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Existence of a closed geodesic on *p*-convex sets

by

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ABSTRACT. — The existence of a non constant closed geodesic on some nonsmooth sets is proved.

Key words : Closed geodesics, Lusternik-Fet theorem, nonsmooth analysis, *p*-convex sets.

RÉSUMÉ. — On montre l'existence d'une géodésique fermée non constante sur certains ensembles non réguliers.

0. INTRODUCTION

A well-known result by Lusternik-Fet (*see*, for instance, [12]) establishes the existence of a non-constant closed geodesic in a compact regular Riemannian manifold without boundary.

In [15], this result is generalized to cover manifolds with boundary.

In both cases, the problem is reduced to a research of critical points for the energy functional $f(\gamma)=\frac{1}{2}\int_0^1 |\gamma'|^2\,ds$ on the space of the admissible

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paths $X=\{\gamma \in W^{1,2}(0,1;\ M);\ \gamma(0)=\gamma(1)\}$ where M is the manifold considered.

In this paper, we shall extend Lusternik-Fet result to cover a more general situation, namely p-convex sets. Such class of sets was introduced in [9] and in a less restrictive version in [2], where is also proved the existence of infinitely many geodesics on M orthogonal to M_0 and M_1, under the hypothesis that M, M_0 and M_1 are p-convex subsets of $\mathbb{R}^n$.

Examples of p-convex sets are $C^{1,1}_{loc}$-submanifolds (possibly with boundary) of a Hilbert space and images under a $C^{1,1}_{loc}$-diffeomorphism of convex sets.

The motivation for considering Lusternik-Fet result in the context of p-convex sets comes from some remarks about regularity of f and X.

In the case handled by Lusternik-Fet, f is a regular functional and X is a regular Riemannian manifold, on the contrary, in [15], even if M is a regular manifold, X has not a natural structure of manifold and f is not regular. All that suggests that the more natural way to deal with this problem is to consider as starting-point irregular sets.

This consideration prompted the present work.

Other typical problems in differential geometry, concerning sets with a certain degree of irregularity, are treated in [17].

For proving our result, we use a variational technique adapted for non regular functionals. We characterize closed geodesics as "critical points" for the energy functional f on the space X of the admissible paths. Then, we prove that f is included in the class of φ-convex functions (*see*, for instance, [10]). For such functions, some adaptations of classical variational methods in critical point theory (such as deformation lemmas) are available (*see*, for instance, [4], [8], [13]).

The present work is divided in 4 sections.

In the first section, we recall the definition of p-convex sets and describe some properties of them. In the second one, we give a variational characterization for closed geodesics. The third section is a topological one. We deduce some homotopic properties of X. They together with a suitable deformation lemma are the basic tools for the proof of the existence of at least a non-constant closed geodesic on a p-convex subset of $\mathbb{R}^n$, in section four.

1. SOME RECALLS ON p-CONVEX SETS

In this section, we shall define p-convex sets and describe their properties.

Before, let us recall some notions of non-smooth analysis (*cf.* [3] to [7], [9], [10]).

From now on, H will be a real Hilbert space, $|.|$ and $(.,.)$ its norm and scalar product, respectively.

DEFINITION 1.1 (see *also* [3] *and* [6]). — *Let* Ω *be an open subset of* H *and* $f : \Omega \to \mathbb{R} \cup \{+\infty\}$ *a map.*

We set

$$D(f) = \{u \in \Omega : f(u) < +\infty\}.$$

Let u belong to $D(f)$. *The function f is said to be subdifferential at u if there exists* $\alpha \in H$ *such that*

$$\liminf_{v \to u} \frac{f(v) - f(u) - (\alpha, v-u)}{|v-u|} \geqq 0.$$

We denote by $\partial^- f(u)$ *the (possibly empty) set of such* α*'s and we set*

$$D(\partial^- f) = \{u \in D(f) : \partial^- f(u) \neq \varnothing\}.$$

It is easy to check that $\partial^- f(u)$ *is convex and closed* $\forall u \in D(f)$.

If $u \in D(\partial^- f)$, $\operatorname{grad}^- f(u)$ *will denote the element of minimal norm of* $\partial^- f(u)$. *Moreover, let* M *be a subset of* H. *We denote by* I_M *the function:*

$$I_M(u) = \begin{cases} 0, & u \in M \\ +\infty, & u \in H \setminus M. \end{cases}$$

It is easy to check that $\partial^- I_M(u)$ *is a cone* $\forall u \in M$.

We will call normal cone to M *at u the set* $\partial^- I_M(u)$ *and tangent cone to* M *at u its negative polar* $(\partial^- I_M(u))^-$, *i. e.,*

$$(\partial^- I_M(u))^- = \{v \in H : (v, w) \leqq 0, \forall w \in \partial^- I_M(u)\}.$$

DEFINITION 1.2. — *A point* $u \in D(f)$ *is said to be critical from below for f if* $0 \in \partial^- f(u)$; $c \in \mathbb{R}$ *is said to be a critical value of f it there exists* $u \in D(f)$ *such that*

$$0 \in \partial^- f(u) \quad \text{and} \quad f(u) = c.$$

DEFINITION 1.3 (see *also* [5], [10]). — *Let* Ω *be an open subset of* H. *A function* $f : \Omega \to \mathbb{R} \cup \{+\infty\}$ *is said to have a* φ*-monotone subdifferential if there exists a continuous function*

$$\varphi : \quad D(f) \times \mathbb{R}^2 \to \mathbb{R}^+$$

such that:

$$(\alpha - \beta, u - v) \geqq -(\varphi(u, f(u), |\alpha|) + \varphi(v, f(v), |\beta|)) |u - v|^2$$

whenever

$$u, v \in D(\partial^- f), \qquad \alpha \in \partial^- f(u) \qquad \text{and} \qquad \beta \in \partial^- f(v).$$

If $p \geqq 1$, f is said to have a φ-monotone subdifferential of order p if there exists a continuous function

$$\chi: \quad D(f)^2 \times \mathbb{R}^2 \to \mathbb{R}^+$$

such that:

$$(\alpha-\beta, u-v) \geqq -\chi(u, v, f(u), f(v))(1+|\alpha|^p+|\beta|^p)|u-v|^2$$

whenever

$$u, v \in D(\partial^- f), \qquad \alpha \in \partial^- f(u) \qquad \text{and} \qquad \beta \in \partial^- f(v).$$

Now let us give the definition of p-convex sets (*cf.* [2]).

DEFINITION 1.4. — *Let* M *be a subset of* H. M *is said to be a p-convex set if there exists a continuous function $p: M \to \mathbb{R}^+$ such that*

$$(\alpha, v-u) \leqq p(u)|\alpha||v-u|^2$$

whenever $u, v \in M$ and $\alpha \in \partial^- I_M(u)$.

Examples of p-convex sets are the following ones:

(1) the $C^{1,1}_{loc}$-submanifolds (possibly with boundary) of H;

(2) the convex subsets of H;

(3) the images under a $C^{1,1}_{loc}$-diffeomorphism of convex sets;

(4) the subset of $\mathbb{R}^n$: $\{x : \max|x_i| \leqq 1, \sum x_i^2 \geqq 1\}$ [note that it is not included in the classes (1), (2), (3)].

Several properties of p-convex sets are proved in [2]. We recall some of them.

Let us define the following set relatively to a p-convex set M:

DEFINITION 1.5. — *Let us denote by $\hat{A}$ the set of u's $\in$ H with the two properties:*

(i) $\delta_p(u, M) < 1$ *where* $\delta_p(u, M) = \limsup_{\substack{|u-w| \to d(u, M) \\ w \in M}} 2p(w)|u-w|$.

(ii) $\exists r \geqq 0$ *such that* $M \cap \{v \in H : |v-u| \leqq r\}$ *is closed in* H *and not empty.*

Obviously, $M \subset \hat{A}$ and:

PROPOSITION 1.6. — *Let $M \subset H$ be p-convex and locally closed. Then $\hat{A}$ is open and $\forall u \in \hat{A}$ there exists one and only one $w \in M$ such that $|u-w| = d(u, M)$.*

Moreover, if we set $\pi(u) = w$, then

(i) $(u-\pi(u)) \in \partial^- I_M(\pi(u))$ *and* $2p(\pi(u))|u-\pi(u)| < 1$, $\forall u \in \hat{A}$.

(ii) $|\pi(u_1)-\pi(u_2)| \leqq (1-p(\pi(u_1))|u_1-\pi(u_1)| - p(\pi(u_2))|u_2-\pi(u_2)|)^{-1}|u_1-u_2|$, $\forall u_1, u_2 \in \hat{A}$.

(iii) $(t\pi(u)+(1-t)u) \in \hat{A}$, $\forall u \in \hat{A}$, $\forall t \in [0,1]$.

Remark 1.7. — *Let us set $A = \{u \in \hat{A} : 4p(\pi(u))|u-\pi(u)| < 1\}$. Then* A *is an open set containing* M *and one can easily prove that $\pi : A \to M$ is Lipschitz continuous of constant two.*

PROPOSITION 1.8. — *Let* $M \subset H$ *be locally closed and p-convex. Then*

$$\lim_{s \to 0^+} \frac{\pi(u+sv)-u}{s} = P_u(v)$$

$\forall u \in M$ *and* $\forall v \in H$, *where* P_u *is the projection on the tangent cone to* M *at* u, *i. e.* $(\partial^- I_M(u))^-$.

PROPOSITION 1.9. — *Let* $M \subset H$ *be locally closed and p-convex. Let us take* $u \in M$ *and* $B(u, r) = \{ v \in H : |v-u| < r \} \subset \hat{A}$. *Then*

$$|su_1+(1-s)u_0-\pi(su_1+(1-s)u_0)| \leq 2p(\pi(su_1+(1-s)u_0))\,s(1-s)\,|u_0-u_1|^2$$

$\forall s \in [0,1]$ *and* $\forall u_0, u_1 \in B(u, r)$.

PROPOSITION 1.10. — *Let* $M \subset H$ *be locally closed and p-convex. Then* M *is an absolute neighbourhood retract* (see [14] *for the definition of absolute neighbourhood retract*).

Finally, let us point out that the two definitions of tangent cone given in [1] and in [3] coincide in the case of p-convex sets. Indeed:

PROPOSITION 1.11. — *Let* $M \subset H$ *be locally closed and p-convex. Then* $\forall u \in M$

$$C_M(u) = T_M(u) = (\partial^- I_M(u))^-,$$

where $C_M(u)$ *and* $T_M(u)$ *are respectively the tangent cone and the contingent cone to* M *at* u.

2. VARIATIONAL CHARACTERIZATION OF CLOSED GEODESICS

In this section, H will indicate a real Hilbert space, $M \subset H$ a locally closed p-convex set and we will deal with closed geodesics on M, namely:

DEFINITION 2.1. — *A curve* $\gamma : [0,1] \to M$ *is said to be a closed geodesic on* M *if*

(*a*) $\gamma \in W^{2,1}(0,1; H)$;

(*b*) $\gamma''(s) \in \partial^- I_M(\gamma(s))$ *a. e. in*$]0,1[$;

(*c*) $\gamma(0) = \gamma(1)$ *and* $\gamma'_+(0) = \gamma'_-(1)$.

We want to characterize them as critical points for the energy functional

$$f: \quad L^2(0,1; H) \to \mathbb{R} \cup \{+\infty\}$$

defined in such a way:

$$f(\gamma)=\begin{cases}\dfrac{1}{2}\displaystyle\int_0^1 |\gamma'|^2\,ds, & \gamma\in X\\ +\infty, & \gamma\in L^2(0,1;\,H)\setminus X\end{cases}$$

where

$$X=\{\gamma\in W^{1,2}(0,1;\,H):\gamma(s)\in M,\ \forall s,\ \gamma(0)=\gamma(1)\}$$

is the so called space of the admissible paths.

For this purpose, let us state:

THEOREM 2.2. — *Let us take* $\gamma\in X$. *Then* $\partial^- f(\gamma)\neq\varnothing$ *if and only if*

$$\gamma\in W^{2,2}(0,1;\,H)\qquad\text{and}\qquad \gamma'_+(0)=\gamma'_-(1);$$

in such a case

$$\|\operatorname{grad}^- f(\gamma)\|_{L^2}\leqq\|\gamma''\|_{L^2}\leqq\theta(\bar p, f(\gamma))(1+\|\operatorname{grad}^- f(\gamma)\|_{L^2})$$

where $\bar p=\max_{[0,1]}(p\circ\gamma)$ *and* $\theta:\mathbb{R}^2\to\mathbb{R}^+$ *is a continuous function.*

Moreover, if $0\in\partial^- f(\gamma)$ *then* $\gamma\in W^{2,\infty}(0,1;\,H)$.

Before the proof, we give some lemmas which are essentially contained in [2].

If $\gamma\in X$ and $\delta\in L^2(0,1;\,H)$, we set:

$$(P_\gamma\delta)(s)=P_{\gamma(s)}\delta(s)$$

where $P_{\gamma(s)}$ is the projection on the tangent cone to M at $\gamma(s)$.

By Proposition 1.8, $P_\gamma\delta\in L^2(0,1;\,H)$.

LEMMA 2.3 (see [2], *Lemma* 3.3). — *Let us take* $\delta\in W^{1,2}(0,1;\,H)$ *and* $\gamma\in W^{1,2}(0,1;\,H)$ *such that* $\gamma(s)\in M$, $\forall s\in[0,1]$. *Then*

$$\liminf_{t\to0^+}\frac{\dfrac{1}{2}\displaystyle\int_0^1|(\gamma+t\delta)'|^2\,ds-\dfrac{1}{2}\displaystyle\int_0^1|\pi(\gamma+t\delta)'|^2\,ds}{t}$$

$$\geqq-2\int_0^1 p(\gamma)\,|\delta-P_\gamma\delta|\,.\,|\gamma'|^2\,ds.$$

LEMMA 2.4. — *Let us take* $\gamma\in X$ *and* $\alpha\in\partial^- f(\gamma)$. *Then*

$$\int_0^1(\gamma',\delta')\,ds\geqq\int_0^1(\alpha,P_\gamma\delta)\,ds-2\int_0^1 p(\gamma)\,|\delta-P_\gamma\delta|\,.\,|\gamma'|^2\,ds$$

$\forall\delta\in W^{1,2}(0,1;\,H)$ *with* $\delta(0)=\delta(1)$.

Proof. — Let us take $\delta\in W^{1,2}(0,1;\,H)$ with $\delta(0)=\delta(1)$.

We observe that, if $t>0$ is sufficiently small, we can define $\pi(\gamma+t\delta)$ and:

$$\pi(\gamma+t\delta)(s)\in M,\qquad \pi[(\gamma+t\delta)(0)]=\pi[(\gamma+t\delta)(1)].$$

Then

$$\frac{1}{2}\int_0^1 |\pi(\gamma+t\,\delta)'|^2 = f(\pi(\gamma+t\,\delta)).$$

Now, let us consider $\alpha \in \partial^- f(\gamma)$. By Proposition 1.8, we have:

$$\int_0^1 (\gamma', \delta')\,\mathrm{d}s - \int_0^1 (\alpha, \mathrm{P}_\gamma \delta)\,ds$$
$$= \lim_{t\to 0^+} \frac{1}{t}\int_0^1 \left\{\frac{1}{2}|(\gamma+t\,\delta)'|^2 - \frac{1}{2}|\gamma'|^2 - \alpha(\pi(\gamma+t\,\delta)-\gamma)\right\} ds$$
$$\geqq \liminf_{t\to 0^+} \frac{1}{t}\int_0^1 \left\{\frac{1}{2}|\pi(\gamma+t\,\delta)'|^2 - \frac{1}{2}|\gamma'|^2 - \alpha(\pi(\gamma+t\,\delta)-\gamma)\right\}$$
$$+ \liminf_{t\to 0^+} \frac{1}{2t}\int_0^1 \{|(\gamma+t\,\delta)'|^2 - |\pi(\gamma+t\,\delta)'|^2\}\,ds.$$

Recalling that $\left(\dfrac{\pi(\gamma+t\,\delta)-\gamma}{t}\right)$ is bounded in $\mathrm{L}^2(0,1;\mathrm{H})$, the thesis is a consequence of Definition 1.1 and Lemma 2.3. ■

LEMMA 2.5 (see [2], *Lemma* 3.5). — *Let* $\alpha \in \mathrm{L}^2(0,1;\mathrm{H})$ *and* $\gamma \in \mathrm{W}^{1,2}(0,1;\mathrm{H})$ *be such that* $\gamma(s) \in \mathrm{M}$, $\forall s \in [0,1]$.

Let us suppose that:

$$\int_0^1 (\gamma', \delta')\,ds \geqq \int_0^1 (\alpha, \mathrm{P}_\gamma \delta)\,ds - 2\int_0^1 p(\gamma)\,|\delta - \mathrm{P}_\gamma \delta|\,.\,|\gamma'|^2 ds$$
$$\forall \delta \in \mathrm{W}_0^{1,2}(0,1;\mathrm{H}).$$

Then

$$\gamma \in \mathrm{W}^{2,2}(0,1;\mathrm{H}), \qquad \gamma''(s)+\alpha(s) \in \partial^- \mathrm{I_M}(\gamma(s)) \quad \text{a. e.},$$

and

$$\|\gamma''\|_{\mathrm{L}^2} \leqq \left[1+2\bar{p}\left(\int_0^1 |\gamma'|^2\,ds\right)^{1/2}\right]\left(2\bar{p}\int_0^1 |\gamma'|^2\,ds + \|\alpha\|_{\mathrm{L}^2}\right)$$

where $\bar{p} = \max_{[0,1]} p \circ \gamma$.

LEMMA 2.6. — *Let us take* $\gamma \in \mathrm{X} \cap \mathrm{W}^{2,1}(0,1;\mathrm{H})$ *with* $\gamma'_+(0) = \gamma'_-(1)$ *and* $\alpha \in \mathrm{L}^1(0,1;\mathrm{H})$ *such that* $\alpha + \gamma'' \in \partial^- \mathrm{I_M}(\gamma)$ *a. e. Then* $\forall \eta \in \mathrm{X}$,

$$f(\eta) \geqq f(\gamma) + \int_0^1 (\alpha, \eta - s)\,ds - \theta_1(\bar{p})(1+\|\gamma''\|_{\mathrm{L}^1}^2 + \|\alpha\|_{\mathrm{L}^1}^2)\,\|\eta-\gamma\|_{\mathrm{L}^2}^2$$

where $\bar{p} = \max_{[0,1]} p \circ \gamma$ *and* $\theta_1 : \mathbb{R} \to \mathbb{R}^+$ *is a continuous function.*

Proof. — If $\eta \in X$, then:

$$f(\eta)-f(\gamma)-\int_0^1 (\alpha, \eta-\gamma)\,ds$$
$$=\frac{1}{2}\int_0^1 |\eta'-\gamma'|^2\,ds+\int_0^1 (\gamma', \eta'-\gamma')\,ds-\int_0^1 (\alpha, \eta-\gamma)\,ds$$
$$=\frac{1}{2}\int_0^1 |\eta'-\gamma'|^2\,ds-\int_0^1 (\alpha+\gamma'', \eta-\gamma)\,ds.$$

By p-convexity of M, we have:

$$\frac{1}{2}\int_0^1 |\eta'-\gamma'|^2\,ds-\int_0^1 (\alpha+\gamma'', \eta-\gamma)\,ds$$
$$\geqq \frac{1}{2}\int_0^1 |\eta'-\gamma'|^2\,ds-\int_0^1 p(\gamma)\,|\alpha+\gamma''|.|\eta-\gamma|^2\,ds$$
$$\geqq \frac{1}{2}\int_0^1 |\eta'-\gamma'|^2\,ds-\bar{p}\,\|\alpha+\gamma''\|_{L^1}\|\eta-\gamma\|_{L^\infty}^2. \quad (2.6.1)$$

Using in (2.6.1) the following estimate:

$$\|\eta-\gamma\|_{L^\infty}^2 \leqq \|\eta-\gamma\|_{L^2}^2+2\|\eta-\gamma\|_{L^2}\|\eta'-\gamma'\|_{L^2}$$

and then applying Young's inequality to the factor

$$2\|\eta-\gamma\|_{L^2}\|\eta'-\gamma'\|_{L^2},$$

we obtain:

$$\frac{1}{2}\int_0^1 |\eta'|^2\,ds-\frac{1}{2}\int_0^1 |\gamma'|^2\,ds-\int_0^1 (\alpha, \eta-\gamma)\,ds$$
$$\geqq \frac{1}{2}\int_0^1 |\eta'-\gamma'|^2\,ds-\bar{p}\,\|\alpha+\gamma''\|_{L^1}(\|\eta-\gamma\|_{L^2}^2+2\|\eta-\gamma\|_{L^2}\|\eta'-\gamma'\|_{L^2})$$
$$\geqq \frac{1}{2}\int_0^1 |\eta'-\gamma'|^2\,ds-2\bar{p}^2\|\alpha+\gamma''\|_{L^1}^2\|\eta-\gamma\|_{L^2}^2$$
$$-\bar{p}\,\|\alpha+\gamma''\|_{L^1}\|\eta-\gamma\|_{L^2}^2-\frac{1}{2}\int_0^1 |\eta'-\gamma'|^2\,ds$$

which gives the thesis. ■

Now we come back to the

Proof of theorem 2.2. — If $\partial^- f(\gamma)\neq\emptyset$, as a consequence of Definition 1.1 and Lemmas 2.4, 2.5, we get:

$$\gamma \in W^{2,2}(0,1;\,H)$$

and

$$\|\gamma''\|_{L^2} \leqq (1+2\bar{p}\sqrt{2f(\gamma)})(4\bar{p}\,f(\gamma)+\|\alpha\|_{L^2}).$$

If $0 \in \partial^- f(\gamma)$, from Lemma 2.4, we obtain $\forall \delta \in W_0^{1,2}(0,1; H)$:

$$\int_0^1 (\gamma', \delta')\, ds \geqq -2 \int_0^1 p(\gamma) \,|\delta - P_\gamma \delta| . |\gamma'|^2\, ds \qquad (2.2.1)$$

Since

$$\gamma' \in L^\infty(0,1; H),$$

$$\left| \int_0^1 (\gamma', \delta')\, ds \right| \leqq 2\bar{p} \,\|\gamma'\|_{L^\infty}^2 \|\delta\|_{L^1}, \qquad \forall \delta \in W_0^{1,2}(0,1; H)$$

and by duality:

$$\gamma'' \in L^\infty(0,1; H).$$

Now, let us prove that $\gamma'_-(1) = \gamma'_+(0)$.

Let us consider $v \in H$ and $\forall n \in \mathbb{N}$, $\rho_n \in W^{1,2}(0,1)$ such that

$$0 \leqq \rho_n \leqq 1, \qquad \rho_n(0) = \rho_n(1) = 1,$$
$$\rho_n = 0 \quad \textit{in} \quad \left[\frac{1}{2n}, 1 - \frac{1}{2n}\right].$$

Then, let us define the following functions:

$$\delta_n = \rho_n v, \qquad \forall n \in \mathbb{N}.$$

Again, from Lemma 2.4, we have:

$$\int_0^1 (\gamma', \delta'_n)\, ds \geqq \int_0^1 (\alpha, P_\gamma \delta_n)\, ds - 2 \int_0^1 p(\gamma) \,|\delta_n - P_\gamma \delta_n| . |\gamma'|^2\, ds \quad (2.2.2)$$

Integrating by parts and passing to the limit as $n \to \infty$, we obtain:

$$(\gamma'_-(1) - \gamma'_+(0), v) \geqq 0, \qquad \forall v \in H$$

and then

$$\gamma'_-(1) = \gamma'_+(0).$$

Now suppose that $\gamma \in W^{2,2}(0,1; H)$ and $\gamma'_+(0) = \gamma'_-(1)$. By applying Lemma 2.6 with $\alpha = -\gamma''$, we get $-\gamma'' \in \partial^- f(\gamma)$, so that

$$\|\operatorname{grad}^- f(\gamma)\|_{L^2} \leqq \|\gamma''\|_{L^2}. \quad \blacksquare$$

THEOREM 2.7. — *Let us consider* $\gamma \in X \cap W^{2,2}(0,1; H)$ *with* $\gamma'_+(0) = \gamma'_-(1)$ *and* $\alpha \in L^2(0,1; H)$.

Then $\alpha \in \partial^- f(\gamma)$ if and only if $\alpha(s) + \gamma''(s) \in \partial^- I_M(\gamma(s))$ *a. e.*

Moreover $\operatorname{grad}^- f(\gamma) = -P_\gamma(\gamma'')$.

Proof. — If $\alpha \in \partial^- f(\gamma)$, by Lemmas 2.4 and 2.5 we get

$$\alpha(s) + \gamma''(s) \in \partial^- I_M(\gamma(s)) \quad \text{a. e.}$$

Viceversa, if $\alpha(s) + \gamma''(s) \in \partial^- I_M(\gamma(s))$ a. e., we apply Lemma 2.6 obtaining $\alpha \in \partial^- f(\gamma)$.

Now, since $-P_\gamma \gamma'' \in L^2$ and $-P_\gamma \gamma'' \in \partial^- f(\gamma)$, if $\alpha \in \partial^- f(\gamma)$ then $\int_0^1 (\alpha+\gamma'', P_\gamma \gamma'')\, ds \leqq 0$.

This means:

$$\int_0^1 (P_\gamma \gamma'', \gamma'')\, ds \leqq -\int_0^1 (\alpha, P_\gamma \gamma'')\, ds.$$

So that,

$$\| P_\gamma \gamma'' \|_{L^2}^2 \leqq \| \alpha \|_{L^2} \| P_\gamma \gamma'' \|_{L^2}. \quad \blacksquare$$

Now, we are ready to state the desired characterization:

THEOREM 2.8. — *Let us consider* $\gamma \in X$. *Then:* $0 \in \partial^- f(\gamma)$ *if and only if* γ *is a closed geodesic on* M; *in this case* $\gamma \in W^{2,\infty}(0,1; H)$ *and the function* $s \to |\gamma'(s)|$ *is constant.*

Proof. — If γ is a closed geodesic on M, we can apply Lemma 2.6 with $\alpha = 0$ obtaining $0 \in \partial^- f(\gamma)$.

Vice versa, if $0 \in \partial^- f(\gamma)$, from Theorem 2.2 we get:

$$\gamma \in W^{2,\infty}(0,1; H) \qquad \text{and} \qquad \gamma'_+(0) = \gamma'_-(1).$$

Moreover, by Theorem 2.7 we get

$$\gamma''(s) \in \partial^- I_M(\gamma(s)) \quad \text{a.e.}$$

so that, γ is a closed geodesic on M.

Finally, since $|\gamma'|^2$ is Lipschitz continuous, in order to prove that the function $s \to |\gamma'(s)|$ is constant, we will show that

$$(|\gamma'|^2)' = 0 \quad \text{a.e.}$$

Let us consider

$$\alpha \in \partial^- I_M(\gamma(s)).$$

From Definition 1.1, we have:

$$(\alpha, \gamma(t) - \gamma(s)) \leqq |\gamma(t) - \gamma(s)|\, \varepsilon(\gamma(t) - \gamma(s)) \tag{2.8.1}$$

where

$$\lim_{\substack{v \to 0 \\ v \in L^2}} \varepsilon(v) = 0.$$

Dividing by $(t-s)$ and passing to the limit as $t \to s^+$ and $t \to s^-$ in (2.8.1), we get:

$$(\alpha, \gamma'(s)) = 0, \qquad \forall \alpha \in \partial^- I_M(\gamma(s)), \quad \forall s \in]0, 1[$$

which gives the thesis recalling that

$$(|\gamma'(s)|^2)' = 2(\gamma'(s), \gamma''(s)) \qquad \text{and} \qquad \gamma''(s) \in \partial^- I_M(\gamma(s)) \quad \text{a.e.} \quad \blacksquare$$

At this point, the proof of the existence of closed geodesics on M is reduced to the research of critical points for f.

The method we want to use for this aim is based on the evolution theory, as developed in [5], [6], [7], [9] and [10]. Therefore we need to prove that f has a φ-monotone subdifferential of order two:

THEOREM 2.9. — *Let* M *be closed in* H. *Then f is l. s. c. and there exists a continuous function*

$$\varphi_0: \quad L^2 \times \mathbb{R} \to \mathbb{R}^+$$

such that:

$$f(\eta) \geqq f(\gamma) + \int_0^1 (\alpha, \eta-\gamma)\, ds - \varphi_0(\gamma, f(\gamma))(1+\|\alpha\|_{L^2}^2)\|\eta-\gamma\|_{L^2}^2$$

whenever $\eta, \gamma \in X$ *and* $\alpha \in \partial^- f(\gamma)$.

In particular, f has a φ-monotone subdifferential of order two.

Proof. — First we will prove that f is l. s. c.

Let us take $\{\gamma_n\}_n \in X$ such that:

$$\lim_n \gamma_n = \gamma \quad \text{in } L^2(0,1;\, H) \qquad \text{and} \qquad f(\gamma_n) \leqq c.$$

By definition of f, $\{\gamma_n\}_n$ converges weakly to γ in $W^{1,2}(0,1;\, H)$ and

$$\frac{1}{2}\int_0^1 |\gamma'|^2 \leqq c.$$

So, we have only to prove that $\gamma \in X$.

But, since $\{\gamma_n\}_n$ converges uniformly to γ in $[0, 1]$ and M is closed, we deduce that

$$\gamma(s) \in M, \qquad \forall s \in [0, 1]$$

and from $\gamma_n(1) = \gamma_n(0)$, $\forall\, n \in \mathbb{N}$, we have: $\gamma(0) = \gamma(1)$.

So, $\gamma \in X$.

Now, using Theorem 2.2, Theorem 2.7 and Lemma 2.6, we obtain the existence of a continuous function $\theta_2: \mathbb{R}^2 \to \mathbb{R}^+$ such that

$$f(\eta) \geqq f(\gamma) + \int_0^1 (\alpha, \eta-\gamma)\, ds - \theta_2(\bar{p}, f(\gamma))(1+\|\alpha\|_{L^2}^2)\|\eta-\gamma\|_{L^2}^2$$

whenever $\eta, \gamma \in X$, $\alpha \in \partial^- f(\gamma)$ and were $\bar{p} = \max_{[0,1]} p \circ \gamma$.

By paracompactness and partition of unity, we obtain the existence of φ_0. ■

3. HOMOTOPICAL PROPERTIES OF THE SPACE OF THE ADMISSIBLE PATHS

In this section, we want to deduce some "homotopical" properties of the space of the admissible paths X endowed with the $W^{1,2}$-topology. To this aim, let us recall the following result contained in [16] (*see* Theorem 8.14, page 189).

THEOREM 3.1. – *Let $p: X \to B$ be a fibration. Let $x_0 \in X$, $b_0 = p(x_0)$, $F = p^{-1}(b_0)$. If p has a cross section, then*

$$\pi_q(X, x_0) \approx \pi_q(F, x_0) \oplus \pi_q(B, b_0), \qquad \forall q \geqq 2$$

while $\pi(X, x_0)$ is a semi-direct product of $\pi_1(F, x_0)$ by $\pi_1(B, b_0)$.

From now on, if M is a metric space and $u_0 \in M$, we will denote by $\Omega(M, u_0)$ its loop space with base point u_0 and we will set:

$$X^* = \{ \gamma \in C([0, 1]; M) \text{ such that } \gamma(0) = \gamma(1) \}$$

endowed with the topology of the uniform convergence.

Remark 3.2. – *The map $p: X^* \to M$ defined by $p(\gamma) = \gamma(0)$ is a fibration and*

$$\text{if } u_0 \in M, \text{ then } p^{-1}(u_0) = \Omega(M, u_0).$$

Moreover, the map $\lambda: M \to X^*$ defined by

$$\lambda(u_0)(s) = u_0, \qquad \forall s \in [0, 1]$$

is a cross section.

As a consequence of Theorem 3.1, let us prove:

THEOREM 3.3 (see, *also, Lemmas* 2.11 *and* 2.12 *in* [11])*Let $M \subset \mathbb{R}^n$ be compact, p-convex, connected and non-contractible in itself. Then, there exists $k \in \mathbb{N}$ such that:*

(i) *There exists a continuous map $g: S^k \to X^*$ which is not homotopic to a constant.*

(ii) *Every continuous map $\tilde{g}: S^k \to M$ is homotopic to a constant.*

Proof. – First of all, let us observe that, by Proposition 1.10, M is also arcwise connected. If M is not simply connected, then X^* is not arcwise connected, so that there exists a continuous map $g: S^0 \to X^*$ which is not homotopic to a constant. On the other hand, M is arcwise connected, then every continuous map $\tilde{g}: S^0 \to M$ is homotopic to a constant.

If M is simply connected, then X^* and $\Omega(M)$ are arcwise connected. Since by Proposition 1.10, M is an A.N.R., $\pi_h(M)$ is not trivial for some h (*cf.* [14]). Let $k+1$ be the first integer such that $\pi_{k+1}(M)$ is not trivial ($k \geqq 1$). Applying Theorem 3.1, we have:

$$\pi_k(X^*) \approx \pi_k(\Omega(M)) \approx \pi_{k+1}(M).$$

Then $\pi_k(X^*)$ is not trivial, on the contrary $\pi_k(M)$ is trivial, so that the theorem is proved.

THEOREM 3.4.. — *Let $M \subset \mathbb{R}^n$ be compact and p-convex. If there exists $k \geqq 0$ and a continuous map $g: S^k \to X^*$ which is not homotopic to a constant, then there exists a continuous map $\tilde{g}: S^k \to X$ which is not homotopic to a constant.*

For the proof of this theorem, we need the following result contained in [8] (*see* Theorem 3.17).

THEOREM 3.5. — *Let W be an open subset of a real Hilbert space V and $g: W \to \mathbb{R} \cup \{+\infty\}$ be a l.s.c. function with a φ-monotone subdifferential of order 2. Then there exists a map $j: D(g) \to D(g)$ such that:*

(i) $j(g^b) \subset g^b$, $\forall b \in \mathbb{R}$, *where* $g^b = \{u \in \Omega : g(u) \leqq b\}$;

(ii) $j: (g^b, |.|_V) \to (g^b, d^*)$ *where*

$$d^*(u, v) = |u-v| + |g(u)-g(v)|, \qquad \forall u, v \in D(g)$$

is continuous and it is a homotopy inverse of the identity function: Id: $(g^b, d^*) \to (g^b, |.|_V)$.

Proof of theorem 3.4. — Let k be a natural number and $g: S^k \to X^*$ a continuous map which is not homotopic to a constant.

Let us set

$$X^*_A = \{\gamma \in C([0, 1]; A);\ \gamma(0) = \gamma(1)\}$$

endowed with the topology of the uniform convergence, where A is the set defined in Remark 1.7.

By Proposition 1.6, X^* is a deformation retract of X^*_A. Then the map $g: S^k \to X^*_A$ is not homotopic to a constant.

Moreover, since X^*_A is an open subset of the Bánach space:

$$X^*_{\mathbb{R}^n} = \{\gamma \in C([0, 1]; \mathbb{R}^n);\ \gamma(0) = \gamma(1)\},$$

by [14], we deduce that X^*_A is homotopically equivalent to

$$X_A = \{\gamma \in W^{1,2}(0, 1; \mathbb{R}^n);\ \gamma(0) = \gamma(1);\ \gamma(s) \in A\}$$

endowed with $W^{1,2}$-topology.

Therefore, there exists a continuous map $f_1: S^k \to X_A$ which is not homotopic to a constant.

Now, let a be a real number such that

$$\frac{1}{2}\int_0^1 |\gamma'|^2\, ds \leqq a, \qquad \forall \gamma \in f_1(S^k).$$

Then, setting

$$X^b_A = \left\{\gamma \in X_A \text{ such that } \frac{1}{2}\int_0^1 |\gamma'|^2\, ds \leqq b\right\},$$

we have that $f_1 : S^k \to X_A^b$ is not homotopic to a constant $\forall b \geqq a$.

At this point, let us remark the following: $\forall \gamma \in X_A^b$ there exists $r(\gamma) > 0$ such that if

$$\eta \in W^{1,2}(0,1;\ \mathbb{R}^n),$$

$$\frac{1}{2}\int_0^1 |\eta'|^2\,ds \leqq b \qquad \text{and} \qquad \|\eta - \gamma\|_{L^2} < r(\gamma)$$

then $\eta(s) \in A$, $\forall s \in [0, 1]$.

Now, let us set

$$V = L^2(0,1;\ \mathbb{R}^n); \qquad W = \bigcup_{\gamma \in X_A^b} B(\gamma, r(\gamma))$$

where $B(\gamma, r(\gamma))$ is the open ball in L^2 of center γ and radius $r(\gamma)$ and let us define a function $g : W \to \mathbb{R} \cup \{+\infty\}$ in such a way:

$$g(\gamma) = \begin{cases} \dfrac{1}{2}\displaystyle\int_0^1 |\gamma'|^2\,ds & \text{if } \gamma \in X_A^b \\ +\infty & \text{if } \gamma \in W \setminus X_A^b \end{cases}$$

Obviously, g is the restriction to W of a convex and l. s. c. function on $L^2(0, 1;\ \mathbb{R}^n)$.

Since $X_A^b = g^b$, by Theorem 3.5 we deduce that

$$i:\quad X_A^b \to \tilde{X}_A^b,$$

where $\tilde{X}_A^b$ is defined as the space X_A^b endowed with the L^2-topology, is a homotopy equivalence $\forall b \geqq a$.

Therefore, $f_1 : S^k \to \tilde{X}_A^b$ is not homotopic to a constant $\forall b \geqq a$.

Now, let us consider the following homotopy H defined on $f_1(S^k) \times [0, 1]$, in such a way:

$$H(\gamma, t)(s) = t\,\pi(\gamma(s)) + (1-t)\,\gamma(s).$$

By Remark 1.7, we have:

$$|H(\gamma, t)'(s)| \leqq 2t\,|\gamma'(s)| + (1-t)\,|\gamma'(s)| \leqq 2\,|\gamma'(s)|.$$

So that $H : f_1(S^k) \times [0, 1] \to \tilde{X}_A^b$ where $b \geqq 4a$.

Let us take $f_2 = H(\cdot, 1) \circ f_1$.

The map $f_2 : S^k \to \tilde{X}_A^b$ is not homotopic to a constant, moreover $f_2(S^k) \subset \tilde{X}^b$ where

$$\tilde{X}^b = \left\{ \gamma \in X : \frac{1}{2}\int_0^1 |\gamma'|^2\,ds \leqq b \right\}$$

endowed with the L^2-topology.

Then, $f_2 : S^k \to \tilde{X}^b$ is not homotopic to a constant $\forall b \geqq 4a$. Now, applying Theorem 3.5 to

$$V = W = L^2(0, 1;\ \mathbb{R}^n) \qquad \text{and} \qquad g \equiv f$$

where f is the energy functional defined in section 2, we deduce the existence of a map $j:\tilde{X}\to X$ where $\tilde{X}$ denotes the space X endowed with the L^2-topology such that $\forall b, j(\tilde{X}^b)\subset X^b$. Moreover j is continuous and it is a homotopy inverse of the identity function.

Finally, let us consider the continuous map $f_3: S^k\to X^b$ defined by $f_3=j\circ f_2$. It is not homotopic to a constant $\forall b\geqq 4a$ and then $f_3: S^k\to X$ is not homotopic to a constant. ■

THEOREM 3.6. — *Let $M\subset\mathbb{R}^n$ be compact and p-convex and f the functional defined in section 2. Then there exists $a>0$ such that*

$$f^a=\{\gamma:\gamma\in X \quad \text{and} \quad f(\gamma)\leqq a\}$$

endowed with the $W^{1,2}$-topology is homotopically equivalent to M.

For the proof of this theorem we will need the following lemma:

LEMMA 3.7. — *Let f^0 be the set of the constant curves. Then there exists $a>0$ such that f^0 is a strong deformation retract of f^a endowed with the L^2-topology.*

Proof. — Since M is compact, we can suppose that M is p-convex with $p\equiv$Const. Let us take $\gamma\in f^a$ and let us consider

$$t\gamma(0)+(1-t)\gamma(s) \quad \text{with} \quad t\in[0, 1].$$

We remark that:

$$d(t\gamma(0)+(1-t)\gamma(s), M)\leqq|t\gamma(0)+(1-t)\gamma(s)-\gamma(0)|$$
$$=(1-t)|\gamma(s)-\gamma(0)|\leqq\left(\int_0^1|\gamma'|^2\,ds\right)^{1/2}\leqq\sqrt{2a}. \quad (3.7.1)$$

Therefore, taking a such that $4p\sqrt{2a}<1$, by (3.7.1), we have that

$$t\gamma(0)+(1-t)\gamma(s)\in A$$

where A is defined in Remark 1.7.

Now we can consider the map H defined on $f^a\times[0, 1]$ in this way:

$$H(\gamma, t)(s)=\pi(t\gamma(0)+(1-t)\gamma(s)).$$

Let us observe that by Proposition 1.9:

$$d(t\gamma(0)+(1-t)\gamma(s), M)$$
$$=|t\gamma(0)+(1-t)\gamma(s)-\pi(t\gamma(0)+(1-t)\gamma(s)|$$
$$\leqq 2pt(1-t)|\gamma(0)-\gamma(s)|^2\leqq 4pat(1-t). \quad (3.7.2)$$

By (3.7.2) and (ii) of Proposition 1.6, we have:

$$\left|\frac{d}{ds}\mathrm{H}(\gamma, t)(s)\right| \leqq (1-8p^2 at(1-t))^{-1}(1-t)\left|\gamma'(s)\right| \leqq \left|\gamma'(s)\right|$$

so that we deduce:

$$\int_0^1 \left|\frac{d}{ds}\mathrm{H}(\gamma, t)(s)\right|^2 ds \leqq 2a.$$

Therefore,

$$\mathrm{H}(\gamma, t)(s): f^a \times [0, 1] \to f^a.$$

Moreover,

$$\mathrm{H}(\gamma, 0)(s) = \gamma(s) \quad \text{and} \quad \mathrm{H}(\gamma, 1)(s) = \gamma(0), \qquad \forall s \in [0, 1]$$

To conclude the proof it is enough to point out that if we endowe f^a with the L^2-topology, H is a continuous map. ■

Proof of Theorem 3.6. — By applying Theorem 3.5 to

$$\mathrm{W} = \mathrm{L}^2(0, 1; \mathbb{R}^n) \qquad \text{and} \qquad g \equiv f$$

where f is the functional defined in section 2, we obtain that f^a endowed with the $\mathrm{W}^{1,2}$-topology is homotopically equivalent to f^a with the L^2-topology.

On the other hand, M is homeomorphic to f^0 with the L^2-topology. Using lemma 3.7 we get the thesis. ■

THEOREM 3.8. — *There exists* $a > 0$ *such that* f^a *and* X (*both endowed with the* $\mathrm{W}^{1,2}$-*topology*) *are not homotopically equivalent.*

Proof. — Obvious from Theorems 3.3, 3.4 and 3.6. ■

4. THE MAIN RESULT

After Theorem 2.8, the problem to establish the existence of a non-constant closed geodesic on M, compact, connected and p-convex subset of $\mathbb{R}^n$, is reduced to find critical points for the energy functional f on the space of the admissible paths X (*see* section 2 for the Definition of f and X).

To this aim, we need a deformation lemma like the one contained in [13]. We shall use a version included in [8] (*see* Lemma 4.4).

LEMMA 4.1. — *Let* V *be a real Hilbert space and* $g: \mathrm{V} \to \mathbb{R} \cup \{+\infty\}$ *a l. s. c. function with a* φ-*monotone subdifferential of order* 2. *We set*

$$d^*(u, v) = |u - v| + |g(u) - g(v)|, \qquad \forall u, v \in \mathrm{D}(g).$$

Let $-\infty < a < b \leqq +\infty$ *be such that:*

(i) $0 \notin \partial^- g(u)$ *whenever* $u \in D(g)$ *and* $a \leqq g(u) \leqq b$;

(ii) $\forall c \in [a, b[$ *and* $\forall \{u_n\}_n \subset D(\partial^- g)$ *with* $\lim_n g(u_n) = c$ *and* $\lim_n \operatorname{grad}^- g(u_n) = 0$, $\{u_n\}_n$ *has a converging subsequence in* V.

Then g^a *is a strong deformation retract of* g^b *in* g^b, *where* g^a *and* g^b *are endowed with the metric* d^*.

Combining this lemma with the topological results in section 3, we can state the desired result:

THEOREM 4.2. — *Let* $M \subset \mathbb{R}^n$ *be compact, p-convex, connected and non-contractible in itself.*

Then, there exists at least a non-constant closed geodesic on M.

Proof. — Let us consider the energy functional f defined in section 2. By Theorem 2.9, f is l.s.c. and it has a φ-monotone subdifferential of order 2.

Moreover, by Theorem 2.8, the thesis is equivalent to state that there exists $\gamma \in X$ such that $0 \in \partial^- f(\gamma)$, and $f(\gamma) > 0$. So, if, by contradiction, the thesis is not true, we can apply Lemma 4.1 with

$$V = L^2(0, 1; \mathbb{R}^n), \qquad g \equiv f, \qquad b = +\infty$$

and a given by Theorem 3.8.

We recall that condition (ii) is satisfied because M is compact and the metric d^* induces the $W^{1,2}$-topology on $X = f^b$.

Then, by Lemma 4.1 we deduce that X and f^a are homotopically equivalent, which is impossible by Theorem 3.8. ■

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