & \text{for } \beta \notin \mathbf{N}, \end{cases} \tag{4.15}$$

where f_1 and f_2 are analytic functions and where $c_1(\delta, m)$ and $c_2(\delta, \beta)$ do not have zeros in a punctured neighborhood of $\delta = 0$. Compare remark *i)* to theorem 4.4. Clearly it is sufficient to prove this last statement.

In order to do this first observe that I diverges at $t = 0$ for $\beta \geq 1$. Therefore we consider the case $0 < \beta < 1$ and then use the fact that I is analytic in β.

If $0 < \operatorname{Re} \beta < 1$ we have $\mathrm{I}(t, \delta, \beta) = \mathrm{I}_1(t, \delta, \beta) + \mathrm{I}_2(t, \delta, \beta)$ where

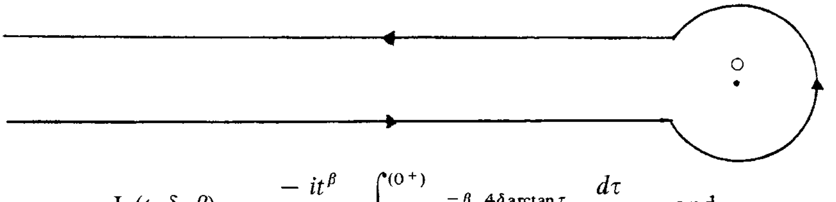
$$\mathrm{I}_1(t, \delta, \beta) = t^\beta \int_0^\infty \tau^{-\beta} e^{-4\delta \arctan \tau} \frac{d\tau}{1 + \tau^2} \quad \text{and}$$
$$\mathrm{I}_2(t, \delta, \beta) = -t^\beta \int_0^t \tau^{-\beta} e^{-4\delta \arctan \tau} \frac{d\tau}{1 + \tau^2}. \tag{4.16}$$

Now I_2 is analytic in t for $|t| < 1$ and

$$\mathrm{I}_1(t, \delta, \beta) = c_2(\delta, \beta) t^\beta \quad \text{with} \quad c_2(0, \beta) = \left\{ 2 \cos\left(\frac{1}{2}\beta\pi\right) \right\}^{-1}. \tag{4.17}$$

Note that it follows that if $0 < \beta < 1$ then I is of class C^0, but not C^1.

In order to get the analytic continuation of I_1 and I_2 we replace the integrals in (4.16) by contour integrals which avoid the point 0:



$$\mathrm{I}_1(t, \delta, \beta) = \frac{-it^\beta}{2 \sin(\pi\beta)} \int_{-\infty}^{(0+)} \tau^{-\beta} e^{4\delta \arctan \tau} \frac{d\tau}{1 + \tau^2} \quad \text{and}$$
$$\mathrm{I}_2(t, \delta, \beta) = \frac{it}{2 \sin(\pi\beta)} \int_{-1}^{(0+)} s^{-\beta} e^{4\delta \arctan(st)} \frac{ds}{1 + s^2 t^2}. \tag{4.18}$$

By analytic continuation the formulae (4.18) hold for all $\beta \in \mathbf{C} \backslash \mathbf{Z}$.

Now if $\beta > 0$ and $\beta \notin \mathbf{N}$ then (4.17) remains valid and, recalling that I_2 is analytic for $|t| < 1$, we conclude that the second line of (4.15) has been proven.

Finally we consider the other case of (4.15), where $\beta = m \in \mathbf{N}$. Since $I = I_1 + I_2$ is analytic in β we obtain by differentiating $(I_1 + I_2) \sin \pi\beta$ with respect to β

$$I(t, \delta, m) = \frac{(-1)^m}{2\pi i} \int_{-\infty}^{(0+)} \left(\frac{t}{\tau}\right)^m \log\left(\frac{t}{\tau}\right) c^{4\delta \arctan \tau} \frac{d\tau}{1+\tau^2} + \\ + \frac{(-1)^m}{2\pi i} \int_{-1}^{(0+)} s^{-m} \log (s) e^{4\delta \arctan (st)} \frac{ds}{1 + s^2 t^2}.$$

i. e. (4.15) with

$$c_1(\delta, m) = \frac{(-1)^m}{2\pi i} \int_{(0+)} \tau^{-m} e^{4\delta \arctan \tau} \frac{d\tau}{1 + \tau^2}.$$

By the Cauchy integral formula we see that $(-1)^m c_1(\delta, m)$ is the $(m-1)^{th}$ Taylor coefficient of $(1 + \tau^2)^{-1} e^{4\delta \arctan \tau}$ in $\tau = 0$. Now if $m - 1$ is even it follows that $|c_1(0, m)| = 1$, and if on the other hand m is even that

$$\frac{d^{m-1}}{d\delta^{m-1}} c_1(0, m) = 4^{m-1}.$$

Hence $c_1(\delta, m)$ has no zeros in a punctured neighborhood of $\delta = 0$. This finishes the proof. Q. E. D.

§ 5. PROOF OF THE NORMAL FORM THEOREM

a. Frame of the proof.

a1. *We start presenting a plan.*

The map $\Phi : \mathbf{T}^n \times \mathbf{C} \times \Lambda_\gamma^M \to \mathbf{T}^n \times \mathbf{C} \times \Lambda$ of the form (2.3), conjugating (2.1) and (2.2), will be found iteratively as the limit of a sequence $\Phi_0, \Phi_1, \Phi_2, \ldots$ of maps, all of the same form (2.3). These maps Φ_j are analytic diffeomorphisms, approximating Φ as a Whitney-differentiable map, see the appendix.

Let us consider the j^{th} step of the iteration. Using vector field notation, we write X for the system (2.1) and define $X_j := \Phi_j^* X$. We take $\Phi_0 = id$, so $X_0 = X$. Now if

$$\Phi_j(\theta, z_j, \lambda_j) = (\theta, z(\theta, z_j, \lambda_j), \lambda(\lambda_j)), \tag{5.1}$$

then in the (θ, z_j, λ_j) variables X_j shall have the system form

$$(5.2)\qquad \begin{cases} \dot\theta = \omega \\ \dot z_j = \lambda_j z_j + \Sigma_{l=1}^{N} \alpha_l^j(\lambda_j) z_j \,|\, z_j \,|^{2l} + g^j(\theta, z_j, \lambda_j)\,. \end{cases}$$

Note that $\alpha_l^0 \equiv 0$ for $1 \leq l \leq N$, $g^0 = g$.

Both Φ_j and X_j are defined on domains, extended into the complex, where all component-functions are complex analytic. Note that we suppressed, as far as possible, all conjugate variables and equations.

The idea of the proof is the following. If in (5.1) we write

$$z(\theta, z_j, \lambda_j) = z_j + V^j(\theta, z_j, \lambda_j) \quad \text{with}$$

$$(5.3)\qquad V^j(\theta, z_j, \lambda_j) = \sum_{k+l=0}^{M-1} V_{kl}^j(\theta, \lambda_j) z_j^k \bar z_j^l \quad \text{and}$$

$$\lambda(\lambda_j) = \lambda_j + U^j(\lambda_j)\,,$$

then we apply the Whitney approximation technique to the sequences

$$(5.4)\qquad \{\, \alpha_l^j \,\}_{j=0}^{\infty}, \qquad \{\, V_{kl}^j \,\}_{j=0}^{\infty} \quad \text{and} \quad \{\, U^j \,\}_{j=0}^{\infty}$$

of complex functions, defined on a corresponding sequence $\{\, W_j \,\}_{j=0}^{\infty}$ of complexified neighborhoods of $T^n \times \Lambda_\gamma^M$. Here it is important that W_j in the λ_{j2}-direction, i. e. the Cantor-direction, shrinks in a geometric way. We shall ensure that the expression

$$|\, \alpha_l^j - \alpha_l^{j+l} \,|_{W_{j+1}}, \qquad |\, V_{kl}^j - V_{kl}^{j+1} \,|_{W_{j+1}} \quad \text{and} \quad |\, U^j - U^{j+1} \,|_{W_{j+1}}$$

exhibit exponential decay as $j \to \infty$ and therefore on the long run are dominated by any geometric sequence. By the inverse approximation lemma, see the appendix, we then conclude that all the sequences (5.4) have Whitney-C^∞ limits. At the same time we consider a domain W_j^e, $0 \leq j \leq \infty$, which is the product of W_j and a neighborhood O_j of 0 in $\mathbf{C}^2$, the $(z_j, \bar z_j)$-direction, on which g^j is defined as a complex analytic function. We shall have that $W_0^e \subseteq \Omega$ and that the $(z_j, \bar z_j)$-component O_j of W_j^e shrinks to zero as $j \to \infty$. Also we can ensure that the « error » $|\, g^j \,|_{W_j^e}$, for $j \to \infty$, is reduced in such a way that in the limit only a term remains which is of order M in z_∞ and $\bar z_\infty$.

a2. *Relation between* X_j *and* X_{j+1}.

We first examine the relationship between X_j and X_{j+1} in any iteration process as described above. So let $\Phi_{j+1} = \Phi_j \circ \Psi_j$ for $\Psi_j : W_{j+1}^e \to W_j^e$ with the property that $\Psi_j^* X_j = X_{j+1}$. For simplicity we write (z, λ) instead of (z_j, λ_j) and (ζ, μ) instead of (z_{j+1}, λ_{j+1}). Also we replace g^j by g, g^{j+1} by g^+, α_l^j by α_l, α_l^{j+1} by α_l^+, $W_j^{(e)}$ by $W^{(e)}$, $W_{j+1}^{(e)}$ by $W_+^{(e)}$, Ψ_j by Ψ, etc.

Now $\Psi : W^e_+ \to W^e$ shall have the form

$$\Psi : (\theta, \zeta, \mu) \to (\theta, z, \lambda) = (\theta, \zeta + v(\theta, \zeta, \mu), \mu + u(\mu)) \quad \text{with}$$

$$v(\theta, \zeta, \mu) = \sum_{k+l=0}^{M-1} v_{kl}(\theta, \mu)\zeta^k \bar{\zeta}^l \tag{5.5}$$

and the fact that $\Psi_j^* X_j = X_{j+1}$ translates to

$$\begin{aligned} \text{(5.6)} \quad & D_\theta v(\omega) - (\mu + u)v - u\zeta + \mu \frac{\partial v}{\partial \zeta}\zeta + \bar{\mu}\frac{\partial v}{\partial \bar{\zeta}}\bar{\zeta} = \\ & = g(\theta, \zeta + v, \mu + u) - g^+ - \frac{\partial v}{\partial \zeta} g^+ - \frac{\partial v}{\partial \bar{\zeta}} \overline{g^+} + \\ & + \sum_{l=1}^{N} \{ \alpha_l(\mu + u)(\zeta + v) \,|\, \zeta + v \,|^{2l} - \alpha_l^+(\mu)\zeta \,|\, \zeta \,|^{2l} \} + \\ & - \frac{\partial v}{\partial \zeta} \sum_{l=1}^{N} \alpha_l^+(\mu)\zeta \,|\, \zeta \,|^{2l} - \frac{\partial v}{\partial \bar{\zeta}} \sum_{l=1}^{N} \overline{\alpha_l^+(\mu)\zeta} \,|\, \zeta \,|^{2l}, \end{aligned}$$

where everything is expressed in the (θ, ζ, μ)-variables.

a3. *We now determine* Ψ *and* $\alpha_l - \alpha_l^+$.

In order to get a well-defined iteration process we are going to define Ψ as in (5.5). To this purpose we consider (5.6) and recall that it is one of our aims to have g^+ small compared to g. Therefore we determine the functions u and v in (5.5) from an equation that does not differ much from (5.6): in (5.6) we neglect uv, $\frac{\partial v}{\partial \zeta}\alpha_l^+$, $\frac{\partial v}{\partial \bar{\zeta}}\overline{\alpha_l^+}$ *and* g^+ and also we linearize all expressions by replacing the arguments $\zeta + v$ and $\mu + u$ by ζ and μ respectively. Finally we truncate g in the $(\zeta, \bar{\zeta})$-variables at the order M. So if

$$^M g(\theta, \zeta, \mu) = \sum_{k+l=0}^{M-1} g_{kl}(\theta, \mu)\zeta^k \bar{\zeta}^l \quad \text{with}$$

$$g_{kl}(\theta, \mu) = \frac{1}{k!\,l!} \frac{\partial^{k+l}}{\partial \zeta^k \partial \bar{\zeta}^l} g(\theta, 0, \mu),$$

then we consider the equation

$$D_\theta v(\omega) - \mu v - u\zeta + \mu \frac{\partial v}{\partial \zeta}\zeta + \bar{\mu}\frac{\partial v}{\partial \bar{\zeta}}\bar{\zeta} = {}^M g + \sum_{l=1}^{N} (\alpha_l - \alpha_l^+)\zeta \,|\, \zeta \,|^{2l}$$

in u and v.

Comparing coefficients of $\zeta^k\bar{\zeta}^l, 0 \leq k + l \leq M - 1$, equivalently yields *for* $k = l + 1$

$$\begin{aligned} & D_\theta v_{10}(\omega) = g_{10} + u && (l = 0) \\ (5.7) \qquad & D_\theta v_{kl}(\omega) + l(\mu + \bar{\mu})v_{kl} = g_{kl} + (\alpha_l - \alpha_l^+) - && (l > 0) \end{aligned}$$

and for $k \neq l + 1$:

$$D_\theta v_{kl}(\omega) + \{ (k - 1)\mu + l\bar{\mu} \} v_{kl} = g_{kl}.$$

In fact we shall solve the equations (5.7) only approximately by truncating the Fourier series which form the exact formal solution of (5.7). Compare [Ar1]. To be more precise we expand the g_{kl} as a Fourier series

$$g_{kl}(\theta, \mu) = \sum_{h \in Z^n} g_{kl,h}(\mu) e^{i(\theta,h)}$$

and for $m = m_j \in \mathbf{N}$, to be chosen appropriately later on, we consider the truncation

$$_m g_{kl}(\theta, \mu) = \sum_{|h| \leq m} g_{kl,h}(\mu) e^{i(\theta,h)}$$

and we replace (5.7) by

$k = l + 1$:

$$\begin{aligned} & D_\theta v_{10}(\omega) = {}_m g_{10} + u && (l = 0) \\ (5.8) \qquad & D_\theta v_{kl}(\omega) + l(\mu + \bar{\mu})v_{kl} = {}_m g_{kl} + (\alpha_l - \alpha_l^+) && (l > 0) \end{aligned}$$

$k \neq l + 1$:

$$D_\theta v_{kl}(\omega) + \{ (k - 1)\mu + l\bar{\mu} \} v_{kl} = {}_m g_{kl}.$$

First observe that u and $\alpha_l - \alpha_l^+$ are determined by an integrability condition: since in (5.8) the averages on both sides have to be equal, we take

$$\begin{aligned} (5.9) \qquad & u(\mu) = - [g_{10}(\cdot, \mu)] \quad \text{and} \\ & \alpha_l(\mu) - \alpha_l^+(\mu) = - [g_{l+1l}(\cdot, \mu)] \end{aligned}$$

(note that μ may be purely imaginary, in which case $\mu + \bar{\mu} = 0$). Now (5.8) has as a solution for the v_{kl}, $0 \leq k + l \leq M - 1$, the trigonometric polynomials:

$k = l + 1$:

$$\begin{aligned} & v_{10}(\theta, \mu) = v_{10,0}(\mu) + \sum_{0 < |h| \leq m} \frac{g_{10,h}(\mu)}{i(\omega, h)} e^{i(\theta,h)} && (l = 0) \\ (5.10) \qquad & v_{kl}(\theta, \mu) = \sum_{0 < |h| \leq m} \frac{g_{kl,h}(\mu)}{i(\omega, h) + l(\mu + \bar{\mu})} e^{i(\theta,h)} && (l > 0) \end{aligned}$$

$k \neq l + 1$:

$$v_{kl}(\theta, \mu) = \sum_{|h| \leq m} \frac{g_{kl,h}(\mu)}{i(\omega, h) + \{ (k - 1)\mu + l\bar{\mu} \}} e^{i(\theta,h)}.$$

provided that the denominators in these expressions do not vanish in the domain $W_+ = W_{j+1}$. We shall take $v_{10,0}(\mu) = 0$.

Later on the complex neighborhoods W_j and the orders of truncation m_j will be chosen in such a way that the denominators in (5.10) are even larger than cst. $|h|^{-\tau}$.

a4. *Reduction to the case $\gamma = 1$.*

Before making our choices more explicit and gradually filling in the necessary details, we first simplify the matter by reducing to the case where $\gamma = 1$. To this purpose we introduce a stretched time $\tilde{t} = \gamma t$ and denote differentiation with respect to $\tilde{t}$ by $'$. Now the original system (2.1) in this stretched time takes the form

$$\begin{cases} \theta' = \dfrac{\omega}{\gamma} \\ z' = \dfrac{\lambda}{\gamma} z + \dfrac{1}{\gamma} g(\theta, z, \lambda). \end{cases}$$

So if we rescale λ to $\tilde{\lambda} = \dfrac{\lambda}{\gamma}$ and define

$$\tilde{\omega} = \frac{\omega}{\gamma} \quad \text{and} \quad \tilde{g}(\theta, z, \tilde{\lambda}) = \frac{1}{\gamma} g(\theta, z, \gamma\tilde{\lambda}),$$

then (2.1) can be rewritten as

$$\begin{cases} \theta' = \tilde{\omega} \\ z' = \tilde{\lambda} z + \tilde{g}(\theta, z, \tilde{\lambda}). \end{cases}$$

Furthermore we observe that

$$|(\omega, h) - l\lambda_2| \geq \gamma |h|^{-\tau}$$

if and only if

$$|(\tilde{\omega}, h) - l\tilde{\lambda}_2| \geq |h|^{-\tau},$$

while

$$|g| < \gamma\delta \quad \text{if and only if} \quad |\tilde{g}| < \delta.$$

Here the norms are taken on complexified domains that have been stretched in the λ-direction in accordance to the above. In fact $\Lambda_1 \times \Lambda_2 + (\rho_1, \rho_2)$ has been replaced by

$$\frac{1}{\gamma}\Lambda_1 \times \frac{1}{\gamma}\Lambda_2 + \left(\frac{\rho_1}{\gamma}, \frac{\rho_2}{\gamma}\right),$$

compare remark *i*) to the normal form theorem. Since one of the conditions of the normal form theorem is that $0 < \gamma \leq \min\{\rho_1, \rho_2\}$, it appears to be sufficient to give the proof for $\gamma = 1$. Also we may assume that $\rho_1 = \rho_2 = 1$. Finally we see that it is also sufficient to prove remark *i*) to the theorem only for $\gamma = 1$.

b. Estimates for the iteration step.

We now specify the orders of truncation m_j and, to some extent, also the complexified neighborhoods W_j and W_j^e. This is going to enable us to estimate

$$| g^{j+1} |_{W^e_{j+1}}, \ | \alpha_l^j - \alpha_l^{j+1} |_{W_{j+1}}, \ | U^j - U^{j+1} |_{W_{j+1}} \quad \text{and} \quad | V^j_{kl} - V^{j+1}_{kl} |_{W_{j+1}}$$

in terms of $| g^j |_{W^e_j}$, compare *a*1 above.

Since we have stretched the λ-variable with the factor γ^{-1} the domain Λ_2 may be assumed large enough to contain the nonempty Cantor set Λ_2^c, consisting of those points $\lambda_2 \in \Lambda_2$, having distance not less than 2 from the boundary $\partial\Lambda_2$, that satisfy the inequalities

$$| (\omega, h) - l\lambda_2 | \geq 2 \, | h |^{-\tau}$$

for all $h \in \mathbf{Z}^n \setminus \{ 0 \}$ and for $l = 0, 1, 2, \ldots, M$. (Note that in terms of (2.5) we have $\Lambda_2^c = \Lambda_{2,2}^M$.)

Moreover we assume $g = g^0$ to be complex analytic on a domain

$$\Omega = (\mathbf{T}^n + \sigma) \times O \times (\Lambda_1 + 1) \times (\Lambda_2 + 1)$$

(recall that $\rho_1 = \rho_2 = 1$) and that finally we shall control the whole process by taking $| g^0 |_\Omega < \delta$ for some $\delta > 0$ (recall that $\gamma = 1$).

In the next subsection we shall choose $\delta = \delta_0$ and other constants in such a way that the induction on *j* works, meaning that the iteration converges.

b1. *Definition of* W_j *and* W_j^e.

First we define

$$W_j := \mathbf{T}^n \times \Lambda_1 \times \Lambda_2^c + \left(\frac{1}{2}\sigma + s_j, \frac{1}{2} + r_j, r_j \right)$$

where $\{ s_j \}_{j=0}^\infty$ and $\{ r_j \}_{j=0}^\infty$ are sequences of positive numbers satisfying

$$(5.11a) \qquad \begin{aligned} & s_{j+1} < \frac{1}{2} s_j \quad \text{and} \quad r_{j+1} < \frac{1}{2} r_j \quad \text{for all } j \geq 0 \quad \text{while} \\ & s_0 < \min \left\{ \frac{1}{2}\sigma, \frac{1}{2} \right\} \quad \text{and} \quad r_0 < \frac{1}{2}. \end{aligned}$$

Next we define

$$W_j^e := \left(\mathbf{T}^n + \frac{1}{2}\sigma + s_j\right) \times O_j \times \left(\Lambda_1 + \frac{1}{2} + r_j\right) \times (\Lambda_2^c + r_j), \quad \text{where}$$

$$O_j := \{ (z, \bar{z}) \in \mathbf{C}^2 \mid |z| < \varepsilon_j \quad \text{and} \quad |\bar{z}| < \varepsilon_j \}$$

(note that $W_j^e = W_j \times O_j$), for a sequence $\{ \varepsilon_j \}_{j=0}^{\infty}$ of positive numbers, now with

$$(5.11b) \qquad \begin{aligned} &\varepsilon_{j+1} < \frac{1}{2}\varepsilon_j \text{ for all } j \geq 0, \quad \text{while} \\ &\varepsilon_0 < \frac{1}{2} \text{ is sufficiently small to ensure that } O_0 \subseteq O. \end{aligned}$$

In the next subsection $\{ s_j \}_{j=0}^{\infty}$ and $\{ r_j \}_{j=0}^{\infty}$ will be fixed as geometric sequences and $\{ \varepsilon_j \}_{j=0}^{\infty}$ as an exponential sequence.

If no confusion seems possible we often again adopt the $+$-notation introduced in $a2$ above, and write

$$g = g^j, \qquad g^+ = g^{j+1}, \qquad W = W_j, \qquad W_+ = W_{j+1},$$
$$r = r_j, \qquad r_+ = r_{j+1}, \quad \text{etc.}$$

Finally we introduce

$$r_* = \frac{1}{2}(r + r_+), \qquad s_* = \frac{1}{2}(s + s_+) \qquad \text{and} \qquad \varepsilon_* = \frac{1}{2}(\varepsilon + \varepsilon_+)$$

and define the « intermediate » sets

$$W_* = \left(\mathbf{T}^n + \frac{1}{2}\sigma + s_*\right) \times \left(\Lambda_1 + \frac{1}{2} + r_*\right) \times (\Lambda_2 + r_*) \quad \text{and}$$
$$W_*^e = W_* \times \{ (z, \bar{z}) \in \mathbf{C}^2 \,\|\, z| < \varepsilon_* \quad \text{and} \quad |\bar{z}| < \varepsilon_* \}.$$

b2. *Estimate of* $|g^+|_{W_+^e}$ *and of* $|\alpha_l - \alpha_l^+|_{W_+}$.

We proceed in estimating the derivatives of g on W_*^e as well as the component-functions of Ψ and their derivatives, expressing everything in $|g|_{W^e}$. As an immediate consequence we obtain sufficient conditions to ensure that $\Psi(W_\pm^e) \subseteq W_*^e (\subseteq W^e)$. In the next subsection we show by a direct argument that the Φ_j can all be made diffeomorphisms. As another immediate consequence we get an estimate on $|\alpha_l - \alpha_l^+|_{W_+}$.

In the estimates positive constants show up which only depend on n, τ, M or σ. If there is no need to give such a constant a name, the corresponding inequality sign is denoted by $\lessdot$. The constants which have to be remembered for later use are called c_0 to c_4.

We start with a proposition on the derivatives of g, which can be easily proven using the Cauchy estimate.

5.1. PROPOSITION. — *On the domain* W_*^e, *for* $0 \leq k + l \leq M$ *we have*

$$\varepsilon^{k+l}\left|\frac{\partial^{k+l}}{\partial z^k \partial \bar{z}^l} g\right|, \qquad s\varepsilon^{k+l}\left|\frac{\partial^{k+l+1}}{\partial\theta\partial z^k \partial \bar{z}^l} g\right|, \qquad r\varepsilon^{k+l}\left|\frac{\partial^{k+l+1}}{\partial\lambda\partial z^k \partial \bar{z}^l} g\right| \leq c_0 \mid g \mid_{W^e}.$$

In the last estimate $\dfrac{\partial}{\partial\lambda}$ *may be replaced by* $\dfrac{\partial}{\partial\bar{\lambda}}$.

In order to be able to estimate Ψ we have to specify the order of truncation m of the Fourier series of g. We choose

$$m := [(Mr)^{-1/\tau}]. \tag{5.12}$$

where $[-]$ denotes the integral part. We then have

5.2. PROPOSITION. — *For all* $\lambda_2 \in \Lambda_2^c + r$ *and all* $0 < |h| \leq m, 0 \leq l \leq M$ *we have*

$$|(\omega, h) - l\lambda_2| \geq |h|^{-\tau}.$$

Proof. — For $\lambda_2 \in \Lambda_2^c + r$ there exists $\lambda_2^* \in \Lambda_2^c$ such that $|\lambda_2 - \lambda_2^*| < r$. It then follows for $0 \leq l \leq M$ and all $h \neq 0$:

$$|(\omega, h) - l\lambda_2| \geq |(\omega, h) - l\lambda_2^*| - l|\lambda_2 - \lambda_2^*| \geq 2|h|^{-\tau} - Mr.$$

So for $0 < |h| \leq m$ we have

$$|(\omega, h) - l\lambda_2| \geq 2|h|^{-\tau} - m^{-\tau} \geq |h|^{-\tau}. \qquad \text{Q. E. D.}$$

We now formulate our estimates on Ψ and some of its derivatives. A proof will be given after two lemmata.

5.3. PROPOSITION. — *On the domain* W_*, *for* $0 \leq k+l \leq M-1$ *we have*

$$\varepsilon|u|,\ s^{2\tau}\varepsilon^{k+l}|v_{kl}|,\ s^{2\tau+1}\varepsilon^{k+l}\left|\frac{\partial v_{kl}}{\partial\theta}\right|,\ r\varepsilon\left|\frac{\partial u}{\partial\mu}\right|,\ rs^{2\tau}\varepsilon^{k+l}\left|\frac{\partial v_{kl}}{\partial\mu}\right| \leq c_1 \mid g \mid_{W^e}$$

$\left(\textit{similarly for } \dfrac{\partial}{\partial\bar{u}}\right)$; *and on the domain* W_*^e, *for* $0 \leq h + l \leq M - 1$ *we have*

$$s^{2\tau+1}\left|\frac{\partial v}{\partial\theta}\right|,\quad s^{2\tau}\varepsilon^{k+l}\left|\frac{\partial^{k+l}}{\partial\zeta^k\partial\bar{\zeta}^l} v\right|,\quad rs^{2\tau}\varepsilon^{k+l+1}\left|\frac{\partial^{k+l+1}}{\partial\mu\partial\zeta^k\partial\bar{\zeta}^l} v\right| \leq c_1 \mid g \mid_{W^e}$$

$\left(\textit{similarly for } \dfrac{\partial}{\partial\bar{\mu}}\right)$.

For the proof of this we need for the coefficient g_{kl}, as defined in $a3$

5.4. LEMMA. — *If* $\lambda \in \Lambda_1 \times \Lambda_2^c + \left(\frac{1}{2} + r, r\right)$, *then for* $0 \leq k + l \leq M$ *we have*

$$|g_{kl,h}(\lambda)| \leq .\,| g |_{W^e}\varepsilon^{-(k+l)}e^{-|h|\left(\frac{1}{2}\sigma+s\right)}.$$

This expresses the familiar exponential decay of the Fourier coefficients of real analytic functions (Paley-Wiener). A proof can be easily given using the Cauchy integral theorem, the Cauchy integral formula and proposition 5.1. Furthermore we need

5.5. LEMMA. — *Let $\beta > 0$. Then a constant $c(\beta) > 0$ exists, such that for all* $0 < x \leq 1$

$$\sum_{j=1}^{\infty} j^{\beta} e^{-jx} \leq c(\beta) x^{-(\beta+1)}.$$

Proof.

$$\sum_{j=1}^{\infty} j^{\beta} e^{-jx} = x^{-(\beta+1)} \sum_{j=1}^{\infty} x(jx)^{\beta} e^{-jx} \leq x^{-(\beta+1)} \left(\int_0^{\infty} t^{\beta} e^{-t} dt + x \max_{t \geq 0} t^{\beta} e^{-t} \right)$$

Q. E. D.

Proof of Proposition 5.3. — First, as a corollary to lemma 5.5 we have that for $0 < x \leq 1$

$$\sum_{h \in Z^n} | h |^{\tau} e^{-|h|x} \leq .x^{2\tau}.$$

This can be seen as follows. The number of n-vectors $h \in \mathbf{Z}^n$ with $| h | = j$ can be estimated by $2^n j^{n-1}$, so

$$\sum_{h \in Z^n} | h |^{\tau} e^{-|h|x} \leq . \sum_{j=0}^{\infty} j^{n-1} j^{\tau} e^{-jx} \leq . \sum_{j=0}^{\infty} j^{2\tau-1} e^{-jx},$$

where we use that $\tau > n$. Now apply lemma 5.5 for $\beta = 2\tau - 1$.

Remain the estimates. We start with the domain W_*. The inequalities involving u are a direct consequence of the definition in (5.9) and proposition 5.1.

Now v_{kl}. By (5.10), proposition 5.2 and lemma 5.4, we know that on $\mathbf{T}^n \times \Lambda_1 \times \Lambda_2^c + \left(\frac{1}{2}\sigma + s_*, \frac{1}{2} + r, r\right)$

$$| v_{kl} | \leq \sum_{|h| \leq m} |h|^{\tau} | g_{kl,h}(\mu) | e^{|\mathrm{Im}\,\theta|\,|h|} \leq \varepsilon^{-(k+l)} | g |_{W^e} \sum_{|h| \leq m} | h |^{\tau} e^{|h|\left(|\mathrm{Im}\,\theta| - \left(\frac{1}{2}\sigma + s\right)\right)} \leq$$

$$\leq \varepsilon^{-(k+l)} | g |_{W^e} \sum_{h \in Z^n} | h |^{\tau} e^{-\frac{1}{4}|h|s} \leq . s^{-2\tau} \varepsilon^{-(k+l)} | g |_{W^e},$$

where we used that $s - s^* > \dfrac{1}{4} s$ and the above corollary to lemma 5.5 for the last estimate, similarly for $\frac{\partial v_{kl}}{\partial \theta}$. The inequality on $\left|\frac{\partial v_{kl}}{\partial \mu}\right|$ on W_* now can be derived with Cauchy's estimate.

In order to prove the estimates on W_*^e consider the polynomial

$$v(\theta, \zeta, \mu) = \sum_{p+q=0}^{M-1} v_{pq}(\theta, \mu)\zeta^p\bar{\zeta}^q \quad \text{which gives}$$

$$\frac{\partial^{k+l}}{\partial\zeta^k\partial\bar{\zeta}^l} v(\theta, \zeta, \mu) = \Sigma\, c(p, q, k, l) v_{pq}(\theta, \mu)\zeta^{p-k}\bar{\zeta}^{q-l},$$

so on W_*^e it follows

$$\left|\frac{\partial^{k+l} v}{\partial\zeta^k\partial\bar{\zeta}^l}\right| \leq \Sigma c(p, q, k, l) \mid v_{pq} \mid_{W_*} \varepsilon_*^{(p+q)-(k+l)} \leq$$
$$\leq .\, s^{-2\tau}\varepsilon^{-(k+l)} \mid g \mid_{W^e} \Sigma c(p, q, k, l)\left(\frac{\varepsilon_*}{\varepsilon}\right)^{p+q} \leq .\, s^{-2\tau}\varepsilon^{-(k+l)} \mid g \mid_{W^e},$$

where we used that by (5.11 *b*) one has $\frac{1}{2}\varepsilon < \varepsilon_* < \frac{3}{4}\varepsilon$.

The proof of the last statement of proposition 5.3 is similar, using Cauchy's estimate. Q. E. D.

A direct consequence of proposition 5.3 is

5.6. COROLLARY. — *If* $\varepsilon^{-1} \mid g \mid_{W^e} \max \{ r^{-1}, s^{-2\tau} \} \leq \frac{1}{4c_1}$, *then* $\Psi(W_+^e) \subseteq W_*^e$.

We proceed estimating $\mid g^+ \mid_{W_+^e}$. It follows from (5.6) and (5.8) that

$$\begin{aligned}(5.13)\qquad \left(1 + \frac{\partial v}{\partial \zeta}\right)g^+ + \frac{\partial v}{\partial \bar{\zeta}}\,\bar{g}^+ &= g(\theta, \zeta + v, \mu + u) - g(\theta, \zeta, \mu) + \\ &+ g(\theta, \zeta, \mu) - {}^{M}g(\theta, \zeta, \mu) + {}^{M}g(\theta, \zeta, \mu) - {}^{M}_{m}g(\theta, \zeta, \mu) + uv + \\ &+ \sum_{l=1}^{N} \{ \alpha_l(\mu + u)(\zeta + v) \mid \zeta + v \mid^{2l} - \alpha_l(\mu)\zeta \mid \zeta \mid^{2l} \} + \\ &- \frac{\partial v}{\partial \zeta}\sum_{l=1}^{N} \alpha_l^+(\mu)\zeta \mid \zeta \mid^{2l} - \frac{\partial v}{\partial \bar{\zeta}}\sum_{l=1}^{N} \overline{\alpha_l^+(\mu)\zeta} \mid \zeta \mid^{2l}.\end{aligned}$$

Now, if we abbreviate $\mid \alpha \mid_W := \max_{1 \leq l \leq N} \mid \alpha_l \mid_W$, then we have

5.7. PROPOSITION. — *Under the assumptions of corollary 5.6, one has for* $1 \leq l \leq N$ *that*

$$\mid \alpha_l - \alpha_l^+ \mid_{W_+} \leq c_0\varepsilon^{-(2l+1)} \mid g \mid_{W^e}.$$

And if, moreover, ms > 2n then

$$| g^+ |_{W^e_+} \leq c_2(\mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} + \mathrm{V} + \mathrm{VI}),$$

where

$$\mathrm{I} = \varepsilon^{-1} | g |^2_{W^e} \max \{ r^{-1}, s^{-2\tau} \}, \qquad \mathrm{II} = \left(\frac{\varepsilon_+}{\varepsilon}\right)^{\mathrm{M}} | g |_{W^e},$$

$$\mathrm{III} = m^n e^{-\frac{ms}{2}} | g |_{W^e}, \qquad \mathrm{IV} = s^{-2\tau}\varepsilon^{-1} | g |^2_{W^e},$$

$$\mathrm{V} = | \alpha |_{W} | g |_{W^e} \max \{ s^{-2\tau}\varepsilon^2, r^{-1}\varepsilon^{-1}\varepsilon^3_+ \}, \qquad \mathrm{VI} = s^{-2\tau}\varepsilon^{-1}\varepsilon^3_+ | \alpha^+ |_{W_+} | g |_{W^e}.$$

Proof. — By (5.9) and proposition 5.1 we know

$$| \alpha_l - \alpha_l^+ |_{W_+} \leq | g_{l+1l} |_{W_+} \leq c_0 \varepsilon^{-(2l+1)} | g |_{W^e}.$$

Next consider (5.13). First observe that by proposition 5.3

$$\left| \frac{\partial v}{\partial \zeta} \right|_{W^e_+}, \quad \left| \frac{\partial v}{\partial \bar{\zeta}} \right|_{W^e_+} \leq c_1 s^{-2\tau} \varepsilon^{-1} | g |_{W^e} \leq \frac{1}{4}$$

by our assumption. Therefore

$$\left| \left(1 + \frac{\partial v}{\partial \zeta}\right) g^+ + \frac{\partial v}{\partial \bar{\zeta}} \overline{g^+} \right|_{W^e_+} \geq \left(1 - 2 \left| \frac{\partial v}{\partial \zeta} \right|_{W^e_+}\right) | g^+ |_{W^e_+} \geq \frac{1}{2} | g^+ |_{W^e_+}.$$

Now we estimate separately each of the six lines in the right hand side of (5.13).

By the mean value theorem and the propositions 5.1 and 5.3 we have

$$| g(\theta, \zeta + v, \mu + u) - g(\theta, \zeta, \mu) |_{W^e_+} \leq . \max \left\{ \left| \frac{\partial g}{\partial \zeta} \right|_{W^e_*} | v |_{W^e_+}, \left| \frac{\partial g}{\partial \mu} \right|_{W^e_*} | u |_{W_+} \right\} \leq$$

$$\leq . \varepsilon^{-1} | g |^2_{W^e} \max \{ r^{-1}, s^{-2\tau} \} = \mathrm{I};$$

Similarly by the Taylor formula and proposition 5.1 (for definitions cf. *a*3)

$$| g(\theta, \zeta, \mu) - {}^{\mathrm{M}}g(\theta, \zeta, \mu) |_{W^e_+} \leq . \sum_{k+l \geq \mathrm{M}} \left| \frac{\partial^{k+l} g}{\partial \zeta^k \partial \bar{\zeta}^l} \right|_{W^e_+} \varepsilon_+^{k+l} \leq . \left(\frac{\varepsilon_+}{\varepsilon}\right)^{\mathrm{M}} | g |_{W^e} = \mathrm{II};$$

In order to estimate the tail of the Fourier series of ${}^{\mathrm{M}}g$ we first observe that by proposition 5.1

$$| {}^{\mathrm{M}}g |_{W^e} \leq . \sum_{k+l=0}^{\mathrm{M}-1} \left| \frac{\partial^{k+l}}{\partial \zeta^k \partial \bar{\zeta}^l} g(., 0,.) \right|_{W} \varepsilon^{k+l} \leq . | g |_{W^e}.$$

Then by lemma 5.4, analogous to the proof of lemma 5.5 (cf. $a3$)

$$|{}^{\mathrm{M}}g - {}^{\mathrm{M}}_{m}g|_{\mathrm{W}^e_+} \leq \sum_{|h|>m} |{}^{\mathrm{M}}g_h e^{i(\theta,h)}|_{\mathrm{W}^e_+} \leq$$

$$\leq . |{}^{\mathrm{M}}g|_{\mathrm{W}^e} \sum_{k=m+1}^{\infty} k^{n-1} e^{-\frac{1}{2}ks} \leq . |g|_{\mathrm{W}^e} \int_m^{\infty} x^{n-1} e^{-\frac{1}{2}xs} dx \leq$$

$$\leq . |g|_{\mathrm{W}^e} m^{n-1} e^{-\frac{1}{2}ms} \int_m^{\infty} e^{\left(\frac{n-1}{m} - \frac{s}{2}\right)(x-m)} dx \leq$$

$$\leq . |g|_{\mathrm{W}^e} m^{n-1} e^{-\frac{1}{2}ms} \int_m^{\infty} e^{-\frac{x-m}{m}} dx = |g|_{\mathrm{W}^e} m^n e^{-\frac{ms}{2}} = \mathrm{III},$$

where we used that $s_+ < \frac{1}{2}s$ and our assumption that $ms > 2n$.

Next we have by proposition 5.3

$$|uv|_{\mathrm{W}^e_+} \leq . s^{-2\tau}\varepsilon^{-1} |g|^2_{\mathrm{W}^e} = \mathrm{IV}.$$

Remain the last two lines concerning the α_l. First we have by the mean value theorem, the Cauchy estimate and proposition 5.3, for $(\theta, \zeta, \mu) \in \mathrm{W}^e_+$

$$\left|\sum_{l=1}^{\mathrm{N}} \{ \alpha_l(\mu + u)(\zeta + v) |\zeta + v|^{2l} - \alpha_l(\mu)\zeta |\zeta|^{2l} \} \right| \leq$$

$$\leq \sum_{l=1}^{\mathrm{N}} |\alpha_l(\mu+u)| |(\zeta+v)|\zeta+v|^{2l} - \zeta|\zeta|^{2l}| + \sum_{l=1}^{\mathrm{N}} |\alpha_l(\mu+u) - \alpha_l(\mu)| |\zeta|^{2l+1} \leq$$

$$\leq . \sum_{l=1}^{\mathrm{N}} \left\{ |\alpha|_{\mathrm{W}} \varepsilon_*^{2l} |v|_{\mathrm{W}^e_+} + \frac{1}{r} |\alpha|_{\mathrm{W}} |u|_{\mathrm{W}^e_+} \varepsilon_+^{2l+1} \right\} \leq$$

$$\leq |\alpha|_{\mathrm{W}} |g|_{\mathrm{W}^e} \max \{ s^{-2\tau}\varepsilon^2, r^{-1}\varepsilon^{-1}\varepsilon_+^3 \} = \mathrm{V},$$

where we estimated by the terms for $l = 1$ and again used that $\varepsilon_* < \frac{3}{4}\varepsilon$. Similarly for the last line on W^e_+ we have

$$\left| \frac{\partial v}{\partial \zeta} \sum_{l=1}^{\mathrm{N}} \alpha_l^+(\mu)\zeta |\zeta|^{2l} - \frac{\partial v}{\partial \bar{\zeta}} \sum_{l=1}^{\mathrm{N}} \overline{\alpha_l^+(\mu)} \bar{\zeta} |\zeta|^{2l} \right| \leq$$

$$\leq . |\alpha^+|_{\mathrm{W}_+} \left| \frac{\partial v}{\partial \zeta} \right|_{\mathrm{W}^e_+} \varepsilon_+^3 \leq . s^{-2\tau}\varepsilon^{-1}\varepsilon_+^3 |\alpha^+|_{\mathrm{W}_+} |g|_{\mathrm{W}^e} = \mathrm{VI},$$

where we estimated by the contribution for $l = 1$ and used proposition 5.3.

Q. E. D.

b3. *Estimate of* $|\Phi - \Phi_+|_{W^e_+}$.

We have

5.8. PROPOSITION. — *Under the assumptions of corollary 5.6 the following holds:*

$$|D\Phi_+|_{W^e_+} \le |D\Phi|_{W^e}(1 + c_3 |g|_{W^e}\varepsilon^{-1}r^{-1}s^{-(2\tau+1)});$$
$$|U - U^+|_{W_+}, \ |V_{00} - V^+_{00}|_{W_+} \le c_3 |D\Phi|_{W^e} |g|_{W^e} \max\{s^{-2\tau}, \varepsilon^{-1}\};$$

and for $1 \le k + l \le M - 1$

$$|V_{kl} - V^+_{kl}|_{W_+} \le c_3(1 + |D\Phi|_{W^e})|g|_{W^e}\varepsilon^{-1}\varepsilon_*^{-(k+l-1)} \max\{r^{-1}, s^{-2\tau}\}.$$

Proof. — Since $\Phi_+ = \Phi \circ \Psi$ we have by the chain rule

$$D\Phi_+(\zeta) = D\Phi(z).D\Psi(\zeta)$$

where

$$D\Psi = id + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \frac{\partial v}{\partial \theta} & \frac{\partial v}{\partial \zeta} & \frac{\partial v}{\partial \overline{\zeta}} & \frac{\partial v}{\partial \mu} & \frac{\partial v}{\partial \overline{\mu}} \\ \frac{\partial \overline{v}}{\partial \theta} & \frac{\partial \overline{v}}{\partial \zeta} & \frac{\partial \overline{v}}{\partial \overline{\zeta}} & \frac{\partial \overline{v}}{\partial \mu} & \frac{\partial v}{\partial \overline{\mu}} \\ 0 & 0 & 0 & \frac{\partial u}{\partial \mu} & \frac{\partial u}{\partial \overline{\mu}} \\ 0 & 0 & 0 & \frac{\partial \overline{u}}{\partial \mu} & \frac{\partial \overline{u}}{\partial \overline{\mu}} \end{bmatrix}$$

The first assertion of the proposition now directly follows from proposition 5.3.

Observe that in view of (5.3) for the second conclusion of the proposition it is sufficient to prove that

$$|\Phi - \Phi_+|_{W^e_+} \le \ . \ |D\Phi|_{W^e} |g|_{W^e} \max\{s^{-2\tau}, \varepsilon^{-1}\}.$$

In order to show this again write $\Phi_+ = \Phi \circ \Psi$, which by the mean value theorem for all points $(\theta, \zeta, \mu) \in W^e_+$ gives

$$|(\Phi - \Phi_+)(\theta, \zeta, \mu)| = |\Phi(\theta, \zeta, \mu) - \Phi(\theta, \zeta + v(\theta, \zeta, \mu), \mu + u(\mu))| \le$$
$$\le \ . \ |D\Phi|_{W^e_*} \max\{|v|_{W^e_+}, |u|_{W_+}\}.$$

Now we use the fact that $W^e_+ \subseteq W^e_* \subseteq W^e$ and proposition 5.3.

The last inequality of the proposition needs some more care.

We first prove that

$$\left|\frac{\partial}{\partial\zeta}(V-V_+)\right|_{W_+^e} \leq .(1 + |D\Phi|_{W^e})\,|g|_{W^e}\,\varepsilon^{-1}\max\{r^{-1}, s^{-2\tau}\}$$

and similarly for $\frac{\partial}{\partial\bar\zeta}$.

In fact by the mean value theorem we have for $(\theta, \zeta, \mu) \in W_+^e$

$$\left|\frac{\partial}{\partial\zeta}(V - V_+)(\theta, \zeta, \mu)\right| =$$

$$= \left|\frac{\partial V}{\partial\zeta}(\theta, \zeta, \mu) - \frac{\partial V}{\partial\zeta}(\theta, \zeta + v(\theta, \zeta, \mu), \mu + u(\mu)).\left(1 + \frac{\partial v}{\partial\zeta}(\theta, \zeta, \mu)\right)\right| \leq$$

$$\leq .\left|\frac{\partial^2 V}{\partial z^2}\right|_{W_*^e} |v|_{W_+^e} + \left|\frac{\partial^2 V}{\partial\lambda\partial z}\right|_{W_*^e} |u|_{W_+} + \left|\frac{\partial V}{\partial z}\right|_{W_*^e}\left|\frac{\partial v}{\partial\zeta}\right|_{W_+^e},$$

where we suppressed some bars in the notation. Now clearly

$$\left|\frac{\partial V}{\partial z}\right|_{W^e} \leq 1 + |D\Phi|_{W_+^e}$$

and by the Cauchy estimate we conclude that on W_*^e

$$\varepsilon\left|\frac{\partial^2 V}{\partial z^2}\right|, \qquad r\left|\frac{\partial^2 V}{\partial\lambda\partial z}\right| \leq .\,(1 + |D\Phi|_{W^e}).$$

With proposition 5.3 we then directly have (5.14). From this we conclude the proof as follows

$$|V_{kl} - V_{kl}^+|_{W_+} \leq .\left|\frac{\partial^{k+l}}{\partial\zeta^k\partial\bar\zeta^l}(V - V^+)(\theta, 0, \mu)\right|_{W_+} \leq$$
$$\leq .\,\varepsilon^{-1}\varepsilon_*^{-(k+l-1)}(1 + |D\Phi|_{W^e})\,|g|_{W^e}\max\{r^{-1}, s^{-2\tau}\},$$

applying the Cauchy estimate for a derivative of order $k+l-1$. Q.E.D.

c. Induction and convergence.

In this, final subsection, we complete our proof. We recall from § $a1$, above, that the error $|g^j|_{W_j^e}$ will be reduced in an exponential way, at the same time ensuring that the sequences

$$|\alpha_l^j - \alpha_l^{j+1}|_{W_{j+1}}, \qquad |V_{kl}^j - V_{kl}^{j+1}|_{W_{j+1}} \qquad \text{and} \qquad |U^j - U^{j+1}|_{W_{j+1}}$$

also decrease exponentially. The domains W_j are going to collapse geometrically on their intersection $\cap_{j=0}^{\infty} W_j$. By the inverse approximation lemma, see the appendix, the limit functions α_l^∞, V_{kl}^∞ and U^∞ then are of class C^∞

in the sense of Whitney, i. e. extendible as C^∞-functions to an open neighborhood of this intersection in $\mathbf{T}^n \times \mathbf{C}$.

Note that in the $\theta_1, \theta_2, \ldots, \theta_n$ and μ_1-direction these limits will be even real analytic, since on the complex domain $\mathbf{T}^n \times \Lambda_{1,\gamma} + \left(\frac{1}{2}\sigma, \frac{1}{2}\right)$ they are the uniform limits of complex analytic functions.

Finally we come to the point that the limit Φ_∞ be a diffeomorphism. In the ζ-direction we just use the inverse function theorem in the point $\zeta = 0$. In the μ-direction we apply the Whitney extension theorem, see the appendix, in order to conclude that there exists a C^∞-extension with a good control of the derivative. Such an extension is easily proved to be a diffeomorphism in the μ-direction.

c1. *Specification of the constants.*

We choose the geometric sequences of positive numbers

$$s_j = bv^j, \qquad r_j = s_j^{2\tau} \qquad (j = 0, 1, 2, \ldots),$$

where, in accordance with (5.11), $v \in (0, 1/2)$ and $b \in \left(0, \min\left\{\frac{1}{2}\sigma, \frac{1}{2}\right\}\right)$.

Also we choose two exponential sequences of positive numbers

$$\delta_{j+1} = \delta_j^{1+p}, \quad \varepsilon_j = \delta_j^q \qquad (j = 0, 1, 2, \ldots)$$

with $p, q \in (0, \infty)$, $\delta_0^{pq} \in \left(0, \frac{1}{2}\right)$ and $\varepsilon_0 = \delta_0^q$ sufficiently small, again in accordance with (5.11). The sequence $\{ \delta_j \}_{j=0}^{\infty}$ will serve to dominate the « error » $| g^j |_{W^e}$.

c2. *Exponential decay.*

We have

5.9. PROPOSITION. — *Assume that the constants $p, q \in (0, \infty)$ and $v \in (0, 1/2)$ are fixed with*

$$0 < p < \min\left\{1 - q, 2q, \frac{1}{M - 2}\right\},$$

$$\frac{1}{M} < q < \frac{1}{M - 1 + (M - 2)p} \qquad \text{and} \qquad v < \frac{1}{1 + p}.$$

Then, for sufficiently small $\delta_0 \in (0, \infty)$ there exists $b \in \left(0, \min\left\{\frac{1}{2}\sigma, \frac{1}{2}\right\}\right)$, such that if $| g^0 |_{W_0^e} \leq \delta_0$, then

i) $\qquad | g^j |_{W_j^e} \leq \delta_j \quad$ for all $j = 0, 1, 2, \ldots$;

ii) The sequences

$$\{ | \alpha_l^j - \alpha_l^{j+1} |_{W_{j+1}} \}_{j=0}^{\infty} (1 \leq l \leq N), \qquad \{ | U^j - U^{j+1} |_{W_{j+1}} \}_{j=0}^{\infty}$$

and $$\{ | V_{kl}^j - V_{kl}^{j+1} |_{W_{j+1}} \}_{j=0}^{\infty} (0 \leq k + l \leq M - 1)$$

decrease exponentially as $j \to \infty$, *with initial terms tending to* 0 *as* $\delta_0 \to 0$.

In our proof of this proposition we use a lemma, which serves to estimate III in proposition 5.7.

5.10. LEMMA. — *Given* $\eta > 0$, $A > 0$ *and* $n \in \mathbb{N}$, *we have* $x^n e^{-Ax} \leq \eta$ *for* $x \geq -\frac{2n}{A} \log\left(\frac{A}{n} \eta^{1/n}\right)$ and $x > 0$.

Proof. — First we consider the case where $A = n = 1$. We have to show that $xe^{-x} \leq \eta$ as soon as $x \geq -2 \log \eta$. To this end for $x > 0$ consider the function $f(x) = -x + \log x$. Observe that f is maximal for $x = 1$, $f(1) = -1$ and that f decreases on $(1, \infty)$. Also observe that $xe^{-x} \leq \eta$ if and only if $f(x) \leq \log \eta$. Now if $\log \eta > -1$ then obviously $f(x) < \log \eta$ for all $x > 0$. So conversely assume that $\log \eta \leq -1$. In that case $x_0 \geq 1$ exists with $f(x_0) = \log \eta$. Since always $\log x < \frac{1}{2} x$ we see that $-\frac{1}{2} x_0 > \log \eta$, so $x_0 < -2 \log \eta$. Because of the monotonicity of f we conclude that $f(x) \leq \log \eta$ for $x \geq x_0$, so in particular for $x \geq -2 \log \eta$.

The case of arbitrary A and n now simply is a matter of scaling: Put $y = \frac{A}{n} x$, then $x^n e^{-Ax} \leq \eta$ is equivalent to $ye^{-y} \leq \frac{A}{n} \eta^{1/n}$, which by the first part of this proof is implied by $y \geq -2 \log\left(\frac{A}{n} \eta^{1/n}\right)$ or equivalently by $x \geq -\frac{2n}{A} \log\left(\frac{A}{n} \eta^{1/n}\right)$. Q. E. D.

Proof of Proposition 5.9. — We are going to apply the propositions 5.7 and 5.8. To this purpose we have to satisfy the assumptions of corollary 5.6, which are implied by $\delta^{1-q} \leq \frac{1}{4c_1} s^{2\tau}$. Sufficient for this is that

$$\delta_0^{1-q} \leq \frac{1}{4c_1} b^{2\tau} \quad \text{and} \quad \delta_0^{(1-q)p} \leq v^{2\tau}. \tag{5.15}$$

We now start considering the sequences $\{ \alpha_l^j \}_{j=0}^{\infty}$, $1 \leq l \leq N$. First observe from proposition 5.7 that $| \alpha_l^j - \alpha_l^{j+1} |_{W_{j+1}}$ has exponential decay for $j \to \infty$, as soon as $\delta_j^{1-(2N+1)q}$ does. This is equivalent to

$$(2N + 1)q < 1, \tag{5.16}$$

so this is sufficient for one of the assertions in part *ii*) of the proposition.

Secondly we wish to bound the series $\Sigma_{j=0}^{\infty}\delta_j^{1-(2N+1)q}$ by a constant c_4 which is universal, so in particular independent of δ_0 and q. In that case the sequence $\{|\alpha^j|_{W_j}\}_{j=0}^{\infty}$ is bounded by c_0c_4. Sufficient for this is e. g. that

$$(5.17)\qquad \delta_0^{1-(2N+1)q} < \frac{1}{2} \quad \text{and} \quad \delta_0^{\{1-(2N+1)q\}p} < \frac{1}{2}.$$

Next we look for sufficient conditions for part i) of the proposition. In fact we want to ensure that (cf. prop. 5.7)

$$\text{I, II, III, IV, V, VI} \leq \frac{1}{6c_2}\delta_+ \quad \text{and that } ms > 2n.$$

In order to have $\text{I} \leq \frac{1}{6c_2}\delta_+$ it is sufficient that $\delta^{1-(p+q)} \leq \frac{1}{6c_2}s^{2\tau}$, which is implied by

$$p + q < 1$$

$$\delta_0^{1-(p+q)} \leq \frac{1}{6c_2}b^{2\tau} \quad \text{and} \quad \delta_0^{\{1-(p+q)\}p} \leq v^{2\tau}.$$

Similarly for $\text{II} \leq \frac{1}{6c_2}\delta_+$ it is sufficient to have $\delta^{(Mq-1)p} \leq \frac{1}{6c_2}$ which is implied by

$$(5.19)\qquad Mq > 1 \quad \text{and} \quad \delta_0^{(Mq-1)p} \leq \frac{1}{6c_2}.$$

Now $\text{III} \leq \frac{1}{6c_2}\delta_+$ and the condition $ms > 2n$ are taken care of as follows. Recall that by (5.12) we have $m = [(Mr)^{-1/\tau}]$. This implies that

$$s(Mr)^{-1/\tau} - 1 < ms \leq s(Mr)^{-1/\tau}, \qquad \text{and hence}$$

$$m^n e^{-\frac{ms}{2}} \leq \frac{e}{(Mr)^{n/\tau}}\, e^{-\frac{s}{2(Mr)^{1/\tau}}}.$$

The fact that $\text{III} \leq \frac{1}{6c_2}\delta_+$ therefore can be established by applying lemma 5.10 for

$$x = \frac{1}{(Mr)^{1/\tau}}, \qquad A = \frac{s}{2}, \qquad \eta = \frac{1}{6ec_2}\delta^p.$$

This yields as a sufficient condition that for all j

$$\frac{1}{(Mr)^{1/\tau}} \geq -\frac{4n}{s}\log\left\{\frac{s}{2n}\left(\frac{1}{6ec_2}\delta^p\right)^{1/n}\right\},$$

which is equivalent to

$$4M^{1/\tau}s \le \frac{(1+p)^{-j}}{p\log\left(\frac{1}{\delta_0}\right)+n(1+p)^{-j}\log\left(\frac{2n}{bv^j}\right)+(1+p)^{-j}\log(6ec_2)}.$$

Sufficient for this and for the fact that $ms > 2n$ now is

$$(5.20)\quad v \le \frac{1}{1+p},$$
$$b \le \frac{1}{M^{1/\tau}}\min\left\{\frac{1}{2n+1},\frac{1}{4\left\{p\log\left(\frac{1}{\delta_0}\right)+n\log\left(\frac{1}{v}\right)/\log(1+p)\right\}}\right\}.$$

For $\mathrm{IV} \le \frac{1}{6c_2}\delta_+$ it is sufficient to have $\delta^{1-(p+q)} \le \frac{1}{6c_2}s^{2\tau}$, which is already taken care of by (5.18).

In order to have $\mathrm{V} \le \frac{1}{6c_2}\delta_+$ it is sufficient that $\delta^{2q-p}, \delta^{2q-p+3pq} \le \frac{1}{6c_0c_2c_4}s^{2\tau}$ which is implied by

$$2q > p,$$

$$(5.21)\quad \delta_0^{2q-p},\quad \delta_0^{2q-p+3pq} \le \frac{1}{6c_0c_2c_4}b^{2\tau},\quad \delta_0^{(2q-p)p},\quad \delta_0^{(2q-p+3pq)p} \le v^{2\tau}.$$

And for $\mathrm{VI} \le \frac{1}{6c_2}\delta_+$ similarly it is sufficient to have that

$$\delta^{2q-p+3pq} \le \frac{1}{6c_0c_2c_4}s^{2\tau},$$

which is already contained in (5.21).

Remains part *ii*) of the proposition. Recall that the exponential decay of $|\alpha_l^j - \alpha_l^{j+1}|_{W_{j+1}}$ is taken care of by (5.16). First we consider the sequence $\{|D\Phi|_{W_j^c}\}_{j=0}^{\infty}$, which we want to bound by a universal constant. Sufficient for this is that $\Sigma_{j=0}^{\infty}\delta_j^{1-q}s_j^{-(4\tau+1)}$ has a universal bound, see proposition 5.8. This, in turn, is implied by e. g.

$$(5.22)\qquad \delta_0^{1-q}b^{-(4\tau+1)} < \frac{1}{2} \qquad \text{and} \qquad \delta_0^{(1-q)p}v^{-(4\tau+1)} < \frac{1}{2},$$

compare (5.17). Note that by (5.16) we already have that $q < 1$.

Now by proposition 5.8 we see that the sequences $|U^j - U^{j+1}|_{W_{j+1}}$ and $|V_{00}^j - V_{00}^{j+1}|_{W_{j+1}}$ decay exponentially as soon as $\delta_j s_j^{-2\tau}$ and δ_j^{1-q} do. So the only important thing is that $q < 1$ which we have by (5.16).

Finally we conclude that the sequences $|V_{kl}^{j} - V_{kl}^{j+1}|_{W_{j+1}}$ have exponential decay as soon as $\delta_j^{1-q\{M-1+(M-2)p\}} s_j^{-2\tau}$ does. This is equivalent to

$$\{M - 1 + (M - 2)p\}\, q < 1 .$$

We conclude our proof by considering the harvest of the conditions (5.15) to (5.23). Let us start with the conditions on p, q and ν. First note that, since $M = 2N + 2$ or $2N + 3$, we have that (5.23) implies (5.16). It then follows from (5.19) and (5.23) that

$$\frac{1}{M} < q < \frac{1}{M - 1 + (M - 2)p},$$

which can be fulfilled if $(M - 2)p < 1$. Together with (5.18) and (5.21) this yields for p the condition

$$0 < p < \min\left\{ 1 - q, 2q, \frac{1}{M - 2} \right\},$$

which can be easily satisfied. The remaining condition on ν comes from (5.20):

$$\nu \leq \frac{1}{1 + p}.$$

The last three formulae are assumptions of our proposition.

Now assume that p, q and ν are chosen as above, then we are left with conditions on δ_0 and b. As a consequence all remaining inequalities —also see $c1$ — are of, or can be replaced by, the following type

$$\delta_0 \leq d_0, \qquad \delta_0^{d_1} \leq d_2 b, \qquad b \leq d_3 \quad \text{or}$$

$$b \leq \frac{1}{d_4 \log\left(\dfrac{1}{\delta_0}\right) + d_5}$$

(the last one e. g. comes from (5.20)). Here $d_0, d_1, \ldots, d_5$ are constants, which may depend on p, q and ν. These conditions can all be satisfied: by choosing both δ_0 and b small, but in an interdependent way. This follows from the fact that $\lim_{\delta_0 \to 0} \delta_0^{\xi} \log \delta_0 = 0$ for any $\xi > 0$. Q. E. D.

The following corollary to the above proposition states that, provided that the transformations converge to a « good » limit, the function g^{∞} disappears up to order $M - 1$ in $(z_\infty, \bar{z}_\infty)$ in the origin.

5.11. Corollary. — *Under the same assumptions as in proposition 5.9 we have for* $0 \leq k + l \leq M - 1$ *that*

$$\left| \frac{\partial^{k+l}}{\partial z_j^k \partial \bar{z}_j^l} g^j \right|_{W_j^{e*}} \to 0 \quad \text{as} \quad j \to \infty .$$

Proof. — See proposition 5.1 and use that $q < \frac{1}{M-1}$. Q. E. D.

c3. *The limit.*

We now finish the proof of the normal form theorem by showing that the α_l^j $(1 \le l \le N)$ and the Φ_j converges as $j \to \infty$.

5.12. PROPOSITION. — *Let p, q and v be as in proposition 5.9. Then for sufficiently small $\delta_0 > 0$ there exists $b \in \left(0, \min\left\{\frac{1}{2}\sigma, \frac{1}{2}\right\}\right)$ such that the sequences*

$$\{\alpha_l^j\}_{j=0}^{\infty}\ (1 \le l \le N),\quad \{U^j\}_{j=0}^{\infty}\quad \textit{and}\quad \{V_{kl}^j\}_{j=0}^{\infty}\ (0 \le k+l \le M-1)$$

on $T^n \times \Lambda_1 \times \Lambda_2^c$ have limits α_l^∞, U^∞ and V_{kl}^∞ respectively, which are real analytic in $\theta \in T^n$ and $\lambda_{1\infty} \in \Lambda_1$ and which are of class C^∞ in the sense of Whitney in $\lambda_{2\infty} \in \Lambda_2^c$.

Moreover these limits have C^∞-extensions to $T^n \times \Lambda_1 \times \Lambda_2$, also to be denoted by α_l^∞, U^∞ and V_{kl}^∞, such that

$$z_\infty \to z_\infty + \sum_{k+l=0}^{M-1} V_{kl}^\infty(\theta, \lambda_\infty) z_\infty^k \bar{z}_\infty^l$$

is a C^∞-diffeomorphism near $z_\infty = 0$ and

$$\lambda_\infty \to \lambda_\infty + U^\infty(\lambda_\infty)$$

is a C^∞-diffeomorphism onto its image. These extensions remain analytic in θ and $\lambda_{1,\infty}$.

Proof. — Compare the introduction to this subsection and to subsection $a1$. We shall apply the inverse approximation lemma and the Whitney-extension theorem to our sequences, see the appendix. To this purpose we check that for $\beta > 0$, $\beta \notin \mathbf{N}$

$$|\alpha_l^j - \alpha_l^{j+1}|_{W_{j+1}},\ |U^j - U^{j+1}|_{W_{j+1}},\ |V_{kl}^j - V_{kl}^{j+1}|_{W_{j+1}} = 0(1) r_j^\beta \quad \text{as} \quad \delta_0 \to 0,$$

since this implies that the limits are of class C^β for all β and hence of class C^∞. The estimates obviously hold as a consequence of proposition 5.9.

The inverse approximation lemma now implies the existence of the limits α_l^∞, U^∞ and V_{kl}^∞ together with estimates on their first derivatives. Choosing δ_0 and b sufficiently small we obtain the assertions concerning the diffeomorphisms in the proposition. In fact for δ_0 and b sufficiently small we have that e. g. $\left|\frac{\partial U^\infty}{\partial \lambda_\infty}\right| < 1/2$. The mean value theorem then implies that $\lambda_\infty \to \lambda_\infty + U^\infty(\lambda_\infty)$ is injective, while the inverse function theorem then moreover ensures that this map is a diffeomorphism onto its image. Similarly it follows by the inverse function theorem that the map

$$z_\infty \to z_\infty + \sum_{k+l=0}^{M-1} V_{kl}^\infty(\theta, \lambda_\infty) z_\infty^k \bar{z}_\infty^l$$

is a diffeomorphism near $z_\infty = 0$.

Remark. — From the inverse approximation lemma we conclude that in the C^β-norm ($\beta \notin \mathbf{N}$)

$$\| \alpha_l \|_\beta, \qquad \| \Phi - id \|_\beta \quad \text{tend to zero with } | g |_\Omega.$$

In fact $\| \alpha_l \|_\beta$ and $\| \Phi - id \|_\beta$ tend to zero as fractional powers of δ_0. The scaling procedure from subsection *a*4 now directly yields the estimates (2.6) and (2.7) in the remarks to the normal form theorem.

APPENDIX

For the sake of completeness we include a formulation of the inverse approximation lemma used to obtain the Whitney differentiable conjugacy of the normal form theorem. Compare [Stei, Ze].

Let $\beta > 0$ be a fixed order of differentiability and let $r_j = a\kappa^j$ be a fixed geometric sequence with $a = r_0 > 0$ and $0 < \kappa < 1$. Also let $\Omega \subseteq \mathbf{R}$ be a closed set and consider

$$W_j = \Omega + r_j$$

where, as in § 2*b* and § 5, $\Omega + r_j = \cup_{x\in\Omega} \{ z \in \mathbf{C} \mid | z - x | < r_j \}$.

The following lemma states when the limit U^∞ of a sequence $\{ U^j \}_{j=0}^{\infty}$, where U^j is a real analytic map with complex domain W_j, is of class C^β.

INVERSE APPROXIMATION LEMMA. — *Assume that $\beta \notin \mathbf{N}$. Let $\{ U^j \}_{j=0}^{\infty}$ be a sequence of functions such that U^j is real analytic on W^j, $U^0 \equiv 0$ and that for $j \geq 1$*

$$| U^j - U^{j-1} |_{W_j} \leq M r_j^\beta$$

for some constant M. *Then there exists a unique function* U^∞, *defined on* Ω, *which is of class* C^β *and such that*

$$\| U^\infty \|_\beta \leq M c_\beta,$$

where the constant c_β only depends on β and κ. Moreover for all $\alpha < \beta$

$$\| U^\infty - U^j \|_\alpha \to 0 \quad as \quad j \to \infty .$$

Here $\| - \|_\alpha$ denotes the C^α-norm on Ω.

Explanatory remarks (we abbreviate $l = [\beta]$ and $\varepsilon = \beta - [\beta]$):

i) The lemma is also valid for *open* Ω, in which case the concepts used are easier to understand. The class $C^\beta(\Omega)$ then consists of C^l-functions with bounded derivatives up to order l and where the l^{th} derivative satisfies a Hölder condition with exponent ε. Now the norm $\| \ \|$ is the usual Hölder norm, defined as the infimum of all positive M with for all x, $y \in \Omega$ and all $0 \leq k \leq l$

$$| D^k U(x) | \leq M \qquad \text{and}$$

$$| D^l U(x) - D^l U(y) | \leq M | x - y |^\varepsilon .$$

ii) In our present case of *closed* Ω the elements of $C^\beta(\Omega)$ are families $U = \{ U_k \}_{k=0}^{l}$ of functions defined on Ω. The U_k play the role of the derivatives of U_0, satisfying corresponding compatibility conditions. In fact for some positive constant M, all x, $y \in \Omega$ and all $0 \leq k \leq l$ one has

$$| U_k(x) | \leq M \qquad \text{and}$$

$$| U_k(x) - P_k(x, y) | \leq M | x - y |^{\beta - k} .$$

Here $P_k(x, y) = \Sigma_{j=0}^{l-k} \frac{1}{j!} U_{k+j}(y)(x - y)^j$, the analogue of the k^{th} Taylor polynomial. In this case the norm $\| U \|_\beta$ is defined as the infimum of all M for which the above inequalities are valid.

The Whitney extension theorem, see [Wh, Hö, Stei], now states that a linear extension operator

$$E : C^\beta(\Omega) \to C^\beta(\mathbf{R}^n)$$

exists, such that for all $0 \leq k \leq l$

$$D^k(EU)|_\Omega = U_k \quad \text{while}$$
$$\| EU \|_{\beta,\mathbf{R}^n} \leq c \| U \|_{\beta,\Omega}$$

for a constant c that only depends on β. So $U \in C^\beta(\Omega)$ may be regarded as a restriction of a function in $C^\beta(\mathbf{R}^n)$ to be closed set Ω, and the U_k indeed play the role of derivatives.

In our application we have $\Omega = \Lambda_2^c$ and $x = \lambda_2$. We consider the sequences

$$\{ \alpha_l^j \}_{j=0}^\infty \ (1 \leq l \leq N), \qquad \{ U^j \}_{j=0}^\infty \quad \text{and} \quad \{ V_{kl}^j \}_{j=0}^\infty \ (0 \leq k + l \leq M - 1)$$

of functions which also depend analytically on other variables, like λ_1 or λ_1 and $\theta_1, \theta_2, \ldots, \theta_n$. The dependence on the θ_i $(1 \leq i \leq n)$ is periodic with period 2π. From the present point of view these extra variables are regarded as parameters. The analyticity, as well as the periodicity in the θ-variables, can be preserved as one passes to the respective extensions.

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