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Convergence of convex-concave saddle functions: applications to convex programming and mechanics

by

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ABSTRACT. — It is shown that operation of partial conjugation (the partial Legendre-Fenchel transform) of bivariate convex-concave functions has bicontinuity properties with respect to the extended epi/hypo-convergence of saddle functions and the epi-convergence of the partial conjugate (convex) functions. The results are applied to study the stability of the optimal solutions and associated multipliers of convex programs, and to a couple of problems in mechanics.

Key words : Epi-convergence, variational convergence, epi/hypo-convergence, Legendre-Fenchel transform, conjugate, convex functions, homogenization, Lagrangians, Reisner functional.

RÉSUMÉ. — On montre que la conjuguée partielle (la transformation de Legendre-Fenchel partielle) de fonctions de selles convexes-concaves a des propriétés de bicontinuité par rapport à l'épi/hypo-convergence des

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fonctions de selles et l'épi-convergence des conjuguées partielles. On applique les résultats à l'étude de la stabilité des solutions et des multiplicateurs (de Lagrange associés à ces solutions) de problèmes d'optimisation convexe, ainsi qu'à des problèmes en mécanique qui proviennent de l'homogénéisation et du renforcement de matériaux.

One of the main results of the theory of epi-convergence is that on the space of proper, lower semicontinuous, convex functions defined on $\mathbb{R}^n$, the Legendre-Fenchel transform (conjugation) is bicontinuous with respect to the epi-topology. This result generalizes to reflexive Banach spaces, provided one works with a strengthened version of epi-convergence involving both the strong and the weak topologies of the underlying space and of its dual. In this paper we extend these results to the partial conjugation of bivariate convex functions that generate convex-concave functions, also called saddle functions. It is shown that appropriate notion of convergence for saddle function is that of epi/hypo-convergence introduced and studied earlier by Attouch and Wets [10]. The main results are proved in Section 3 (Theorems 3.1, 3.2 and 3.5). In Sections 1 and 2 we review the key definitions and derive some preliminary results. Section 4 is devoted to the application of the main results in the context of constrained convex programming. In this context, epi-convergence of the sequence of convex perturbation functions (and hence epi/hypo convergence of corresponding convex-concave Lagrangians) is obtained as a consequence of a general result (Theorem 4.1) concerning the epi-convergence of the sum of two sequences of closed convex functions. We prove a stability result that guarantees the convergence of the solutions as well as that of the associated dual multipliers (Theorem 4.3). In section 5 we sketch out a couple of applications in mechanics. First, we develop a unified approach to the study of the homogenization of composite materials in mechanics, that relies on the convergence of the associated Lagrangians, to obtain the convergence of the strain tensor fields as well as that of the stress tensor fields (Proposition 5.2 and Corollary 5.3). And second, we study a reinforcement problem when the thickness of the reinforced zone goes to zero.

1. EPI-CONVERGENCE: THE CONVEX CASE

We review the main features of the theory of epi-convergence of convex functions to set the stage for the latter investigation of convex-concave bivariate functions. We emphasize some aspects of the univariate theory that has been glossed over in earlier presentation but whose counterparts play a significant role in the bivariate case, in particular the notion of a closed convex functions.

The concept of epi-convergence was first utilized by R. A. Wijsman [1]. U. Mosco [2] was responsible for bringing to the fore the important relationship between epi-convergence and the convergence of solutions to variational inequalities. E. DeGiorgi *et al.* [3] extended widely the study of epi-convergence, under the name of Γ-convergence, in their study of integral functionals that arise in the Calculus of Variations. There is now a rich literature, consult [4], dealing with the theory (convex and nonconvex) as well as with the applications of epi-convergence. We have chosen to deal with epi-convergence rather than hypo-convergence. Obviously, every epi-result has his counterpart in the hypo-setting.

Let us review some definitions. The *effective domain* of a function $F : X \to \bar{\mathbb{R}}$ is the set

$$\operatorname{dom} F = \{ x \in X : F(x) < +\infty \}.$$

(X, τ) being a topological space, the *lower closure* (or lower semicontinuous regularization) of the function F is the function

$$(\mathrm{cl}_\tau F)(x) = \liminf_{y \to x} F(y) = \sup_{V \in N_\tau(x)} \inf_{y \in V} F(y) \tag{1.1}$$

where $N_\tau(x)$ denotes the system of neighborhoods of x with respect to the topology τ. A function F is τ-lower semicontinuous (τ-l. sc. or simply l. sc.) at x if

$$\mathrm{cl}_\tau F(x) = F(x). \tag{1.2}$$

It is τ-l. sc. if (1.2) holds at all $x \in X$ or equivalently if its *epigraph*

$$\operatorname{epi} F = \{ (x, \alpha) \in X \times \mathbb{R},\ \alpha \geqq f(x) \}$$

is a closed subset of $X \times \mathbb{R}$ with respect to the product topology of τ and the natural topology of $\mathbb{R}$. It is well known that

$$\operatorname{epi}(\mathrm{cl}_\tau F) = \mathrm{cl}_\tau(\operatorname{epi} F). \tag{1.3}$$

The *extended lower closure* of F is the function τ-$\underline{\mathrm{cl}}\, F$, also denoted $\underline{\mathrm{cl}}\, F$ if there is no risk of confusion, defined as follows

$$\underline{\mathrm{cl}}\, F = \begin{cases} \mathrm{cl}\, F & \text{if } \mathrm{cl}\, F > -\infty \\ -\infty & \text{otherwise.} \end{cases}$$

We say that F is *closed* if $F = \underline{\mathrm{cl}}\, F$. Note that the closure operation is basically of local character, as is evident from (1.1), whereas the extended closure involves the whole function. As a direct consequence of (1.1) and the definition of $\underline{\text{cl}}\, F$ we have for all open set G

(1.4) $\inf_{x \in G} F(x) = \inf_{x \in G} (\text{cl}\, F)(x)$, and $\inf_X (F) = \inf_X (\text{cl}\, F) = \inf_X (\underline{\text{cl}}\, F)$

The above closure operations arise naturally when considering biconjugation operations.

We consider now two linear spaces X and X* paired through a bilinear form $\langle .,. \rangle$. The weak topology $\sigma(X, X^*)$ is the coarsest one for which the linear forms $x \to \langle x, x^* \rangle$ are continuous. The topological dual space of X equipped with the topology $\sigma(X, X^*)$ is X^*. A locally convex topology τ is called consistent with the pairing $\langle .,. \rangle$ if the topological dual space of (X, τ) is X^*. For any convex set $C \subset X$, the closure is the same for all topologies consistent with the pairing, the same being true, from (1.3) for the l. sc. regularization of convex functions defined on X with extended real values. If $F: X \to \overline{\mathbb{R}}$ is a convex function, its conjugate is obtained via the *Legendre-Fenchel transformation*: $F^*: X^* \to \overline{\mathbb{R}}$ defined by

(1.5) $$F^*(x^*) = \sup_{x \in X} [\langle x, x^* \rangle - F(x)].$$

It is easy to show that F^* is convex and $\sigma(X^*, X)$-lower semicontinuous. The biconjugate F^{**} is $(F^*)^*$

(1.6) $$F^{**}(x) = \sup_{x^* \in X^*} [\langle x, x^* \rangle - F^*(x^*)].$$

It is clear that $F^{**} \leqq F$. Moreover, it is not difficult to see that

(1.7) $$F^{**} = F \text{ if and only if } F = \underline{\text{cl}}\, F \textit{ is convex.}$$

Let us now review the main topological features of epi-convergence (for more details *see* [4]). Let $\{F^n, n \in \mathbb{N}: X \to \overline{\mathbb{R}}\}$ be a sequence of functions defined on a topological space (X, τ). The τ-*epi-limit inferior* of the sequence $\{F^n, n \in \mathbb{N}\}$ is denoted by $\tau\text{-li}_e F^n$ and is defined by

(1.8) $$\tau\text{-li}_e F^n(x) = \sup_{G \in N_\tau(x)} \liminf_n \inf_{x \in G} F^n(x).$$

The τ-*epi-limit superior* is denoted by $\tau\text{-ls}_e F^n$, and is defined similarly

(1.9) $$\tau\text{-ls}_e F^n(x) = \sup_{G \in N_\tau(x)} \limsup_n \inf_{x \in G} F^n(x).$$

Both $\tau\text{-li}_e F^n$ and $\tau\text{-ls}_e F^n$, are τ-lower semicontinuous. A function F is said to be τ-*epi-limit* of the sequence F^n, and we write $F = \tau\text{-lm}_e F_n$, if

(1.10) $$\tau\text{-li}_e F^n = \tau\text{-ls}_e F^n = F$$

or equivalently

(1.11) $$\tau\text{-ls}_e F^n \leqq F \leqq \tau\text{-li}_e F^n$$

the converse inequality follows from the definition. In the case when X is a linear space, and the functions F^n are convex we have that

$\tau\text{-ls}_e F^n$ is convex, but

$\tau\text{-li}_e F^n$ is not necessarily convex.

Therefore in the convex case $\tau\text{-lm}_e F^n$, when it exists, is a convex l. sc. function. When the space (X, τ) is first countable, one can work with sequences and we have

$$\tau\text{-li}_e F^n = \inf_{\{x_n \xrightarrow{\tau} x\}} \liminf_n F^n(x_n) \tag{1.12}$$

$$\tau\text{-ls}_e F^n = \inf_{\{x_n \xrightarrow{\tau} x\}} \limsup_n F^n(x_n) \tag{1.13}$$

(*cf.* [4], Theorem 1.13). Moreover, these two infima are in fact attained.

In the sequel we deal with weak topologies on Banach spaces, and thus for the weak epi-limit *we use* (1.12) *and* (1.13) *as definition* and work with sequential epi-limits rather than topological epi-limits defined in (1.8) and (1.9). Note however, that, in general, topological and sequential epi-limits do not coincide. We write $w\text{-lm}_e$ for a weak epi-limit $s\text{-lm}_e$, for a strong epi-limit, $w^*\text{-lm}_e$ for the weak epi-limit of functions defined on the dual of X equipped with its weak topology, etc.

We now review, *cf.* [1], [5], [6], [7], the continuity properties of the Legendre-Fenchel transform. A sequence $\{F^n, n \in \mathbb{N} : X \to \bar{\mathbb{R}}\}$ of functions defined on a Banach space is said to be *upper modulated* if

(1.14) there exists a bounded sequence (x_n) in X such that

$$\limsup_n F^n(x_n) < +\infty.$$

The sequence is said to be *equi-coercive* if

$$\limsup_n (F^n)(x_n) < +\infty \text{ implies } (x_n) \text{ bounded.} \tag{1.15}$$

The following theorems can be found in [6] and [4]:

THEOREM 1.1. — *Let* X *be a reflexive Banach space,* $\{F^n, n \in \mathbb{N} : X \to \bar{\mathbb{R}}\}$ *a sequence of upper modulated convex, l. sc. functions, then*

$$(w\text{-li}_e F^n)^* = s\text{-ls}_e (F^n)^*.$$

THEOREM 1.2. — *Let* X *be a separable Banach space,* $\{F^n, n \in \mathbb{N} : X \to \bar{\mathbb{R}}\}$ *a sequence of convex, l. sc. proper functions such that the sequence* $(F^n)^*$ *is equi-coercive, then*

$$(s\text{-li}_e F^n)^* = w\text{-ls}_e (F^n)^*.$$

Theorems 1.1 and 1.2 suggest the following strengthened notion of epi-limit. Let X be a Banach space. A function $F: X \to \bar{\mathbb{R}}$ is the *Mosco-epi-limit* of the sequence $\{ F^n, n \in \mathbb{N} : X \to \bar{\mathbb{R}} \}$ and we write $F = M\text{-lm}_e F^n$, if

$$(1.16) \qquad F = w\text{-lm}_e F^n = s\text{-lm}_e F^n$$

or equivalently, for all $x \in X$

$$(1.17(a)) \qquad \text{there exists } x_n \xrightarrow{s} x \text{ such that } \limsup_n F^n(x_n) \leqq F(x)$$

$$(1.17(b)) \qquad \text{for all } x_n \xrightarrow{w} x, \quad F(x) \leqq \liminf_n F^n(x_n).$$

The Legendre-Fenchel transform is bicontinuous with respect to the Mosco-epi-topology on the space of l. sc. convex functions; this follows from the following theorem.

THEOREM 1.3 ([1], [5], [6]). — *Let* X *be a reflexive separable Banach space and* $\{ F; F^n, n \in \mathbb{N} : X \to \bar{\mathbb{R}} \}$ *be l. sc., proper, convex functions. Then*

$$F = M\text{-lm}_e F^n \textit{ if and only if } F^* = M\text{-lm}_e (F^n)^*.$$

Let us now consider bivariate functions defined on a product space $(X, \tau) \times (Y, \sigma)$. We write $(\tau \times \sigma)\text{-lm}_e F^n$ for the $\tau \times \sigma$-epi-limit of a sequence $\{ F^n, n \in \mathbb{N} \}$, assuming it exists, if the calculation of the epi-limit at (x, y) is made with sequences $\{ (x_n, y_n), n \in \mathbb{N} \}$ such that the (x_n) τ-converge to x and the (y_n) σ-converge to y. We are particularly interested in the following situation:

X is a Banach space,

Y is a separable Banach space,

$F^n : X^* \times Y \to \bar{\mathbb{R}}, n \in \mathbb{N}$,

$\langle (x^*, y), (x, y^*) \rangle = \langle x^*, x \rangle + \langle y, y^* \rangle$ pairing of $X^* \times Y$ with $X \times Y^*$.

We say that the sequence $\{ F^n : X^* \times Y \to \bar{\mathbb{R}}, n \in \mathbb{N} \}$ satisfies *assumption* $\mathscr{H}$ if

$$(1.18) \quad \begin{cases} \exists\, r > 0 \text{ such that for every sequence } (y_n) \text{ in } Y \text{ with } \| y_n \| \leqq r, \\ \exists \text{ a bounded sequence } (x_n^*) \text{ in } X^* \text{ with } \limsup_n F^n(x_n^*, y_n) < +\infty. \end{cases}$$

In the framework of the duality theory for bivariate functions, *cf.* Rockafellar [8] and Ekeland-Temam [9], assumption $\mathscr{H}$ corresponds to the classical regularity qualification: there exist $x_n^* \in X^*$, $r_n > 0$ and $M_n \in \mathbb{R}$ such that

$$\sup_{\| y \| \leqq r_n} F^n(x_n^*, y) \leqq M_n.$$

If $r=\inf r_n>0$ and $M=\sup M_n<+\infty$, and the functions $\{F_n(., 0), n\in\mathbb{N}\}$ are equi-coercive, then assumption $\mathscr{H}$ is clearly satisfied. Combining the proofs of Theorems 1.1 and 1.2, as done in [40], Theorem 3.3, we obtain the following:

THEOREM 1.4. — *Let* $\{F; F^n, n\in\mathbb{N}: X^*\times Y\to\bar{\mathbb{R}}\}$ *be a collection of convex, l. sc. functions such that the sequence* (F^n) *satisfies assumption* $\mathscr{H}$ (1.18), *then*

$$F\leqq(w^*\times s)\text{-li}_e F^n$$

implies

$$F^*\geqq(s\times w^*)\text{-ls}_e(F^n)^*.$$

In Section 2 we briefly review the main definitions and properties of the variational notion of convergence for saddle functions introduced in [10], [11]: epi/hypo-convergence. This notion is well adapted to our purposes, since in Section 3 we show that the epi-convergence of convex bivariate functions is equivalent to the epi/hypo-convergence of their partial Legendre-Fenchel transform.

2. EPI/HYPO-CONVERGENCE OF BIVARIATE FUNCTIONS

We review the definition and the main properties of epi/hypo-convergence (for further details *see* [10], [11]). Let us consider topological spaces (X, τ) and (Y, σ) and a sequence $\{F^n: X\times Y\to\bar{\mathbb{R}}, n\in\mathbb{N}\}$. The *hypo/epi-limit inferior* of the sequence (F^n) is the function denoted by $h_\sigma/e_\tau\text{-li}\, F^n$ and defined by

$$(2.1)\quad (h_\sigma/e_\tau\text{-li}\, F^n)(x, y)=\inf_{V\in N_\sigma(y)}\ \sup_{U\in N_\tau(x)}\ [\liminf_n(\inf_{u\in U}\ \sup_{v\in V} F^n(u, v))].$$

The *epi/hypo-limit superior* denoted by $e_\tau/h_\sigma\text{-ls}\, F^n$ is defined by

$$(2.2)\quad (e_\tau/h_\sigma\text{-ls}\, F^n)(x, y)=\sup_{U\in N_\tau(x)}\ \inf_{V\in N_\sigma(y)}\ [\limsup_n(\sup_{v\in V}\ \inf_{u\in U} F^n(u, v))].$$

A bivariate function F is said to be an epi/hypo-limit of the sequence (F^n) if

$$(2.3)\qquad e_\tau/h_\sigma\text{-ls}\, F^n\leqq F\leqq h_\sigma/e_\tau\text{-li}\, F^n.$$

Thus, in general epi-hypo-limits are not unique. This is not the only type of convergence of bivariate functions that could be defined. In fact our two limit functions are just two among many possible limits of bivariate functions [10]. The choice of these two functions is in some sense minimal (*see* [11], Section 2) to obtain convergence of saddle points as made clear in Section 4 of [10]. Other definitions have been proposed by

Cavazutti [12], [13], Greco [14] (*see* also Sonntag [15]). They all imply epi/hypo-convergence, but unduly restrict the domain of applications.

Finally, observe that when the F^n do not depend on y, then the definition of epi/hypo-convergence specializes to the classical definition of epi-convergence (with respect to the variable x). On the other hand, if the F^n do not depend on x, then epi/hypo-convergence is simply hypo-convergence. Thus the theory contains both the theory of epi and hypo-convergence.

When the topological spaces (X, τ) and (Y, σ) are metrizable, it is possible to give a representation of the limits in terms of sequences that turns out to be very useful in verifying epi/hypo-convergence, *cf.* [10], Corollary 4.4. The formulas that we give here in terms of sequences rather than subsequences. They complement those given earlier in [10].

THEOREM 2.1. — *Suppose* (X, τ) *and* (Y, σ) *are two metrizable spaces and* $\{F^n : X \times Y \to \bar{\mathbb{R}}, n \in \mathbb{N}\}$ *is a sequence of functions. Then for every* $(x, y) \in X \times Y$

$$e/h\text{-ls}\, F^n(x, y) = \sup_{y_n \xrightarrow{\sigma} y} \min_{x_n \xrightarrow{\tau} x} \limsup_n F^n(x_n, y_n) = \sup_{\substack{\{n_k\} \subset \mathbb{N} \\ y_k \xrightarrow{\sigma} y}} \min_{x_k \xrightarrow{\tau} x} \limsup_k F^{n_k}(x_k, y_k) \tag{2.4}$$

and

$$h/e\text{-li}\, F^n(x, y) = \inf_{x_n \xrightarrow{\tau} y} \max_{y_n \xrightarrow{\sigma} y} \liminf_n F^n(x_n, y_n) = \inf_{\substack{\{n_k\} \subset \mathbb{N} \\ x_k \xrightarrow{\tau} x}} \max_{y_k \xrightarrow{\sigma} y} \liminf_k F^{n_k}(x_k, y_k). \tag{2.5}$$

These characterizations of the limits functions yield directly the following criterion for epi/hypo-convergence.

COROLLARY 2.2. — *Suppose* (X, τ) *and* (Y, σ) *are metrizable, and* $\{F^n : X \times Y \to \bar{\mathbb{R}}, n \in \mathbb{N}\}$ *is a sequence of functions. Then the following assertions are equivalent.*

$$F = e_\tau/h_\sigma\text{-lim}\, F^n; \tag{2.6}$$

(2.7) *for all* $(x, y) \in X \times Y$

(i) *to every* $y_n \xrightarrow{\sigma} y$ *there corresponds* $x_n \xrightarrow{\tau} x$ *such that*

$$\limsup_n F^n(x_n, y_n) \leqq F(x, y),$$

and

(ii) *to every* $x_n \xrightarrow{\tau} x$, *there corresponds* $y_n \xrightarrow{\sigma} y$ *such that*

$$F(x, y) \leqq \liminf_n F^n(x_n, y_n).$$

Formulas (2.4) and (2.5) define epi/hypo-limits in terms of sequences. As in the case of epi/convergence [*see* (1.12), (1.13)], when applying the theory, one often has to work with weak-topologies on a (nonnecessarily reflexive) Banach space, the topological definitions of epi-limits are then not easy to handle. This leads us to introduce sequential notions of epi/hypo-limits which coincide with the topological ones when the underlying spaces are metrizable. For a sequence $\{F^n : X \times Y \to \mathbb{R};\ n \in \mathbb{N}\}$ we define

$$(2.8)\qquad \text{seq-}h_\sigma/e_\tau\text{-li}\, F^n(x, y) = \inf_{x_n \xrightarrow{\tau} x}\ \sup_{y_n \xrightarrow{\sigma} y}\ \liminf_n F^n(x_n, y_n)$$

$$(2.9)\qquad \text{seq-}e_\tau/h_\sigma\text{-ls}\, F^n(x, y) = \sup_{y_n \xrightarrow{\sigma} y}\ \inf_{x_n \xrightarrow{\tau} x}\ \limsup_n F^n(x_n, y_n).$$

To simplify notations, we shall henceforth omit the prefix "seq". The reader, however, should stay aware of the difference in the general (i. e. non-metrizable) case. Note also that when (X, τ) and (Y, σ) are linear spaces and the F^n are convex-concave for all $n \in \mathbb{N}$

e/h-ls F^n is convex in the variable x,
h/e-li F^n is concave in the variable y.

We introduce now one class of limit functions involving extended closure (*see* Section 1). If the bivariate functions are convex-concave, then the extended closures are generated by conjugacy operations and continuity of the partial Legendre-Fenchel transform leads us to work with the following notion of *extended epi/hypo-convergence* introduced in [10].

DEFINITION 2.3. – *A sequence* $\{F^n : X \times Y \to \bar{\mathbb{R}};\ n \in \mathbb{N}\}$, *where* (X, τ) *and* (Y, σ) *are topological spaces, is said to epi/hypo-converge in the extended sense to a function* $F : X \times Y \to \bar{\mathbb{R}}$ *if*

$$(2.10)\qquad \underline{\text{cl}}_x(e_\tau/h_\sigma\text{-ls}\, F^n) \leqq F \leqq \overline{\text{cl}}^y(h_\sigma/e_\tau\text{-li}\, F^n);$$

where $\underline{\text{cl}}_x = \tau - \underline{\text{cl}}_x$ *means the extended lower closure with respect to* x *for fixed* y *and* $\overline{\text{cl}}_y = \sigma - \overline{\text{cl}}_y$ *the extended upper closure with respect to* y *for fixed* x. *The interval of extended epi/hypo-limits is in general greater than that defined in* (2.3).

The following theorem (compare with [10], Theorem 3.10) shows that this notion of convergence is a variational convergence. Recall that $(\bar{x}, \bar{y}) \in X \times Y$ is a *saddle point* of the bivariate function $F : X \times Y \to \bar{\mathbb{R}}$ if

(2.11) $$\text{for all } (x, y) \in X \times Y, \quad F(\bar{x}, y) \leqq F(\bar{x}, \bar{y}) \leqq F(x, \bar{y})$$

or equivalently

(2.12) $$\text{for all } (x, y) \in X \times Y, \quad F(\bar{x}, y) \leqq F(x, \bar{y}).$$

THEOREM 2.4. – *Let us assume that* $\{ F^n, F : (X, \tau) \times (Y, \sigma) \to \bar{\mathbb{R}}, n \in \mathbb{N} \}$ *are such that*

(2.13) $$\underline{\mathrm{cl}}_x (e_\tau / h_\sigma - \mathrm{ls}\, F^n) \leqq F \leqq \overline{\mathrm{cl}}^y (h_\sigma / e_\tau - \mathrm{li}\, F^n)$$

(2.14) $$\begin{cases} (\bar{x}_k, \bar{y}_k) \text{ is a saddle point of } F^{n_k} \text{ for all } k \in \mathbb{N}, \\ \{ n_k \} \text{ is an increasing sequence of integers,} \end{cases}$$

(2.15) $$\bar{x}_k \xrightarrow{\tau} \bar{x} \quad \text{and} \quad \bar{y}_k \xrightarrow{\sigma} \bar{y},$$

then $(\bar{x}, \bar{y})$ *is a saddle point of* F *and*

$$F(\bar{x}, \bar{y}) = \lim_k F^{n_k} (\bar{x}_k, \bar{y}_k).$$

Proof. – Let $\{ \xi_n, n \in \mathbb{N} \}$ and $\{ \bar{\eta}_n, n \in \mathbb{N} \}$ be two sequences such that $\bar{\xi}_n \xrightarrow{\tau} \bar{x}$, $\bar{\eta}_n \xrightarrow{\sigma} \bar{y}$, and for $k \in \mathbb{N}$, $\bar{\xi}_{n_k} = \bar{x}_k$, $\bar{\eta}_{n_k} = \bar{y}_k$. Let us consider $y \in Y$ and $y_n \xrightarrow{\sigma} y$. Since $(\bar{x}_k, \bar{y}_k)$ is a saddle point of F^{n_k} we have

$$\text{for all } (x, y) \in X \times Y, \quad F^{n_k} (\bar{x}_k, y) \leqq F^{n_k} (\bar{x}_k, \bar{y}_k) \leqq F^{n_k} (x, \bar{y}_k),$$

from which it follows that

$$\begin{aligned} \liminf_n F^n (\bar{\xi}_n, y_n) &\leqq \liminf_k F^{n_k} (\bar{x}_k, y_k) \\ &\leqq \liminf_k F^{n_k} (\bar{x}_k, \bar{y}_k). \end{aligned}$$

Hence

$$\sup_{y_n \xrightarrow{\sigma} y} \liminf_n F^n (\bar{\xi}_n, y_n) \leqq \liminf_k F^{n_k} k (\bar{x}_k, \bar{y}_k),$$

and using the fact that $\bar{\xi}_n \xrightarrow{\tau} \bar{x}$, we see that

(2.16) $$(h_\sigma / e_\tau - \mathrm{li}\, F^n)(\bar{x}, y) \leqq \liminf_k F^{n_k} (\bar{x}_k, \bar{y}_k).$$

A symmetric argument shows that for all $x \in X$

(2.17) $$\limsup_k F^{n_k} (\bar{x}_k, \bar{y}_k) \leqq (e_\tau / h_\sigma - \mathrm{ls}\, F^n)(x, \bar{y}).$$

Inequality (2.16) being true for all $y \in \mathrm{Y}$, *we obtain that for all* $y \in Y$

$$(2.18) \qquad \mathrm{F}(\bar{x}, y) \leqq \overline{\mathrm{cl}}^y (h_\sigma/e_\tau - \mathrm{li}\,\mathrm{F}^n)(\bar{x}, y) \leqq \liminf_k \mathrm{F}^{n_k}(\bar{x}_k, \bar{y}_k),$$

similarly from (2.17) we derive that for all $x \in \mathrm{X}$

$$(2.19) \qquad \limsup_k \mathrm{F}^{n_k}(\bar{x}_k, \bar{y}_k) \leqq \underline{\mathrm{cl}}_x (e_\tau/h_\sigma - \mathrm{ls}\,\mathrm{F}^n)(x, \bar{y}) \leqq \mathrm{F}(x, \bar{y}).$$

From (2.18) and (2.19) we have for all $(x, y) \in \mathrm{X} \times \mathrm{Y}$

$$\mathrm{F}(\bar{x}, y) \leqq \liminf_k \mathrm{F}^{n_k}(\bar{x}_k, \bar{y}_k) \leqq \limsup_k \mathrm{F}^{n_k}(\bar{x}_k, \bar{y}_k) \leqq \mathrm{F}(x, \bar{y})$$

and this completes the proof of the theorem. □

The variational character of this convergence notion is stressed by the following result whose proof is straightforward.

THEOREM 2.5. – *Suppose* X, Y, $\{\mathrm{F}^n, \mathrm{F} : \mathrm{X} \times \mathrm{Y} \to \bar{\mathbb{R}}, n \in \mathbb{N}\}$ *are as in Theorem* 2.4 *with*

$$\underline{\mathrm{cl}}_x (e/h - \mathrm{ls}\,\mathrm{F}^n) \leqq \mathrm{F} \leqq \overline{\mathrm{cl}}^y (h/e - \mathrm{li}\,\mathrm{F}^n).$$

Then for any continuous function $\mathrm{G} : (\mathrm{X}, \tau) \times (\mathrm{Y}, \sigma) \to \mathbb{R}$

$$\underline{\mathrm{cl}}_x (e/h\text{-}\mathrm{ls}(\mathrm{F}^n + \mathrm{G})) \leqq \mathrm{F} + \mathrm{G} \leqq \overline{\mathrm{cl}}^y (h/e - \mathrm{li}(\mathrm{F}^n + \mathrm{G})).$$

3. CONTINUITY PROPERTIES OF THE PARTIAL LEGENDRE-FENCHEL TRANSFORM

We study here the continuity properties of the partial Legendre-Fenchel transform that establishes a natural correspondence between convex and convex-concave bivariate functions. The argumentation is surprisingly complex. In part this comes from the fact that the functions can take on the values $+\infty$ and $-\infty$, and that the Legendre-Frenchel transformation then looses its local character and it is only the global properties of the operations that are preserved. An elegant study of this phenomena and its implications has been made by Rockafellar ([8], [17] and [18]) and further analyzed by McLinden ([19], [20]); *see* also Ekeland-Temam [9], J. P. Aubin [21] and Auslender [42].

Convex-concave bivariate functions are related to convex bivariate functions through partial conjugation, i. e. conjugation with respect to one of the two variables. We are led to introduce equivalence classes of convex-concave saddle functions. For the sake of the noninitiated reader we review quickly the motivations and the main features of Rockafellar's scheme ([8], [17], [18]). We begin with an example.

Let K_0 be a convex-concave continuous function on $[-1,1]\times[-1,1]$. We associate to K_0 the two functions

$$K_1(x, y)=\begin{cases} +\infty & \text{if } |x|>1 \\ K_0(x, y) & \text{on } [-1,1]\times[-1,1] \\ -\infty & \text{if } |x|\leqq 1 \text{ and } |y|>1 \end{cases}$$

and

$$K_2(x, y)=\begin{cases} +\infty & \text{if } |x|>1 \text{ and } |y|\leqq 1 \\ K_0(x, y) & \text{on } [-1,1]\times[-1,1] \\ -\infty & \text{if } |y|>1. \end{cases}$$

Then both K_1 and K_2 have the same saddle points (and values) as K_0, although they differ on substantial portions of the plane. However, not only do these two functions have the same saddle points, but so do all linear perturbations of these two functions. So from a variational viewpoint they appear to be undistinguishable. It is thus natural when studying limits of the variational character that we need to deal with equivalent classes whose members have similar saddle point properties.

Let (X, X^*), (Y, Y^*) be two pairs of linear spaces, paired by the bilinear forms $\langle x, x^*\rangle$ and $\langle y, y^*\rangle$. The space $X^*\times Y$ is then paired with $X\times Y^*$ in the obvious fashion.

Let $K : X^*\times Y^*\to\bar{\mathbb{R}}$ be a convex-concave function. We associate with K its *convex and concave parents* defined by

$$F(x^*, y)=\sup_{y^*\in Y^*}[K(x^*, y^*)+\langle y, y^*\rangle] \tag{3.1}$$

$$G(x, y^*)=\inf_{x^*\in X^*}[K(x^*, y^*)-\langle x, x^*\rangle]. \tag{3.2}$$

Thus we have the following relations between these functions:

$$\begin{array}{ccc} \underset{\text{convex}-\text{concave}}{K : X^*\times Y^*\to\bar{\mathbb{R}}} & \overset{*2}{\to} & \underset{\text{convex parent}}{F : X^*\times Y\to\bar{\mathbb{R}}} \\ {}^{*1}\downarrow & \nearrow^{*} & \\ \underset{\text{concave parent}}{G : X\times Y^*\to\bar{\mathbb{R}}} & & \end{array}$$

In the example above K_1 and K_2 have the same parents; they cannot be distinguished as coming from different bivariate convex or concave functions. Two convex-concave bivariate functions K_1 and K_2, are said to be equivalent if they have the same parents.

A bivariate convex-concave function is said to be closed if its parents are the conjugates of each other, i. e., if the above diagram can be closed through the classical Legendre-Fenchel transform (with respect to both variables), i. e.

$$-G(x, y^*)=\sup_{\substack{x^*\in X^*\\ y\in Y}}[\langle x, x^*\rangle+\langle y, y^*\rangle-F(x^*, y)]. \tag{3.3}$$

One can prove [8] that for closed convex-convace functions, the associated equivalence class is an interval of functions denoted by $[\underline{K}, \bar{K}]$ with

$$(3.4)\qquad \underline{K}(x^*, y^*)=(\underline{cl}_{x^*}K)(x^*, y^*)=\sup_{x\in X}[G(x, y^*)+\langle x, x^*\rangle]$$

and

$$(3.5)\qquad \bar{K}(x^*, y^*)=(\overline{cl}^{y^*}K)(x^*, y^*)=\inf_{y\in Y}[F(x^*, y)-\langle y, y^*\rangle],$$

where $\underline{cl}_{x^*}K$ denotes the extended lower closure with respect to x^* and $\overline{cl}^{y^*}=-\underline{cl}_{y^*}(-K)$ is the extended upper closure with respect to y^*.

We recall that a convex function $F: X^*\times Y\to\bar{\mathbb{R}}$ is closed if $\underline{cl}_{(x^*, y)}F=F$ or equivalently $F^{**}=F$. The following elegant result is proved by Rockafellar [8]:

The map $K\overset{*2}{\to}F$ *establishes a one-to-one correspondence between closed convex-concave (equivalence) classes and closed convex functions.*

This correspondence has continuity properties that are made explicit here below. Given a sequence $(F^n)_{n\in\mathbb{N}}$ of closed convex bivariate functions that τ-epi-converges to F, we study the induced convergence for the associated classes of convex-concave bivariate functions associated to (F^n) through the partial Legendre-Fenchel transform. We show that the appropriate notion of convergence for this class is, for our purpose, the extended epi/hypo-convergence introduced in [10] and reviewed in Section 2.

The next two theorems, with Theorem 3.6 about the convergence of subdifferentials, summarize our main results about the continuity properties of the partial Legendre-Fenchel transform.

THEOREM 3.1. — *Suppose* X *and* Y *are reflexive Banach spaces,* $\{F, F^n: X^*\times Y\to\bar{\mathbb{R}}, n\in\mathbb{N}\}$ *is a collection of proper,* l.sc., *convex functions such that at least one of the two sequences* (F^n) *or* $(F^n)^*$ *is upper modulated* [see (1.14)]. *Then the following statements are equivalent:*

(i) *the sequence* F^n *Mosco-epi-converges to* F;

(ii) *for all* $K\in[\underline{K}, \bar{K}]$ *and* $K^n\in[\underline{K}^n, \bar{K}^n]$, *we have:*

$$(3.6)\qquad \underline{cl}_{x^*}(e_s/h_w-\text{ls}\,K^n)\leqq K\leqq\overline{cl}^{y^*}(h_s/e_w-\text{li}\,K^n).$$

Proof. — (i) $\Rightarrow$ (ii). We apply part (i) of Lemma 3.3 below with $\tau^*=s_{X^*}$ and $\sigma=s_Y$ to obtain

$$\underline{cl}_{x^*}(e_w/h_s-\text{ls}\,K^n)\leqq\underline{K}\leqq K.$$

Since the conjugate functions $(F^n)^*$ Mosco-epi-converge to F^* (Theorem 1.3), we can apply Lemma 3.3 (ii), with $\tau=s_X$, $\sigma^*=s_{Y^*}$ to obtain the second inequality in (3.6).

(ii) $\Rightarrow$ (i). Parts (iii) and (iv) of Lemma 3.3 yield

$$\mathrm{F} \leqq w \times w - \mathrm{li}_e \mathrm{F}^n,$$
$$\mathrm{F}^* \leqq w \times w - \mathrm{li}_e (\mathrm{F}^n)^*.$$

To complete the proof, we can apply Theorem 1.1 since by assumption one of the sequences (F^n) or $(\mathrm{F}_n)^*$ is upper modulated. □

Remark. — In Theorem 3.1 the implication (ii) $\Rightarrow$ (i) cannot be obtained without the upper modulated assumption as is shown by the following counterexample. Let $\mathrm{X} = \mathrm{Y} = \mathbb{R}$, $\mathrm{F}^n(x, y) = nx + \psi_{[y=n]}(x, y)$, where ψ_{C} denotes the indicator function of C. We have $\bar{\mathrm{K}}^n(x, y) = \underline{\mathrm{K}}^n(x, y) = nx - ny$. Thus if

$$x - y > 0 \quad (e/h - \mathrm{ls}\,\mathrm{K}^n)(x, y) = (h/e - \mathrm{li}\,\mathrm{K}^n)(x, y) = +\infty$$
$$x - y < 0 \quad (e/h - \mathrm{ls}\,\mathrm{K}^n)(x, y) = (h/e - \mathrm{li}\,\mathrm{K}^n)(x, y) = -\infty$$

and hence

$$\overline{\mathrm{cl}}^y (h/e - \mathrm{li}\,\mathrm{K}^n) \equiv +\infty$$
$$\underline{\mathrm{cl}}_x (e/h - \mathrm{ls}\,\mathrm{K}^n) \equiv -\infty$$

hence (ii) holds but not (i). □

Another case of practical interest is when the sequence of saddle points is bounded in the space $\mathrm{X}^* \times \mathrm{Y}^*$. The natural choice of topologies is then $\sigma^* = w_{\mathrm{Y}^*}$ and $\tau^* = w_{\mathrm{X}^*}$. Epi/hypo-convergence of the saddle functions K^n is then related to the epi-convergence of the sequence (F^n) for the $w_{\mathrm{X}^*} \times s_{\mathrm{Y}}$ topology.

The connection between the $w_{\mathrm{X}^*} \times s_{\mathrm{Y}}$-epi-convergence of a collection of convex functions and the $s_{\mathrm{X}} \times w_{\mathrm{Y}^*}$-epi-convergence of their conjugates (Theorem 1.4) relies on assumption $\mathscr{H}$ (1.18). It is also this assumption that we use when dealing with weak-strong (or strong-weak) epi-convergent sequences, and partial conjugation. Assumption $\mathscr{H}^*$ is $\mathscr{H}$ for the sequence $\Phi^n : \mathrm{Y}^* \times \mathrm{X} \to \bar{\mathbb{R}}$ defined by $\Phi^n(y^*, x) = (\mathrm{F}^n)^*(x, y^*)$. As the corollary of Theorem 1.4 and Lemma 3.3 below, we have:

THEOREM 3.2. — *Suppose* X *and* Y *are separable Banach spaces (not necessarily reflexive),* $\{\mathrm{F}, \mathrm{F}^n : \mathrm{X}^* \times \mathrm{Y} \to \bar{\mathbb{R}}\}$ *is a collection of proper, l. sc., convex functions, and*

$$(3.7) \qquad \begin{cases} \mathrm{F} = w^* \times s\text{-}\mathrm{lm}_e \mathrm{F}^n \\ \mathscr{H} \quad \textit{is verified.} \end{cases}$$

Then, for every $\mathrm{K} \in [\underline{\mathrm{K}}, \bar{\mathrm{K}}]$ *and* $\mathrm{K}^n \in [\underline{\mathrm{K}}^n, \bar{\mathrm{K}}^n]$

$$(3.8) \qquad \underline{\mathrm{cl}}_{x^*} (e_{w^*}/h_{w^*}\text{-}\mathrm{ls}\,\mathrm{K}^n) \leqq \mathrm{K} \leqq \overline{\mathrm{cl}}^{y^*} (h_{w^*}/e_{w^*}\text{-}\mathrm{li}\,\mathrm{K}^n).$$

Conversely, if we assume that (3.8) *holds and that* $\mathscr{H}^*$ *is satisfied, then*

$$(3.9) \qquad \mathrm{F} = w^* \times s\text{-}\mathrm{lm}_e \mathrm{F}^n.$$

Proof. – (3.7) ⇒ (3.8) is a straightforward consequence of parts (i) and (ii) of Lemma 3.3, relying on Theorem 1.4 to guarantee

$$F^* = s \times w^*\text{-lm}_e (F^n)^*.$$

(3.8) + $\mathscr{H}^*$ ⇒ (3.9). From parts (iii) and (iv) of Lemma 3.3 we have

$$F \leqq w^* \times s\text{-li}_e F^n$$
$$F^* \leqq s \times w^*\text{-li}_e (F^n)^*.$$

From $\mathscr{H}^*$, by Theorem 1.4, we know that

$$w^* \times s\text{-ls}_e (F^n)^{**} \leqq F^{**}$$

and hence

$$w^* \times s\text{-ls}_e F^n \leqq F$$

which, when combined with the inequalities above, yields the desired result. □

The key ingredient in both proofs is the next lemma that is concerned with the effect of partial conjugation on epi-convergence.

LEMMA 3.3. – *Suppose* X *and* Y *are Banach spaces,* $\{F, F^n : X^* \times Y \to \bar{\mathbb{R}}, n \in \mathbb{N}\}$ *are the corresponding classes of bivariate closed convex-concave functions. Then*

(i) $\tau^* \times \sigma\text{-ls}_e F^n \leqq F$ *implies* $\text{cl}_{x^*} (e_{\tau^*}/h_{\sigma^*}\text{-ls}\, \bar{K}^n) \leqq \underline{K}$;

(ii) $\tau \times \sigma^*\text{-ls}_e (F^n)^* \leqq F^*$ *implies* $\bar{K} \leqq \overline{\text{cl}}^{y^*} (h_{\sigma^*}/e_{\tau^*}\text{-li}\, \underline{K}^n)$;

(iii) $\bar{K} \leqq \overline{\text{cl}}^{y^*} (h_{\sigma^*}/e_{\tau^*}\text{-li}\, \underline{K}^n)$ *implies* $F \leqq \tau^* \times \sigma\text{-li}_e F^n$;

(iv) $\underline{\text{cl}}_{x^*} (e^*_{\tau}/h_{\sigma^*}\text{-ls}\, \bar{K}^n) \leqq \underline{K}$ *implies* $F^* \leqq \tau \times \sigma^*\text{-li}_e (F^n)^*$

where (τ, σ) [*respectively* (τ^*, σ^*)] *are topologies on* X *and* Y (*respectively* X* *and* Y*) *such that the pairings* $\langle .,. \rangle_{(X, \tau) \times (X^*, \tau^*)}$ *and* $\langle .,. \rangle_{(Y, \sigma) \times (Y^*, \sigma^*)}$ *are sequentially continuous.*

Proof. – (i) Let $x^* \in X^*$, $y^* \in Y^*$ and $\alpha \in R$ be such that

$$\bar{K}(x^*, y^*) < \alpha < \infty.$$

From the definition of $\bar{K}$, there exists $y \in Y$ such that $F(x^*, y) - \langle y, y^* \rangle < \alpha$. Let us consider $y^*_n \xrightarrow{\sigma^*} y^*$. In view of (1.13) for all $\beta \in \mathbb{R}$ with $F(x^*, y) < \beta$, there exists

$$x^*_n \xrightarrow{\tau^*} x^*$$

$$y_n \xrightarrow{\sigma} y$$

such that

$$\limsup_n F^n (x^*_n, y_n) < \beta.$$

It follows that

$$\limsup_n \bar{K}^n (x^*_n, y^*_n) \leqq \limsup_n [F^n (x^*_n, y_n) - \langle y_n, y^*_n \rangle] \leqq \beta - \langle y, y^* \rangle.$$

Letting β decrease to $F(x^*, y)$, this yields

$$\inf_{\{x_n^* \overset{\tau^*}{\to} x^*\}} (\limsup_n \bar{K}^n(x_n^*, y_n^*)) \leqq F(x^*, y) - \langle y, y^* \rangle < \alpha.$$

Letting α decrease to $\bar{K}(x^*, y^*)$, we see that

$$\inf_{\{x_n^* \overset{\tau^*}{\to} x^*\}} (\limsup_n \bar{K}^n(x_n^*, y_n^*)) \leqq \bar{K}(x^*, y^*).$$

This inequality being true for all sequences (y_n^*) σ^*-converging to y^*, it implies

$$e_{\tau^*}/h_{\sigma^*}\text{-ls}\, \bar{K}^n \leqq \bar{K}. \tag{3.10}$$

Since $\underline{\text{cl}}_{x^*} \bar{K} = \underline{K}$, it follows that

$$\underline{\text{cl}}_{x^*} (e_{\tau^*}/h_{\sigma^*}\text{-ls}\, \bar{K}^n) \leqq \underline{K}.$$

(ii) follows directly from (i), if we apply (i) to the following collection $\{\Phi; \Phi_n, n \in \mathbb{N}: Y^* \times X \to \bar{\mathbb{R}}\}$ where

$$\begin{cases} \Phi^n(y^*, x) = (F^n)^*(x, y^*) \\ \Phi(y^*, x) = F^*(x, y^*). \end{cases} \tag{3.11}$$

If we denote by $[\underline{L}^n, \bar{L}^n]$, $[\underline{L}, \bar{L}]$ the corresponding classes of bivariate convex-concave functions, using the fact that

$$(\sigma^* \times \tau)\text{-ls}_e\, \Phi^n \leqq \Phi$$

we have from (i) that

$$(e_{\sigma^*}/h_{\tau^*}\text{-ls}\, \bar{L}^n) \leqq \bar{L}.$$

Since

$$\bar{L}^n = -\underline{K}^n$$
$$\bar{L} = -\underline{K}$$
$$e_{\sigma^*}/h_{\tau^*}\text{-ls}\, \bar{L}^n = -h_{\sigma^*}/e_{\tau^*}\text{-li}\, \underline{K}^n,$$

we obtain

$$\underline{K} \leqq h_{\sigma^*}/e_{\sigma^*}\text{-li}\, \underline{K}^n$$

and

$$\bar{K} = \overline{\text{cl}}^{y^*} \underline{K} \leqq \overline{\text{cl}}^{y^*} (h_{\sigma^*}/e_{\tau^*}\text{-li}\, \underline{K}^n).$$

(iii) Let $(x^*, y) \in X^* \times Y$, $x_n^* \overset{\tau^*}{\to} x^*$ and $y_n \overset{\sigma}{\to} y$, we want to prove that

$$\liminf_n F_n(x_n^*, y_n) \geqq F(x^*, y). \tag{3.12}$$

We have

$$\begin{aligned} F(x^*, y) &= \sup_{y^* \in Y^*} [\langle y, y^* \rangle + \bar{K}(x^*, y^*)] \\ &\leqq \sup_{y^* \in Y^*} [\langle y, y^* \rangle + \overline{\text{cl}}^{y^*} (h_{\sigma^*}/e_{\tau^*}\text{-li}\, K^n)(x^*, y^*)] \\ &\leqq \sup_{y^* \in Y^*} [\langle y, y^* \rangle + (h_{\sigma^*}/e_{\tau^*}\text{-li}\, K^n)(x^*, y^*)]. \end{aligned}$$

Thus to prove (3.12) it suffices to show that for any sequence $(x_n^*, y_n) \overset{\tau^* \times \sigma}{\to} (x^*, y)$ and any $y^* \in Y^*$

(3.13) $$\langle y, y^* \rangle + (h_{\sigma^*}/e_{\tau^*}\text{-li}\, K^n)(x^*, y^*) \leqq \liminf_n F^n(x_n^*, y_n).$$

Using the definition of h_{σ^*}/e_{τ^*}-li K^n, we see that to every $\alpha \in \mathbb{R}$ with $\alpha < (h_{\sigma^*}/e_{\tau^*}\text{-li}\, K^n)$ (x^*, y^*), and to every $x_n^* \overset{\tau^*}{\to} x^*$, we can associate a sequence $y_n^* \overset{\sigma^*}{\to} y^*$ such that

$$\langle y, y^* \rangle + \alpha \leqq \langle y, y^* \rangle + \liminf_n K^n(x_n^*, y_n^*).$$

Using the fact that

$$F^n(x_n^*, y_n) \geqq \langle y_n, y_n^* \rangle + K^n(x_n^*, y_n^*)$$

we have

$$\liminf_n F^n(x_n^* y_n) \geqq \langle y, y^* \rangle + \liminf_n K^n(x_n^*, y_n^*)$$
$$\geqq \langle y, y^* \rangle + \alpha.$$

Letting α increase to $(h_{\sigma^*}/e_{\tau^*}\text{-li}\, K^n)(x^*, y^*)$ yields the inequality (3.12).

(iv) is obtained by applying (iii) to the collection of functions defined by (3.11). □

For sequences that $w^* \times w$-epi-converge, again by relying on Lemma 3.3 (and Theorem 1.1), we obtain convergence result for the associated saddle functions:

COROLLARY 3.4. — *Let* X *and* X *be reflexive Banach spaces and* $\{F, F^n : X^* \times Y \to \bar{\mathbb{R}}, n \in \mathbb{N}\}$ *a collection of upper modulated, l.sc., convex functions. If we assume that*

(3.14) $$F = w^* \times w\text{-lm}_e F^n,$$

then for every $K \in [\underline{K}, \bar{K}]$ *and* $K^n \in [\underline{K}^n, \bar{K}^n]$, *we have*

$$\underline{\mathrm{cl}}_{x^*}(e_{w^*}/h_s\text{-ls}\, K^n) \leqq K \leqq \overline{\mathrm{cl}}^{y}{}_{*}(h_s/e_{w^*}\text{-li}\, K^n).$$

The *subgradient-set* of a convex-concave function $K : X^* \times Y^* \to \bar{\mathbb{R}}$, denoted by ∂K, is by definition

$$\partial K(x^*, y^*) := \partial_{x^*} K(x^*, y^*) \times (-\partial_{y^*}(-K)(x^*, y^*))$$

where $\partial_{x^*} K(x^*, y^*)$ is the subgradient set of the convex function $K(., y^*)$, *cf.* [17]. With F, the convex parent of K, we have that

$$(u, v) \in \partial K(x^*, y^*) \quad \Leftrightarrow \quad (u, y^*) \in \partial F(x^*, v)$$

and

$$(0,0) \in \partial K(x^*, y^*) \quad \Leftrightarrow \quad (x^*, y^*) \text{ is a saddle point of } K.$$

We need the following result from [4] that relates the epi-convergence of a sequence of convex functions to the graph-convergence of the associated (subdifferential) maps.

THEOREM 3.5. — *Suppose* $\{F; F^n \times Y \to \bar{\mathbb{R}}, n \in \mathbb{N}\}$ *is a collection of proper convex functions. Then, the following are equivalent*

(i) $F = \text{Mosco-}epi\text{-}lim_{n \to \infty} F^n$;

(ii) $\text{gph}\, \partial F = \text{Lim}_{n \to \infty} \text{gph}\, \partial F^n$ *plus a normalizing condition:*

$$(\text{NC}) \quad \begin{cases} \exists [((x^*, y), (x, y^*)) \in \text{gph}\, \partial F, ((x_n^*, y_n), (x_n, y_n^*)) \in \text{gph}\, \partial F_n, n \in \mathbb{N}] \\ \textit{such that} \\ \qquad (x^*, y) = s\text{-}\lim_{n \to \infty} (x_n^*, y_n), (x, y^*) = s^*\text{-}\lim_{n \to \infty} (x_n, y_n^*) \\ \textit{and} \\ \qquad F(x^*, y) = \lim_{n \to \infty} F^n(x_n^*, y_n). \end{cases}$$

Here

$$\text{gph}\, \partial F := \{((x^*, y), (x, y^*)) \mid (x, y^*) \in \partial F(x^*, y)\}$$

and

$$\text{Lim}_{n \to \infty} \text{gph}\, \partial F^n = \text{Lim inf}_{n \to \infty} \text{gph}\, \partial F^n = \text{Lim sup}_{n \to \infty} \text{gph}\, \partial F^n$$

is the (Kuratowski) set limit of the graphs of the subdifferentials with respect to the strong topologies.

THEOREM 3.6. — *Suppose* $\{K; K^n : X^* \times Y^* \to \bar{\mathbb{R}}, n \in \mathbb{N}\}$ *is a collection of convex-concave saddle functions with proper convex parents* $\{F; F^n : X^* \times Y \to \bar{\mathbb{R}}, n \in \mathbb{N}\}$. *Suppose that the sequence* (F^n) *satisfies the normalizing condition* (NC). *Then, the following are equivalent*

(i) $\text{gph}\, \partial K = \lim_{n \to \infty} \text{gph}\, \partial K^n$;

(ii) *for all* $K \in [\underline{K}, \bar{K}]$ *and* $K^n \in [K_n, K^n]$, *we have*

$$\underline{\text{cl}}_{x^*} (e_s/h_w\text{-ls}\, K^n) \leq \overline{\text{cl}}^{y^*} (h_s/e_w\text{-li}\, K^n).$$

Proof. — First observe that the normalizing condition (NC) implies the upper modulated condition (1.14). From Theorems 3.1 and 3.5, it follows that

$$\text{(ii)} \iff F = \text{Mosco-epi-lim}_{n \to \infty} F^n \iff \text{gph}\, \partial F = \lim_{n \to \infty} \text{gph}\, \partial F^n.$$

To complete the argument, observe that with respect to set-convergence, the map

$$(x^*, y, v, u) \in X \times Y \times X^* \times Y^* \to (x^*, u, v, -y) \in X^* \times Y^* \times X \times Y$$

is continuous, and thus from the definition of the subgradients, we have

$$\operatorname{gph} \partial F = \operatorname*{Lim}_{n \to \infty} \operatorname{gph} \partial F^n \iff \operatorname{gph} \partial K = \operatorname*{Lim}_{n \to \infty} \operatorname{gph} \partial K^n. \quad \square$$

This last result has important implications in nonsmooth analysis where the second order epi-derivative [49] is defined as the function whose subdifferential is the tangent cone to the graph of the subdifferential.

4. CONVEX PROGRAMMING

Our first example is intended to illustrate some problems that arise in connection with Lagrangians in mathematical programming. Our results are direct applications of Theorem 3.1.

We consider the following classes of problems:

(i) X is a reflexive separable Banach space;

(ii) $\{f; f^n : X \to \bar{\mathbb{R}},\ n \in \mathbb{N}\}$, a collection of closed convex proper functions;

(iii) $\{g, g_i^n : X \to \mathbb{R} \cup \{+\infty\},\ n \in \mathbb{N},\ i = 1, \ldots, m\}$, a collection of closed convex proper functions.

We are interested in the asymptotic behaviour of the following sequence of optimization problems

$$(4.1)_n \qquad \begin{cases} \text{minimize} f^n(x) \\ \text{subject to } g_i^n(x) \leqq 0, \quad i = 1, \ldots, m. \end{cases}$$

A classical perturbation scheme is to consider for $y \in \mathbb{R}^m$, the problems

$$(4.2)_n \qquad \begin{cases} \text{minimize} f^n(x) \\ \text{subject to } x \in D_n(y) \end{cases}$$

where

$$D_n(y) = \{ x \in X;\ g^n(x) \leqq y \}$$

and $g^n(x) \leqq y$ means $g_i^n(x) \leqq y_i$, for $i = 1, \ldots, m$. Let

$$(4.3) \qquad D^n = \{ (x, y) \in X \times \mathbb{R}^m,\ g^n(x) \leqq y \}.$$

The associated *perturbation function* F^n is given by

$$(4.4)_n \qquad F^n(x, y) = f^n(x) + \psi_{D^n}(x, y),$$

and the *Lagrangian function* $\bar{K}^n(x, y)$, *see* [22],

$$(4.5)_n \qquad \bar{K}^n(x, y) = \begin{cases} f^n(x) - \sum_{i=1}^m y_i g_i^n(x) & \text{if } y \leqq 0 \\ -\infty & \text{otherwise} \end{cases}$$

here ψ_C is the *indicator function* ($=0$ on C, $+\infty$ otherwise) of the set C. We think of the problems $(4.2)_n$ and their Lagrangians $(4.5)_n$ as approximations of some *limit problem*:

$$(4.6) \qquad \begin{cases} \text{minimize } f(x) \\ \text{subject to } g(x) \leqq 0 \end{cases}$$

with associated perturbation function

$$(4.7) \qquad F(x, y) = f(x) + \psi_D(x, y)$$

and Lagrangian function

$$(4.8) \qquad \bar{K}(x, y) = \begin{cases} f(x) - \sum_{i=1}^{m} y_i g_i(x) & \text{if } y \leqq 0 \\ -\infty & \text{otherwise.} \end{cases}$$

In (4.7), D is defined by

$$(4.9) \qquad D = \{(x, y) \in X \times \mathbb{R}^m,\ g(x) \leqq y\}$$

and for all $y \in \mathbb{R}^m$ we also define

$$(4.10) \qquad D(y) = \{x \in X;\ g(x) \leqq y\}.$$

A typical situation is when the problems $(4.2)_n$ are obtained from (4.6) as the result of adding penalization or barrier terms to the objective, or when the $(4.2)_n$ are the restriction of (4.6) to finite dimensional subspaces of X, and so on. In particular dealing with numerical procedures, one is naturally interested by convergence of solutions, but also by the convergence of multipliers, for reason of stability ([22], [23] and [24]) or to be able to calculate rates of convergence such as in augmented Lagrangian methods. Our objective is to give some conditions which will ensure the epi/hypo-convergence of the Lagrangians $\bar{K}^n$ to $\bar{K}$, and under suitable compactness of the saddle points of $\bar{K}^n$, the convergence of these saddle points to a saddle point of $\bar{K}$.

In Section 3 epi/hypo-convergence of the Lagrangian function is derived from the epi-convergence of the sequence of perturbation functions (F^n). In this setting theses functions take the form of a sum of two functions $(4.4)_n$. In order to obtain the Mosco-epi-convergence of the sequence (F^n) to F, the following theorem gives a sufficient condition for the Mosco-epi-convergence of the sum of two closed convex functions defined on a reflexive Banach space, that extends a result of McLinden and Bergstrom [25].

THEOREM 4.1. — *Let $(\varphi^n)_{n\in\mathbb{N}}$, φ, $(\psi^n)_{n\in\mathbb{N}}$, ψ be l. sc. proper, convex functions defined on a reflexive Banach space, such that $\varphi = \text{M-lm}_e\, \varphi^n$ and*

$\psi = \text{M-lm}_e \psi^n$. *Assume that*

(4.11) *there exists* $r>0$ *such that, for every* $\xi \in B_r(0)$, *there exist two sequences* (x_n) *and* (y_n) *of elements of* X *verifying:*

$$(x_n) \text{ is bounded,}$$
$$(y_n) \text{ is bounded,}$$
$$\xi = x_n - y_n,$$
$$\limsup_n \varphi^n(x_n) < +\infty,$$
$$\limsup_n \psi^n(y_n) < +\infty.$$

then

(4.12) *there exists* $n_0 \in \mathbb{N}$ *such that* $\varphi^n + \psi^n$ *is proper for* $n \geqq n_0$ *and* $\varphi + \psi = \text{M-lm}_e(\varphi^n + \psi^n)$, *moreover* $\varphi + \psi$ *is proper.*

Before we turn to the proof of this theorem, we show that it implies the finite dimentional result of McLinden and Bergstrom:

COROLLARY 4.2 [25]. — *Suppose* X *is finite dimentional and* $(\varphi_n)_{n \in \mathbb{N}}$, φ, $(\psi_n)_{n \in \mathbb{N}}$, ψ, *are l. sc., proper, convex functions defined on* X *such that* $\varphi = \text{lm}_e \varphi^n$ *and* $\psi = \text{lm}_e \psi^n$. *If*

(4.13) $$0 \in \text{int}(\text{dom}\,\varphi - \text{dom}\,\psi)$$

then there exists n_0 *such that* $\varphi^n + \psi^n$ *is proper for* $n \geqq n_0$ *and* $\varphi + \psi = \text{lm}_e(\varphi^n + \psi^n)$, *moreover* $\varphi + \psi$ *is proper.*

Proof of Corollary 4.2. — It is sufficient to prove that (4.13) implies (4.11). From (4.13) there exists some $r>0$ such that

$$\text{dom}\,\varphi - \text{dom}\,\psi \supset B_r(0).$$

Taking advantage of the fact that X is a finite dimensional space, we can find a finite number of vector $(\xi_1, \xi_2, \ldots, \xi_N)$ of X such that

$$B_{r/2}(0) \subset \text{co}(\xi_1, \xi_2, \ldots, \xi_N) \subset B_r(0)$$

where $\text{co}(\xi_1, \xi_2, \ldots, \xi_N)$ is the closed convex hull of $(\xi_1, \ldots, \xi_N)$. Without loss of generality we may assume that the closure of $B_{r/2}(0)$, is included in the interior of $\text{co}(\xi_1, \ldots, \xi_N)$ and that the closure of $\text{co}(\xi_1, \ldots, \xi_N)$ is included in the interior of $B_r(0)$. From the above, for every $i \in \{1, 2, \ldots, N\}$, ξ_i belongs to $\text{dom}\,\varphi - \text{dom}\,\psi$, thus there exist some $x_i \in \text{dom}\,\varphi$, $y_i \in \text{dom}\,\psi$ such that

$$\xi_i = x_i - y_i.$$

Using the epi-convergence of (φ^n) and (ψ^n) we derive the existence of sequences $(x_i^n)_{n \in \mathbb{N}}$ and $(y_i^n)_{n \in \mathbb{N}}$, $i \in \{1, \ldots, N\}$ converging respectively to

x_i and y_i such that for $i=1, \ldots, N$

$$\begin{aligned} \varphi(x_i) &= \lim_{n \to +\infty} \varphi^n(x_i^n) \\ \psi(y_i) &= \lim_{n \to +\infty} \psi^n(y_i^n). \end{aligned} \tag{4.14}$$

For n sufficiently large we obtain that

$$\begin{gathered} \varphi^n(x_i^n) \text{ and } \psi^n(x_i^n) \text{ are bounded above for } i=1, \ldots, N, \\ B_{r/2}(0) \subset \operatorname{co}(\xi_1^n, \ldots, \xi_N^n) \subset B_r(0) \quad \text{with} \quad \xi_i^n = x_i^n - y_i^n, \\ i=1, \ldots, N. \end{gathered} \tag{4.15}$$

Now, for any $\xi \in B_{r/2}(0)$ and $n \in \mathbb{N}$, there exist weights $\{t_i^n, i=1, \ldots, N\}$ such that

$$\xi = \sum_{i=1}^{N} t_i^n \xi_i^n, \quad t_i^n \geqq 0, \qquad \sum_{i=1}^{N} t_i^n = 1$$

$$\xi = \sum_{i=1}^{N} t_i^n x_i^n - \sum_{i=1}^{N} t_i^n y_i^n.$$

So we choose $x_n = \sum_{i=1}^{N} t_i^n x_i^n$, $y_n = \sum_{i=1}^{N} t_i^n y_i^n$ and we have that $\xi = x_n - y_n$. Cleary the sequences (x_n) and (y_n) are bounded, lie in dom φ_n and dom ψ_n respectively, and from the convexity of the functions φ^n and ψ^n and (4.15) it follows that $\psi^n(x_n)$ and $\psi^n(y_n)$ are bounded from above. Thus, we have shown that the conditions (4.11) are satisfied. The assertion now follows from Theorem 4.1 since in finite dimension Mosco-epi-convergence coincides with the standard definition of epi-convergence. □

Proof of Theorem 4.1. – Since φ^n and ψ^n are proper, $-\infty < \varphi^n + \psi^n$, moreover from (4.11) with $\xi = 0$ we see that for n sufficiently large $\varphi^n(x_n) + \psi^n(y_n) < +\infty$. Hence for n sufficiently large $(\varphi^n + \psi^n)$ is proper. Let us now notice that

$$B_r(0) \subset \operatorname{dom} \varphi - \operatorname{dom} \psi. \tag{4.16}$$

To see this, pick any ξ in $B_r(0)$, then (4.11) yields the existence of bounded sequences (x_n) and (y_n) satisfying $\xi = x_n - y_n$ such that for n sufficiently large the sequences $\{\varphi^n(x_n), n \in \mathbb{N}\}$ and $\{\psi^n(y_n), n \in \mathbb{N}\}$ are bounded from above. Passing to a subsequence if necessary, let x and y be weak-limit points such that $x_n \overset{w}{\to} x$ and $y_n \overset{w}{\to} y$. Then

$$\begin{gathered} \xi = x - y, \\ \varphi(x) \leqq \liminf_n \varphi^n(x_n) < +\infty \\ \psi(y) \leqq \liminf_n \psi^n(y_n) < +\infty \end{gathered}$$

as follows from Mosco-epi-convergence, *see* [1.17 (*b*)]. Observe that (4.16) implies that $\varphi+\psi$ is proper.

To prove Mosco-epi-convergence we use the following characterization [46], Proposition 1.19: *A sequence* (F^n) *of proper, l. sc., convex functions defined on a reflexive Banach space* X, *Mosco-epi-converges to the proper, l. sc. convex function* F *if and only if:*

(∗) *the sequence* (F^n) *is upper modulated* [see (1.14)];

(α) *for all* $x \in X$ *and* $x_n \xrightarrow{w} x$, $\liminf_n F^n(x_n) \geqq F(x)$;

(β) *for all* $x^* \in X^*$ *and* $x_n^* \xrightarrow{w^*} x^*$, $\liminf_n (F^n)^*(x_n^*) \geqq F^*(x^*)$.

Let us apply the above result to the sequence $F^n = \varphi^n + \psi^n$ and $F = \varphi + \psi$. In order to verify (∗) we argue as above and use (4.11) with $\xi=0$, $x_n = y_n$ which implies that

$$\limsup_n (\varphi^n + \psi^n)(x_n) < \infty.$$

For any x in X and $x_n \xrightarrow{w} x$, from Mosco-epi-convergence of the sequences (φ^n) and (ψ^n), in particular [1.17 (*b*)] it follows that

$$\begin{aligned}\liminf_n (\varphi^n+\psi^n)(x_n) &\geqq \liminf_n \varphi^n(x_n) + \liminf_n \psi^n(x_n)\\ &\geqq \varphi(x)+\psi(x)\end{aligned}$$

and this yields (α).

There only remains to establish (β), i. e.

$$(4.20)\qquad \begin{cases} \text{for every } x^* \in X^* \text{ and } x_n^* \xrightarrow{w^*} x^*, \\ \liminf_n (\varphi^n+\psi^n)^*(x_n^*) \geqq (\varphi+\psi)^*(x^*). \end{cases}$$

For n sufficiently large, the function $\varphi^n+\psi^n$ is proper and thus (*see* [26] for example)

$$(\varphi^n+\psi^n)^* = [(\varphi^n)^* \,\square\, (\psi^n)^*]^{**} = \mathrm{cl}\,[(\varphi^n)^* \,\square\, (\psi^n)^*]$$

where $\square$ denotes inf-convolution (epi-sum) and cl denotes closure with respect to the strong topology of X* [*see* (1.7)]. Since otherwise there is nothing to prove, we may as well assume that the sequence $x_n^* \xrightarrow{w} x^*$ is such that $\liminf_n (\varphi^n+\psi^n)^*(x_n^*) < +\infty$, and thus passing to a subsequence if necessary, that the sequence $\{(\varphi^n+\psi^n)^*(x_n^*),\ n \in \mathbb{N}\}$ is bounded from above. From this, and what precedes, follows the existence of (z_n^*) such that

$$[(\varphi^n)^* \,\square\, (\psi^n)^*](z_n^*) < (\varphi^n + \psi^n)^* (x_n^*) + 1/n,$$
$$\| z_n^* - x_n^* \| \leqq 1/n$$

and the definition of inf-convolution then yields a sequence (ξ_n^*) in X^* such that

$$(4.17) \qquad (\varphi^n)^* (\xi_n^*) + (\psi^n)^* (z_n^* - \xi_n^*) \leqq (\varphi^n + \psi^n)^* (x_n^*) + 1/n.$$

Let us consider $\xi \in B_r(0)$ with $r > 0$ and the two bounded sequences (x_n) and (y_n) defined in (4.11), then

$$\langle \xi_n^*, x_n \rangle \leqq \varphi^n (x_n) + (\varphi^n)^* (\xi_n^*)$$
$$\langle z_n^* - \xi_n^*, y_n \rangle \leqq \psi^n (y_n) + (\psi^n)^* (z_n^* - \xi_n^*)$$

and hence, since $\xi = x_n - y_n$,

$$\langle \xi_n^*, \xi \rangle \leqq \varphi^n (x_n) + \psi^n (y_n) + (\varphi^n)^* (\xi_n^*) + (\psi^n)^* (z_n^* - \xi_n^*) - \langle z_n^*, y_n \rangle.$$

From the above, and (4.11) it follows that:

$$\limsup_n \langle \xi_n^*, \xi \rangle < +\infty.$$

Since this holds for every $\xi \in B_r(0)$, the Banach-Steinhaus Theorem tells us that the sequence $(\xi_n^*) \subset X^*$ is bounded, and thus has at least one weak-cluster point, say ξ^*. We now use the continuity of the Legendre-Fenchel transform with respect to the Mosco-epi-convergence (Theorem 1.3) to conclude that

$$\varphi^* = \text{M-lm}_e (\varphi^n)^* \qquad \text{and} \qquad \psi^* = \text{M-lm}_e (\psi^n)^*.$$

Taking lim inf on both sides of (4.17), we obtain:

$$\begin{aligned} \liminf_n (\varphi^n + \psi^n)^* (x_n^*) &\geqq \liminf_n [(\varphi^n)^* (\xi_n^*) + (\psi^n)^* (z_n^* - \xi_n^*] \\ &\geqq \varphi^* (\xi^*) + \psi^* (x^* - \xi^*) \\ &\geqq (\varphi^* \,\square\, \psi^*) (x^*) \\ &\geqq (\varphi + \psi)^* (x^*) \end{aligned}$$

and this completes the proof. □

Remark 1. — Theorem 4.1 also extends a result of Joly [27], p. 96, which relies on the following assumption:

$$(\text{J}) \qquad \begin{cases} \textit{there exists } x_0 \in \operatorname{dom} \varphi \cap \operatorname{dom} \psi \textit{ and } V \in N_s(x_0) \\ \textit{such that either } \varphi^n \textit{ or } \psi^n \textit{ is uniformly bounded above on } V. \end{cases}$$

It is not difficult to see that under Mosco-epi-convergence, (J) implies (4.11). Suppose (φ^n) is bounded on V. Let $r > 0$ be such that $B_{2r}(x_0) \subset V$. From [1.18 (*a*)] we know there exists a sequence $z_n \to x_0$ such that $\limsup_n \psi^n (z_n) \leqq \psi (x_0)$. For n sufficiently large $\psi^n (z_n)$ is bounded from above and $z_n \in B_r(x_0)$. Pick $\xi \in B_r(0)$ and set

$$x_n = z_n + \xi \quad \text{and} \quad y_n = z_n.$$

Then $x_n - y_n = \xi$, the sequences (x_n) and (y_n) are bounded, $\limsup_n \psi^n(y^n) < \infty$, and since $(x_n) \subset B_{2r}(x_0) \subset V$, $\limsup_n \varphi^n(x^n) < \infty$.

Remark 2. — In [44] one can find related results concerning the convergence of the sum of two maximal monotone operators. Constraint qualification conditions of the type

$$0 \in \operatorname{int}(\operatorname{dom} \varphi - \operatorname{dom} \psi)$$

are due to Aubin [28], condition (4.11) is the "equi"-version of this condition.

Theorem 4.1 and 3.2 lead us to the following stability result for infinite dimensional convex programs. The functions f^n, g_i^n and K^n are as defined earlier, $(4.1)_n$ to (4.6).

THEOREM 4.3. — *Suppose* X *is a reflexive Banach space, and*

$$f = \text{M-lm}_e\, f^n$$
$$g_i = \text{M-lm}_e\, g_i^n, \qquad i = 1, \ldots, m$$

and

(4.18) *there exists* $r > 0$ *such that for all* $\xi \in B_r(0)$, *there exists a bounded sequence* (x_n) *in* X *that verifies*

$$\limsup_n f^n(x_n) < +\infty,$$
$$\limsup_n g_i^n(x_n + \xi) < +\infty, \qquad i = 1, \ldots, m.$$

Then any sequence (K^n), *defined by* $(4.5)_n$ *Mosco-epi/hypo-converges in the extended sense to* K *defined by* (4.8), *for any* $K^n \in [\underline{K}^n, \bar{K}^n]$ *and* $K \in [\underline{K}, \bar{K}]$; *by this one means that*

$$\underline{\text{cl}}_x(e_s/h_w\text{-ls } K^n) \leqq K \leqq \overline{\text{cl}}^y(h_s/e_w\text{-li } K^n).$$

(*Note that in finite dimension, in particular on* $\mathbb{R}^m$, $w = s$.)

Proof. — It suffices to show that $F = \text{M-lm}_e F^n$ [*see* $(4.4)_n$ and (4.7)] and then apply Theorem 3.1. To begin with, let us prove that

$$\psi_D = \text{M-lm}_e\, \psi_{D^n} \tag{4.19}$$

where ψ_C is the indicator function of the set C. Consider $(x, y) \in X \times \mathbb{R}^m$ and $x_n \overset{w}{\to} x$, $y_n \to y$, we need to show first that

$$\liminf_n \psi_{D^n}(x_n, y_n) \geqq \psi_D(x, y).$$

If the lim inf is $+\infty$, the result is immediate. Extracting some subsequence, we can assume that $(x_n, y_n) \in D^n$, i.e. $g_n(x_n) \leqq y_n$. Taking the limit inferior on both sides and using the fact that the (g_i^n) Mosco-epi-converge, we see that $g(x) \leqq y$, which yields the desired inequality. Consider now $(x, y) \in D$ and y_δ decreasing componentwise to y when $\delta > 0$ goes to zero. There exists $x_n \to x$ (strong convergence in X) such that

$$\limsup_n g^n(x_n) \leqq g(x) \leqq y < y_\delta.$$

Using the definition of the lim sup operation we derive that for each $\delta > 0$, there exists $n(\delta)$ such that $(x_n, y_\delta) \in D^n$ for $n \geqq n(\delta)$. Thus

$$\limsup_n \psi_{D^n}(x_n, y_\delta) \leqq \psi_D(x, y)$$

and

$$\limsup_{\delta \downarrow 0} (\limsup_n \psi_{D^n}(x_n, y_\delta)) \leqq \psi_D(x, y).$$

A diagonalization argument [10], Lemma A-3, yields a sequence $\delta(n) \downarrow 0$ such that

$$\limsup_n \psi_{D^n}(x_n, y_{\delta(n)}) \leqq \psi_D(x, y)$$

and this with the lim inf inequality yields (4.19).

Let us denote by $\bar{f}^n$ and $\bar{f}$ the functions defined on $X \times \mathbb{R}^m$ with values in $\mathbb{R} \cup \{+\infty\}$ by

$$\bar{f}^n(x, y) = f^n(x),$$
$$\bar{f}(x, y) = f(x),$$

we have that

$$\bar{f} = \text{M-lm}_e \bar{f}^n$$
$$\psi_D = \text{M-lm}_e \psi_{D^n}.$$

In order to apply Theorem 4.1 it suffices to verify assumption (4.11). So let $r > 0$ be given by assumption (4.18), and $\xi \in B_r(0)$; there exists a bounded sequence (x_n) in X such that

$$\limsup_n f^n(x_n) < +\infty$$
$$\limsup_n g^n(x_n + \xi) < y - re$$

for some $y \in \mathbb{R}^m$ and $e = (1, \ldots, 1)$. Let us consider $\eta \in \mathbb{R}^m$ with $\sup_{1 \leqq i \leqq m} |\eta_i| \leqq r$; we have

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} x_n + \xi \\ y + \eta \end{pmatrix} - \begin{pmatrix} x_n \\ y \end{pmatrix}$$

and

$$\limsup_n \tilde{f}^n(x_n, y) = \limsup_n f^n(x_n) < +\infty$$
$$g^n(x_n+\xi) - y - \eta \leqq -\mathrm{re} + \mathrm{re} \leqq 0$$

for n sufficiently large. Thus

$$(x_n+\xi,\ y+\eta) \in D^n$$

and

$$\limsup_n \psi_{D^n}(x_n+\xi,\ y+\eta) < +\infty.$$

Since (4.11) is satisfied, we can apply Theorem 4.1 to obtain

$$\tilde{f} + \psi_D = \text{M-lm}_e(\tilde{f}^n + \psi_{D^n})$$
$$F = \text{M-lm}_e F^n.$$

The assertion now follows from Theorem 3.1. □

Remarks. — 1. This theorem is related to [11], Proposition 1.17, and is close to the results obtained by T. Zolezzi in [20] under the stronger assumption of continuous convergence of the constraints (*see* also McLinden [30] and Lucchetti-Patrone [31]). The above result and the variational properties of extended epi/hypo-convergence (Theorem 2.4) guarantees that any cluster point (x, y), of the sequence of saddle points (x_n, y_n) of the Lagrangian $\bar{K}^n$, is a saddle point of $\bar{K}$. A related question—if each saddle point of the limit problem can be obtained as a limit of a sequence of ε_n-saddle point for $\bar{K}_n$ when $\varepsilon_n \downarrow 0$, is settled (in the affirmative) by Azé [32] (*see* also [10]). "Equi-" versions of the constraint qualification and coercive assumptions naturally appear in order to guarantee existence and boundedness of saddle points, *cf.* [40].

2. In the finite dimensional case $X = \mathbb{R}^p$, when the functions $\{g_i^n;\ i=1, \ldots, m\}$ and g are finitely valued (and hence continuous, since convex) let us stress the fact that assumption (4.18) is then automatically satisfied. Indeed, in this situation, epi-convergence turns to be equivalent to pointwise convergence and to uniform convergence on bounded sets. As a corollary, we obtain in this convex setting a result similar to that of Attouch-Wets [11], Proposition 1.17.

5. APPLICATIONS IN MECHANICS

Theorem 3.2 is aimed at *simultaneous* weak convergence of primal and dual variables. Let us describe two typical situations where this kind of problem does occur.

The first one concerns *homogenization* of composite material where the physical parameters rapidly oscillate between the different values of each component. For pedagogical reasons we consider here a classical situation, namely linear elasticity, for which the convergence of primal and dual variables has already been obtained by other methods. Indeed, because of its flexibility, the same method can be used to solve various primal/dual homogenization problems, especially nonlinear problems (possibly involving constraints) where the Euler equation is much more difficult to deal with (and even to formulate!).

The second example deals with the convergence of the primal/dual solutions in a *reinforcement problem* in mechanics when the thickness of the reinforced zone goes to zero. This is balanced with the fact that the physical parameter goes to $+\infty$ in the same region. In this situation epi-convergence techniques are the only ones that, at present, provide a proof of the convergence of the saddle points.

5.1. The *homogenization* approach consists of replacing a composite material by a homogenous, ideal one, obtained by letting ε go to zero in the governing equations where the parameter ε describes the periodicity, and hence the tightness, of the structure. In the case of elasticity, primal and dual variables are respectively equal to *displacement vector fields and stress tensor fields* (internal forces).

Because of its technological importance, an abundant literature has been devoted to this subject in recent years. The energy methods provide a sharp and flexible mathematical approach to these convergence problems and can be subdivided into two categories: The so-called *direct energy methods* (compactness by compensation...) introduced by Bensoussan, Lions et Papanicoulaou [33], Murat and Tartar ([34], [45]), etc., considers the convergence (as $\varepsilon \downarrow 0$) of the Euler equations, i. e., of the operators governing these equations. On the other hand, the *epi-convergence* approach introduced by DeGiorgi [35], Marcellini [36], Marchenko-Hruslov [37], Attouch [38], relies on the formulation of the problems as minimization problems and studies the convergence of the energy functionals.

Recently a dual version of these results expressed uniquely in terms of dual variables (stress tensors) has been developed by Suquet [39] and Azé [40]. In this section, we present a *unified approach*, which, via the introduction of the associated *Lagrangians*, known in mechanics as *Reisner functionals*, and the study of their epi/hypo-convergence, provides the convergence of their saddle points (and saddle values), i. e. of both primal and dual variables.

Let us introduce the notations used in the sequel of this paragraph.

(5.1) $\begin{cases} \Omega \text{ is an open regular bounded set in } \mathbb{R}^N \\ (N=1, 2, 3 \text{ in the applications}) \end{cases}$

(5.2) $u(x)=(u_1(x), \ldots, u_N(x))$ is the displacement vector field

(5.3) $(e(u))_{(i,j)\in[1,N]^2}$ is the deformation tensor, where

$$(e(u))_{i,j}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right).$$

A well known result, the Korn's inequality, asserts that

$$\{u\in(L^2(\Omega))^N;\ e(u)\in(L^2(\Omega))^{N^2}\}=(H^1(\Omega))^N$$

see [41] for instance.

In elasticity (linear or more generally nonlinear hyperelasticity) the structural equation relating deformation and stress tensors in given by

(5.4) $$\sigma(x)\in\partial j(x, e(u(x))),$$

where $j:\Omega\times\mathbb{R}_S^{N^2}\to\mathbb{R}$ (the letter S stands for symmetric) is a function which is convex with respect to the second variable and measurable with respect to the first and $\partial j(x, .)$ denotes the subdifferential of the convex function $j(x, .)$. For simplicity of the presentation, we assume that no displacement takes place on the boundary $\partial\Omega$. At equilibrium, the displacement field is then the solution of:

(5.5) $$\min_{u\in(H_0^1(\Omega))^N}\int_\Omega j(x, e(u))\,dx-\int_\Omega f.u\,dx$$

where $f\in(H^{-1}(\Omega))^N$ is the density of forces acting on the material. To have problem (5.5) well-posed, we assume that

(5.6) $$\begin{cases} \text{for every } (x, z)\in\Omega\times\mathbb{R}_S^{N^2} \\ \lambda_0|z|^2\leqq j(x, z)\leqq\Lambda_0(1+|z|^2), \qquad 0<\lambda_0\leqq\Lambda_0<+\infty. \end{cases}$$

The Euler equation associated with (5.5) is:

(5.7) $$\begin{gathered} \sigma(x)\in\partial j(x, e(u(x)))\text{ a. e. on }\Omega \\ -\operatorname{div}\sigma=f\text{ on }\Omega \\ u\in(H_0^1(\Omega))^N \end{gathered}$$

where div σ is the vector field defined by:

$$(\operatorname{div}\sigma)_i=\sum_{j=1}^{n}\frac{\partial\sigma_{i,j}}{\partial x_j}.$$

A classical way to perturb (5.5) is as follows: let F, the perturbation function, be defined by

$$F : (H_0^1(\Omega))^N \times (L^2(\Omega))_S^{N2} \to \mathbb{R}$$
$$(u, \tau) \to \int_\Omega j(x, e(u)+\tau)\,dx - \int_\Omega f.u\,dx.$$

The associated Lagrangian (the Reisner functional) takes the form

$$\bar{K}(u, \sigma) = \int_\Omega \sigma.e(u)\,dx - \int_\Omega f.u\,dx - \int_\Omega j^*(x, \sigma)\,dx. \tag{5.8}$$

In the case of homogenization, we are concerned with a sequence $(j^\varepsilon)_{\varepsilon\downarrow 0}$ of functions which are defined in the following way.

We consider $Y = (]0, 1[)^N$ as the unit cell and the function

(5.9) $j : \mathbb{R}^N \times \mathbb{R}_S^{N^2} \to \mathbb{R}$
$(x, z) \to j(x, z)$
which is Y-periodic, measurable in x, convex in z and satisfies the growth conditions (5.6),

and

$$F^\varepsilon(u, \tau) = \int_\Omega j^\varepsilon(x, e(u)+\tau)\,dx - \int_\Omega f.u\,dx \quad \text{where} \quad j^\varepsilon(x, z) = j\left(\frac{x}{\varepsilon}, z\right). \tag{5.10}$$

It is clear, from the growth conditions imposed on j, that for each $\varepsilon > 0$, the Lagrangian $(5.8)_\varepsilon$ admits a saddle point $(u_\varepsilon, \sigma_\varepsilon)$ characterized by

$$\begin{aligned} &\sigma_\varepsilon \in \partial j^\varepsilon(x, e(u_\varepsilon))\ \text{a. e.} \\ &-\operatorname{div}\sigma_\varepsilon = f \\ &u_\varepsilon \in (H_0^1(\Omega))^N \end{aligned} \tag{5.11}$$

and that the corresponding solutions $(u_\varepsilon, \sigma_\varepsilon)$ remain bounded in $(H_0^1(\Omega))^N \times (L^2(\Omega))_S^{N^2}$. The epi-convergence of the unperturbed functionals $F^\varepsilon(., 0)$ was first proved by P. Marcellini [36], see also [4], and is the following

$$\begin{aligned} &w_{(H_0^1(\Omega))^N}\text{-lm}_e\, F^\varepsilon(., 0) = F^{\text{hom}}(., 0) \\ &\text{where} \\ &F^{\text{hom}}(u, \tau) = \int_\Omega j^{\text{hom}}(e(u)+\tau)\,dx - \int_\Omega f.u\,dx; \\ &\text{and} \\ &j^{\text{hom}}(z) = \min_{w \in (H_p^1(Y))^N} \int_Y j(y, e(w)+z)\,dy; \end{aligned} \tag{5.12}$$

$H_p^1(Y)$ denoting the set of periodic functions belonging to $H^1(Y)$ (having the same traces on opposite faces of ∂Y). We have that

PROPOSITION 5.1 :

$$w_{(H_0^1(\Omega))^N} \times s_{(L^2(\Omega))_S^{N^2}}\text{-lm}_e F^\varepsilon = F^{hom} \tag{5.13}$$

Proof. — The proof, whose details appear in [4], is sketched out below. We can neglect the effect of the continuous perturbation $\int_\Omega f.u\,dx$ and first restrict the vector field $\tau(x)$ to be piecewise constant on an open paving of Ω. We then apply (5.12) on each open set on which $\tau(x)$ is constant, observing that $e(u)+z=e(u+\langle z, .\rangle)$, from which the lim inf condition of epi-convergence (1.11) follows. For the lim sup condition, we proceed in a similar manner after stiching together the approximated sequences in order to obtain approximate sequence in $H^1(\Omega)$. The proof is then completed by a density and continuity argument using the upper growth condition on $j(.,.)$. □

As a consequence of Proposition 5.1 and Theorem 3.2, we obtain

PROPOSITION 5.2. — *The Lagrangian sequence*

$$\overline{K}^\varepsilon(u, \tau)=\int_\Omega \tau.e(u)\,dx-\int_\Omega f.u\,dx-\int_\Omega (j^\varepsilon)^*(x, \tau)\,dx \tag{5.14$_\varepsilon$}$$

epi/hypo-converges in the extended sense for the topology

$$w-(H_0^1(\Omega))^N \times w-(L^2(\Omega))_S^{N^2}$$

to the Lagrangian

$$\overline{K}^{hom}(u, \tau)=\int_\Omega \tau.e(u)\,dx-\int_\Omega f.u\,dx-\int_\Omega (j^{hom})^*(\tau)\,dx. \tag{5.14}$$

Moreover we have, for every $\eta \in \mathbb{R}_S^{N^2}$

$$\begin{cases} (j^{hom})^*(\eta)=\min \int_Y j^*(y, \sigma+\eta)\,dy \\ \quad \begin{cases} \operatorname{div}\sigma=0 \\ \int_y \sigma\,dy=0 \\ \sigma.n \textit{ takes opposite values on opposite faces of } \partial Y. \end{cases} \end{cases} \tag{5.15}$$

Proof. — It suffices to apply Theorem 3.2 and Proposition 5.2, taking into account that $F^\varepsilon(0, \tau)=\int_\Omega j^\varepsilon(x, \tau)\,dx \leqq \Lambda_0 \int_\Omega (1+|\tau|^2)\,dx$, as follows from (5.16), and thus assumption $\mathscr{H}$ is verified. The arguments that yield (5.15) are straightforward (*see* [4] for instance). □

Using now the variational properties of extended epi/hypo-convergence (Theorem 2.6) and the boundedness of the sequences (u_ε) and (σ_ε), we obtain

COROLLARY 5.3. — *The sequence* $(u_\varepsilon, \sigma_\varepsilon)$ *of saddle points of* $\bar{K}^\varepsilon$ *converges for* $w-(H_0^1(\Omega))^N \times w-(L^2(\Omega))_S^{N^2}$ *to the unique saddle point* (u, σ) *of* $\bar{K}_{hom}$, *moreover, we obtain*

$$\lim_{\varepsilon \to 0} \bar{K}^\varepsilon(u_\varepsilon, \sigma_\varepsilon) = \bar{K}^{hom}(u, \sigma).$$

5.2. We now consider the convergence of the primal/dual solution in a *reinforcement* problem when the zone of thickness goes to 0. We consider a bounded regular open set $\Omega \subset \mathbb{R}^3$ split into two open subsets Ω_1, Ω_2 by a surface Σ. For the sake of simplicity, we shall assume that Σ is the plane $x_3 = 0$. The surface Σ is surrounded by a thin layer of size ε, $\Sigma_\varepsilon = \left\{ x \in \Omega;\ d(x, \Sigma) \leqq \frac{\varepsilon}{2} \right\}$. We consider in Ω the problem

$$\left\{ \begin{array}{ll} -\operatorname{div}(a_\varepsilon(x) \operatorname{D} u(x)) = f(x) & \text{in } \Omega \\ u \,|\, \partial\Omega = 0 & \end{array} \right. \tag{5.16}$$

where $a_\varepsilon(x) = 1$ on $\Omega \setminus \Sigma_\varepsilon$, $a_\varepsilon(x) = \lambda(\varepsilon) > 0$ on Σ_ε and $f \in L^2(\Omega)$. We assume that

$$\lim_{\varepsilon \to 0} \varepsilon \lambda(\varepsilon) = k > 0 \tag{5.17}$$

and we are interested in the behaviour as $\varepsilon \to 0$ of the couple $(u_\varepsilon, a_\varepsilon \operatorname{D} u_\varepsilon)$ where u_ε is the solution of (5.16). The behaviour of u_ε is well known (*see* [4], [47]). The convergence of the dual variable $\sigma_\varepsilon = a_\varepsilon \operatorname{D} u_\varepsilon$ has been studied in [43] by using epi-convergence methods. Here we give an alternate and simpler proof of the convergence result based on Theorem 3.2. The solution of (5.16) can be characterized as the unique minimum point of the functional $\frac{1}{2} \int_\Omega a_\varepsilon |\operatorname{D} u|^2 dx - \int_\Omega f u \, dx$ on the Sobolev space $H_0^1(\Omega)$. By using simple estimates for which we refer to [4] and [47], we obtain that (u_ε) is bounded in $H_0^1(\Omega)$ and $\sigma_\varepsilon = a_\varepsilon \operatorname{D} u_\varepsilon$ is bounded in $(L^1(\Omega))^3$. So it is reasonable to study the convergence of (σ_ε) for the w^* topology of bounded measures on Ω. Denote by $\mathscr{C}_0(\Omega)$ the set of continuous functions from Ω ino $\mathbb{R}$ that vanish on the boundary $\partial\Omega$ of Ω and define for $u \in H_0^1(\Omega)$ and $\varphi = (\varphi_1, \varphi_2, \varphi_3) \in (\mathscr{C}_0(\Omega))^3$

$$F_\varepsilon(u, \varphi) = \frac{1}{2} \int_\Omega a_\varepsilon |\operatorname{D} u + \varphi|^2 dx - \int_\Omega f u \, dx. \tag{5.18}$$

Let $\mathcal{M}(\Omega)=(\mathscr{C}_0(\Omega))^*$ be the space of bounded measures on Ω viewed as the dual space of $\mathscr{C}_0(\Omega)$ endowed with the norm $\|\,.\,\|_\infty$ of uniform convergence. The upper Lagrangian $\bar{K}_\varepsilon$ associated with $F_\varepsilon(.,\,.)$ is easy to compute by taking into account the fact that, when extended to $(L^2(\Omega))^3$, the function $F_\varepsilon(u,\,.)$ is continuous and that $\mathscr{C}_0(\Omega)$ is dense in $L^2(\Omega)$. For any $(u,\,\mu)\in H_0^1(\Omega)\times\mathcal{M}(\Omega))^3$, we obtain

$$(5.19)\quad \bar{K}_\varepsilon(u,\,\mu)=\begin{cases} -\infty & \text{if } \mu\notin(L^2(\Omega))^3\\ \displaystyle\int_\Omega \sigma.\,Du\,dx-\int_\Omega f.\,u\,dx-\frac{1}{2}\int_\Omega \frac{1}{a_\varepsilon}|\sigma|^2\,dx & \\ & \text{if } \mu=\sigma\in(L^2(\Omega))^3). \end{cases}$$

In order to apply Theorem 3.2, we need to analyze the epi-convergence of the sequence $F_\varepsilon(.,\,.)$ for the topology $w\text{-}H_0^1(\Omega)\times s\text{-}(\mathscr{C}_0))^3$. Let us begin by some notations. Define for $\varphi\in(\mathscr{C}(\Omega))^3$, $\|\varphi\|^2:=\|\varphi\|^2_{(L^2(\Omega))^3}+k\,\|\varphi^t\|^2_{(L^2(\Sigma))^2}$ where $\varphi^t(x_1,\,x_2,\,0):=(\varphi_1(x_1,\,x_2,\,0),\,\varphi_2(x_1,\,x_2,\,0))$. We denote by H the completion of $(\mathscr{D}(\Omega))^3$ for the norm $\|\,.\,\|$, and by V the completion of $\mathscr{D}(\Omega)$ for the norm $\|Du\|$. One has $V\subset H_0^1(\Omega)$ and $Dv\in H$ when $v\in V$. For an element $\tau\in H$, the notation $\tau^t(x_1,\,x_2,\,0)=(\tau_1(x_1,\,x_2,\,0),\,\tau_2(x_1,\,x_2,\,0))$ makes sense and $\tau^t\in(L^2(\Sigma))^2$. We observe that H is a Hilbert space endowed with the inner product

$$(5.20)\qquad [\sigma,\,\tau]=\int_\Omega \sigma.\,\tau\,dx+k\int_\Sigma \sigma^t.\,\tau^t\,ds.$$

PROPOSITION 5.4 :

$$w\text{-}H_0^1(\Omega)\times s\text{-}(\mathscr{C}_0(\Omega))^3\text{-}\mathrm{lm}_e\,F_\varepsilon(.,\,.)=F(.,\,.)$$

where

$$(5.21)\quad F(u,\,\varphi)=\begin{cases} +\infty & \textit{if } u\notin V\\ \displaystyle\frac{1}{2}\|Du+\varphi\|^2-\langle f,\,u\rangle=\frac{1}{2}\int_\Omega |Du+\varphi|^2\,dx & \\ \displaystyle+\frac{k}{2}\int_\Sigma |Du^t+\varphi^t|^2\,ds-\int_\Omega f\,u\,dx & \textit{if } u\in V. \end{cases}$$

Proof. — In [43], Section 4.2, it is shown that, for each $\varphi\in(\mathscr{D}(\Omega))^3$

$$w\text{-}H_0^1(\Omega)\text{-}\mathrm{lm}_e\,F_\varepsilon(.,\,\varphi)=F(.,\,\varphi).$$

The result then follows by using the following observation whose proof is a simple exercice; for each sequence $\tau_\varepsilon,\,\tau'_\varepsilon\in(L^2(\Omega))^3$ such that $\|\tau_\varepsilon-\tau'_\varepsilon\|_\infty\to 0$ as $\varepsilon\to 0$ one has

$$\liminf_{\varepsilon\to 0}\int_\Omega a_\varepsilon|\tau'_\varepsilon|^2\,dx=\liminf_{\varepsilon\to 0}\int_\Omega a_\varepsilon|\tau_\varepsilon|^2\,dx$$

and the same for the lim sup. □

Let us now compute the upper Lagrangian $\bar{K}$ associated with F.

LEMMA 5.5. — *For each* $(u, \mu) \in H_0^1(\Omega) \times (\mathcal{M}(\Omega))^3$

$$(5.22)\quad \bar{K}(u, \mu) = \begin{cases} +\infty & \text{if } u \notin V \\ -\infty & \text{if } u \in V \text{ and } \mu \notin H \\ [\sigma, Du] - \langle f, u \rangle - \frac{1}{2}\|\sigma\|^2 & \text{if } u \in V \text{ and } \mu = \sigma \in H. \end{cases}$$

Proof. — If $u \notin V$, $F(u, .) \equiv +\infty$ then $\bar{K}(u, .) \equiv +\infty$. Consider now $u \in V$. We observe that $(\mathcal{C}_0(\Omega))^3$ is continuously and densely imbedded in H, thus H identified with H* can be viewed as a subspace of $\mathcal{M}(\Omega))^3$. For any $\sigma \in H$, the associated measure is defined by $\langle \sigma, \varphi \rangle = [\sigma, \varphi] = \int_\Omega \sigma . \varphi \, dx + k \int_\Sigma \sigma^t . \varphi^t \, ds$. We can extend the functional F to a functional F_0 defined on $V \times H$ by $F_0(u, \tau) = \frac{1}{2}\|Du + \tau\|^2 - \langle f, u \rangle$. As $F_0(u, .)$ is clearly continuous on H and $(\mathcal{C}_0(\Omega))^3$ is dense in H, it follows that $(F(u, .))^*(\mu) = +\infty$ if $\mu \notin H$ and $(F(u, .))^*(\mu) = (F_0(\mu, .))^*(\sigma)$ if $\mu = \sigma \in H$. From which the result follows. □

The convergence result is the following.

PROPOSITION 5.6. — *The sequence of Lagrangians* $\bar{K}_\varepsilon : H_0^1(\Omega) \times (\mathcal{M}(\Omega))^3 \to \bar{\mathbb{R}}$ *defined in* (5.19) *epi/hypo-converges in the extended sense with respect to the topology* $w\text{-}H_0^1(\Omega) \times w^*\text{-}(\mathcal{M}(\Omega))^3$ *to the Lagrangian* $\bar{K} : H_0^1(\Omega) \times (\mathcal{M}(\Omega))^3 \to \bar{\mathbb{R}}$ *defined in* (5.22). *Moreover, the sequence of saddle points* $(u_\varepsilon, a_\varepsilon D u_\varepsilon)$ *of* $\bar{K}_\varepsilon$ *converges with respect to the topology* $w\text{-}H_0^1(\Omega) \times w^*\text{-}(\mathcal{M}(\Omega))^3$ *to* (u, μ) *the unique saddle point of* $\bar{K}$, *where u is the unique solution to* $\min\left\{\frac{1}{2}\|Dv\|^2 - \langle f, v \rangle, v \in V\right\}$ *and* $\mu = Du$ *in* H.

Proof. — By appealing to Proposition 5.4 we can apply Theorem 3.2. Observe that assumption $\mathcal{H}$ is fulfilled since

$$F_\varepsilon(0, \varphi) = \frac{1}{2}\int_\Omega a_\varepsilon |\varphi|^2 \, dx \leqq \frac{1}{2}\|a_\varepsilon\|_{L^1(\Omega)} \|\varphi\|_\infty^2 \leqq C \|\varphi\|_\infty^2.$$

The convergence of the saddle points then follows from the variational properties of extended epi/hypo-convergence and from the unicity of the saddle point (u, μ) of $\bar{K}$. The fact that $\mu = Du$ in H follows from Lemma 5.5. That tells us that the Lagrangian $\bar{K}_0$ (associated with F_0) and

$\bar{K}$ have the same saddle points and that (u, Du) is clearly the unique saddle-point of $\bar{K}_0$. □

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