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Differential inclusions with free time

by

Philip D. LOEWEN

Department of Mathematics,
University of British Columbia,
Vancouver B.C., Canada V6T 1Y4

Frank H. CLARKE

Centre de Recherches Mathématiques,
Université de Montréal,
Montréal, Canada H3C 3J7

and

Richard B. VINTER

Electrical Engineering Department,
Imperial College of Science and Technology,
London SW7 2BT, United Kingdom

ABSTRACT. — We prove Hamiltonian necessary conditions for state-constrained differential inclusion problems in which the basic time interval is one of the unknowns. Previous approaches typically reduce this problem to one based on a fixed time interval by transforming the time variable into an auxiliary state—a device which unfortunately requires that the data exhibit rather smooth t-dependence. Here we use proximal analysis to avoid this transformation, and offer the first complete treatment of free-time problems whose dynamics are assumed to be merely measurable in t.

Key words : Differential inclusions, free time, state constraints, essential values, proximal analysis.

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RÉSUMÉ. — Nous démontrons des conditions nécessaires d'optimalité sous forme hamiltonienne pour des problèmes de contrôle avec contraintes sur l'état lorsque l'horizon fait partie des inconnues. Notre étude utilise l'analyse proximale et s'applique des situations où la dépendance en t est simplement mesurable.

I. INTRODUCTION

Consider a standard differential inclusion problem in Mayer form, namely to minimize the objective $l(T, x(T))$ over all times $T>0$ and arcs $x : [0, T] \to \mathbb{R}^n$ satisfying

$$\dot{x}(t) \in F(t, x(t)) \text{ a. e. } [0,T]. \tag{1.1}$$

The endpoint conditions on x and T are given by

$$x(0)=x_0, \qquad (T, x(T)) \in S. \tag{1.2}$$

When $S=\{T\} \times D$ for some $T>0$ and $D \subseteq \mathbb{R}^n$, we have a *fixed-time* problem. Any solution x of the problem obeys a well-known set of necessary conditions built around the Hamiltonian inclusion [2], Ch. III. Among these conditions is the terminal transversality relation, which asserts for the fixed-time problem above that the adjoint variable p is transverse to the effective objective function at $x(T)$:

$$(-p(T), -1) \in N_{\text{epi}\,(l+\Psi_D)}(x(T), l(x(T))), \tag{1.3}$$

where $N_D(z)$ is the cone of normals to D at $z \in D$. The hypotheses under which the Hamiltonian necessary conditions apply require that the multifunction F be Lipschitz in x, but perhaps only measurable in t. Our focus here is upon the nature of the t-dependence: we emphasize that measurability is a workable and natural hypothesis in the derivation of necessary conditions for fixed-time problems.

This contrasts sharply with the situation in which the terminal time T may vary. To clarify the issue, consider the *free-time* problem in which the set S above equals $(0, +\infty) \times D$ and l does not depend on T. It is clear that any solution (T, x) to this free-time problem is also a solution to the fixed-time problem whose terminal constraint set is $\{T\} \times D$. Thus the Hamiltonian necessary conditions, including (1.3), certainly pertain, assuming only measurable t-dependence. However these conditions fail to account for the additional degree of freedom arising from the variability of T. One more condition is required. The nature of this additional condition has long been known (*see* Pontryagin *et al.* [11]). It is formulated in terms of the problem's *Hamiltonian*

$$H(t, x, p) = \sup\{\langle p, v\rangle : v \in F(t, x)\}, \tag{1.4}$$

and relies upon Lipschitzian t-dependence of the multifunction F. When the data exhibit this extra regularity, the additional necessary condition for (T, x) to solve the free-time problem is [2], Thm. 3.6.1

$$H(T, x(T), p(T)) = 0. \tag{1.5}$$

Now it is the case that the extra necessary condition (1.5) has previously been proved only under strong assumptions on the regularity of the data with respect to the time variable. For problems where the state constraint is not operative and a differential equation formulation is adopted, hypotheses have been imposed requiring continuous dependence in a neighbourhood of the optimal terminal time (*see*, e. g., [8] and [1]). As for problems where the state constraints enter in a nontrivial way, or the dynamics are described by means of differential inclusions, free time necessary conditions have been proved, at best, under assumptions of Lipschitz continuous dependence (*see*, e. g. [15], [2]). An explanation of the difficulties encountered in trying to weaken the continuity hypotheses was offered by Ioffe and Tihomirov ([9], p. 237) as follows.

We can not, apparently, reduce the free time problem to a standard optimization problem over a Banach space of functions, and then deduce necesary conditions from an abstract multiplier rule, "without some transformation connected in particular, with treating the time as a phase coordinate. In so doing, the requirement of differentiability with respect to time becomes unavoidable." [The free time problem can, of course, be posed over other Banach spaces. For example, the editor suggests the change of variable $t = Ts$, which displays the problem's domain as the space of pairs (T, y (.)) consisting of a parameter $T \in \mathbb{R}$ and an absolutely continuous function $y : [0,1] \to \mathbb{R}^n$. However, this reduction yields a dynamic constraint $y'(s) \in TF(Ts, y(s))$ which exhibits at best measurable dependence upon the parameter T, and this irregularity again places the problem beyond the reach of standard methods.]

In this article we obtain complete necessary conditions for free-time problems with measurable time dependence. The exact formulation of the problem (*see* Section 2) is more general than that given above, but let us remain with the free-time problem for purposes of illustration. The first issue concerns the very interpretation (or extension) of (1.5) when F, and hence H, depends only measurably on t.

Let $f: \mathbb{R} \to \mathbb{R}^n$ be measurable. We define the set of *essential values of f at* T, denoted $\operatorname*{ess}_{t \to T} f(t)$, as follows:

$$\operatorname*{ess}_{t \to T} f(t) = \{ y \in \mathbb{R}^n : \forall \varepsilon > 0,$$
$$m \{ t \in (T-\varepsilon, T+\varepsilon) : |f(t) - y| < \varepsilon \} > 0 \}. \quad (1.6)$$

When f is real-valued and essentially bounded near T, we have the following relation (co denotes convex hull):

$$\operatorname{co} \operatorname*{ess}_{t \to T} f(t) = [\operatorname*{ess\,lim\,inf}_{t \to T} f(t), \operatorname*{ess\,lim\,sup}_{t \to T} f(t)]. \quad (1.7)$$

Furthermore, if g is defined by $g(t) = \int_0^t f(s)\,ds$, then the set in (1.7) coincides with $\partial g(T)$, the generalized gradient of g at T [2], Example 2.2.5. The calculus of generalized gradients is at the heart of the methodology of this article.

For the free time problem, the necessary conditions that we prove incorporate the condition

$$0 \in \operatorname{co} \operatorname*{ess}_{t \to T} H(t, x(T), p(T)). \quad (1.8)$$

This recaptures (1.5) when F is Lipschitz in t, since in the case H is also Lipschitz in t and the right side of (1.8) reduces to $\{H(T, x(T), p(T))\}$. In fact, this reduction relies only upon the continuity of H in t, so that (1.5) holds even when F is only assumed to be continuous. This is a new result.

The special case of our necessary conditions, treating problems where the state constraint is not operative, has been proved in [6]. An illustration of the application of optimality conditions of this nature to free time problems with data discontinuous in the time variable is provided in [7].

Our proof is a finite-dimensional application of proximal analysis. This approach, introduced in [2], Thms. 3.4.3 and 6.5.2, relies upon the characterization of the (Clarke) normal cone in terms of analytically simpler "proximal normals" or "perpendiculars". Recall that for a given closed set $C \subseteq \mathbb{R}^m$ and point $c \in C$, a vector v is *proximal normal to* C *at* c [written $v \in PN_C(c)$] if one has

$$\exists M > 0 \quad \text{s.t.} \quad 0 \leqq \langle -v, c' - c \rangle + M |c' - c|^2, \qquad \forall c' \in C. \quad (1.9)$$

Associated with the set C is the *distance function* d_C, defined by

$$d_C(x) = \inf \{ |x - c'| : c' \in C \}; \quad (1.10)$$

the distance function is Lipschitz of rank 1 everywhere, and its generalized gradient at a point $c \in C$ can be computed using the *proximal normal* *formula*

$$\partial d_C(c) = \text{co}\,[\{0\} \cup \{v : v = \lim v_i/|v_i|,\ v_i \in PN_C(c_i),\ c_i \to c\}]. \quad (1.11)$$

The latter set generates the *normal cone to* C *at* c :

$$N_C(c) = \bigcup_{\lambda \geq 0} \lambda\, \partial d_C(c). \quad (1.12)$$

Proximal analysis usually consists of studying inequality (1.9) in sufficient detail to evaluate the right side of (1.11), from which both $\partial d_C(c)$ and $N_C(c)$ are then obtained. In this paper, however, we use (1.9) and (1.11) somewhat differently. Thanks to our finite-dimensional context, $\partial d_C(c)$ contains nonzero points whenever c lies on the boundary of C:

$$c \in bdy\, C \quad \Rightarrow \quad \partial d_C(c) \setminus \{0\} \neq \varnothing. \quad (1.13)$$

It follows from (1.11) that there exists at least one convergent sequence of proximal normal unit vectors, and this will turn out to be all we need. A suitable scaling of the terms in this sequence leads to a new convergent sequence whose limit furnishes the desired necessary conditions.

The theory underlying the previous paragraph is all in [2], where one can also find applications of proximal analysis to differential inclusion and mathematical programming problems. The technique has since been applied in a variety of other situations—*see* [4], [5], [3], for example.

Essential Values

The developments to follow rely upon some elementary properties of essential values. Given an integrable real valued function f, we define

$$\bar{f}(t) = \lim_{\varepsilon \to 0} \sup (1/\varepsilon) \int_t^{t+\varepsilon} f(s)\, ds,$$

$$\underline{f}(t) = \lim_{\varepsilon \to 0} \inf (1/\varepsilon) \int_t^{t+\varepsilon} f(s)\, ds.$$

One-sided analogues $\bar{f}(t-)$, $\bar{f}(t+)$, $\underline{f}(t-)$, $\underline{f}(t+)$ are defined in terms of the appropriate one-sided limits.

LEMMA 1.1. — *For any integrable function* f, *we have*

$$\operatorname*{ess\,inf}_{t \to T} f(t) \leq \underline{f}(T) = \min\{\underline{f}(T-), \underline{f}(T+)\},$$

$$\operatorname*{ess\,sup}_{t \to T} f(t) \geq \bar{f}(T) = \max\{f(T-), \bar{f}(T+)\}.$$

In particular, $\text{co}\operatorname*{ess}_{t \to T} f(t)$ *contains the intervals* $[\bar{f}(T+), \underline{f}(T-)]$ *and* $[\bar{f}(T-), \underline{f}(T+)]$.

LEMMA 1.2. — *Let* $f(t, x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ *be essentially bounded in a neighbourhood of some point* $(\bar{t}, \bar{x})$. *Suppose that on this neighbourhood* $f(., x)$ *is measurable and* $f(t, .)$ *is continuous, uniformly in t. Then the multifunction* $(t, x) \to \operatorname{co} \underset{s \to t}{\operatorname{ess}} f(s, x)$ *has closed graph at* $(\bar{t}, \bar{x})$.

Proof. — Let (t_i, x_i) be any sequence converging to $(\bar{t}, \bar{x})$, and suppose $v_i \in \underset{s \to t_i}{\operatorname{ess}} f(s, x_i)$, $\forall i$ while v_i converges to some limit v. We shall first establish $v \in \underset{s \to \bar{t}}{\operatorname{ess}} f(s, \bar{x})$. This is equivalent to showing $m(S(\varepsilon)) > 0$, $\forall \varepsilon > 0$, where

$$S(\varepsilon) = \{ s \in (\bar{t} - \varepsilon, \bar{t} + \varepsilon) : |v - f(s, \bar{x})| < \varepsilon \}.$$

Fix $\varepsilon > 0$. For each i, the inclusion $v_i \in \underset{s \to t_i}{\operatorname{ess}} f(s, x_i)$ implies $m(S_i(\varepsilon/3)) > 0$, where

$$S_i(\delta) = \{ s \in (t_i - \delta, t_i + \delta) : |v_i - f(s, x_i)| < \delta \}.$$

If we now choose I so large that $i \geqq I$ implies

$$\begin{gathered} |t_i - \bar{t}| < \varepsilon/3, \\ |f(s, x_i) - f(s, \bar{x})| < \varepsilon/3, \qquad \forall s \in (\bar{t} - \varepsilon, \bar{t} + \varepsilon), \\ |v_i - v| < \varepsilon/3, \end{gathered}$$

then $i \geqq I$ also implies

$$\begin{aligned} S_i(\varepsilon/3) &= \{ s \in (t_i - \varepsilon/3, t_i + \varepsilon/3) : |v_i - f(s, x_i)| < \varepsilon/3 \} \\ &\subseteqq \{ s \in (t_i - \varepsilon/3, t_i + \varepsilon/3) : |v - f(s, \bar{x})| < \varepsilon \} \\ &\subseteqq S(\varepsilon). \end{aligned}$$

The desired conclusion follows. Otherwise stated, $(t, x) \to \underset{s \to t}{\operatorname{ess}} f(s, x)$ has closed graph at $(\bar{t}, \bar{x})$. That $(t, x) \to \operatorname{co} \underset{s \to t}{\operatorname{ess}} f(s, x)$ also has closed graph follows from this result and Caratheodory's theorem, in view of our assumption that f is essentially bounded near $(\bar{t}, \bar{x})$. ////

II. STATEMENT OF THE MAIN RESULT

We consider the state- and endpoint-constrained differential inclusion problem (P) defined by

$$\begin{aligned} \min_{(a, b, x)} \{ l(a, x(a), b, x(b)) : \dot{x}(t) \in F(t, x(t)) \text{ a. e. } [a, b], \\ (a, x(a), b, x(b)) \in S, \\ g(t, x(t)) \leqq 0, \ \forall t \in [a, b] \cap J, \\ |a - \bar{a}| < \omega, \ |b - \bar{b}| < \omega, \ \|x - \bar{x}\|_\infty < \omega \}. \qquad \text{(P)} \end{aligned}$$

Among the given quantities defining problem (P), the nondegenerate interval $[\bar{a}, \bar{b}]$ and arc $\bar{x} : [a, b] \to \mathbb{R}^n$ are central. In the language of the classical calculus of variations, the open set specified by the last line of (P) is a *strong neighbourhood* of $(\bar{a}, \bar{b}, \bar{x})$: our analysis below proceeds under the assumption that $(\bar{a}, \bar{b}, \bar{x})$ solves (P)—and hence remains valid whenever such a triple provides a *strong local solution* to a dynamic optimization problem of the same form. The results to be derived hold for any $\omega > 0$, but for convenience we assume that $\bar{a}+\omega<\bar{b}-\omega$. To make sense of expressions like $\|x-\bar{x}\|_\infty$ for arcs defined on different time intervals, we extend the domain of any x defined on any $[a, b]$ by setting

$$x(t)=x(a), \quad \forall\, t<a, \qquad x(t)=x(b), \quad \forall\, t>b.$$

Given the quantities $(\bar{a}, \bar{b}, \bar{x})$ and $\omega \in (0, (\bar{b}-\bar{a})/2)$, we make the following hypotheses. Our notation is $B=\{x \in \mathbb{R}^n : |x|<1\}$

$$\Omega=\{(t, x) : t \in (\bar{a}-\omega, \bar{b}+\omega), |x-\bar{x}(t)|<\omega\},$$
$$\Omega_t=\{x : (t, x) \in \Omega\}=\bar{x}(t)+\omega B.$$

(H1) The objective function $l : D_l \to \mathbb{R}$ is Lipschitz of rank K_l on

$$D_l=\{(a, x, b, y) : |a-\bar{a}|<\omega, |x-\bar{x}(\bar{a})|<\omega, |b-\bar{b}|<\omega, |y-\bar{x}(\bar{b})|<\omega\};$$

also the endpoint constraint set $S \subseteq \mathbb{R}^{1+n+1+n}$ is closed.

(H2) The state constraint mapping $g : \Omega \to \mathbb{R}$ is continuous and the set $J \subseteq \mathbb{R}$ of times when the state constraint applies is closed; moreover there is a constant $K_g \geqq 0$ such that

$$|g(t, y)-g(t, x)| \leqq K_g |y-x|, \qquad \forall\, t \in (\bar{a}-\omega, \bar{b}+\omega), \quad \forall\, x, y \in \Omega_t.$$

(H3) For each $(t, x) \in \Omega$, the set $F(t, x)$ is nonempty, compact, and convex.

(H4) For each $(t, x) \in \Omega$, the multifunction $t' \to F(t', x)$ is measurable on some neighbourhood of t; also, there is a nonnegative $\varphi_F \in L^1(\bar{a}-\omega, \bar{b}+\omega)$ which is essentially bounded on $(\bar{a}-\omega, \bar{a}+\omega) \cup (\bar{b}-\omega, \bar{b}+\omega)$ and obeys

$$F(t, x) \subseteq \varphi_F(t) B, \qquad \forall (t, x) \in \Omega.$$

(H5) There is a nonnegative $k_F \in L^1(\bar{a}-\omega, \bar{b}+\omega)$ which is essentially bounded on $(\bar{a}-\omega, \bar{a}+\omega) \cup (\bar{b}-\omega, \bar{b}+\omega)$ and obeys

$$F(t, y) \subseteq F(t, x)+k_F(t)|y-x| B, \qquad \forall\, t \in (\bar{a}-\omega, \bar{b}+\omega), \quad \forall\, x, y \in \Omega_t.$$

(H6) The triple $(\bar{a}, \bar{b}, \bar{x})$ solves (P), and at each of the points $\tau=\bar{a}, \bar{b}$ one of the following conditions holds: either $g(\tau, \bar{x}(\tau))<0$, or else the τ-component of the endpoint constraint set S is the single point $\{\tau\}$.

In the statement of necessary conditions given below as Theorem 2.1, we use the notation

$$\partial_x^+ g(t, x)=\text{co}\{\lim \gamma_i : \gamma_i \in \partial_x g(t_i, x_i), (t_i, x_i) \to (t, x), g(t_i, x_i)>0, \forall\, i\}.$$

Note that $\partial_x^+ g(t, x) = \varnothing$ if $g(t, x) < 0$, while $\partial_x^+ g(t, x) \supseteqq \partial_x g(t, x)$ if $g(t, x) > 0$. The latter inclusion can be proper, since the multifunction $\partial_x^+ g$ evidently has closed graph with respect to joint variations in t and x, a feature not enjoyed by its subset $\partial_x g$. When $g(t, x) = 0$, $\partial_x^+ g(t, x)$ contains first-order information gathered only from directions violating the state constraint and can be strictly smaller than $\partial_x g(t, x)$. [Take $g(t, x) = \max\{0, x\}$ at $x = 0$ to see this.]

THEOREM 2.1. — *Assume* (H1)-(H6), *and write* $\bar{s} = (\bar{a}, \bar{x}(\bar{a}), \bar{b}, \bar{x}(\bar{b}))$. *Then there exist constants* $\lambda \geqq 0$, h, *and* k, *an arc* $p : [\bar{a}, \bar{b}] \to \mathbb{R}^n$, *a measurable* $\gamma : [\bar{a}, \bar{b}] \to \mathbb{R}^n$, *and a nonnegative measure* μ *such that* $\lambda + \|p\|_\infty + \mu[\bar{a}, \bar{b}] = 1$ *and for every*

$$\mathrm{R} \geqq (\mathrm{K}_l^2 + 1)^{1/2} \left| \left(-h, p(\bar{a}), -\lambda, -k, -p(\bar{b}) - \int_{[\bar{a}, \bar{b}]} \gamma \, d\mu, -\lambda\right) \right|,$$

one has

$$(-\dot{p}(t), \dot{\bar{x}}(t)) \in \partial \mathrm{H}\left(t, \bar{x}(t), p(t) + \int_{[\bar{a}, t)} \gamma \, d\mu\right) a.e. [\bar{a}, \bar{b}] \tag{2.1}$$

$$\left(-h, p(\bar{a}), k, -p(\bar{b}) - \int_{[\bar{a}, \bar{b}]} \gamma \, d\mu\right) \in \lambda \partial l(\bar{s}) + \mathrm{R}\, \partial d_{\mathrm{S}}(\bar{s}) \tag{2.2}$$

$$h \in \operatorname{co} \operatorname*{ess}_{t \to \bar{a}} \mathrm{H}(t, \bar{x}(\bar{a}), p(\bar{a})) \tag{2.3}$$

$$k \in \operatorname{co} \operatorname*{ess}_{t \to \bar{b}} \mathrm{H}\left(t, \bar{x}(\bar{b}), p(\bar{b}) + \int_{[\bar{a}, \bar{b}]} \gamma \, d\mu\right) \tag{2.4}$$

$$\gamma(t) \in \partial_x^+ g(t, \bar{x}(t))\, \mu\text{-}a.e. [\bar{a}, \bar{b}] \tag{2.5}$$

$$\operatorname{Supp}(\mu) \subseteqq \{t \in [\bar{a}, \bar{b}] \cap \mathrm{J} : \partial_x^+ g(t, \bar{x}(t)) \neq \varnothing\}. \tag{2.6}$$

Note that the state constraint function $g(t, x) \equiv -1$ obeys (H2), so that out theory applies to state-constraint-free problems as well. Since $g \equiv -1$ forces $\partial_x^+ g(t, x(t)) = \varnothing$, $\forall\, x, t$, condition (2.6) implies that Supp $(\mu) = \varnothing$, i. e., μ is the zero measure. Thus we obtain the following simplied conditions for problems without state constraints.

COROLLARY 2.2. — *Take* $g \equiv -1$ *and assume* (H1)-(H6). *Then there exist constants* $\lambda \geqq 0$, h, *and* k, *and an arc* $p : [\bar{a}, \bar{b}] \to \mathbb{R}^n$, *such that* $\lambda + \|p\|_\infty = 1$ *and for every*

$$\mathrm{R} \geqq (\mathrm{K}_l^2 + 1)^{1/2} |(-h, p(\bar{a}), -\lambda, -k, -p(\bar{b}), -\lambda)|,$$

one has

$$(-\dot{p}(t), \dot{\bar{x}}(t)) \in \partial \mathrm{H}(t, \bar{x}(t), p(t)) \text{ a. e. } [\bar{a}, \bar{b}] \tag{2.7}$$

$$(-h, p(\bar{a}), k, -p(\bar{b})) \in \lambda\, \partial l(\bar{s}) + \mathrm{R}\, \partial d_{\mathrm{S}}(\bar{s}) \tag{2.8}$$

$$h \in \operatorname{co} \operatorname*{ess}_{t \to \bar{a}} \mathrm{H}(t, \bar{x}(\bar{a}), p(\bar{a})) \tag{2.9}$$

$$k \in \operatorname{co} \operatorname*{ess}_{t \to \bar{b}} \mathrm{H}(t, \bar{x}(\bar{b}), p(\bar{b})) \tag{2.10}$$

III. A SUBSTANTIAL SPECIAL CASE

We begin our proof of Theorem 2.1 by analysing the special case arising when the objective function is simply $\langle l, x(b) \rangle$ for some fixed $l \in \mathbb{R}^n$, the state constraints are continually in force (i.e., $\mathrm{J} = \mathbb{R}$), and $(\bar{a}, \bar{b}, \bar{x})$ is the problem's unique solution. We will derive a set of necessary conditions similar to those of Theorem 2.1 for the reduced problem; these will form the backbone of our proof of Theorem 2.1 itself in Section 4.

Our approach relies upon finite-dimensional perturbations of the endpoint set and state constraint to produce a value function amenable to proximal analysis. Our perturbations of the state constraint rely on the observation that the joint continuity of g implies

$$g(t, x(t)) \leqq 0, \qquad \forall t \in [a, b] \quad \Leftrightarrow \quad \int_a^b g^+(t, x(t))\, dt \leqq 0,$$

where $g^+ = \max\{0, g\}$. It is the right-hand formulation we adopt here. (Infinite-dimensional perturbations of the left side were studied in [5].) Thus we fix any $\bar{\omega} \in (0, \omega)$ and vector $\theta = (\alpha, \xi, \beta, \eta, \rho)$ in $\mathbb{R}^{1+n+1+n+1}$, and consider the problem

$$\min_{(a, b, x)} \Big\{ \langle l, x(b) \rangle : \dot{x} \in \mathrm{F}(t, x(t)) \text{ a.e. } [a, b],$$
$$(a, x(a), b, x(b)) \in \mathrm{S} + (\alpha, \xi, \beta, \eta),$$
$$\int_a^b g^+(t, x(t))\, dt + \rho \leqq 0,$$
$$|a - \bar{a}| \leqq \bar{\omega}, \ |b - \bar{b}| \leqq \bar{\omega}, \ |x - \bar{x}|_\infty \leqq \bar{\omega} \Big\}. \qquad \mathrm{P}(\theta)$$

The *value function* $\mathrm{V} : \mathbb{R}^{1+n+1+n+1} \to \mathbb{R} \cup \{+\infty\}$ arising from this family of problems is simply $\mathrm{V}(\theta) = \inf \mathrm{P}(\theta)$. Since $(\bar{a}, \bar{b}, \bar{x})$ solves problem (P) it also solves P(0), whence $\mathrm{V}(0) = \langle l, \bar{x}(\bar{b}) \rangle$ is finite. Conversely, whenever $\mathrm{V}(\theta)$ is finite [i.e., whenever there exists a triple (a, b, x) satisfying the constraints of $\mathrm{P}(\theta)$], problem $\mathrm{P}(\theta)$ has a solution. The proof of this fact relies upon a sequential compactness property of F-trajectories [2], Thm. 3.1.7 (*cf.* [4], Prop. 1.2]) and the nonstrict inequalities involving $\bar{\omega}$ in the statement of the perturbed problem. In the present setting these properties yield the following result, in which we refer to the uniqueness hypothesis

(H6+) Hypothesis (H 6) holds, and the triple $(\bar{a}, \bar{b}, \bar{x})$ is the unique solution of P(0).

LEMMA 3.1. – *Assume* (H 1)-(H 5).

(*a*) *If* $V(\theta) < +\infty$ *then* $P(\theta)$ *has a solution.*

(*b*) *If* θ_i *is any sequence converging to some point* θ *along which* $V(\theta_i) < +\infty$, $\forall i$, *then any sequence* (a_i, b_i, x_i) *of solutions to* $P(\theta_i)$ *has a subsequence converging uniformly to a limit* (a, b, x) *which is feasible for* $P(\theta)$. *In particular,* V *is lower semicontinuous everywhere.*

(*c*) *Suppose* (H 6+) *also holds. Then for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *so small that the following holds. Whenever* $|\theta| < \delta$ *and* $V(\theta) < V(0) + \delta$, *any solution* (a, b, x) *to* $P(\theta)$ *obeys* $|a-\bar{a}|+|b-\bar{b}|+|x-\bar{x}|_\infty < \varepsilon$. *In particular, if a sequence* θ_i *tends to* 0 *and* $V(\theta_i) \to V(0)$, *then the solution sequence described in* (*b*) *has a subsequence which actually tends to* $(\bar{a}, \bar{b}, \bar{x})$.

Note that the function V is certain not to be smooth at 0-whenever θ is chosen so that $\rho > 0$, one has $V(\theta) = +\infty$. Nonetheless, the lower semicontinuity of V implies that the epigraph set

$$\operatorname{epi} V = \{ (\theta, v) : v \geqq V(\theta) \}$$

is closed. Since (0, V(0)) lies on the boundary of epi V, (1.13) ensures that $\partial d_{\operatorname{epi} V}(0, V(0)) \setminus \{0\} \neq \varnothing$. According to (1.11) there exists some unit vector $\nu = (h, \varphi, k, \psi, \zeta, -\lambda)$ in $\mathbb{R}^{1+n+1+n+1+1}$ which can be realized as the limit of a sequence of unit vectors proximal normal to epi V at base points approaching (0, V(0)). We begin, therefore, with the study of a single proximal normal vector based near (0, V(0)).

LEMMA 3.2. – *Assume* (H 1)-(H 6+). *Let* $\nu = (h, \varphi, k, \psi, \zeta, -\lambda)$ *be a nonzero vector which is proximal normal to* epi V *at some point* (θ, V) *obeying* $|\theta| < \delta$ *and* $V < V(0) + \delta$, *where* δ *is given by Lemma* 3.1 (*c*) *corresponding to the choice* $\varepsilon = \bar{\omega}$. *Then problem* $P(\theta)$ *has a solution* (a, b, x) *to which there correspond an arc* $p : [a, b] \to \mathbb{R}^n$ *and measurable functions* $\gamma : [a, b] \to \mathbb{R}^n$, $m : [a, b] \to [0, 1]$ *such that* $\lambda \geqq 0$, $\zeta \geqq 0$, *and the following conditions hold.*

$$(-\dot{p}(t), \dot{x}(t)) \in \partial H\left(t, x(t), p(t) + \int_a^t \gamma(s) \zeta m(s)\, ds\right) a.e.\ [a, b] \quad (3.1)$$

$$(-h, -\varphi, -k, -\psi) \in |(h, \varphi, k, \psi)|\, \partial d_S(a-\alpha, x(a)-\xi, b-\beta, x(b)-\eta) \quad (3.2)$$

$$h+\zeta g^{+}(a, x(a)) \in \operatorname{co} \operatorname*{ess}_{t \to a} \mathrm{H}(t, x(a), p(a)) \tag{3.3}$$

$$-k+\zeta g^{+}(b, x(b)) \in \operatorname{co} \operatorname*{ess}_{t \to b} \mathrm{H}\left(t, x(b), p(b)+\int_a^b \gamma(s)\zeta m(s)\,ds\right) \tag{3.4}$$

$$\gamma(t) \in \partial_x^{+} g(t, x(t)) \text{ a. e. } \{t \in [a, b] : \partial_x^{+} g(t, x(t)) \neq \varnothing\} \tag{3.5}$$

$$\{t \in [a, b] : m(t) > 0\} \subseteqq \{t \in [a, b] : \partial_x^{+} g(t, x(t)) \neq \varnothing\} \tag{3.6}$$

$$-p(b)-\int_a^b \gamma(s)\zeta m(s)\,ds = \lambda l - \psi \tag{3.7}$$

$$p(a) = -\varphi \tag{3.8}$$

Proof. — The nature of an epigraph set implies that $\lambda \geqq 0$, and allows us to assume without loss of generality that $\mathrm{V}=\mathrm{V}(\theta)$. The condition $\mathrm{V}<+\infty$ implies that problem $\mathrm{P}(\theta)$ has a solution (a, b, x) by Lemma 3.1 (*a*); in fact, Lemma 3.1 (*c*) guarantees that $|a-\bar{a}|+|b-\bar{b}|+|x-\bar{x}|_{\infty}<\bar{\omega}$. Thus there is a constant $\sigma>0$ such that for any triple (a', b', y) obeying

$$\dot{y}(t) \in \mathrm{F}(t, y(t)) \text{ a. e. } [a', b'], \tag{3.9a}$$

$$|b'-b|<\sigma, \qquad |a'-a|<\sigma, \qquad \|y-x\|_{\infty}<\sigma \tag{3.9b}$$

as well as the state and terminal constraints of $\mathrm{P}(\theta)$, one has $\langle l, x(b)\rangle \leqq \langle l, y(b')\rangle$.

Now $\mathrm{V}=\langle l, x(b)\rangle$, and the constraints in $\mathrm{P}(\theta)$ imply that $(s, u, t, v) \in \mathrm{S}$ and $r \geqq 0$, where

$$(s, u, t, v)=(a, x(a), b, x(b))-(\alpha, \xi, \beta, \eta),$$
$$r=-\rho-\int_a^b g^{+}(t, x(t))\,dt.$$

Conversely, let (a', b', y) be any triple obeying (3.9), and let any $(s', u', t', v') \in \mathrm{S}$ and $r' \geqq 0$ be given. Then upon defining

$$(\alpha', \xi', \beta', \eta')=(a', y(a'), b', y(b'))-(s', u', t', v')$$
$$\rho'=-r'-\int_a^b g^{+}(t, y(t))\,dt,$$
$$\theta'=(\alpha', \xi', \beta', \pi', \rho'),$$

we have $\mathrm{V}(\theta') \leqq \langle l, y(b')\rangle$. In other words, $(\theta', \langle l, y(b')\rangle) \in \operatorname{epi} \mathrm{V}$. According to the proximal normal inequality (1.9), there is a constant $\mathrm{M}>0$

such that

$$\Big\langle -(h,\ \varphi,\ k,\ \psi,\ \zeta,\ -\lambda),\ \Big(a'-s',\ y(a')-u',\ b'-t',\ y(b')-v',\ -r'-\int_{a'}^{b'} g^+(\tau,\ y(\tau))\,d\tau,\ \langle l,\ y(b')\rangle\Big)\Big\rangle$$
$$+M\Big|\Big(a'-a,\ y(a')-x(a),\ b'-b,\ y(b')-x(b),\ \int_a^b g^+(\tau,\ x(\tau))\,d\tau-\int_{a'}^{b'} g^+(\tau,\ y(\tau))\,d\tau,\ \langle l,\ y(b')-x(b)\rangle\Big)\Big|^2$$
$$+\mathrm{M}\,|(s'-s,\ u'-u,\ t'-t,\ v'-v,\ r'-r)|^2$$
$$\geqq \Big\langle -(h,\ \varphi,\ k,\ \psi,\ \zeta,\ -\lambda),\ \Big(a-s,\ x(a)-u,\ b-t,\ x(b)-v,\ -r-\int_a^b g^+(\tau,\ x(\tau))\,d\tau,\ \langle l,\ x(b)\rangle\Big)\Big\rangle \tag{3.10}$$

for all $(a',\ b',\ y)$ obeying (3.9), all $(s',\ u',\ t',\ v')\in \mathrm{S}$, and all $r'\geqq 0$.

Now certainly $(a,\ b,\ x)$ obeys (3.9) and $r\geqq 0$, so choosing these values for $(a',\ b',\ y)$ and r' in (3.10) leads to the inequality

$$\langle (h,\ \varphi,\ k,\ \psi),\ (s',\ u',\ t',\ v')-(s,\ u,\ t,\ v)\rangle + \mathrm{M}\,|(s',\ u',\ t',\ v')-(s,\ u,\ t,\ v)|^2 \geqq 0$$

for all $(s',\ u',\ t',\ v')\in \mathrm{S}$. It follows that $(s,\ u,\ t,\ v)$ minimizes the left side over all $(s',\ u',\ t',\ v')\in \mathrm{S}$, from which we deduce [2], Prop. 2.4.3, p. 51

$$0\in(h,\ \varphi,\ k,\ \psi)+|(h,\ \varphi,\ k,\ \psi)|\,\partial d_{\mathrm{S}}(s,\ u,\ t,\ v). \tag{3.11}$$

This verifies (3.2).

If we next take $(s',\ u',\ t',\ v')=(s,\ u,\ t,\ v)$ and $(a',\ b',\ y)=(a,\ b,\ x)$ in (3.10), then we obtain $\langle \zeta,\ r'-r\rangle+\mathrm{M}\,|r'-r|^2\geqq 0,\ \forall\, r'\geqq 0$. It follows that $\zeta\geqq 0$.

Finally, we fix $(s',\ u',\ t',\ v')=(s,\ u,\ t,\ v)$ and $r'=r$ in (3.10) to obtain

$$\Big\langle -(h,\ \varphi,\ k,\ \psi-\lambda l,\ \zeta),\ \Big(a',\ y(a'),\ b',\ y(b'),\ -\int_{a'}^{b'} g^+(t,\ y(t))\,dt\Big)\Big\rangle$$
$$+\mathrm{M}\Big|\Big(a'-a,\ y(a')-x(a),\ b'-b,\ y(b')-x(b),\ \int_a^b g^+(t,\ x(t))\,dt-\int_{a'}^{b'} g^+(t,\ y(t))\,dt,\ \langle l,\ y(b')-x(b)\rangle\Big)\Big|^2$$
$$\geqq \Big\langle -(h,\ \varphi,\ k,\ \psi-\lambda l,\ \zeta),\ \Big(a,\ x(a),\ b,\ x(b),\ -\int_a^b g^+(t,\ x(t))\,dt\Big)\Big\rangle \tag{3.12}$$

for all $(a',\ b',\ y)$ obeying (3.9). Choosing $a'=a$ and $b'=b$ in this inequality implies that x locally solves the fixed-time differential inclusion problem of minimizing the objective

$$\langle -\varphi, y(a)\rangle + \langle \lambda l - \psi, y(b)\rangle + \zeta \int_a^b g^+(t, y(t))\, dt$$
$$+ \mathrm{M} \left| \left(y(a) - x(a),\ y(b) - x(b),\ \int_a^b (g^+(t, x) - g^+(t, y))\, dt,\ \langle l, y(b) - x(b)\rangle \right) \right|^2$$

over all y obeying (3.9). From the state- and endpoint-constraint free, hence normal, case of [2], Thm. 3.2.6, we deduce that there is an arc q on $[a, b]$ obeying

$$(-\dot{q}(t), \dot{x}(t)) \in \partial \mathrm{H}(t, x(t), q(t)) - \zeta\, \partial_x g^+(t, x(t)) \times \{0\} \text{ a. e. } [a, b] \quad (3.13)$$
$$q(a) = -\varphi \quad (3.14)$$
$$-q(b) = \lambda l - \psi. \quad (3.15)$$

We pause to verify the following inclusion:

$$\partial_x g^+(t, x) \subseteq \mathrm{co}(\{0\} \cup \partial_x^+ g(t, x)).$$

This is obviously true if $g(t, x) \neq 0$, since $g(t, .)$ is continuous. So we may limit attention to the case $g(t, x) = 0$. Now $\partial_x g^+(t, x)$ is the convex hull of limits $s = \lim_i \nabla g^+(t_i, x_i)$, where $\nabla g^+(t_i, x_i)$ exists $\forall i$ and $(t_i, x_i) \to (t, x)$. We may assume that $g^+(t_i, x_i) = 0$, $\forall i$, for otherwise we deduce via subsequence extraction that $s = 0$ or else belongs to $\partial_x^+ g(t, x)$. But if $g^+(t_i, x_i) = 0$, then $\nabla g^+(t_i, x_i) = 0$ since g^+ achieves a minimum at (t_i, x_i). It follows that $\nabla g^+(t_i, x_i) = 0$, $\forall i$, so $s = 0$. The inclusion is proved.

We define multifunctions Γ, Σ by

$$\Gamma(t) = \begin{cases} \partial_x^+ g(t, x(t)) & \text{if } \partial_x^+ g(t, x(t)) \neq \varnothing, \\ \{0\} & \text{if } \partial_x^+ g(t, x(t)) = \varnothing; \end{cases}$$
$$\Sigma(t) = \begin{cases} \{0\} & \text{if } g(t, x(t)) < 0 \text{ or } \partial_x^+ g(t, x(t)) = \varnothing, \\ [0, 1] & \text{if } g(t, x(t)) = 0 \text{ and } \partial_x^+ g(t, x(t)) \neq \varnothing, \\ \{1\} & \text{if } g(t, x(t)) > 0. \end{cases}$$

The Hamiltonian inclusion and the earlier inclusion result imply that $\partial_x g^+(t, x(t)) \subseteq \Sigma(t)\Gamma(t)$ a. e., and (3.13) gives

$$(-\dot{q}(t), \dot{x}(t)) \in \partial \mathrm{H}(t, x(t), q(t)) - \zeta \Sigma(t) \Gamma(t) \times \{0\} \text{ a. e. } [a, b].$$

Now standard measurable selection theorems, e. g. [15], Thm. I.7.10, imply that there exist measurable functions $\gamma(t) \in \Gamma(t)$ and $m(t) \in \Sigma(t)$ a. e. on $[a, b]$ such that in fact

$$(-\dot{q}(t), \dot{x}(t)) \in \partial \mathrm{H}(t, x(t), q(t)) - \zeta m(t) \gamma(t) \times \{0\} \text{ a. e. } [a, b]. \quad (3.16)$$

Note that $\gamma(t)$ and $m(t)$ obey (3.5)-(3.6). Upon defining the arc

$$p(t)=q(t)-\int_a^t \gamma(s)\zeta m(s)\,ds, \tag{3.17}$$

we obtain from (3.16), (3.14), and (3.15) the conditions (3.1), (3.7), and (3.8):

$$(-\dot p(t),\ \dot x(t))\in\partial H\left(t,\ x(t),\ p(t)+\int_a^t \gamma(s)\zeta m(s)\,ds\right) \text{ a. e. } [a,\ b],$$

$$p(a)=-\varphi,$$

$$-p(b)-\int_a^b \gamma(s)\zeta m(s)\,ds=\lambda l-\psi.$$

This completes the verification of all the assertions of Lemma 3.2 except for the two dealing with the free times a and b. To these we now turn.

Let us fix a small $\varepsilon>0$, and consider any measurable selection

$$f(t)\in\arg\max\{\langle\psi-\lambda l,\ v\rangle : v\in F(t,\ x(b))\},\ \text{a. e. } (b-\varepsilon,\ b+\varepsilon).$$

We may then define the arc y_0 on $[a,\ b+\varepsilon]$ via

$$y_0(t)=\begin{cases} x(t), & t\in[a,\ b];\\ x(b)+\displaystyle\int_b^t f(s)\,ds, & t\in(b,\ b+\varepsilon].\end{cases}$$

Since $|f(s)|\leqq\varphi_F(s)$ a. e., y_0 lies in the tube specified in (3.9 b), provided ε is small enough. Of course y_0 may fail to be an F-trajectory on $[b,\ b+\varepsilon]$, but it cannot fail to an excessive extent—we estimate

$$\begin{aligned}\rho_F(y_0)&=\int_b^{b+\varepsilon}\operatorname{dist}(\dot y_0(t),\ F(t,\ y_0(t)))\,dt\\ &\leqq\int_b^{b+\varepsilon}k_F(t)\,|y_0(t)-x(b)|\,dt\\ &\leqq\int_b^{b+\varepsilon}k_F(t)\int_b^t\varphi_F(r)\,dr\,dt.\end{aligned}$$

Since φ_F and k_F are essentially bounded in $(b-\varepsilon,\ b+\varepsilon)$, [2], Thm. 3.1.6, applies to give an F-trajectory y on $[b,\ b+\varepsilon]$ with $y(b)=y_0(b)=x(b)$ and

$$\int_b^{b+\varepsilon}|\dot y(t)-f(t)|\,dt\leqq c\,\varepsilon^2 \tag{3.18}$$

for some $c\geqq 0$ depending only on φ_F and k_F. When extended to $[a,\ b+\varepsilon]$ by setting $y(t)=x(t)$ on $[a,\ b]$, the F-trajectory y obeys (3.9), and must therefore satisfy [by (3.12)]

$$0 \leqq \int_b^{b+\varepsilon} (-k + \langle \lambda l - \psi, \dot{y}(t) \rangle) \, dt + \zeta \int_b^{b+\varepsilon} g^+(t, y(t)) \, dt + \mathrm{M}\, \Delta(\varepsilon)^2, \quad (3.19\,a)$$

$$\Delta(\varepsilon) = \left| \left(\varepsilon,\, y(b+\varepsilon) - x(b),\, \int_b^{b+\varepsilon} g^+(t, y(t)) \, dt,\, \langle l, y(b+\varepsilon) - x(b) \rangle \right) \right|. \quad (3.19\,b)$$

Now $\Delta(\varepsilon)^2/\varepsilon \to 0$ as $\varepsilon \to 0^+$, while

$$(\zeta/\varepsilon) \int_b^{b+\varepsilon} g^+(t, y(t)) \, dt \to \zeta g^+(b, x(b))$$

by continuity of g. Moreover, (3.18) gives

$$\left| (1/\varepsilon) \int_b^{b+\varepsilon} \langle \lambda l - \psi, \dot{y}(t) - f(t) \rangle \, dt \right| \leqq |\lambda l - \psi| \, c \, \varepsilon,$$

so we deduce from (3.19) that

$$0 \leqq \liminf_{\varepsilon \to 0+} (1/\varepsilon) \int_b^{b+\varepsilon} (-k + \langle \lambda l - \psi, f(t) \rangle) \, dt + \zeta g^+(b, x(b))$$

whence

$$-k + \zeta g^+(b, x(b)) \geqq \limsup_{\varepsilon \to 0+} (1/\varepsilon) \int_b^{b+\varepsilon} \mathrm{H}(t, x(b), \psi - \lambda l) \, dt. \quad (3.20)$$

This is the contribution of the possibility $b' > b$ to condition (3.4). [Note (3.7).]

Next consider $b' = b - \varepsilon < b$ for some fixed $\varepsilon > 0$. We consider the F-trajectory $y = x|_{[a,\, b-\varepsilon]}$. In this case (3.12) gives [for a slightly different $\Delta(\varepsilon)$]

$$\begin{aligned} 0 &\leqq \int_{b-\varepsilon}^{b} [k + \langle \psi - \lambda l, \dot{x}(t) \rangle] \, dt - \zeta \int_{b-\varepsilon}^{b} g^+(t, x(t)) \, dt + \mathrm{M}\, \Delta(\varepsilon)^2 \\ &\leqq \int_{b-\varepsilon}^{b} [k + \mathrm{H}(t, x(t), \psi - \lambda l)] \, dt - \zeta \int_{b-\varepsilon}^{b} g^+(t, x(t)) \, dt + M\, \Delta(\varepsilon)^2. \end{aligned}$$

Dividing through this inequality by ε and letting $\varepsilon \to 0^+$, we obtain

$$-k + \zeta g^+(b, x(b)) \leqq \liminf_{\varepsilon \to 0+} (1/\varepsilon) \int_{b-\varepsilon}^{b} \mathrm{H}(t, x(b), \psi - \lambda l) \, dt. \quad (3.21)$$

In view of Lemma 1.1, conditions (3.20) and (3.21) combine to give (3.4). Similar arguments give rise to (3.3). ////

Convergence analysis

Suppose now that the unit vector $\nu=(h, \varphi, k, \psi, \zeta, -\lambda)$ is obtained as the limit of a sequence ν_i of unit vectors proximal normal to epi V at base points (θ_i, V_i) converging to $(0, V(0))$. [The existence of such a sequence follows from (1.11) and (1.13).] Then Rockafellar's proximal subgradient formula ([12], *see* also [10]) implies that by adjusting (θ_i, V_i) and ν_i slightly, the same limit can be obtained under the additional assuption that $\lambda_i>0$, $\forall i$. Let us assume that this has been done. Then for each i there is a solution (a_i, b_i, x_i) of $P(\theta_i)$ together with corresponding quantities p_i, γ_i, m_i as in Lemma 3.2. Since $\lambda_i>0$, $\forall i$, the quantities $\varepsilon_i=\lambda_i+|h_i|+|\varphi_i|+|k_i|+|\psi_i|+\zeta_i\|m_i\|_1$ are all positive, so we may divide the quantities ν_i and p_i appearing in Lemma 3.2 by ε_i (without renaming them) to obtain the conditions below.

$$(-\dot{p}_i(t), \dot{x}_i(t))\in\partial H\left(t, x_i(t), p_i(t)+\int_{a_i}^{t}\gamma_i(s)\,d\mu_i(s)\right) \text{ a.e. } [a_i, b_i] \quad (3.22)$$

$$(-h_i, -\varphi_i, -k_i, -\psi_i)\in|(h_i, \varphi_i, k_i, \psi_i)| \times\partial d_S(a_i-\alpha_i, x_i(a_i)-\xi_i, b_i-\beta_i, x_i(b_i)-\eta_i) \quad (3.23)$$

$$h_i+\zeta_i g^+(a_i, x_i(a_i))\in\operatorname{co}\operatorname*{ess}_{t\to a_i} H(t, x_i(a_i), p_i(a_i)) \quad (3.24)$$

$$-k_i+\zeta_i g^+(b_i, x_i(b_i))\in \operatorname{co}\operatorname*{ess}_{t\to b_i} H\left(t, x_i(b_i), p_i(b_i)+\int_{a_i}^{b_i}\gamma_i(s)\,d\mu_i(s)\right) \quad (3.25)$$

$$\gamma_i(t)\in\partial_x^+ g(t, x_i(t))\ \mu_i\text{-a.e.} \quad (3.26)$$

$$\operatorname{Supp}(\mu_i)\subseteq\{t\in[a_i, b_i]: \partial_x^+ g(t, x_i(t))\neq\varnothing\} \quad (3.27)$$

$$-p_i(b_i)-\int_{a_i}^{b_i}\gamma_i(s)\,d\mu_i(s)=\lambda_i l-\psi_i \quad (3.28)$$

$$p_i(a_i)=-\varphi_i \quad (3.29)$$

$$\lambda_i+|h_i|+|\varphi_i|+|k_i|+|\psi_i|+\mu_i[a_i, b_i]=1. \quad (3.30)$$

According to Lemma 3.1 (*c*), we may pass to a subsequence (without relabelling) along which (a_i, b_i, x_i) converges uniformly to $(\bar{a}, \bar{b}, \bar{x})$. Here we have written $\mu_i(ds)=\zeta_i m_i(s)\,ds$; note that these measures are nonnegative because both $\zeta_i\geqq 0$ and $m_i(t)\geqq 0$ a.e., while $\mu_i(\mathbb{R})\leqq 1$, $\forall i$ by (3.30). Consequently the sequence $\{\mu_i\}$ has a subsequence converging weak* to a measure μ supported on $[\bar{a}, \bar{b}]$. Likewise, the condition $|\gamma_i(t)|\leqq K_g$ a.e.

$\forall i$ implies that the $\mathbb{R}^n$-valued measures $\gamma_i(t)\, d\mu_i(t)$ are bounded in total variation, so they too have a weak*-convergent subsequence. The limit of this subsequence is a vector measure supported on $[\bar{a}, \bar{b}]$ which is absolutely continuous with respect to μ, and hence has a representation as $\gamma(t)\, d\mu(t)$ for some μ-integrable mapping γ. Vinter and Pappas [14], Lemma 4.5, show that γ actually satisfies the limiting version of (3.26). As for (3.27), let K_i denote the set on the right-hand side, and let $K = \{t \in [\bar{a}, \bar{b}] : \partial_x^+ g(t, \bar{x}(t)) \neq \varnothing\}$. The closure and uniform boundedness of the multifunction $\partial_x^+ g(.,.)$ readily imply that for any $\varepsilon > 0$, one has $K_i \subseteqq K + (-\varepsilon, \varepsilon)$ for all i sufficiently large. Consequently the limiting measure μ is supported on K.

Along a further subsequence we may assume that $(h_i, \varphi_i, k_i, \psi_i) \to (h, \varphi, k, \psi)$ and $\lambda_i \to \lambda$, where (3.30) holds in the limit. In particular the initial points $p_i(a_i)$ form a convergent sequence and hence [2], Prop. 3.1.7, implies that (along yet another subsequence) the arcs (x_i, p_i) converge uniformly to some arc $(\bar{x}, p)$ satisfying both (3.22) and (3.23). We turn finally to inclusions (3.24) and (3.25). If $g(\bar{a}, \bar{x}(\bar{a})) < 0$, then it follows that $g^+(a_i, x_i(a_i)) = 0$ for all i sufficiently large, and the limiting validity of (3.24) follows from Lemma 1.2. The limiting form of (3.25) is proven likewise, assuming that $g(\bar{b}, \bar{x}(\bar{b})) < 0$. We may now present the desired necessary conditions for P(0).

THEOREM 3.5. — *Assume* (H 1)-(H 6+). *Then there exist constants* $\lambda \geqq 0$, h, *and* k, *an arc* $p \in AC[\bar{a}, \bar{b}]$, *a measurable mapping* $\gamma : [\bar{a}, \bar{b}] \to \mathbb{R}^n$, *and a nonnegative measure* μ *such that* $\lambda + \|p\|_\infty + \mu[\bar{a}, \bar{b}] > 0$ *and for any*

$$R \geqq \left| \left(h,\ -p(\bar{a}),\ k,\ \lambda l + p(\bar{b}) + \int_{[\bar{a}, \bar{b}]} \gamma\, d\mu \right) \right|,$$

$$(-\dot{p}(t), \dot{\bar{x}}(t)) \in \partial H\left(t, \bar{x}(t), p(t) + \int_{[\bar{a}, t)} \gamma\, d\mu \right) a.e.\ [\bar{a}, \bar{b}] \tag{3.31}$$

$$\left(-h, p(\bar{a}), -k, -p(\bar{b}) - \int_{[\bar{a}, \bar{b}]} \gamma\, d\mu \right) \in \lambda(0, 0, 0, l) + R\, \partial d_S(\bar{a}, \bar{x}(\bar{a}), \bar{b}, \bar{x}(\bar{b})) \tag{3.32}$$

$$h \in \operatorname{co} \operatorname*{ess}_{t \to \bar{a}} H(t, \bar{x}(\bar{a}), p(\bar{a})) \tag{3.33}$$

$$-k \in \operatorname{co} \operatorname*{ess}_{t \to \bar{b}} H\left(t, \bar{x}(\bar{b}), p(\bar{b}) + \int_{[\bar{a}, \bar{b}]} \gamma\, d\mu \right) \tag{3.34}$$

$$\gamma(t) \in \partial_x^+ g(t, \bar{x}(t))\ \mu\text{-}a.e.\ [\bar{a}, \bar{b}] \tag{3.35}$$

$$\operatorname{Supp}(\mu) \subseteqq \{ t \in [\bar{a}, \bar{b}] : \partial_x^+ g(t, \bar{x}(t)) \neq \varnothing \}. \tag{3.36}$$

Proof. — We assert that the limiting quantities λ, h, k, p, γ, and μ identified in the previous paragraph satisfy the theorem's conclusions. Conditions (3.31)-(3.36) have been checked above. Only the nontriviality condition remains to verify. Suppose it were false. Then $\lambda=0$, $\|p\|_\infty=0$, and $\mu[\bar{a}, \bar{b}]=0$ by assumption, so $h=k=0$ by (3.33)-(3.34), while

$$-\varphi=p(\bar{a})=0 \qquad \text{and} \qquad \psi=\lambda l+p(\bar{b})+\int_{[\bar{a},\, \bar{b}]} \gamma\, d\mu=0.$$

Consequently

$$\lambda+|h|+|\varphi|+|k|+|\psi|+\mu[\bar{a}, \bar{b}]=0.$$

This contradicts the limiting version of (3.30), and hence is impossible. ////

IV. PROOF OF THEOREM 2.1

We turn now to the proof of Theorem 2.1, for which the following technical result is necessary.

LEMMA 4.1. — *Let* $S \subseteqq \mathbb{R}^\nu$ *be a closed set containing a point* $\bar{s}$. *Suppose there is a constant* $\delta>0$ *and a function* $l: \bar{s}+\delta B \to \mathbb{R}$ *such that* l *is Lipschitz of rank* K_l *on* $\bar{s}+\delta B$. *Then for all* $R \geqq (K_l^2+1)^{1/2}$, *one has*

$$\partial d_{\text{epi}\,(l+\Psi_S)}(\bar{s}, l(\bar{s})) \subseteqq \{(\zeta, -\varepsilon): \zeta \in \varepsilon\, \partial l(\bar{s})+R\, \partial d_S(\bar{s}),\ \varepsilon \geqq 0\}.$$

[*Here* $\Psi_S(s)$ *equals* 0 *if* $s \in S$, $+\infty$ *otherwise.*]

Proof. — Let E denote the set on the right side of the desired inclusion. Observe that the set E is convex and contains 0, so it suffices to show that E contains all limits of unit proximal normals as described in (1.11).

Let $(\zeta, -\varepsilon)$ be proximal normal to $\text{epi}(l+\Psi_S)$ at (x, v), where (x, v) is so near to $(\bar{s}, l(\bar{s}))$ that $|x-\bar{s}|<\delta$. Then $\varepsilon \geqq 0$ and, for some $M>0$, x minimizes the following functional over S:

$$x' \to \varepsilon\, l(x')-\langle \zeta, x' \rangle+M\,|(x', l(x'))-(x, l(x))|^2.$$

Now for any constant ρ obeying $\rho>\varepsilon K_l+|\zeta|$, there is a neighbourhood of x on which the Lipschitz rank of this functional is majorized by ρ. On this neighbourhood, [2], Prop. 2.4.3, asserts that x provides a local minimum for the penalized functional

$$x' \to \varepsilon\, l(x')-\langle \zeta, x' \rangle+M\,|(x', l(x'))-(x, l(x))|^2+\rho\, d_S(x').$$

Hence zero belongs to the functional's generalized gradient at x, i.e.

$$\zeta \in \varepsilon\, \partial l(x)+\rho\, \partial d_S(x).$$

Since this holds for all $\rho > \varepsilon K_l + |\zeta|$, it holds also for $\rho = \varepsilon K_l + |\zeta|$, and we deduce that

$$\frac{\zeta}{|(\zeta, -\varepsilon)|} \in \frac{\varepsilon}{|(\zeta, -\varepsilon)|} \partial l(x) + \frac{\varepsilon K_l + |\zeta|}{|(\zeta, -\varepsilon)|} \partial d_S(x)$$
$$\subseteq \frac{\varepsilon}{|(\zeta, -\varepsilon)|} \partial l(x) + (K_l^2 + 1)^{1/2} \partial d_S(x).$$

[The last inclusion holds because $(K_l^2+1)^{1/2}$ is the largest possible coefficient of $\partial d_S(x)$, attained when $\varepsilon = K_l |\zeta|$.] It follows readily that E contains all limits of proximal normal unit vectors, as required. ////

Let us now reduce the general case of Theorem 2.1 to an application of Theorem 3.5: Given a problem (P) satisfying (H 1)-(H 6), we formulate a modified problem $(\tilde{P})$ whose state (x, y, z) evolves in $\mathbb{R}^{n+1+1}$. The data governing $(\tilde{P})$ are

$$\tilde{F}(t, x, y, z) = \{ (v, |x - \bar{x}(t)|^2, 0) : v \in F(t, x) \},$$
$$\tilde{S} = \{ (a, x_0, y_0, z_0, b, x_1, y_1, z_1) :$$
$$(a, x_0, b, x_1) \in S,\ y_0 = 0,\ y_1 \in \mathbb{R},\ z_0 \in \mathbb{R},\ z_1 \geqq l(a, x_0, b, x_1) \},$$
$$\tilde{g}(t, x, y, z) = g(t, x) - g^+(t, \bar{x}(t)) - d_J(t),$$
$$\tilde{J} = \mathbb{R},$$
$$\tilde{l}(a, x_0, y_0, z_0, b, x_1, y_1, z_1) = y_1 + z_1.$$

The resulting problem $(\tilde{P})$ bears a simple relationship to the given problem (P). First, any (a, b, x, y, z) admissible for $(\tilde{P})$ gives rise to a triple (a, b, x) admissible for (P). For x is certainly an F-trajectory on $[a, b]$, and one has

$$g(t, x(t)) \leqq g^+(t, \bar{x}(t)) + d_J(t) = 0, \quad \forall t \in J.$$

The objective value of (a, b, x) in the original problem (P) is majorized by the constant value of z in $(\tilde{P})$, and the terminal value $y(b)$ in $(\tilde{P})$ equals $\int_a^b |x(t) - \bar{x}(t)|^2 \, dt$. Consequently the value of (a, b, x, y, z) in $(\tilde{P})$ is greater than or equal to the value of (a, b, x) in (P), and this inequality is strict unless $x \equiv \bar{x}$. Indeed, the arc

$$(\bar{x}(t), \bar{y}(t), \bar{z}(t)) = (\bar{x}(t), 0, l(\bar{a}, \bar{x}(\bar{a}), \quad \bar{b}, \bar{x}(\bar{b})))$$

is the unique solution of problem $(\tilde{P})$.

Let us apply the necessary conditions of Theorem 3.5, whose hypotheses are clearly in force. We obtain scalars $\lambda \geqq 0$, h, and k, an arc $(p, q, r) : [\bar{a}, \bar{b}] \to \mathbb{R}^{n+1+1}$, a measurable $\tilde{\gamma} : [\bar{a}, \bar{b}] \to \mathbb{R}^{n+1+1}$, and a nonnegative measure μ for which conditions (3.31)-(3.36) hold. First, note that the state constraint function $\tilde{g}$ does not depend on either y or z, so that the measurable selection $\tilde{\gamma}$ of $\partial^+_{(x, y, z)} \tilde{g}$ generated by Theorem 3.5 actually has the form $(\gamma(t), 0, 0)$ for some selection $\gamma(t)$ of $\partial^+_x \tilde{g}(t, \bar{x}(t))$. Now for any (t, x).

$$\tilde{g}(t, x)>0 \quad \Rightarrow \quad g(t, x)>g^{+}(t, \bar{x}(t))+d_{\mathrm{J}}(t)\geqq 0.$$

Consequently $\partial_x^+ \tilde{g}(t, x) \subseteqq \partial_x^+ g(t, x)$ for all (t, x). Moreover, if $t \notin \mathrm{J}$ then $d_{\mathrm{J}}(t)>0$ and $\tilde{g}(t, \bar{x}(t))=g(t, \bar{x}(t))-g^{+}(t, \bar{x}(t))-d_{\mathrm{J}}(t)<0$. Therefore $\partial_x^+ \tilde{g}(t, \bar{x}(t))=\varnothing$ for $t \notin \mathrm{J}$, and the selection conditions of Theorem 3.5 imply

$$\operatorname{supp}(\mu) \subseteqq \{t \in [\bar{a}, \bar{b}] \cap \mathrm{J} : \partial_x^+ g(t, \bar{x}(t)) \neq \varnothing\},$$
$$\gamma(t) \in \partial_x^+ g(t, \bar{x}(t)) \ \mu\text{-a. e. } [\bar{a}, \bar{b}].$$

The Hamiltonian for problem $(\tilde{\mathrm{P}})$ is

$$\tilde{\mathrm{H}}(t, x, y, z, p, q, r)=\mathrm{H}(t, x, p)+q\,|x-\bar{x}(t)|^2.$$

Since it is independent of y and z, the Hamiltonian inclusion (3.31) implies that $\dot{q}=\dot{r}=0$ while

$$(-\dot{p}(t), \dot{\bar{x}}(t)) \in \partial \mathrm{H}\left(t, \bar{x}(t), p(t)+\int_{[\bar{a}, t)} \gamma\, d\mu\right) \text{ a. e. } [\bar{a}, \bar{b}].$$

The transversality condition (3.32) implies

$$\left(-h, p(\bar{a}), q, r, -k, -p(\bar{b})-\int_{[\bar{a}, \bar{b}]} \gamma\, d\mu, -q, -r\right)-\lambda(0,0,0,0,0,0,1,1)$$
$$\in \mathrm{K}\, \partial d_{\tilde{\mathrm{S}}}(\bar{a}, \bar{x}(\bar{a}), 0, l(\bar{s}), \bar{b}, \bar{x}(\bar{b}), 0, l(\bar{s})),$$

where K is the magnitude of the left-hand side. [We write $\bar{s}=(\bar{a}, \bar{x}(\bar{a}), \bar{b}, \bar{x}(\bar{b}))$ for simplicity.] According to the general formula $\partial d_{\mathrm{C}\times\mathrm{D}}(c, d) \subseteqq \partial d_{\mathrm{C}}(c) \times \partial d_{\mathrm{D}}(d)$, the right side is a subset of

$$\{\mathrm{K}(\alpha, \xi, x, 0, \beta, \eta, 0, \zeta) :$$
$$(\alpha, \xi, \beta, \eta, \zeta) \in \partial d_{(\mathrm{S}\times\mathbb{R}) \cap (\mathrm{epi}\, l)}(\bar{s}, l(\bar{s})),\ x \in [-1, 1]\}.$$

Consequently $r=0$, $q=-\lambda$, and one has

$$\left(-h, p(\bar{a}), -k, -p(\bar{b})-\int_{[\bar{a}, \bar{b}]} \gamma\, d\mu, -\lambda\right) \in \mathrm{K}\, \partial d_{(\mathrm{S}\times\mathbb{R}) \cap (\mathrm{epi}\, l)}(\bar{s}, l(\bar{s}))$$

In view of Lemma 4.1, the right side is contained in

$$\{(\zeta, -\varepsilon) : \varepsilon \geqq 0, \zeta \in \varepsilon\, \partial l(\bar{s})+\mathrm{K}\,(\mathrm{K}_l^2+1)^{1/2}\, \partial d_{\mathrm{S}}(\bar{s})\}.$$

Thus we obtain

$$\left(-h, p(\bar{a}), \quad -k, -p(\bar{b})-\int_{[\bar{a}, \bar{b}]} \gamma\, d\mu\right) \in \lambda\, \partial l(\bar{s})+\mathrm{R}\, \partial d_{\mathrm{S}}(\bar{s})$$

for any R as described in the statement of Theorem 2.1. As for the inclusions (2.3) and (2.4), these follow readily from (3.33) and (3.34) in view of the simple relationship between $\tilde{\mathrm{H}}$ and H. Thus conclusions (2.1)-(2.6) all hold.

Finally, we consider the nontriviality condition. From Theorem 3.5, we have $\lambda+\|(p, -\lambda, 0)\|_\infty+\mu[\bar{a}, \bar{b}]>0$: this certainly forces

$$\lambda+\|p\|_\infty+\mu[\bar{a}, \bar{b}]>0.$$

We may therefore divide the quantities h, k, p, λ, and μ appearing in (2.1)-(2.6) by this positive number: the desired conclusions remain valid, and the scaled quantities also satisfy $\lambda+\|p\|_\infty+\mu[\bar{a}, \bar{b}]=1$. This completes the proof of Theorem 2.1.

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