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Looking for the Bernoulli shift

by

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ABSTRACT. — We prove a result on the topological entropy of a large class of Hamiltonian systems. This result is obtained variationally by constructing "multibump" homoclinic solutions.

Key words : Hamiltonian systems, convexity, dual variational methods, concentration-compactness, homoclinic orbits, Bernoulli shift, topological entropy, chaos.

RÉSUMÉ. — On démontre un résultat sur l'entropie topologique d'une grande classe de systèmes hamiltoniens. Ce résultat est obtenu par une méthode variationnelle qui permet de construire des solutions homoclines « multi-bosses ».

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I. INTRODUCTION

1. Some history

Homoclinic orbits were first introduced by H. Poincaré (*see* [M] for a modern exposition). Considering a hyperbolic fixed point p of a diffeomorphism φ in $\mathbb{R}^{2N}$, we say that a point $r \neq p$ is homoclinic if it belongs to the intersection of the unstable and stable manifolds W^u, W^s associated to (p, φ); the orbit of r is called a homoclinic orbit. Assuming that W^u, W^s intersect transversally at r, and that φ is symplectic, Poincaré proved that there are infinitely many homoclinic orbits, geometrically distinct in the following sense:

(the orbits of r, r' are geometrically distinct) $\Leftrightarrow$ $(\forall n \in \mathbb{Z} : \varphi^n(r) \neq r')$.

Birkhoff, Smale and other authors also studied homoclinic orbits, and their relation with Bernoulli shifts. We state here a result of Smale on homoclinics (*see* [M]): if $r \neq p$ is a point of transverse intersection of W^u, W^s, then there are $l \in \mathbb{N}^*$ and a homeomorphism $\tau : \{0, 1\}^{\mathbb{Z}} \to I$, where I is an invariant set for φ^l, such that $\varphi^l \circ \tau = \tau \circ \sigma$. Here, $\sigma((a_n)) = (b_n)$ with $b_n = a_{n+1}$ and $\{0, 1\}^{\mathbb{Z}}$ is endowed with the standard metric

$$d(a, b) = \frac{1}{3} \sum_{n \in \mathbb{Z}} \frac{|b_n - a_n|}{2^{|n|}}.$$

This structure is called a Bernoulli shift.

Bernoulli shifts are an important tool in the study of chaotic behavior. For instance, Smale's result given above implies that the topological entropy of φ, $h_{\text{top}}(\varphi)$, is greater than $\frac{\text{Ln}\, 2}{l}$. This is a direct consequence of the following definition (*see* [O], p. 182-183):

$$h_{\text{top}}(\varphi) = \sup_{R > 0} \lim_{e \to 0} \left(\limsup_{n \to \infty} \frac{\text{Log}\, s(n, e, R)}{n} \right),$$

where

$$s(n, e, R) = \max \{ \text{Card}(E) : E \subset B(0, R), \\ (\forall x \neq y \in E)(\exists k \in [\![0, n]\!]) : |\varphi^k(x) - \varphi^k(y)| \geq e \}.$$

2. Variational approach

The results described in the preceding section were proved by dynamical systems methods, with a transversality assumption on W^u, W^s. The question examined in this paper is the following one:

We assume that φ is the time-one map of a Hamiltonian system $x' = J \nabla_x H(t, x)$, H being one-periodic in time. Is it possible to say something about Bernoulli shifts and topological entropy, using a variational method? We will see that this approach has several advantages:

- The existence of a homoclinic point r is not an assumption any more, but follows from global hypotheses on H that we call (hA), (hR).
- The classical transversality hypothesis can be replaced by a weaker condition, denoted $(\mathscr{H})$.

3. Main results

We work with the same Hamiltonian system as in the paper [CZ-E-S]:

$$x' = \mathrm{J}\mathrm{A}x + \mathrm{J}\nabla_x \mathrm{R}(t, x), \qquad x \in \mathbb{R}^{2\mathrm{N}}, \quad t \in \mathbb{R}, \quad \mathrm{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{1}$$

We are looking for non-zero solutions satisfying $x(\pm\infty) = 0$, *i.e.* solutions homoclinic to 0.

We make the following assumptions on A, R:

$$\left.\begin{array}{c} \mathrm{A}^* = \mathrm{A}, \text{ and } \mathrm{JA} = \mathrm{E} \text{ is a constant matrix,} \\ \text{all eigenvalues of which have a non-zero real part.} \end{array}\right\} \tag{hA}$$

- $\mathrm{R}(.+1, .) = \mathrm{R}(., .)$, and R is C^2.
- $(\forall\, t \in \mathbb{R})$, $\mathrm{R}(t, .)$ is strictly convex.
- for some $\alpha > 2$, $0 < k_1 < k_2 < +\infty$, we have

$$\forall (t, x) \in \mathbb{R} \times \mathbb{R}^{2\mathrm{N}}, \quad \mathrm{R}(t, x) \leqq \frac{1}{\alpha}(\nabla_x \mathrm{R}, x), \tag{hR}$$
$$k_1 |x|^\alpha \leqq \mathrm{R}(t, x) \leqq k_2 |x|^\alpha.$$

In [CZ-E-S], it was proved under these assuptions that there are at least two homoclinic orbits x, y, geometrically distinct, *i.e.* such that $\forall n \in \mathbb{Z} : n * x \neq y$, where $n * x(t) = x(t-n)$. One of them was obtained by a mountain-pass argument on a dual action functional. This paper has motivated some related work.

Concerning the existence of at least one homoclinic solution, the convexity assumption was relaxed in [H-W] and [T], by two different methods.

Concerning multiplicity, a novel variational argument was introduced in [S], and the following result was proved:

THEOREM I. — *Assume* (hA), (hR) *are true. Then there are infinitely many orbits homoclinic to* 0, *geometrically distinct in the sense*

$$x_1 \neq x_2 \quad \Leftrightarrow \quad (\forall n : n * x_1 \neq x_2).$$

The idea in [S] was to look for solutions near $(-n) * x + n * x$, where x is the homoclinic orbit found in [CZ-E-S] by mountain-pass, and n is large enough. We call them "solutions with two bumps distant of $2n$".

The existence of such solutions is a well-known fact of classical dynamical systems theory, in many particular situations. Let describe briefly one of them (*see* [W]):

Consider the autonomous system associated to the Hamiltonian

$$H(p, q)=p^2-q^2+p^4+q^4, \qquad (p, q)\in\mathbb{R}^2.$$

It is integrable, and does not have any solution with two (or more) bumps. But in the autonomous case, we have a continuum of solutions which are the translates of one of them in time, and Theorem I is not contradicted.

By Melnikov's theory, it is possible to find small non-autonomous perturbations $H(p, q)+\varepsilon K(t, p, q)$ of the Hamiltonian such that W^u, W^s intersect transversally. Then, using the implicit function theorem, multi-bump homoclinic solutions can be constructed.

To give more detailed comments on Theorem I, we need some notations:

f is the dual action functional introduced in [CZ-E-S]. It is defined on the space $L^\beta(\mathbb{R}, \mathbb{R}^{2N})$, with $\frac{1}{\alpha}+\frac{1}{\beta}=1$ (the exact form of f will be given in section II). $f^a=\{x/f(x)\leqq a\}$, $\mathscr{C}$ is the set of non-zero critical points, and $\mathbb{Z}$ acts by integer translations in time.

L: $L^\beta\to W^{1,\beta}$ is an isomorphism such that, if $u\in\mathscr{C}$, then Lu is a homoclinic orbit (*see* §II).

c is the mountain-pass level, let us define it precisely:

0 is a strict local minimum for f, and $f(0)=0$. Moreover, f is not bounded from below (*see* [CZ-E-S]. So we consider

$$\Gamma=\{\gamma\in C^0([0, 1], L^\beta)/\gamma(0)=0,\ f\circ\gamma(1)<0\}.$$

Γ is non-empty, and we choose $c=\inf_{\gamma\in\Gamma}(\max f\circ\gamma)>0$ as mountain-pass level.

In [S], the variational gluing of two bumps was possible under the following assumption:

(*): There is some $c'>c$ such that $(\mathscr{C}\cap f^{c'})/\mathbb{Z}$ is finite.

The following result, which is a more precise version of Theorem I, is an immediate consequence of the arguments given in [S]:

THEOREM I'. — *Assume that* (hA), (hR) *and* (*) *are true. Then there are two critical points u, v such that for any r, $h>0$ and $n\geqq N(r, h)$, exists a critical point u_n, with*

$$\|u_n-[(-n)*u+n*v]\|_{L^\beta}<r \qquad \textit{and} \qquad f(u_n)\in[2c-h, 2c+h].$$

u, v, possibly equal, satisfy $f(u)=f(v)=c$. The homoclinic orbit $y_n=Lu_n$ is called a solution with two bumps distant of $2n$. It satisfies

$$\|y_n-[(-n)*Lu+n*Lv]\|_{W^{1,\beta}}<\|L\|.r.$$

Theorem I is trivial when (*) is not satisfied ("degenerate" situation), and Theorem I′ implies Theorem I when (*) is satisfied ("non-degenerate" situation).

In the later work [CZ-R][1], Coti Zelati and Rabinowitz apply the ideas of [S] to the case of second order systems, and construct, under assumption (*), solutions with m bumps, *i.e.* located in a ball of center $p^1 * x_1 + \ldots + p^m * x_m$ and radius ε, for the norm of the functional space $E = W^{1,2}(\mathbb{R}, \mathbb{R}^N)$. The x_i are in a fixed finite set of critical points of the action functional $\int \frac{\dot{x}^2}{2} - V$ defined on E. They are found thanks to a mountain-pass. Moreover, for any i, $(p^{i+1} - p^i) \geqq K(\varepsilon, m)$. In the construction of [CZ-R][1], the minimal distance K between bumps goes to infinity as m goes to infinity, for ε fixed.

Other applications, in the domain of partial differential equations, are given in [CZ-R][2], [LI][1], [LI][2].

In the paper [C-L] of Chang and Liu, the assumption (*) is replaced by (**) : $\mathscr{C} \cap f^{c'}$ contains only isolated points.

In the present work, (**) is replaced by the weaker assumption

$(\mathscr{H}) : \mathscr{C} \cap f^{c'}$ is at most countable.

Moreover, multibump solutions are constructed for a minimal distance K between bumps independent of m. This last point, whose proof requires many modifications in the arguments of [S], [CZ-R][1], allows to study the topological entropy of the Hamiltonian system. The main theorem that we will prove can be stated as follows:

THEOREM II. – *Assume* (hA), (hR) *and* $(\mathscr{H})$ *are true. Then there exists a homoclinic orbit* x *such that, for any* $\varepsilon > 0$, *and any finite sequence of integers* $\bar{p} = (p^1, \ldots, p^m)$, satisfying

$$(\forall i): \quad (p^{i+1} - p^i) \geqq K(\varepsilon),$$

there is a homoclinic orbit $y_{\bar{p}}$, *with*

$$(\forall t \in \mathbb{R}): \quad \left| y_{\bar{p}}(t) - \sum_{i=1}^{m} x(t - p^i) \right| \leqq \varepsilon.$$

Here, K *is a constant independent of* m.

Remark 1. – The assumption $(\mathscr{H})$ cannot be satisfied in the autonomous situation, where the translates of x in time form a continuum. Now, if W^u, W^s intersect transversally, then their intersection is at most countable, and so is the set of homoclinic solutions; but the converse is false.

Remark 2. – The estimate on $y_{\bar{p}} - \sum_{i=1}^{m} x(t - p^i)$ is given in L^∞ norm. In [S] and [CZ-R][1], it was given in global $W^{1,q}(\mathbb{R})$ norm. Without this change, it seems impossible, or at least very difficult, to choose K independently of m.

Since K does not depend on m, we can study the limit $m \to \infty$, and get solutions with infinitely many bumps (those are not homoclinic orbits any more). We have

COROLLARY II.1. – *With the hypotheses and notations of Theorem II, for any interval* $I \subset \mathbb{Z}$, *finite or infinite, and any sequence of integers* $\bar{p} = (p^i)_{i \in I}$ *such that* $(\forall i): (p^{i+1} - p^i) \geqq K(\varepsilon)$, *there is a solution* $y_{\bar{p}}$ *of* (1) *satisfying*

$$(\forall t \in \mathbb{R}): \quad \left| y_{\bar{p}}(t) - \sum_{i \in I} x(t - p^i) \right| \leqq \varepsilon.$$

If I *is infinite, we say that y has infinitely many bumps.*

As a consequence, we have an "approximate" Bernoulli shift structure:

COROLLARY II.2. – *Under the hypotheses of Theorem II, there is* $x_0 \in \mathbb{R}^{2N} \setminus \{0\}$ *such that, for any* $\varepsilon > 0$, *exist* $K = K(\varepsilon) > 0$ *and*

$$\tilde{\tau} = \tilde{\tau}(\varepsilon) : (\{0, 1\}^{\mathbb{Z}}, d) \to (\mathbb{R}^{2N}, |\,.\,|),$$

with:

- $\tilde{\tau}$ *is injective, and* $\tilde{\tau}^{-1}$ *is uniformly continuous.*
- $(\forall n \in \mathbb{Z}) \| \tilde{\tau} \circ \sigma^n - \varphi^{Kn} \circ \tilde{\tau} \|_\infty < 2\varepsilon.$
- $\begin{cases} s_0 = 1 \Rightarrow |\tilde{\tau}(s) - x_0| < \varepsilon \\ s_0 = 0 \Rightarrow |\tilde{\tau}(s)| < \varepsilon. \end{cases}$

Here, φ *is the time-one flow of* (1), *and* $\sigma(s)_n = s_{n+1}$. *Note that we cannot say that* $\tilde{\tau}$ *is continuous. We call* $(\tilde{\tau}(\{0, 1\}^{\mathbb{Z}}), \varphi^K)$ *an approximate Bernoulli shift structure.*

Corollary II.2 will be proved in section VI.

Now, we are in a position to state the result on topological entropy. Choose $\varepsilon \leqq \frac{|x_0|}{3}$. If two sequences s, s' are such that $s_k \neq s'_k$ for some k, then

$$|\Phi^{K(\varepsilon)k} \circ \tau(s) - \Phi^{K(\varepsilon)k} \circ \tau(s')| \geqq \frac{|x_0|}{3}.$$

So, for $e < \frac{|x_0|}{3}$ and $R > |x_0| + \varepsilon$, we get $s(Kn.\, e, R) \geqq 2^n$, and $h_{\text{top}}(\varphi) \geqq \frac{\text{Ln}\, 2}{K(\varepsilon)}$. So Corollary II.2 implies

COROLLARY II.3. – *With the hypotheses of Theorem I, the flow of* (1) *has a positive topological entropy.*

Note: Independently of the present paper, Bessi in [B] constructs variationally an approximate Bernoulli shift for the one-dimensional pendulum, by a method inspired of [S]. He replaces assumption (*) by a weakening of the classical Melnikov condition, and his result is given for small perturbations of an autonomous system.

II. VARIATIONAL FRAMEWORK AND SKETCH OF PROOF OF THEOREM II

We use a variational formulation based on Clarke's dual action principle (*see* [CZ-E-S], [E]). Define $G(t, y)=\max\{(z.y)-R(t, z)/z\in\mathbb{R}^{2N}\}$. G is 1-periodic in time, strictly convex in y, and satisfies, for $\frac{1}{\alpha}+\frac{1}{\beta}=1$:

$$0\leqq\frac{1}{\beta}(\nabla_y G, y)\leqq G(t, y)\leqq(\nabla_y G, y),$$
$$(\exists c_1, c_2>0)(\forall(y, t))\quad c_1|y|^\beta\leqq G(t, y)\leqq c_2|y|^\beta,$$
$$|\nabla_y G(t.y)|\leqq c_2|y|^{\beta-1}.$$

We define

$$D:\quad W^{1,\beta}(\mathbb{R}, \mathbb{R}^{2N})\to L^\beta(\mathbb{R}, \mathbb{R}^{2N})$$
$$z\mapsto\left(-J\frac{d}{dt}-A\right)z,$$
$$L=D^{-1}.$$

We call $\mathscr{C}$ the set of non-zero critical points of the following functional f:

$$f(u)=\int G(t, u)\,dt-\frac{1}{2}\int(u, Lu)\,dt,\qquad u\in L^\beta(\mathbb{R}, \mathbb{R}^{2N}).$$

We have (*see* [CZ-E-S])

LEMMA 1. — *If* $u\in\mathscr{C}$, *then* $x=Lu$ *is a non-zero solution of* (1) *such that* $x(\pm\infty)=0$, i.e. *an orbit homoclinic to* 0.

Our task will be to find a large class of elements of $\mathscr{C}$.

For this purpose, we need some compactness properties of f. Unfortunately, f does not satisfy the Palais-Smale (PS) condition, because it is invariant for the action of the non-compact group $\mathbb{Z}: n*u=u(.-n)$. To deal with this problem, we use the concentration-compactness theory of P. L. Lions (*see* [LS].

We have (*see* [CZ-E-S])

LEMMA 2. — *Suppose* (hA), (hR) *are true. Then* f *satisfies the following compactness property:*

Let $(u_n)_{n\geqq 0}$ *be a sequence such that*

$$f(u_n)\to a>0,\qquad f'(u_n)\to 0.$$

Then there exist $m>0$, *a subsequence* $(n_p)_{p\geqq 0}$, *and* $u^1, \ldots, u^m$ *in* $\mathscr{C}$, *not necessarily distinct, such that*

$$\left\| u_{n_p} - \sum_{i=1}^{m} k_p^i * u^i \right\| \underset{p \to \infty}{\to} 0,$$

where $k_p^i \in \mathbb{Z}$, $(k_p^j - k_p^i) \to +\infty$ *as* $p \to +\infty$ *if* $i<j$.

To simplify notations, we will write

$$\bar{k}_p = (k_p^1 \ldots k_p^m) \in \mathbb{Z}^m, \qquad \bar{u} = (u^1 \ldots u^m) \in \mathscr{C}^m,$$

$$\bar{k}_p * \bar{u} = \sum_{i=1}^{m} k_p^i * u^i. \qquad \text{Moreover,} \qquad (\lim_{k \to \infty} (k_p^j - k_p^i) = +\infty \text{ if } i<j)$$

will be summarized by

$$(\bar{k}_p \to \Omega \text{ as } p \to +\infty).$$

Now, what is special here is that the splittings $\bar{k} * \bar{u}$ do not vary continuously when $\bar{k}$ varies. This leads to introduce a new compactness condition (*see* [CZ-E-S], [S]).

CONDITION $\overline{\text{PS}}$ (a). – *Let* (u_n) *be a sequence such that* $f(u_n) \leqq a \in \mathbb{R}$, $f'(u_n) \to 0$, $(u_{n+1} - u_n) \to 0$. *Then* (u_n) *is convergent.*

We have:

LEMMA 3. – *Assume* (hA), (hR) *and* $(\mathscr{H})$ *are true. Then* $\overline{\text{PS}}(c')$ *holds.*

Lemma 3 will be proved in section III, and will be used in the proof of Lemma 7, section IV.

The interest of $\overline{\text{PS}}$ is that, if f is bounded on a pseudo-gradient line, then one can find a $\overline{\text{PS}}$ sequence on this line. So $\overline{\text{PS}}$ can give the same kind of deformation lemmas as the Palais-Smale condition. If $\overline{\text{PS}}$ is satisfied under level c', by deforming a particular curve in Γ, one finds at least one critical point u between levels c and c'. When (*) holds, one can impose $f(u)=c$. When only $(\mathscr{H})$ holds, the best that can be done is to take u with $(f(u)-c)$ arbitrarily small.

In [S], under assumption (*), a "product min-max" is constructed at level $2c$, for the "split" functional $\tilde{f}(x) = f(x\chi_{\mathbb{R}_-}) + f(x\chi_{\mathbb{R}_+})$, where χ_I is the caracteristic function of I. Theorems I and I′ are then proved by contradiction, thanks to a deformation argument. This argument works because the differentials f' and $\tilde{f}'$ "look the same" near $(-n)*u+n*v$, where u, v are critical points associated to the mountain-pass, possibly equal.

The proof of Theorem II is based on the same ideas, but contains several technical improvements.

We first construct, for any r, $h>0$, a non-trivial homology class in $H_1(f^{\bar{c}+h}, f^{\bar{c}})$, containing a chain included in $B(u, r)$, thanks to assumption

($\mathscr{H}$). Here, $\bar{c}=f(u)\in[c, c')$, and $u\in\mathscr{C}$, found thanks to the mountain-pass, is independent of r, h (*see* § IV).

Then, roughly speaking, we consider a product of m "copies" of this homology class, and find a "product min-max" in a neighborhood of $\sum_{i=1}^{m} p^i * u$. This is done in section IV thanks to Künneth's formula,

$$H_*(X\times Y, (Z\times Y)\cup(X\times T))=H_*(X, Z)\otimes H_*(Y, T).$$

Note that in [S], [CZ-R][1], a more elementary procedure (without homology) is used to construct the product min-max. It would be possible to use this procedure in the proof of Theorem II. But the method involving homology seems easier to generalize to situations where the min-max is not of mountain-pass type.

Finally, we find a critical point $u_{\bar{p}}$ in a neighborhood of $\sum_{i=1}^{m} p^i * u$, provided $(p^{i+1}-p^i)\geqq K$, K depending only on r, not on m. To do this, we assume that $u_{\bar{p}}$ does not exist, construct a more precise version of the deformation used in [S], and apply it to the "product min-max" to obtain a contradiction (*see* § V).

In the proof of Theorem II, a crucial point is to make a suitable choice of the neighborhood of $\sum_{i=1}^{m} p^i * u$ in which we want to find $u_{\bar{p}}$: this choice allows to control K as m increases. The correct neighborhood will be defined in the statement of Theorem III (*see* the end of section V), after the introduction of some technical notations. Theorem II will be a direct consequence of Theorem III.

III. COMPACTNESS PROPERTIES OF f

We first prove the following result:

LEMMA 4. – *Suppose* (hA), (hR) *and* ($\mathscr{H}$) *are true. Then there is an at most countable compact set* D *such that:*

If $(u_n)_{n\geqq 0}$ *satisfies* $f(u_n)\leqq c'$, $f'(u_n)\to 0$, *then*

$$(\forall r>0)\quad (\exists N>0),\qquad [p>q>N\Rightarrow \|u_p-u_q\|\in B(D, r)].$$

Here, $B(D, r)=\{x\in[0, +\infty)/d(x, D)<r\}$.

Proof. — Consider the set

$$D=\left\{x\in[0,+\infty)/x=\sum_{i=1}^{m}\|u_i-v_i\|,\ m\geqq 1,\ u_i,\ v_i\in\mathscr{C}\cup\{0\},\ \sum_{i=1}^{m}f(u_i)\leqq c',\ \sum_{i=1}^{m}f(v_i)\leqq c'\right\}.$$

From ($\mathscr{H}$), D is at most countable.

Let us prove that D is compact. We know (*see* [CZ-E-S]) that there is $\Lambda>0$ such that

$$(\forall u\in\mathscr{C})\quad f(u)\geqq\Lambda.$$

Consider a sequence (d^n) in D, with

$$d^n=\sum_{i=1}^{M_n}\|u_i^n-v_i^n\|,\quad u_i^n, v_i^n\in\mathscr{C}\cup\{0\},\qquad \sum_{i=1}^{M_n}f(u_i^n)\leqq c',$$
$$\sum_{i=1}^{M_n}f(v_i^n)\leqq c',\quad (u_i^n=0\Rightarrow v_i^n\neq 0).$$

We have $M_n\leqq 2c'/\Lambda$.

So, after extraction, we may assume that $M_n=M$ is constant and, by Lemma 2, that, $\forall i\in[\![1, M]\!]$:

$$\|u_i^n-\bar{k}_i^n * \bar{U}_i\|\to 0,\qquad \bar{U}_i\in\mathscr{C}^{m(i)},\quad \bar{k}_i^n\underset{n\to\infty}{\to}\Omega,$$
$$\|v_i^n-\bar{l}_i^n * \bar{V}_i\|\to 0,\qquad \bar{V}_i\in\mathscr{C}^{m'(i)},\quad \bar{l}_i^n\underset{n\to\infty}{\to}\Omega.$$

One easily sees that

$$d_n\to\sum_{k=1}^{m''}\|\mathscr{U}_k-\mathscr{V}_k\|=d_\infty$$

where $\mathscr{U}_k$, resp. $\mathscr{V}_k$, if non-zero, are of the form $n * \bar{U}_i^j$, resp. $n * \bar{V}_i^j$, and $d_\infty\in D$.

We have thus proved that D is compact. The last step is to study (u_n) such that

$$f(u_n)\leqq c',\qquad f'(u_n)\to 0.$$

Assume there are two subsequences $(u_{p_m})_{m\geqq 0}$, $(u_{q_m})_{m\leqq 0}$ satisfying $\|u_{p_m}-u_{q_m}\|\notin B(D,\rho)$ for some $\rho>0$. After extraction, we may impose

$$\|u_{p_m}-\bar{\kappa}_m * \bar{\mu}\|\to 0,\qquad \bar{\mu}=(\mu^1,\ldots,\mu^r)\in\mathscr{C}^r,$$
$$\kappa_m\to\Omega,\qquad \sum f(\mu^i)\leqq c'$$
$$\|u_{q_m}-\bar{\lambda}_m * \bar{\nu}\|\to 0,\qquad \bar{\nu}=(\nu^1,\ldots,\nu^s)\in\mathscr{C}^s,$$
$$\bar{\lambda}_m\to\Omega,\qquad \sum f(\nu^i)\leqq c'.$$

After a new extraction, each sequence $(\kappa_m^i - \lambda_m^j)$ has a limit $l_{i,j}$ in $\mathbb{Z} \cup \{-\infty, +\infty\}$. Moreover, for each i, $\text{Card}(\{j / |l_{i,j}| < +\infty\}) \leqq 1$.

Hence

$$\| u_{p_m} - u_{q_m} \| \to \sum_{k=1}^{t} \| l_k * w_k - w'_k \|,$$

where $(w_k)_{1 \leqq k \leqq t}$ is a reindexing of

$$(\mu^1, \ldots, \mu^r, \underbrace{0, \ldots, 0}_{(t-r) \text{ terms}}),$$

$(w'_k)_{1 \leqq k \leqq t}$ is a reindexing of

$$(\nu^1, \ldots, \nu^s, \underbrace{0, \ldots, 0}_{(t-s) \text{ terms}}),$$

and $l_k \in \mathbb{Z}$.

Clearly, $\sum f(w_k) = \sum f(\mu^i) \leqq c'$, $\sum f(w'_k) = \sum f(\nu^j) \leqq c'$. So $\sum_{k=1}^{t} \| w_k - w'_k \| \in \mathrm{D}$, which contradicts the assumption $\| u_{p_m} - u_{q_m} \| \notin \mathrm{B}(\mathrm{D}, \rho)$. The last assertion of Lemma 4 is thus proved by contradiction. □

We now give another lemma, that will be used in section V.

LEMMA 5. — *Suppose that f satisfies* (hA), (hR) *and* ($\mathscr{H}$). *Then the set*

$$\mathrm{F} = \left\{ x = \sum_{k=1}^{m} f(u_k)/m \geqq 1, \ (u_1, \ldots, u_m) \in \mathscr{C}^m, \ (\forall k), f(u_k) \leqq c' \right\}$$

is closed and a most countable.

The proof of Lemma 5 is analogous to that of Lemma 4, so we won't give it. Now, we prove Lemma 3 as a consequence of Lemma 4.

Proof. — Consider a sequence (u_n) such that

$$f(u_n) \leqq c', \qquad f'(u_n) \to 0, \quad (u_{n+1} - u_n) \to 0.$$

we want to prove by contradiction that (u_n) is a Cauchy sequence.

Assume the contrary, *i. e.* $\| u_{q_n} - u_{p_n} \| \to \delta > 0$, $p_n < q_n < p_{n+1}$.

The open set $]0, \delta[\setminus \mathrm{D}$ contains an interval $[d_1 - d_2, d_1 + d_2]$. And there is P such that

$$\left(p > \mathrm{P} \Rightarrow \| u_{p+1} - u_p \| \leqq \frac{d_2}{2} \right).$$

So, if $p_n > \mathrm{P}$,

$$\| u_{r_n} - u_{p_n} \| \in \left[d_1 - \frac{d_2}{2}, d_1 + \frac{d_2}{2} \right] \text{ for some } r_n \in [\![p_n, q_n]\!].$$

But this implies $\| u_{r_n} - u_{p_n} \| \notin \mathrm{B}(\mathrm{D}, d_2/2)$, which is impossible by Lemma 4.

So (u_n) is Cauchy, hence convergent. Lemma 3 is thus proved. □

We now study the local compactness of $\mathscr{C}$. We prove

LEMMA 6. — *Assume* (hA) *and* (hR) *are true. There is* $r_0>0$ *such that, if a sequence* (u_n) *satisfies*

$$\begin{cases} f'(u_n)\to 0 \\ (\exists\, \mathrm{R}>0),\ (\forall p,q), \quad \|(u_p-u_q)\chi_{\mathbb{R}\setminus[-\mathrm{R},\mathrm{R}]}\|\leqq 2r_0 \end{cases}$$

then (u_n) *is precompact.*

Proof. — We remark (*see* [CZ-E-S]) that there is $r_0>0$ such that

$$\frac{3r_0}{2}<\|u\| \qquad (\forall u\in\mathscr{C})$$

We now apply Lemma 2 to the sequence (u_n). If $m\geqq 2$ or if $(m=1)$ and $\lim_{p\to\infty}(|k_p^1|=+\infty)$, then for any $\mathrm{P}>0$, there are $p>q>\mathrm{P}$ such that

$$\|(\bar{k}_p*\bar{u}-\bar{k}_q*\bar{u})\chi_{\mathbb{R}\setminus[-\mathrm{R},\mathrm{R}]}\|\geqq 3r_0.$$

This contradicts $\|(u_p-u_q)\chi_{\mathbb{R}\setminus[-\mathrm{R},\mathrm{R}]}\|\leqq 2r_0$, for P large enough.

So $m=1$, and we may extract a subsequence $u_{n_{\varphi(p)}}$ such that $k^1_{\varphi(p)}=k$ is constant, and $u_{n_{\varphi(p)}}\underset{p\to\infty}{\to}k*u^1\in\mathscr{C}$. Lemma 6 is thus proved. □

Lemma 6 will be used in the proof of Lemma 12, section V.

IV. THE PRODUCT MIN-MAX

We want to find a min-max at each level kc, $k\geqq 2$. This will be done thanks to singular homology over $\mathbb{Z}$. We first need to "localize" the min-max

$$\inf_{\gamma\in\Gamma}(\max f\circ\gamma)=c.$$

This will be done thanks to $(\mathscr{H})$.

We recall some notations:

$$f^l=\{x/f(x)\leqq l\},\qquad f^{<l}=\{x/f(x)<l\},$$
$$f_l=(-f)^{-l},\qquad f_a^b=f_a\cap f^b,$$
$$\mathrm{B}(x,\rho)=\{y/\|y-x\|<\rho\},\qquad \mathrm{S}(x,\rho)=\{y/\|y-x\|=\rho\}.$$

We have

LEMMA 7. — *Assume* (hA), (hR) *and* $(\mathscr{H})$ *are true. Choose* $r\in\mathbb{R}_+^*\setminus\mathrm{D}$, *with the notation of Lemma* 4.

Then for any $h>0$, *exist* $p=p(h, r)\in\mathbb{N}^*$, $(u^1, \ldots, u^p)\in(\mathscr{C}\cap \bar{f}_c^{c+h})^p$, *and* $\gamma\in\Gamma$, *with:*

(i) $$\operatorname{Im}(\gamma)\cap f_c\subset \bigcup_{i=1}^{p} \mathrm{B}(u^i, r)$$

(ii) $$\operatorname{Im}(\gamma)\cap f_{c+h}=\varnothing$$

(iii) $$\operatorname{Im}(\gamma)\cap f_c\cap\left[\bigcup_{i=1}^{p} \mathrm{S}(u^i, r)\right]=\varnothing$$

Proof. – Given $r>0$, we just have to prove the result for h small enough. We take $\gamma^h\in\Gamma$ such that $f\circ\gamma^h<c+h$.

We are going to take γ as a deformation of γ^h. We choose $e>0$ such that $[r-2e, r+2e]\cap\mathrm{D}=\varnothing$. For $d\geqq 0$, we define

$$\begin{aligned}
\mathrm{U}^d&=\{x\in f_c^{c+h}/(\forall y\in\mathscr{C}\cap f_c^{c+h})\,\|x-y\|>r+d\}\\
\mathrm{V}^d&=\{x\in f_c^{c+h}/(\exists y\in\mathscr{C}\cap f_c^{c+h})\,\|x-y\|\in[r-d, r+d]\}\\
\mathrm{K}^d&=(\{x\in f_c^{c+h}/(\exists y\in\mathscr{C}\cap f_c^{c+h})\,\|x-y\|<r-d\}\\
&\qquad\cup\{x\in f^{<c}/(\exists y\in\mathscr{C}\cap f^c)\,\|x-y\|<r-d\})\setminus\mathrm{V}^d
\end{aligned}$$

We assume $c+h<c'$. From Lemma 4, there is $\mu>0$, independent of h, and such that $\inf\{\|f'(x)\|/x\in\mathrm{V}^{2e}\}\geqq\mu$. We assume, moreover, that $h<\mu e/2$. We build a locally Lipschitz vector field V on f^{c+h}, such that:

(j) $$x\in\mathrm{K}^{2e}\cup f^{c-h}\ \Rightarrow\ \mathrm{V}(x)=0$$

(jj) $$(\forall x)\quad f'(x).\mathrm{V}(x)\leqq 0,\qquad |\mathrm{V}(x)|\leqq 2|f'(x)|^{-1}$$

(jjj) $$x\in\mathrm{U}^e\cup\mathrm{V}^e\ \Rightarrow\ f'(x).\mathrm{V}(x)\leqq -1$$

Consider the flow φ_t defined by

$$(\forall(t, x)\in\mathbb{R}_+\times f^{c+h})\quad\begin{cases}\varphi_0(x)=x\\ \dfrac{\partial}{\partial t}\varphi_t(x)=\mathrm{V}\circ\varphi_t(x).\end{cases}$$

Assume that for some $x\in f^{c+h}$, the maximal interval of definition of $t\mapsto\varphi_t(x)$ is $[0, \mathrm{L}[$, $\mathrm{L}<+\infty$. Then $\int_0^{\mathrm{L}}\|\mathrm{V}\circ\varphi_t(x)\|\,dt=+\infty$. So we can define a sequence (t_n) by

$$t_0=0$$

$$\int_{t_n}^{t_{n+1}}\|\mathrm{V}\circ\varphi_t(x)\|\,dt=\sqrt{\mathrm{L}-t_n}$$

So we get

(α) $\forall (u, v) \in [t_n, t_{n+1}]^2 \colon \| \varphi_u(x) - \varphi_v(x) \| \leqq \sqrt{L - t_n}$

(β) $\exists s_n \in [t_n, t_{n+1}] \colon \begin{cases} \| f' \circ \varphi_{s_n}(x) \| \leqq 2 \| V \circ \varphi_{s_n}(x) \|^{-1} \leqq 2 \sqrt{L - t_n} \\ \varphi_{s_n}(x) \in f^{c+h} \setminus K^{2e} \end{cases}$

(γ) $\int_0^l \| V \circ \varphi_t(x) \| dt = \sum_{n=0}^{+\infty} \sqrt{L - t_n}$, where $l = \lim_{n \infty} t_n$.

If $l < L$, the left term of (γ) is finite, and the right one infinite. So we have $l = L$, and

$$(\varphi_{s_{n+1}}(x) - \varphi_{s_n}(x)) \to 0, \qquad f' \circ \varphi_{s_n}(x) \to 0.$$

Since f satisfies property $\overline{PS}(c')$, we get

$$u_\infty = \lim_{n \to \infty} \varphi_{s_n}(x) \in (f^{c+h} \setminus K^{2e}) \cap \mathscr{C}.$$

But this intersection is empty. So we have proved that φ_t is defined on $\mathbb{R}_+ \times f^{c+h}$.

Now, suppose that $f(x) < c + h$, and that $\varphi_h(x) \in U^0 \cup V^0$. Then three situations may occur:

- $$(\forall t \in [0, h]), \quad \varphi_t \in U^e \cup V^e$$

apply (jjj), and conclude $f \circ \varphi_h(x) < c$: contradiction.

- $$(\exists y \in \mathscr{C} \cap f_c^{c+h}) \quad (\exists [\alpha, \beta] \subset [0, h]),$$ $$\| \varphi_\alpha(x) - y \| = r - e, \qquad \| \varphi_\beta(x) - y \| = r,$$ $$(\forall t \in [\alpha, \beta]), \quad \| \varphi_t(x) - y \| \in [r - e, r].$$
- $$(\exists y \in \mathscr{C} \cap f_c^{c+h}) \quad (\exists [\alpha, \beta] \subset [0, h]),$$ $$\| \varphi_\alpha(x) - y \| = r + e, \qquad \| \varphi_\beta(x) - y \| = r,$$ $$(\forall t \in [\alpha, \beta]), \quad \| \varphi_t(x) - y \| \in [r, r + e].$$

In the second and third situations, we have $\| \varphi_\beta(x) - \varphi_\alpha(x) \| \geqq e$, and from (jj), (jjj), $f'_y . V_y \leqq -\frac{1}{2} \| f'_y \| . \| V_y \| \leqq -\frac{\mu}{2} \| V_y \|$ if $y \in \varphi_{[\alpha, \beta]}(x) \cap f_{c-h}$. Since $h < \mu e/2$, we also conclude $f \circ \varphi_h(x) < c$: contradiction.

So we have proved that if $f(x) < c + h$, then either $f \circ \varphi_h(x) < c$, or $\varphi_h(x) \in K^0$.

Finally, $\gamma = \varphi_h \circ \gamma^h$ is such that

$$\operatorname{Im} \gamma \cap \Big[\bigcup_{y \in \mathscr{C} \cap f_c^{c+h}} S(y, r) \Big] \cap f_c = \varnothing,$$

$$(\operatorname{Im} \gamma \cap f_c) \subset \bigcup_{y \in \mathscr{C} \cap f_c^{c+h}} B(y, r).$$

Since $\operatorname{Im}\gamma \cap f_c$ is compact, we can extract a finite subcovering:

$$(\operatorname{Im}\gamma \cap f_c) \subset \bigcup_{i=1}^{p} \mathrm{B}(u^i, r). \qquad u^i \in \mathscr{C} \cap f_c^{c+h}.$$

Lemma 7 is thus proved. □

Lemma 7 has a direct consequence:

COROLLARY 7.1. – *Assume* ($\mathscr{H}$) *is true. Choose* $r>0$, $h>0$. *Then there is* $u=u(r, h)\in \mathscr{C} \cap f_c^{c+h}$ *such that* $i_* \neq 0$, *where*

$$i_*: \quad \mathrm{H}_1(f^{<(c+h)} \cap \mathrm{B}(u, r), f^{<c} \cap \mathrm{B}(u, r)) \to \mathrm{H}_1(f^{<(c+h)}, f^{<c})$$

is the morphism induced by the canonical injection

$$i: \quad \mathrm{B}(u, r) \to \mathrm{L}^{\beta}.$$

Proof. – We juste have to prove the result when $r \in \mathbb{R}_+^* \setminus \mathrm{D}$: it will then be true for any $r' \geqq r$.

Let p_0 be the minimal value of p such that there are $(u^1, \ldots, u^p) \in \mathscr{C} \cap (f_c^{c+h})^p$ and $\gamma \in \Gamma$ satisfying the conclusion of Lemma 7. $\operatorname{Im}\gamma \cap \mathrm{B}(u^{p_0}, r)$ is the image of a 1-dimensional complex $\omega \in \mathrm{C}_1(f^{<(c+h)})$, with $\omega \in \bar{\omega}$, for some $\bar{\omega} \in \mathrm{H}_1(f^{<(c+h)} \cap \mathrm{B}(u^{p_0}, r), f^{<c} \cap \mathrm{B}(u^{p_0}, r))$.

If $i_* \bar{\omega} = 0$, then there is a singular 2-dimensional complex $\Omega \in \mathrm{C}_2(f^{<(c+h)})$ such that $\partial\Omega = \omega - \alpha$, with $\alpha \in \mathrm{C}_1(f^{<c})$. So, replacing the curves of ω by curves of α in γ, we get $\bar{\gamma}$ satisfying the conclusion of Lemma 7 with $u^1, \ldots, u^{p_0-1}$. This contradicts the minimality of p_0. So $i_* \bar{\omega} \neq 0$. Corollary 7.1 is thus proved, with $u=u^{p_0}$. □

Corollary 7.1 gives the existence of at least one critical point $u \neq 0$. The hypothesis ($\mathscr{H}$) seems too weak to get u independent of r, h, and we cannot say that $f(u)=c$. The fundamental reason for this is that the Palais-Smale condition is not satisfied. To overcome this difficulty, we shall make use of Lemma 6 which gives a local Palais-Smale condition.

We first choose $\rho^0 \in]0, r_0[$, $d^0>0$, such that $[\rho^0 - d^0, \rho^0 + d^0] \cap \mathrm{D} = \varnothing$, r_0 being defined in Lemma 6.

We define

$$\mu^0 = \frac{1}{2} \inf \{ \| f'(x) \| / x \in f^{c'}, (\exists y \in \mathscr{C} \cap f^{c'}) : \| x-y \| \in [\rho^0, \rho^0 + d^0] \}.$$

We take $0<h<\min(\mu^0 d^0, c'-c)$. By Corollary 7.1, there are

$$u^0 \in \mathscr{C} \cap f_c^{c'}, \qquad \bar{\omega} \in \mathrm{H}_1(\mathrm{B}(u^0, p^0) \cap f^{<c+h}, \mathrm{B}(u^0, \rho^0) \cap f^{<c}),$$

such that $i_* \bar{\omega} \neq 0$, where

$$i_*: \quad \mathrm{H}_1(f^{<c+h} \cap \mathrm{B}(u^0, \rho^0), f^{<c} \cap \mathrm{B}(u^0, \rho^0)) \to \mathrm{H}_1(f^{<c+h}, f^{<c})$$

is the morphism induced by the canonical injection

$$i: \quad \mathrm{B}(u^0, \rho^0) \to \mathrm{L}^{\beta}.$$

We define

$$X=(f^{c+h}\cap B(u^0,\rho^0))$$
$$\cup\left\{x\in L^\beta/\|x-u^0\|\in[\rho^0,\rho^0+d^0[,\ f(x)<c+h\left(1-\frac{\|x-u^0\|-\rho^0}{d^0}\right)\right\},$$
$$Y=f^c\cap B(u^0,\rho^0+d^0).$$

We call

$$j_*:\quad H_1(f^{<c+h}\cap B(u^0,\rho^0),f^{<c}\cap B(u^0,\rho^0))\to H_1(X,Y)$$

the morphism induced by the canonical injections

$$j_+:\quad f^{<c+h}\cap B(u^0,\rho^0)\to X,$$
$$j_-:\quad f^{<c}\cap B(u^0,\rho^0)\to Y.$$

Clearly, we have $j_*\bar{\omega}\neq 0$.

We define $\bar{c}=\inf_{z\in j_*\bar{\omega}}(\max f(z))\in[c,c+h[$.

By arguments similar to those proving Lemma 7 and Corollary 7.1, we find, for any $n\in\mathbb{N}^*$, a critical point $u^n\in\mathscr{C}\cap f_{\bar{c}}^{\bar{c}+(1/n)}\cap B(u^0,\rho^0-d^0)$, such that $i_*^n\neq 0$, where

$$i_*^n:\quad H_1\left(f^{<\bar{c}+(1/n)}\cap B\left(u^n,\frac{d^0}{n}\right),f^{<\bar{c}}\cap B\left(u^n,\frac{d^0}{n}\right)\right)$$
$$\to H_1(f^{<(\bar{c}+(1/n))}\cap B(u^n,d^0),f^{<\bar{c}}\cap B(u^n,d^0))$$

is the morphism induced by the canonical injection

$$i_*^n:\quad B\left(u^n,\frac{d^0}{n}\right)\to B(u^n,d^0).$$

By Lemma 6, the sequence (u^n) is precompact (recall that $\rho^0<r_0$). Considering one of its limit points, and taking $r_1=d^0/2$, we get

LEMMA 8. – *Assume that* (hA), (hR) *and* ($\mathscr{H}$) *are true.*

Then there are $u\in\mathscr{C}$ with $f(u)=\bar{c}\in[c,c')$ and $r_1>0$, such that, for any $r\in]0,r_1]$ and $h>0$, we have $i_*\neq 0$ where

$$i_*:\quad H_1(f^{<(\bar{c}+h)}\cap B(u,r),f^{<\bar{c}}\cap B(u,r))$$
$$\to H_1(f^{<(\bar{c}+h)}\cap B(u,r_1),f^{<\bar{c}}\cap B(u,r_1))$$

is the morphism induced by the canonical injection

$$i:\quad B(u,r)\to B(u,r_1).$$

The great difference with Corollary 7.1 is that u does not depend on r, h any more.

Lemma 8 gives a min-max localized around u. To get our multiplicity result, we are going to make products of several "copies" of this min-max. At each product will be associated a new critical point. We first enounce:

COROLLARY 8.1. — *Assume that* (hA), (hR) *and* ($\mathcal{H}$) *are true. Choose* $r \in]0, r_1[$, $h > 0$.

Then there is $N = N(r, h)$ *such that*

$$(\forall (a, b) \in [N, +\infty]^2): \quad I_* \neq 0,$$

where

$$I_*: \quad H_1(f^{<(\bar{c}+h)} \cap B(u, r) \cap L^\beta_{(-a, b)}, f^{<\bar{c}} \cap B(u, r) \cap L^\beta_{(-a, b)}) \\ \to H_1(f^{<(\bar{c}+h)} \cap B(u, r_1) \cap L^\beta_{(-a, b)}, f^{<\bar{c}} \cap B(u, r_1) \cap L^\beta_{(-a, b)})$$

is the morphism induced by

$$I: \quad B(u, r) \cap L^\beta_{(-a, b)} \to B(u, r_1) \cap L^\beta_{(-a, b)}.$$

and

$$L^\beta_{(-a, b)} = \{ x \in L^\beta / \operatorname{supp}(x) \subset [-a, b] \}.$$

Proof. — We choose $\bar{\omega} \in H_1(f^{<(\bar{c}+h)} \cap B(u, r), f^{<\bar{c}} \cap B(u, r))$ such that

$$i_* \bar{\omega} \neq 0,$$

with the notations of Lemma 8.

The class $\bar{\omega}$ has an element of the form $\sum_{i=1}^{r} \lambda_i \sigma_i$, satisfying

(P) $[\lambda_i \in \mathbb{R}$, and $\sigma_i : S^1 \to L^\beta$ continuous or $\sigma_i : [0, 1] \to L^\beta$ continuous, with $\sigma_i(0), \sigma_i(1) \in f^{<\bar{c}}$, and $\operatorname{Im}(\sigma_i) \subset f^{<(\bar{c}+h)} \cap B(u, r)$ in both cases].

For $t_1, t_2 \in \bar{\mathbb{R}}$, we define

$$K_{t_1, t_2}: \quad L^\beta(\mathbb{R}, \mathbb{R}^{2N}) \to L^\beta(\mathbb{R}, \mathbb{R}^{2N}) \\ x(t) \mapsto \chi_{[t_1, t_2]}(t)\, x(t)$$

We note that $\bigcup_{i=1}^{r} \operatorname{Im} \sigma_i$ is compact, so that

$$\lim_{(t_1, t_2) \to (-\infty, +\infty)} \left(\sup \left\{ \| x - K_{t_1, t_2}(x) \|;\ x \in \bigcup_{i=1}^{r} \operatorname{Im} \sigma_i \right\} \right) = 0.$$

Moreover, $f^{<(\bar{c}+h)} \cap B(u, r)$ and $f^{<\bar{c}} \cap B(u, r)$ are open.

So there is $N = N(r, e, h) \in \mathbb{N}$ such that, if $(a, b) \in [N, +\infty]^2$, then

$$\sum_{i=1}^{r} \lambda_i (K_{-a, b} \circ \sigma_i) \in \bar{\omega}.$$

As a consequence, there is

$$\tilde{\omega} \in H_1(f^{<(\bar{c}+h)} \cap B(u, r) \cap L^\beta_{(-a, b)}, f^{<\bar{c}} \cap B(u, r) \cap L^\beta_{(-a, b)})$$

such that $\sum \lambda_i (K_{-a, b} \circ \sigma_i) \in \tilde{\omega}$, and $i_*(\bar{\omega}) \neq 0$ implies $I_*(\tilde{\omega}) \neq 0$. So I_* cannot be zero.

Corollary 8.1 is thus proved. □

We now have to introduce some notations.

Take $x \in L^\beta$, $\bar{p}=(p^1, \ldots, p^m) \in \mathbb{Z}^m$, $m \geqq 1$, $p^i < p^{i+1}$. Denote

$$x_i = x\,\chi_{[(p^{i-1}+p^i)/2,\,(p^i+p^{i+1})/2]}, \qquad f_i(x)=f(x_i),$$

with χ_I the characteristic function of I, $p^0=-\infty$, $p^{m+1}=+\infty$.

We have $x=\sum_{i=1}^{m} x_i$, but $f \neq \sum_{i=1}^{m} f_i$.

Consider the sets

$$\mathscr{L}_+(h)=\bigcap_{i=1}^{m} (f_i)^{<(\bar{c}+h)}, \qquad \mathscr{L}_-(h)=\bigcup_{i=1}^{m} (f_i)^{<(\bar{c}-h)},$$

and the "product" ball

$$B^u_{\bar{p},\rho}=\{x \in L^\beta/(\forall i)\, \|(x-p^i * u)_i\|_{L^\beta}<\rho\}$$

for $\rho>0$, $u \in \mathscr{C}$.

From Künneth's formula,

$$H_*(X \times Y, (Z \times Y) \cup (X \times T)) = H_*(X, Z) \otimes H_*(Y, T),$$

immediately follows

LEMMA 9. – *Assume that* (hA), (hR) *and* ($\mathscr{H}$) *are true.* u, r_1 *are the same as in Lemma* 8. *Choose* $r \in]0, r_1]$, $h>0$.

Then there is $N=N(r, h)$ *such that, if* $m \geqq 1$ *and* $\bar{p}=(p^1 \ldots p^m)$ *satisfy* $p^{i+1}-p^i \geqq N$ *for* $1 \leqq i \leqq m-1$, *then*

$$J_* \neq 0,$$

where

$$J_*:\ H_m(\mathscr{L}_+(h) \cap B^u_{\bar{p},r},\ \mathscr{L}_-(0) \cap \mathscr{L}_+(h) \cap B^u_{\bar{p},r}) \to H_m(\mathscr{L}_+(h) \cap B^u_{\bar{p},r_1},\ \mathscr{L}_-(0) \cap \mathscr{L}_+(h) \cap B^u_{\bar{p},r_1})$$

is the morphism associated to the canonical injection

$$J:\ B_{\bar{p},r} \to B_{\bar{p},r_1}.$$

Lemma 9 gives the desired product min-max.

V. A DEFORMATION ARGUMENT

In what follows, we assume once again that (hA), (hR) and ($\mathscr{H}$) are true. D, F are the same as in Lemmas 4, 5, r_0 is the same as in Lemma 6, u, $\bar{c}$, r_1 are the same as in Lemmas 8, 9.

5.1. Construction of a vector field

From (hA) (hR), we know that $(\exists\,\theta,\ C_1>0)\ (\forall (X, Y)\in(L^\beta)^2)$:

$$\left|\int (X, LY)\right|\leqq C_1\exp(-\theta\delta(X, Y))\,\|X\|_\beta\,\|Y\|_\beta,$$

for $\delta(X, Y)=\operatorname{dist}(\operatorname{supp} X, \operatorname{supp} Y)$.

From (hR), we know that

$$(\exists\,c_1>0)\quad (\forall (y, t)\in\mathbb{R}^{2N}\times\mathbb{R}),\qquad c_1|y|^\beta\leqq G(y, t)\leqq(\nabla G(y, t), y),$$
$$(\exists\,c_2>0)\quad (\forall (y, t)\in\mathbb{R}^{2N}\times\mathbb{R}),\qquad |\nabla G(y, t)|\leqq c_2|y|^{\beta-1}.$$

We choose $0<r_2<\min(1, r_1)$ such that

$$\frac{c_1}{2}(r_2)^\beta>6\,C_1(r_2)^2,\qquad\text{and}\qquad B(u, r_2)\subset f^{c'}.$$

We are going to use these technical conditions in the proof of the following Lemma:

LEMMA 10. – *Assume that* (hA), (hR) *and* ($\mathscr{H}$) *are true, and to* $0<r<\frac{r_2}{2}$, *associate* $e=e(r)$ *such that*

$$r+2e\leqq\frac{r_2}{2}\qquad\textit{and}\qquad [r-2e, r+2e]\cap D=\varnothing.$$

There are $\mu=\mu(r)>0$, $A=A(r)>0$ *such that:*
If $m\geqq 2$, *and if* $\bar{p}\in\mathbb{Z}^m$ *satisfies* $(\forall i): p^{i+1}-p^i>A$, *then:*
$(\forall x\in B^u_{\bar{p}, r+e}\setminus B^u_{\bar{p}, r-e})(\exists V_x\in B^0_{\bar{p}, 1})$:

1) $$f'(x).V_x>\mu;$$
2) $$(\forall i): (f_i)'(x).V_x\geqq 0;$$
3) $$\|y_i\|\geqq r-e\Rightarrow (f_i)'(x).V_x>\mu,$$

with the notation $y_i=(x-p^i * u)_i$. □

Proof. – Define

$$\bar{\mu}=\frac{1}{2}\inf\{\|f'(x)\|_\alpha / x\in B(u, r+2e(r))\setminus B(u, r-e(r))\}.$$

$\bar{\mu}$ depends only on r, and $\bar{\mu}>0$ by Lemma 4. Let $x\in B^u_{\bar{p}, r+e}\setminus B^u_{\bar{p}, r-e}$, $i\in[\![1, m]\!]$, and $y_i=(x-p^i * u)_i$. Impose $A>64$.

We always have $\|x_i\|\leqq\|u\|+r_2$. So there is $\tau^i\in[2\sqrt{A}, A/2-2\sqrt{A}]$ such that

$$\|x_i\chi_{\{\tau^i-\sqrt{A}\leqq|t-p^i|\leqq\tau^i+\sqrt{A}\}}\|_\beta\leqq\frac{C_2}{A^{1/2\beta}}.$$

Here, C_2 is a constant, but τ^i depends on x, i, A, $\bar{p}$.

Now, impose $\| u\chi_{\{|t|>\sqrt{A}\}}\| \leqq \frac{e}{3}$, and $\frac{C_2}{A^{1/2\beta}} \leqq \frac{e}{3}$, which is possible for $A \geqq A^0(e)$.

Then, three possibilities may occur:

First case:

$$\| x_i \chi_{\{|t-p^i| \geqq \tau^i + \sqrt{A}\}} \| \geqq \frac{e}{3}.$$

We take

$$V_{x,i} = x_i (h_- \chi_{]-\infty, p^i - \tau^i - \sqrt{A}]} + h_+ \chi_{[p^i + \tau^i + \sqrt{A}, +\infty[})$$

with

$$h_+ = 1 \quad \text{if } \| x_i \chi_{[p^i+\tau^i+\sqrt{A}, +\infty[} \| \geqq \frac{e}{6}, \qquad h_+ = 0 \quad \text{otherwise,}$$

$$h_- = 1 \quad \text{if } \| x_i \chi_{]-\infty, p^i-\tau^i-\sqrt{A}]} \| \geqq \frac{e}{6}, \qquad h_- = 0 \quad \text{otherwise.}$$

We have

$$\begin{aligned}
(f_i)'(x) . V_{x,i} \geqq & c_1 \| V_{x,i} \|_\beta^\beta - C_1 \| V_{x,i} \|_\beta^2 \\
& - C_1 \| x \chi_{\{\tau^i - \sqrt{A} \leqq |t-p^i| \leqq \tau^i + \sqrt{A}\}} \|_\beta \cdot \| V_{x,i} \|_\beta \\
& - C_1 \| x \chi_{\{|t-p^i| \leqq \tau^i - \sqrt{A}\}} \|_\beta \cdot \| V_{x,i} \|_\beta \exp(-2\theta\sqrt{A}) \\
\geqq & \frac{3c_1}{4} \| V_{x,i} \|_\beta^\beta - C_1 \frac{e}{3} \| V_{x,i} \|_\beta \\
& - C_1 (\| u \|_\beta + r_2) \| V_{x,i} \|_\beta \exp(-2\theta\sqrt{A}) \\
\geqq & \frac{3c_1}{4} \| V_{x,i} \|_\beta^\beta - C_1 e \| V_{x,i} \|_\beta \qquad \text{for} \quad A \geqq A^1(e) \\
\geqq & \frac{3c_1}{4} \| V_{x,i} \|_\beta^\beta - 6C_1 \| V_{x,i} \|_\beta^2 \\
\geqq & \frac{c_1}{4} \| V_{x,i} \|_\beta^\beta \geqq \frac{c_1}{4} \left(\frac{e}{6} \right)^\beta.
\end{aligned}$$

[We recall that $\frac{e}{6} \leqq \| V_{x,i} \|_\beta \leqq \| u\chi_{\{|t| \geqq \sqrt{A}\}} \| + (r+e) \leqq r_2 < 1$, and that $\frac{c_1}{2}(r_2)^\beta > 6C_1 (r_2)^2$.]

Second case: $\| x_i \chi_{\{|t-p^i| \geqq \tau^i + \sqrt{A}\}} \| < \frac{e}{3}$, and $\| y_i \| < r - e$. Then we take $V_{x,i} = 0$.

Third case: $\|x_i \chi_{\{|t-p^i|\geqq \tau^i+\sqrt{A}\}}\| < \frac{e}{3}$, and $\|y_i\| < r-e$. Then

$$\begin{aligned}\|x \chi_{\{|t-p^i|\leqq \tau^i-\sqrt{A}\}} - p^i * u\| &\geqq \|y_i\| - \|x_i \chi_{\{\tau^i-\sqrt{A}\leqq |t-p^i| \leqq \tau^i+\sqrt{A}\}}\| \\ &\quad - \|u \chi_{|t|\geqq\sqrt{A}}\| - \|x_i \chi_{\{|t-p^i|\geqq \tau^i+\sqrt{A}\}}\| \\ &\geqq r-e-\frac{e}{3}-\frac{e}{3}-\frac{e}{3} = r-2e.\end{aligned}$$

Finally,

$$\begin{aligned} r-2e &\leqq \|x \chi_{\{|t-p^i|\leqq \tau^i-\sqrt{A}\}} - p^i * u\| \\ &\leqq \|y_i \chi_{\{|t-p^i|\leqq \tau^i-\sqrt{A}\}}\| + \|u \chi_{|t|\geqq\sqrt{A}}\| \\ &\leqq r+e+\frac{e}{3} \\ &\leqq r+2e. \end{aligned}$$

So there is $W_{x,i} \in L^\beta$ such that $\|W_{x,i}\| \leqq 1$, and

$$f'(x \chi_{\{|t-p^i|\leqq \tau^i-\sqrt{A}\}}) . W_{x,i} > \bar{\mu}.$$

Now,

$$\begin{aligned} f'(x) &= f'(x_i \chi_{\{|t-p^i|\leqq \tau^i-\sqrt{A}\}} + f'(x_i \chi_{\{\tau^i-\sqrt{A}\leqq |t-p^i| \leqq \tau^i+\sqrt{A}\}}) \\ &\quad + f'(x_i \chi_{\{|t-p^i|\geqq \tau^i+\sqrt{A}\}}) + \sum_{j\neq i} f'(x_j) \\ &= f'(x^a) + f'(x^b) + f'(x^c) + \sum_{j\neq i} f'(x_j). \end{aligned}$$

But $\|x^b\| \leqq \frac{C_2}{A^{1/2\beta}}$, and $\max\{\|x^a\|, \|x^c\|, \|x_j\| (j\neq i)\} \leqq \|u\| + r_2$.

We choose $V_{x,i} = W_{x,i} \chi_{\{|t-p^i|\leqq \tau^i\}}$. Clearly, $\|V_i\| \leqq 1$. Moreover, we have:

$$\begin{aligned} f'(x) . V_{x,i} &\geqq f'(x^a) . W_{x,i} - |f'(x^a) . (V_{x,i} - W_{x,i})| \\ &\quad - |f'(x^b) . V_{x,i}| - |f'(x^c) . V_{x,i}| - \sum_{j\neq i} |f'(x_j) . V_{x,i}| \\ &\geqq \bar{\mu} - C_1 (\|u\| + r_2) \exp(-\theta\sqrt{A}) \\ &\quad - c_2 \left(\frac{C_2}{A^{1/2\beta}}\right)^{\beta-1} - C_1 \frac{C_2}{A^{1/2\beta}} - C_1 (\|u\| + r_2) \exp(-\theta\sqrt{A}) \\ &\quad - \sum_{j\neq i} C_1 (\|u\| + r_2) \exp(-\theta\sqrt{A}) \exp[-\theta(|i-j|-1) A] \\ &\geqq \bar{\mu} - c_2 \left(\frac{C_2}{A^{1/2\beta}}\right)^{\beta-1} - C_1 \frac{C_2}{A^{1/2\beta}} \\ &\quad - C_1 (\|u\| + r_2) . \left(2 + \frac{2}{1-\exp(-\theta A)}\right) \exp(-\theta\sqrt{A}) \\ &\geqq \bar{\mu}/2 \qquad \text{for} \quad A \geqq A^2(r). \end{aligned}$$

Identically,

$$\begin{aligned}(f_i)'(x).V_{x,i} &= f'(x^a+x^b+x^c).V_{x,i}\\ &\geqq \bar{\mu}-c_2\left(\frac{C_2}{A^{1/2\beta}}\right)^{\beta-1}-C_1\frac{C_2}{A^{1/2\beta}}-2C_1(\|u\|+r_2)\exp(-\theta\sqrt{A})\\ &\geqq \bar{\mu}/2 \qquad \text{for} \quad A\geqq A^2.\end{aligned}$$

Conclusion. – We now take $V_x=\sum_i V_{x,i}$. By construction, $V_x\in B^0_{\bar{p},1}$. Denote by I^1, I^2, I^3 the sets of indices i corresponding to Cases 1, 2, 3 respectively. We write

$$\begin{aligned}f'(x).V_x &= \sum_{i\in I^1} f'(x).V_{x,i}+\sum_{i\in I^3} f'(x).V_{x,i}\\ &\geqq \sum_{i\in I^1} f'(x).V_{x,i}+\frac{\bar{\mu}}{2}\operatorname{card}(I^3).\end{aligned}$$

Now, there is a family $J^1\subset[\![0, m]\!]$ such that

$$\sum_{i\in I^1} V_{x,i}=\sum_{j\in J^1} X^j,$$

where

$$\begin{aligned}X^j &= (\xi^j_+\chi_{[((p^j+p^{j+1})/2),\, p^{j+1}-\tau^{j+1}-\sqrt{A}]}+\xi^j_-\chi_{[p^j+\tau^j+\sqrt{A},\, ((p^j+p^{j+1})/2)]})\,x\\ &= \xi^j_+X^j_+ +\xi^j_-X^j_-\end{aligned}$$

with $\xi^j_\pm\in\{0, 1\}$, and

$$(\forall s\in\{+,-\}) \quad (\forall j\in[\![0, m]\!])$$
$$\left(\xi^j_s=1\Rightarrow\|X^j_s\|\geqq\frac{e}{6},\ \xi^j_s=0\Rightarrow\|X^j_s\|<\frac{e}{3}.\right)$$

So there are three possible situations

$$(\xi^j_-=\xi^j_+=1), \quad (\xi^j_-=0 \text{ and } \xi^j_+=1), \quad (\xi^j_-=1 \text{ and } \xi^j_+=0).$$

First situation: $\xi^j_-=\xi^j_+=1$.

Denote

$$Y^j=x\,\chi_{[p^j+\tau^j-\sqrt{A};\, p^j+\tau^j+\sqrt{A}]\,\cup\,[p^{j+1}-\tau^{j+1}-\sqrt{A};\, p^{j+1}-\tau^{j+1}+\sqrt{A}]}$$
$$Z^j=x_j+x_{j+1}-X^j-Y^j.$$

We have

$$\begin{aligned}f'(x).X^j &= f'(X^j).X^j+f'(Y^j).X^j+f'(Z^j).X^j\\ &\quad+\sum_{k\neq j,\, j+1} f'(x_k).X^j\\ &\geqq \frac{3c_1}{4}\|X^j\|^\beta-C_1\frac{2e}{3}\|X^j\|-2C_1\|X^j\|(\|u\|+r_2)\exp(-2\theta\sqrt{A})\\ &\quad-2\|X^j\|(\|u\|+r_2)\sum_{l\geqq 0}\exp(-2\theta\sqrt{A})\exp(-\theta l A)\end{aligned}$$

$$\geqq \frac{3c_1}{4}\|X^j\|^\beta - C_1 e\|X^j\| \qquad \text{for} \quad A \geqq A^3(e)$$

$$\geqq \frac{3c_1}{4}\|X^j\|^\beta - 6C_1\|X^j\|^2$$

$$\geqq \frac{c_1}{4}\|X\|^j\|^\beta \geqq \frac{c_1}{4}\frac{e^\beta}{6^\beta}$$

since $\frac{e}{6} \leqq \|X^j\| \leqq 2\|u\chi_{\{|t| \geqq \sqrt{A}\}}\| + 2(r+e) \leqq r_2$.

Second situation: $\xi_-^j = 0$, $\xi_+^j = 1$.

We now take

$$Y^j = x(\chi_{[p^j + \tau^j + \sqrt{A}, ((p_j + p_{j+1})/2)]} + \chi_{[p^{j+1} - \tau^{j+1} - \sqrt{A}, p^{j+1} - \tau^{j+1} + \sqrt{A}]}),$$

$$Z^j = x_j + x_{j+1} - X^j - Y^j.$$

We have $\|Y^j\| \leqq \frac{e}{3} + \frac{e}{3} = \frac{2e}{3}$, $\operatorname{dist}(\operatorname{supp} Z^j, \operatorname{Supp} X^j) \geqq \sqrt{A}$. As in the first situation, we get

$$f'(x).X^j \geqq \frac{3c_1}{4}\|X^j\|^\beta - C_1 e\|X^j\| \qquad \text{for} \quad A \geqq A^4(u, e)$$

$$\geqq \frac{3c_1}{4}\|X^j\|^\beta - 6C_1\|X^j\|^2$$

$$\geqq \frac{c_1}{4}\|X^j\|^\beta \geqq \frac{c_1}{4}\frac{e^\beta}{6^\beta}.$$

The third situation is identical to the second one. Since $I^1 \cup I^3$ is non-empty, we take

$$A(r) = \max(A^0, A^1, A^2, A^3, A^4) \qquad \text{and} \qquad \mu(r) = \min\left(\bar{\mu}, \frac{c_1}{4}\frac{e^\beta}{6^\beta}\right),$$

and Lemma 10 is proved. □

LEMMA 11. — *Suppose f satisfies* (hA), (hR) *and* ($\mathcal{H}$). *To* $l < c'$, *associate* $\eta = \eta(l) > 0$ *such that* $l + 2\eta \leqq c'$, *and* $[l - 2\eta, l + 2\eta] \cap F = \varnothing$.

Then there are $\mathscr{A} = \mathscr{A}(l)$ *and* $\nu = \nu(l)$ *such that for any* $m \geqq 2$, $\bar{p} \in \mathbb{Z}^m$, *with* $(\forall i)\, p^{i+1} - p^i > \mathscr{A}$, *we have:*

$$\left(\forall x \in B^u_{\bar{p}, (r_2/2)} \cap \bigcup_{i=1}^m (f_i)_{l-\eta}^{l+\eta}\right)(\exists \mathscr{V}_x \in B^0_{\bar{p}, 1}):$$

- $f'(x).\mathscr{V}_x > \nu$;
- $(\forall i \in [\![1, m]\!]) : (x \in (f_i)_{l-\eta}^{l+\eta} \Rightarrow (f_i)'(x).\mathscr{V}_x > \nu)$;
- $(\forall i) : (f_i)'(x).\mathscr{V}_x > 0$.

Proof. – We know that f is uniformly continuous on any bounded part of L^β. So there is $\mathscr{E}(\eta)>0$ such that, if X, Y $\in B(0, \|u\|+r_2)$, then

$$\|X-Y\|\leqq\mathscr{E} \quad\Rightarrow\quad |f(x)-f(y)|\leqq\eta.$$

Now, consider $\bar{\nu}=\dfrac{1}{2}\inf\{\|f'(x)\|;\ x\in f_{l-2\eta}^{l+2\eta}\}$. From Lemma 5, $\bar{\nu}>0$. The proof of Lemma 11 is similar to that of Lemma 10, replacing V by $\mathscr{V}$, $\bar{\mu}$ by $\bar{\nu}$, A by $\mathscr{A}$, e by $\mathscr{E}$. So we just sketch it. The three possibilities are:

First case: $\|x_i\chi_{\{|t-p^i|\geqq\tau^i+\sqrt{\mathscr{A}}\}}\|\geqq\dfrac{\mathscr{E}}{3}$, then

$$\mathscr{V}_{x,i}=x_i(h_-\chi_{]-\infty,\,p^i-\tau^i-\sqrt{\mathscr{A}}]}+h_+\chi_{[p^i+\tau^i+\sqrt{\mathscr{A}},\,+\infty[}),$$

$$(f_i)'(x).\mathscr{V}_{x,i}\geqq\frac{c_1}{2}\frac{\mathscr{E}^\beta}{6^\beta}\qquad\text{for}\quad\mathscr{A}\geqq\max(\mathscr{A}^0,\mathscr{A}^1).$$

Second case: $\|x_i\chi_{\{|t-p^i|>\tau^i+\sqrt{\mathscr{A}}\}}\|<\dfrac{\mathscr{E}}{3}$, *and* $f_i(x)\notin[l-\eta, l+\eta]$, then $\mathscr{V}_{x,i}=0$.

Third case: $\|x_i\chi_{\{|t-p^i|>\tau^i+\sqrt{\mathscr{A}}\}}\|<\dfrac{\mathscr{E}}{3}$, *and* $f_i(x)\in[l-\eta, l+\eta]$, then

$$f(x\chi_{\{|t-p^i|\leqq\tau^i-\sqrt{\mathscr{A}}\}})\in[l-2\eta, l+2\eta]\qquad\text{for}\quad\mathscr{A}\geqq\mathscr{A}^0,$$

hence $f'(x\chi_{\{|t-p^i|\leqq\tau^i-\sqrt{\mathscr{A}}\}}).\mathscr{W}_{x,i}>\bar{\nu}$,

$$\|\mathscr{W}_{x,i}\|\leqq 1,\qquad \mathscr{V}_{x,i}=\mathscr{W}_{x,i}\chi_{\{|t-p^i|\leqq\tau^i\}},$$

$$f'(x).\mathscr{V}_{x,i}\geqq\bar{\nu}/2,\qquad (f_i)'(x).\mathscr{V}_{x,i}\geqq\bar{\nu}/2,\quad\text{for }\mathscr{A}\geqq\mathscr{A}^2.$$

The final study of $f'(x).\mathscr{V}_x$ is the same as in Lemma 10, and 11 is proved with $\mathscr{A}=\max(\mathscr{A}^0,\ \dots,\mathscr{A}^4)$, $\nu=\min\left(\dfrac{\bar{\nu}}{2},\dfrac{c_1}{2}\dfrac{\mathscr{E}^\beta}{6^\beta}\right)$. □

LEMMA 12. – *Suppose f satisfies* (hA), (hR) *and* ($\mathscr{H}$).

r, $e(r)$, $A(r)$, $\mu(r)$ are the same as in Lemma 10. *We impose, moreover, $r<r_0$, with the notation of Lemma* 6.

Choose $\lambda>0$ *such that* $\bar{c}+\lambda<c'$,

$$\text{and}\quad\begin{cases}\bar{c}+\lambda\notin F\\ \bar{c}-\lambda\notin F.\end{cases}$$

Suppose $m\geqq 2$, $\bar{p}\in\mathbb{Z}^m$,

$$(p^{i+1}-p^i)\geqq\max(A(r),\ \mathscr{A}(\bar{c}-\lambda),\ \mathscr{A}(\bar{c}+\lambda))$$
$$=\mathscr{B}(r,\lambda)$$

($\mathscr{A}$ *has been defined in Lemma* 11).

If $\mathscr{C} \cap \mathrm{B}^u_{\bar{p}, r} \cap \mathscr{L}_+(\lambda) \setminus \mathscr{L}_-(\lambda) = \varnothing$, *then there are* $\xi = \xi(\bar{p}, r, \lambda) > 0$ *and a locally Lipschitz vector field* $\mathrm{V}(x)$ *such that:*

(i) $(\forall x) : \mathrm{V}(x) \in \mathrm{B}^0_{\bar{p}, 1}$, *and* $(x \notin \mathrm{B}^u_{\bar{p}, (r_2/2)} \Rightarrow \mathrm{V}(x) = 0)$;

(ii) $\forall x \in [\mathrm{B}^u_{\bar{p}, r} \setminus \mathrm{B}^u_{\bar{p}, (r-e)}]$, $\forall i \in [\![1, m]\!]$,

$$\left(\|y_i\| \in [r-e, r] \Rightarrow (f_i)'(x) \,.\, \mathrm{V}(x) > \frac{\mu(r)}{3} \right).$$

(iii) $(\forall x \in \mathrm{B}^u_{\bar{p}, r} \cap (\mathscr{L}_+(\lambda) \setminus \mathscr{L}_-(\lambda)) : f'(x) \,.\, \mathrm{V}(x) > \xi$.

(iv) $(\forall x \in \mathrm{B}^u_{\bar{p}, (r_2/2)}) (\forall i \in [\![1, m]\!])$:

$$(f_i(x) \in \{\bar{c} + \lambda, \bar{c} - \lambda\} \Rightarrow (f_i)'(x) \,.\, \mathrm{V}(x) > 0).$$

Proof. – In Lemma 6, take $\mathrm{R} = \max(|p^1|, |p^m|)$. Consider a sequence

$$(u_n) \in \mathrm{B}^u_{\bar{p}, r} \cap \mathscr{L}_+(\lambda - \eta(\bar{c} + \lambda)) \setminus \mathscr{L}_-(\lambda - \eta(\bar{c} - \lambda)).$$

(u_n) satisfies

$$(\forall p, q), \quad \|(u_p - u_q)\chi_{\mathbb{R} \setminus [-\mathrm{R}, \mathrm{R}]}\| < 2 r_2 < 2 r_0.$$

So, if $\mathscr{C} \cap \mathrm{B}^u_{\bar{p}, r} \cap \mathscr{L}_+(\lambda) \setminus \mathscr{L}_-(\lambda) = \varnothing$, we cannot have $f'(u_n) \to 0$, and there is $\alpha(\bar{p}, u, r, \lambda) > 0$ such that

$$\forall x \in \mathrm{B}^u_{\bar{p}, r} \cap \mathscr{L}_+(\lambda - \eta(\bar{c} + \lambda)) \setminus \mathscr{L}_-(\lambda - \eta(\bar{c} - \lambda)) : \quad \|f'(x)\| \geqq 2\alpha.$$

Now, if $x \in [\mathrm{B}^u_{\bar{p}, (r+e)} \setminus \mathrm{B}^u_{\bar{p}, (r-e)}]$, we find V_x satisfying the conclusion of Lemma 10, and we choose $\mathrm{V}_x = 0$ otherwise.

For $s \in \{-, +\}$. if $x \in \mathrm{B}^u_{\bar{p}, (r_2/2)} \cap \bigcup_i (f_i)^{\bar{c} + s\lambda + \eta(\bar{c} + s\lambda)}_{\bar{c} + s\lambda - \eta(\bar{c} + s\lambda)}$, we find $\mathscr{V}^s_x$ satisfying the conclusion of Lemma 11 with $l = c + s\lambda$, and we choose $\mathscr{V}^s_x = 0$ otherwise.

If $x \in \mathrm{B}^u_{\bar{p}, r} \cap \mathscr{L}_+(\lambda) \setminus \mathscr{L}_-(\lambda)$ and if $\mathrm{V}_x = \mathscr{V}^+_x = \mathscr{V}^-_x = 0$, we find $\bar{\mathrm{V}}_x \in \mathrm{B}^0_{\bar{p}, 1}$ such that $f'(x) \,.\, \bar{\mathrm{V}}_x > \alpha$, and we choose $\bar{\mathrm{V}}_x = \frac{1}{3}(\mathrm{V}_x + \mathscr{V}^+_x + \mathscr{V}^-_x)$ otherwise.

We take $\xi = \min\left\{ \alpha, \frac{1}{3}(\mu(r) + \nu(\bar{c} + \lambda) + \nu(\bar{c} - \lambda)) \right\}$.

$\bar{\mathrm{V}}_x$ satisfies:

(I) $(\forall x) : \bar{\mathrm{V}}_x \in \mathrm{B}^0_{\bar{p}, 1}$, and $(x \notin \mathrm{B}^u_{\bar{p}, (r_2/2)} \Rightarrow \bar{\mathrm{V}}_x = 0)$.

$$\text{(II)} \quad \begin{cases} \forall x \in [\mathrm{B}^u_{\bar{p}, r+e} \setminus \mathrm{B}^u_{\bar{p}, (r-e)}], \ \forall i \in [\![1, m]\!], \\ \|y_i\| \in [r-e, r+e] \quad \Rightarrow \quad (f_i)'(x) \,.\, \bar{\mathrm{V}}_x > \dfrac{\mu(r)}{3}. \end{cases}$$

$$\text{(III)}\quad \begin{cases} (\forall x \in B^u_{\bar p, r+e} \cap (\mathscr{L}_+ (\lambda+\eta)(\bar c+\lambda)) \setminus \mathscr{L}_- (\lambda+\eta(\bar c-\lambda)))\colon \\ f'(x).\bar V_x > \xi. \end{cases}$$

$$\text{(IV)}\quad \begin{cases} (\forall x \in B^u_{\bar p, (r_2/2)})(\forall i \in [\![1, m]\!])\colon \\ (f_i(x) \in \{\bar c+\lambda, \bar c-\lambda\} \Rightarrow (f_i)'(x).\bar V_x > 0). \end{cases}$$

But $\bar V_x$ is not continuous. A classical pseudo-gradient construction ends the proof. □

5.2. The contradiction

We suppose (hA), (hR) and ($\mathscr{H}$) are true. r, $e(r)$, $\mu(r)$, λ are the same as in Lemma 12. On λ, we impose one more condition:

$$\lambda \leqq \frac{\mu(r)\, e(r)}{6}.$$

As in Lemma 12, we suppose that

$$\mathscr{C} \cap B^u_{\bar p, r} \cap (\mathscr{L}_+ \setminus \mathscr{L}_-)(\lambda) = \varnothing,$$

and we take $m \geqq 2$, $\bar p \in \mathbb{Z}^m$ with

$$(\forall i) \quad (p^{i+1} - p^i) \geqq \mathscr{B}(r, \lambda).$$

We define $\varphi(t, x)$ for $(t, x) \in \mathbb{R} \times L^\beta$ by

$$\varphi(0, x) = x$$

$$\frac{\partial \varphi}{\partial t}(t, x) = -V \circ \varphi(t, x),$$

where $V(x)$ is the vector field of Lemma 12.

We have

LEMMA 13. – *With the notations and hypotheses above, there is* $\mathscr{T} = \mathscr{T}(r, \lambda, \bar p)$ *such that*

$$\varphi(\mathscr{T}, .)[B^u_{\bar p, r-e} \cap \mathscr{L}_+(\lambda)] \subset \mathscr{L}_-(\lambda) \cap \mathscr{L}_+(\lambda).$$

Proof. – Take $x \in B^u_{\bar p, r-e} \cap \mathscr{L}_+(\lambda)$. Then

$$(\forall t \geqq 0), \quad \varphi(t, x) \in B^u_{\bar p, (r_2/2)} \cap \mathscr{L}_+(\lambda),$$

by (i) and (iv) of Lemma 12. Moreover, if $\varphi(t, x) \in \mathscr{L}_-(\lambda)$, then for any $t' \geqq t$, $\varphi(t', x) \in \mathscr{L}_-(\lambda)$, by (iv). Now, define

$$S = S(\bar p) = \sup\{|f(X) - f(Y)|;\ (X, Y) \in (B^u_{\bar p, r_2})^2\}.$$

Define

$$\mathscr{T} = \frac{2\,S(\bar p)}{\xi(\bar p, r, \lambda)}.$$

By (iii) of Lemma 12, there is $t_x \in [0, \mathscr{T}]$ such that

$$\varphi(t_x, x) \notin B^u_{\bar{p}, r} \cap (\mathscr{L}_+(\lambda) \setminus \mathscr{L}_-(\lambda)).$$

By (i), (ii) of Lemma 12, this implies $\varphi(\mathscr{T}, x) \in \mathscr{L}_-(\lambda)$ (we recall that $2\lambda \leqq \mu(r) e(r)/3$).

Lemma 13 is thus proved. □

Now, we impose

$$(\forall i) \quad (p^{i+1} - p^i) \geqq N(r - e(r), \lambda),$$

with the notations of Lemma 9.

The conclusion of Lemma 13 clearly implies $J_* = 0$, which contradicts the conclusion of Lemma 9.

Now, for any $h > 0$, we may choose $\lambda < h$ satisfying all the conditions above.

So, by contradiction, we have proved the following result:

THEOREM III. – *Assume that* (hA), (hR) *and* ($\mathscr{H}$) *are true.*

Then there is $u \in \mathscr{C}$, *with* $f(u) = \bar{c} \in [c, c')$, *and such that for any* $r, h > 0$, *for all* $m \geqq 1$ *and* $\bar{p} = (p^1, \ldots, p^m) \in \mathbb{Z}^m$:

$$[(\forall i) : (p^{i+1} - p^i) \geqq M(r, h)] \quad \Rightarrow \quad [\mathscr{C} \cap U_{\bar{p}, r, h} \neq \varnothing].$$

$M(r, h)$ *is a constant independent of* m, *and* $U_{\bar{p}, r, h}$ *is a neighborhood of* $\sum_{i=1}^{m} p^i * u$ *defined as follows:*

$U_{\bar{p}, r, h} = B^u_{\bar{p}, r} \cap (\mathscr{L}_+(h) \setminus \mathscr{L}_-(h))$, *with the notations of Lemma* 9.

We now prove Theorem II:

We take a fixed value of h, and we write $M(r)$ instead of $M(r, h)$. We may choose $K > M(r)$ large enough to get $\| u \chi_{\{|t| \geqq K/2\}} \| \leqq r$, which implies $\sum_{i=1}^{m} p^i * u \in B^u_{\bar{p}, r}$ for any $m \geqq 2$, and $\bar{p} \in \mathbb{Z}^m$ such that $(\forall i)(p^{i+1} - p^i) \geqq K$. So, from Theorem III, there is $u_{\bar{p}} \in \mathscr{C}$ such that

$$(\forall i \in \mathbb{Z}): \quad \left\| \left(u_{\bar{p}} - \sum_{i=1}^{m} p^i * u \right) \chi_{[((p^{i-1} + p^i)/2);\, ((p^i + p^{i+1})/2)]} \right\|_{\beta} \leqq 2r.$$

So, defining $y_{\bar{p}} = L u_{\bar{p}}$:

$$\left\| y_{\bar{p}} - \sum_{i=1}^{m} p^i * x \right\|_{\infty} \leqq 3 C_3 \sum_{n \geqq 0} 2r \exp[-2\theta' n M(r)]$$

$$= \frac{6 C_3 r}{1 - \exp(-2\theta' K)} \leqq \varepsilon,$$

for $K(\varepsilon)$ large enough. So Theorem II is a direct consequence of Theorem III. □

We are now going to study the limit $(m \to +\infty)$.

VI. THE APPROXIMATE BERNOULLI SHIFT

Our first taks here is to prove Corollary II.1 of Theorem II. We consider a sequence $\bar{p}=(p^i)_{i\in I}$ of integers with $I\subset\mathbb{Z}$ a finite or infinite interval, and $p^{i+1}-p^i\geqq K(\varepsilon)$.

The case $0\leqq \mathrm{Card}(I)<\infty$ is clear. So we just consider the case of an infinite I. We may write $I=\bigcup_{k\geqq 0} I^k$, each I^k being finite. From Theorem II, we get an orbit y^k such that

$$\|y^k-\sum_{i\in I^k} p^i * x\|_\infty\leqq\varepsilon.$$

The y^k's being orbits, $\|y^k\|_\infty+\left\|\frac{d}{dt}y^k\right\|_\infty$ is a bounded sequence. So, after extraction, by Ascoli's theorem. y^k converges to some orbit $y_{\bar{p}}$ in the C^0_{loc} topology, and Corollary II.1 is proved.

Now, we take $s\in\{0, 1\}^{\mathbb{Z}}$ arbitrary (*i.e.* with possibly infinitely many 1's). There are an interval I of integers and a sequence $(q^i)_{i\in I}\subset\mathbb{Z}$, with $(\forall i)\, q^{i+1}>q^i$, and $s_n=\chi_{\{q^i,\, i\in\mathbb{Z}\}}(n)$.

We denote $p^i=K(\varepsilon)\, q^i$, and we define $\mathcal{T}(s)=y_{\bar{p}}$, using Corollary II.1.

We recall that $\{0, 1\}^{\mathbb{Z}}$ may be given the topology associated to the metric $d(s, s')=\frac{1}{3}\sum_{n\in\mathbb{Z}}\frac{|s_n-s'_n|}{2^{|n|}}$.

We define

$$\tilde{\tau}:\ \{0, 1\}^{\mathbb{Z}}\to\mathbb{R}^{2N}$$
$$s\mapsto\mathcal{T}(s)(0).$$

Since

$$\|\mathcal{T}(s)-\sum_n s_n(Kn * x)\|_\infty\leqq\varepsilon,$$

we have $\limsup_{d(s,s')\to 0}|\tilde{\tau}(s')-\tilde{\tau}(s)|\leqq 2\varepsilon$.

Now, we take $\delta>0$. There is $I(\delta)>0$ such that if $d(s, s')\geqq\delta$, then $s^I\neq(s')^I$.

So, taking $K(\varepsilon)$ large enough in Corollary II.1, there is $\rho>0$ independent of s, s', ε, with

$$\|(\sum_n s_n(Kn * x)-\sum_n s'_n(Kn * x))\,\chi_{[-2I,\, 2I]}\|_\infty\geqq 2\rho.$$

So

$$\|(\mathcal{T}(s)-\mathcal{T}(s'))\,\chi_{[-2I,\, 2I]}\|_\infty\geqq\rho$$

for $\varepsilon<\frac{\rho}{2}$.

Now, define

$$\mathcal{O}: \quad \mathbb{R}^{2N} \to C^0([-2I, 2I], \mathbb{R}^{2N})$$
$$x \mapsto \mathcal{O}(x)$$

where

$$\frac{d}{dt}\mathcal{O} - JA\,\mathcal{O} = J\nabla R(t, \mathcal{O})$$
$$\mathcal{O}(x)(0) = x.$$

By the classical continuity results on the Cauchy problem, $\mathcal{O}$ is uniformly continuous on any bounded part of $\mathbb{R}^{2N}$. So there is $\rho'(\delta) > 0$, independent of s, s', r, such that

$$\tilde{d}(s, s') \geqq \delta \quad \Rightarrow \quad \|\tilde{\tau}(s) - \tilde{\tau}(s')\| \geqq \rho'.$$

So $\tilde{\tau}$ is injective, and $\tilde{\tau}^{-1}$ is uniformly continuous. The other assertions of Corollary II.2 are easy to check, if we choose $x_0 = x(0)$. Corollary II.2 is thus proved. One would like $\tilde{\tau}$ to give a Bernoulli shift structure, *i.e.* $\tilde{\tau}$ homeomorphism, and $\tilde{\tau} \circ \sigma = \varphi^K \circ \tilde{\tau}$ (*see* [M], [W]). Unfortunately, this is not the case. We only have the estimate $\|\mathcal{T}(s) - \sum_n s_n(n * x)\|_\infty \leqq \varepsilon$. The points s such that $s_n = 0$ except for a finite number of n's correspond to homoclinic orbits passing through $\tilde{\tau}(s)$ at time 0: there are infinitely many of them.

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