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A note on harmonic maps between surfaces

by

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ABSTRACT. — In this note, the relation between harmonic maps between surfaces and holomorphic quadratic differentials is investigated. Remember that if Σ_1 and Σ_2 are surfaces with conformal metrics $\sigma^2 dz d\bar{z}$ and $\rho^2 du d\bar{u}$, resp., and $u : \Sigma_1 \to \Sigma_2$ is harmonic, then

$$\phi := \rho^2 u_z \bar{u}_z dz^2$$

is a holomorphic quadratic differential on Σ_1 (and ϕ vanishes identically if and only if u is conformal).

It has been an open question to which extent the converse is true, i. e. whether a map with holomorphic ϕ is harmonic.

In the article under consideration, a variational procedure is invented that produces a map with holomorphic ϕ in every homotopy class of maps between closed surfaces. While on one hand, thus conformal selfmaps of the two-sphere are obtained by a variational method, answering a question of Uhlenbeck, contrasting this existence result on the other hand with some nonexistence results for harmonic maps, one is led to a negative answer to the above converse question. An explicit example is displayed as well.

RÉSUMÉ. — Dans cette note, on étudie la relation entre deux définitions pour les applications harmoniques u en dimension deux, l'une étant que la forme différentielle quadratique

(1) $$\omega := | u_x |^2 - | u_y |^2 - 2i \langle u_x, u_y \rangle$$

(associée à u, où $z = x + iy$ est une coordonnée conforme locale) soit holomorphe, l'autre étant l'équation différentielle du deuxième ordre

(2) $$\tau(u) = 0$$

(2) implique que ω soit holomorphe. Nous déterminons, dans quelle mesure l'implication inverse n'est pas vraie, et donnons une réponse négative à une question de Eells-Lemaire. De plus, nous construisons une procédure variationnelle, qui donne des revêtements ramifiés conformes de S^2.

Let Σ_1 and Σ_2 be compact two-dimensional Riemannian manifolds. A theorem of Lemaire [*8*] and Sacks-Uhlenbeck [*9*] asserts that in case $\pi_2(\Sigma_2) = 0$, any homotopy class of maps $f: \Sigma_1 \to \Sigma_2$ contains a harmonic representative. Here, a harmonic map ϕ is a smooth critical point of the energy integral

$$E(\phi) := \frac{1}{2} \int_{\Sigma_1} | d\phi |^2 d\Sigma_1 .$$

Here, as usual, the differential $d\phi$ is considered as a section of $T^*\Sigma_1 \otimes \phi^{-1}T\Sigma_2$, and the norm stems from the natural inner product on the fibers of this bundle.

If $z = x + iy$ and $u = u^1 + iu^2$ are local conformal parameters on Σ_1 and Σ_2, resp., and the metric tensor of Σ_2 is given by

$$\rho^2 du d\bar{u} ,$$

then $u(z)$ is harmonic if and only if

$$\tau(u) := u_{z\bar{z}} + \frac{2\rho_u}{\rho} u_z u_{\bar{z}} = 0 . \tag{1}$$

We have the associated quadratic differential

$$\begin{aligned} \omega &:= (| u_x |^2 - | u_y |^2 - 2i \langle u_x, u_y \rangle) dz^2 \\ &= 4\rho^2 u_z \bar{u}_z dz^2 \end{aligned}$$

We calculate

$$\omega_{\bar{z}} = \rho^2 (\bar{u}_z \tau(u) + u_z \bar{\tau}(u)) . \tag{2}$$

Thus, if u is harmonic,

$$\omega_{\bar{z}} = 0 . \tag{3}$$

Conversely, (3) was used as a definition of harmonicity by Gerstenhaber-Rauch [*4*], and this definition was also adopted by Shibata [*11*]. It was accompanied by some misfortunes. Whereas the program of Gerstenhaber-Rauch is incomplete (and does not seem to be ever completable, since the metric they are seeking probably is singular in general), the paper of Shibata even is outrightly wrong ([1]), and the result he was claiming could only

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([1]) Cf. [*10*] for an examination of Shibata's paper.

be proved twenty years later by completely different methods, see [7].

On the other hand, (3) implies (1), i. e. the nowadays standard definition of harmonic maps, at all points where

$$|u_z|^2 - |u_{\bar{z}}|^2 \neq 0,$$

i. e. where the Jacobian of u does not vanish (at least in case $u \in C^2$).

Now, in contrast to the existence result of Lemaire and Sacks-Uhlenbeck which is proved by variational methods, the case where $\pi_2(\Sigma_2) \neq 0$ presents serious difficulties for a variational procedure since the limit of a minimizing sequence might fall out of the considered homotopy class. Actually, it was even shown by Eells-Wood [3] that there is no harmonic map of degree 1 from a two-dimensional torus T^2 onto the two-sphere S^2.

Also, it was asked by K. Uhlenbeck in [12] whether one can produce conformal branched coverings of S^2 by a variational method.

In the present note we exhibit a variational procedure by which we are able to obtain a map satisfying (3) instead of (1) in any prescribed homotopy class of maps between oriented surfaces. Two cases deserve more discussion. First, we can produce in particular conformal branched coverings of S^2. Since on the other hand, we minimize the energy only in an *a priori* restricted subclass of $H_2^1 \cap C^0(\Sigma_1, \Sigma_2)$, this may not be the answer to Karen Uhlenbeck's question she had in mind. Secondly, if $\Sigma_1 = T^2$ and $\Sigma_2 = S^2$, then our map, in spite of solving (3), cannot be harmonic, because of the non-existence result of Eells-Wood. In order to explain this phenomenon, we construct an explicit example of a map $\phi : T^2 \to S^2$ of degree one which is Lipschitz continuous and solves (3) but not (1).

By slightly modifying this example we can also provide a negative answer to the following question of Sealey and Eells-Lemaire (Problem 2.6 in [2]): if ϕ is a continuous map of finite energy between surfaces with positive Jacobian almost everywhere for which $|\phi_x|^2 - |\phi_y|^2 - 2i\langle \phi_x, \phi_y \rangle$ is holomorphic, is ϕ harmonic? The negative answer to this question also partly explains why Shibata's approach failed.

For some details of the arguments, we frequently refer to [6].

We also make occasional use of conformal automorphisms of Σ_1, in case Σ_1 is homeomorphic to S^2. In order to make the exposition self-contained, one just has to apply our method first to produce a conformal diffeomorphism from S^2 onto Σ_1 and then use the Möbius transformations of S^2.

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1. We first treat the case where Σ_1 is a sphere. In this case, we can also assume that Σ_2 is a sphere, since otherwise all maps $\Sigma_1 \to \Sigma_2$ are homotopically trivial.

Thus, let Σ_1 and Σ_2 be two-dimensional Riemannian manifolds homeomorphic to the standard two-sphere S^2. We distinguish three points p_1, p_2, p_3 on Σ_1 and call a closed topological disc on Σ_1 with smooth boundary small if it contains at most one of these three points. For $\varepsilon > 0$, we define the following class of mappings

$\Gamma(\varepsilon) := \{ \phi \in H^1_2 \cap C^0(\Sigma_1, \Sigma_2) : J(\phi)(p) \geq 0$ for almost all $p \in \Sigma_1$, where $J(\phi)$ is the functional determinant of ϕ, and if $G \subset \Sigma_1$ is small and

$$\phi(\partial G) \subset U(m, \varepsilon) := \{ q \in \Sigma_2 : d(m, q) < \varepsilon \},$$

then also $\phi(G) \subset U(m, \varepsilon) \}$.
Here, $d(.,.)$ is the distance function on Σ_2.

If α is a homotopy class of mappings of positive degree, then we denote by $\Gamma_\alpha(\varepsilon)$ the intersection of $\Gamma(\varepsilon)$ with α, and it is easily seen that for sufficiently small ε, $\Gamma_\alpha(\varepsilon) \neq \phi$. Likewise, if we look at a homotopy class of mappings of negative degree, we require that the mappings in $\Gamma(\varepsilon)$ have nonpositive instead of nonnegative functional determinant almost everywhere.

We wish to minimize the energy integral

$$E(\phi) := \frac{1}{2} \int_{\Sigma_1} | d\phi |^2 d\Sigma_1$$

in $\Gamma_\alpha(\varepsilon)$. Here, as usual the differential $d\phi$ is considered as a section of $T^*\Sigma_1 \otimes \phi^{-1} T\Sigma_2$, and the norm stems from the natural inner product on the fibers of this bundle.

For $K > 0$, let

$$\Gamma_\alpha(\varepsilon, K) := \{ \phi \in \Gamma_\alpha(\varepsilon) : E(\phi) \leq K \}.$$

Of course, $\Gamma_\alpha(\varepsilon, K) \neq \phi$ for sufficiently large K.

LEMMA 1. — $\Gamma_\alpha(\varepsilon, K)$ *is equicontinuous.*

Lemma 1 is an immediate consequence of the definition of $\Gamma_\alpha(\varepsilon, K)$ and the following version of the well-known Courant-Lebesgue Lemma:

LEMMA 2. — *Let* Ω *be an open subset of* $\mathbb{R}^2$, $u \in H^1_2(\Omega, S)$, *where* S *is any Riemannian manifold,*

$$\int_\Omega | du |^2 dx \leq D, \qquad x_0 \in \Omega, \qquad \delta < 1.$$

Then there exists some $r \in (\delta, \sqrt{\delta})$ *for which* $u \mid \partial B(x_0, r) \cap \overline{\Omega}$ *is continuous and*

$$d(u(x_1), u(x_2)) \leq \pi D^{\frac{1}{2}} \left(\log \frac{1}{\delta} \right)^{-\frac{1}{2}}$$

for all $x_1, x_2 \in \partial B(x_0, r) \cap \overline{\Omega}$.

(Of course, $B(x_0, r) := \{ x \in \mathbb{R}^2 : | x - x_0 | \leq r \}$).

Lemma 1 implies that in particular an energy minimizing sequence in $\Gamma_\alpha(\varepsilon)$ is equicontinuous, and hence after selection of a subsequence, converges uniformly to a continuous map u in α. Since H_2^1 is weakly compact, this sequence also has to converge weakly in H_2^1 to u, and by lower semicontinuity of the energy under weak convergence, u minimizes the energy in $\Gamma_\alpha(\varepsilon)$.

We wish to show that u is conformal (and hence a branched covering).

We only indicate the proof and refer to the author's notes [*6*] for details. Let ρ_t be a smooth family of diffeomorphisms of Σ_1, $\rho_0 = d$. Let k_t be a conformal automorphism of Σ_1 with $k_t(p_i) = \rho_t(p_i)$, $i = 1, 2, 3$. Then $u \circ \rho_t^{-1} \circ k_t$ is also in $\Gamma_\alpha(\varepsilon)$ and hence a valid comparison map. Exploiting

$$\frac{d}{dt} \mathrm{E}(u \circ \rho_t^{-1} \circ k_t) |_{t=0} = 0,$$

we infer first as in [*6*], 3.3 that

$$\omega := |u_x|^2 - |u_y|^2 - 2i \langle u_x, u_u \rangle$$

where $z = x + iy$ is a local isothermal parameter on Σ_1, is a holomorphic quadratic differential, and then

$$\omega \equiv 0, \tag{4}$$

by Liouville's theorem, since Σ_1 is conformally S^2.

Since identifying ω as a holomorphic quadratic differential will be of importance later on as well, let us quickly sketch the corresponding argument of [*6*], 3.3. We put

$$\rho_t^{-1} \circ k_t =: \xi + i\eta$$

$$\frac{\partial}{\partial t} (\rho_t^{-1} \circ k_t) |_{t=0} =: v + i\mu$$

and calculate

$$\begin{aligned}\mathrm{E}(u \circ \rho_t^{-1} \circ k_t) = \frac{1}{2} \int \{ & |u_x|^2(\xi_y^2 + \eta_y^2) \\ & - 2 \langle u_x, u_y \rangle (\xi_x \xi_y + \eta_x \eta_y) \\ & + |u_y|^2(\xi_x^2 + \eta_x^2) \} (\xi_x \eta_y - \xi_y \eta_x)^{-1} dxdy,\end{aligned}$$

hence (using $\rho_0^{-1} \circ k_0 = id$)

$$\begin{aligned}0 = \frac{d}{dt} \mathrm{E}(u \circ \rho_t^{-1} \circ k_t) |_{t=0} &= \int (|u_x|^2 - |u_y|^2)(v_x - \mu_y) \\ &\quad + 2 \langle u_x, u_y \rangle (v_y + \mu_x) dxdy \\ &= \mathrm{Re} \int \omega (v + i\mu)_{\bar{z}} dxdy\end{aligned}$$

Since we can use arbitrary smooth μ and v, we see that ω is holomorphic as desired.

Secondly, as in Lemma 3.3 of [6], we see that $J(u) \geq 0$ almost everywhere. For this, we use Lemma 3.2 of [6] which is due to Moorey. It reads

LEMMA 3. — *Let $\phi \in C^0 \cap H_2^1(G, \mathbb{R}^2)$, where G is a twodimensional domain. For every $z_0 \in G$ there exists a set $C(z_0)$ with $H^1(C(z_0)) = 0$, with the property that for all $R \notin C(z_0)$ with $B(z_0, R) cc G$*

$$\int_{B(z_0,R)} J(\phi) dz = \int_{\phi(B(z_0,\mathbb{R}))} m(w, \phi(\partial B(z_0, R)) dw$$

where $m(w, \phi(\partial B(z_0, R)))$ is the winding number of the curve $\phi(\partial B(z_0, R))$ w. r. t. the point w.

We apply this in the following way: Let u_n be a minimizing sequence in $\Gamma_\alpha(\varepsilon)$, let $B(z_0, R)$, $z_0 \in \Sigma_1$, $R \notin C(z_0)$ satisfy the assumptions of Lemma 3 for u and all u_n,

$$\varepsilon_n := \max_{z \in \partial B(z_0,R)} | u_n(z) - u(z) |$$

$$V_n := \{ w : d(w, u(\partial B(z_0, R))) > \varepsilon_n \}$$

For $w \in V_n$, $m(w, u_n(\partial B(z_0, R))) = m(w, u(\partial B(z_0, R)))$. Then, since u_n converges uniformly to u, using Lemma 3

$$\begin{aligned}\lim_{n\to\infty} \int_{B(z_0,R)} J(u_n) dz &= \lim_{n\to\infty} \int_{u_n(B(z_0,R))} m(w, u_n(\partial B(z_0, R))) dw \\ &= \lim_{n\to\infty} \int_{u_n(B(z_0,R))\cap V_n} m(w, u_n(\partial B(z_0, R))) dw \\ &= \lim_{n\to\infty} \int_{u(B(z_0,R))\cap V_n} m(w, u(\partial B(z_0, R))) dw \\ &= \int_{u(B(z_0,R))} m(w, u(\partial B(z_0, R))) dw = \int_{B(z_0,R)} J(u) dz\end{aligned}$$

Since $J(u_n) \geq 0$ and the preceding argument is valid for almost all disks $B(z_0, R)$, we inser $J(n) \geq 0$ as desired.

Combining this with (4), we conclude that u is weakly conformal, i. e. in local coordinates (u^1, u^2) on Σ_2, if (g_{ij}) is the corresponding metric tensor and $g := g_{11} g_{22} - g_{12}^2$,

$$\begin{aligned} u_x^2 &= - g_{22}^{-1}(g_{12} u_x^1 + \sqrt{g} u_y^1) \\ u_y^2 &= g_{22}^{-1}(\sqrt{g} u_x^1 - g_{12} u_y^1) \end{aligned} \tag{5}$$

almost everywhere (more precisely, u solves (5) weakly).

Since (5) is a first-order linear elliptic system, elliptic regularity theory implies that u is regular and solves (5) everywhere. *Thus, u is conformal.* That it is *a branched covering* can be proved in a rather elementary manner with the help of the Hartman-Wintner Lemma as in Lemma 3.4 of [6].

2. If on the other hand, Σ_1 is topologically different from a sphere, we cannot and need not fix three points anymore for defining our class of mappings, since in this case a disc and its complement on Σ_1 are already topologically different. Furthermore, since we are not seeking conformal maps anymore, we don't have to require either that the Jacobian does not change sign (which would not make sense anyway for a nonorientable Σ_1). Hence we define

$$\Delta(\varepsilon) := \{ \phi \in H_2^1 \cap C^0(\Sigma_1, \Sigma_2) :$$

if D is a disc on Σ_1 with $\phi(\partial D) \subset U(m, \varepsilon)$, then also $\phi(D) \subset U(m, \varepsilon)$.

As before, $\Delta_\alpha(\varepsilon)$ is the intersection of $\Delta(\varepsilon)$ with the homotopy class α.

We then minimize the energy as before in $\Delta_\alpha(\varepsilon)$ and obtain a minimum u in α which is stationary under composition with diffeomorphisms of the domain. Hence,

$$\omega = (| u_x |^2 - | u_y |^2 - 2i \langle u_x, u_y \rangle) dz^2$$

again is a holomorphic quadratic differential (cf. [*6*], 3.3), and thus u solves (3). Hence

THEOREM. — *Let Σ_1 and Σ_2 be compact two-dimensional Riemannian manifolds. Then any continuous map $\phi : \Sigma_1 \to \Sigma_2$ is homotopic to a map u for which*

$$(| u_x |^2 - | u_y |^2 - 2i \langle u_x, u_y \rangle) dz^2$$

is a holomorphic quadratic differential.

3. Contrasting this result with the non-existence result for harmonic maps from T^2 onto S^2 one might guess first that in this case $\Delta_\alpha(\varepsilon)$ is empty so that the proof might not be valid after all.

This is not the case, however, as the following construction shows: Let Z_1 be a circular cylinder with circumference a_1 and height b_1 and boundary circles γ_{11} and γ_{12}. Let p_1 and p_2 be the north and south pole of S^2, resp. Take a differentiable map $\psi_1 : Z_1 \to S^2$ mapping γ_{11} onto p_1 and γ_{12} onto p_2 and the interior of Z_1 diffeomorphically onto $S^2 \setminus \{ p_1, p_2 \}$. Let Z_2 be another cylinder with the same circumference as Z_1 and boundary circles γ_{21} and γ_{22}. Let $\psi_2 : Z_2 \to S^2$ map γ_{21} onto p_1, γ_{22} onto p_2 and Z_2 onto a geodesic arc from p_1 to p_2. Identifying γ_{11} with γ_{21} and γ_{12} with γ_{22}, we obtain a map ψ from a torus onto S^2 of degree 1 which obviously lies in some class $\Delta(\varepsilon)$ for suitable ε.

In order to resolve the puzzle, we now want to exhibit an example of a map from T^2 onto S^2 of degree 1 which satisfies (3) but is not harmonic.

By a theorem of the author [*5*] (a similar result was obtained by Brezis-Coron [*1*], both maps $\psi_i : Z_i \to S^2$ $(i = 1, 2)$ are homotopic to harmonic maps ϕ_i with the same boundary values. Since the boundary values are constant, ϕ_i is a stationary point of the energy with respect to composition with any diffeomorphism of Z_i, not necessarily leaving ∂Z_i fixed. Hence (cf. [*6*], 3.3), the corresponding holomorphic quadratic differential ω_i is real on ∂Z_i and therefore constant.

Let

$$\omega_1 \equiv c \in \mathbb{R} .$$

Note that $c \neq 0$, since ϕ_1 cannot be conformal, and that we can also prescribe the sign of c by composing ϕ_1 with a reflection of Z_1 across a plane containing its axis, if necessary.

If b_2 is the height of Z_2 then it is easily seen that

$$\omega_2 \equiv \left(\frac{2\pi}{b_2}\right)^2 .$$

Thus, for a suitable choice of b_2,

$$\omega_1 \equiv \omega_2 ,$$

and the map ϕ patched together from ϕ_1 and ϕ_2 satisfies (5) and is Lipschitz continuous, but not harmonic.

Actually, Sealey and Eells-Lemaire asked the following question (Problem 2.6 of [*2*]): if ϕ is a continuous map of finite energy between surfaces with positive Jacobian almost everywhere for which

$$| \phi_x |^2 - | \phi_y |^2 - 2i \langle \phi_x, \Phi_y \rangle$$

is holomorphic, if ϕ harmonic?

Of course, the answer is yes, if ϕ is of class C^2 or if ϕ is a diffeomorphism of class C^1.

In general, however, the answer is no which can be seen as follows: we take two copies Z_1^1 and Z_1^2, ϕ_1^1 and ϕ_1^2 of the cylinder Z_1 and the harmonic map ϕ_1 constructed before. Identifying the boundaries of Z_1^1 and Z_1^2 via identifying equal angles in standard polar coordinates, we get a flat torus T, and using ϕ_1^1 and ϕ_1^2 on the resp. component, a map ϕ from T onto S^2. If we identify the upper boundary of Z_1^1 with the lower boundary of Z_1^2 and *vice versa*, then the holomorphic quadratic differential associated with ϕ has the same sign on both components and is therefore constant (if we would have identified the boundaries in such a way that a sign change of this differential occurs when passing the boundaries, we could also have remedied this defect alternatively by composing e. g. ϕ_1^2 with a reflection of Z_1^2 across a plane containing its axis as before). If ϕ actually should turn out to be harmonic, then we can compose e. g. ϕ_1^2 with any rotation (not equal to a multiple of 2π) of Z_1^2 around its axis, to get another map from T onto S^2 with constant (and hence holomorphic) associated quadratic differential. This map then can be no more harmonic, for example since harmonic maps are real analytic (as T and S^2 are) and hence determined by their local values, here e. g. on Z_1^1. Alternatively, the new map is no more of class C^2 which a continuous harmonic map would have to be.

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Added in proof: regarding [4], one should also note E. REICH, on the variational principle of Gerstenhaber and Rauch, *Ann. Acad. Sci. Fenn.*, to appear.