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Non-collision orbits for a class of Keplerian-like potentials

by

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ABSTRACT. — We prove the existence of non-collision orbits with large period for a class of Keplerian-like dynamical systems.

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RÉSUMÉ. — Nous prouvons l'existence d'orbites de non-collision avec grand period pour une classe de systèmes dynamiques de type keplerien.

0. INTRODUCTION

In this paper we study the existence of T-periodic solutions for a system of ordinary differential equations of the form

$$-\ddot{y}=V'(y), \tag{1}$$

where V is a Keplerian-like potential, i. e. $V(x)$ behaves like $-|x|^{-\alpha}$ for x close to 0, α being any real number greater than 0. We prove that, for large T, such a system has a non-collision T-periodic solution (i. e. a solution which does not cross the origin) under the only assumption that V attains its maximum on the boundary of an open set which contains the origin.

A potential of such kind arises, for example, if at $x=0$ there are z positive charges sorrounded by $z+k\,(k>0)$ negative ones uniformly distributed on a shell containing $x=0$. Then $V(x)=-z/|x|$ inside the shell, while $V(x)=k/|x|$ at infinity.

The existence of periodic solutions of (1) when $V'\approx-|x|^{-\alpha}$ and $\alpha\geqq 2$ (or, more precisely, the case of strong forces—*see* [6] for a definition) has been studied in [1], [3], [6], [8], *see* also [2] for a review of the results in this and related fields. We notice that in such a case all the periodic solutions are non-collision orbits.

The situation is much more complicated when $\alpha\leqq 2$. For some partial results for $\alpha>1$ we refer to [5] (*see* also [4] for a somewhat different class of potentials). In particular, the results of [5] do not cover the case $\alpha=1$ ([3]), which is known to be quite degenerate. For example, if

([3]) Some results are found also when $\alpha=1$ but under other symmetry conditions.

$V(x)=-|x|^{-1}$, then the T-periodic solutions belong to one parameter families containing collision solutions and all the orbits of each family have the same value of the energy and the same value of the action functional [7].

Actually, in the present paper, we show that Kepler's potential is very sensitive to perturbation, at least in the sense that even a very small perturbation far from the singular set (if it goes in the "right" direction) can assure the existence of non-collision orbits.

The results proved here have been announced in the *C.R. Acad. Sci. Paris* note [0].

1. ASSUMPTIONS AND MAIN EXISTENCE RESULTS

We consider a potential $V \in C^1(\mathbb{R}^N \setminus \{0\}, \mathbb{R})$ satisfying

(V1) $V(x) \to -\infty$ as $|x| \to 0$;

(V2) there exists an open, bounded set $\Omega \subset \mathbb{R}^N$, with smooth boundary Γ such that

(i) $0 \in \Omega$ and Ω is star-shaped with respect to 0;

(ii) letting $b = max\{V(x) : x \in \mathbb{R}^N \setminus \{0\}\}$, one has that $b = V(\xi)$, $\forall \xi \in \Gamma$;

(V3) $\limsup_{|x| \to +\infty} V(x) = \beta < b$.

Given $T > 0$ we look for solutions of

$$-\ddot{y} = V'(y), \qquad y(0) = y(T), \qquad \dot{y}(0) = \dot{y}(T) \qquad (P_T)$$

where V' denotes the gradient of V.

We say that a solution $y(t)$ of (P) is a non-collision orbit if $y(t) \neq 0$, $\forall t$.

Let $S^1 = [0, 1]/\{0, 1\}$, $H = H^1(S^1; \mathbb{R}^N)$, and $\Lambda = \{y \in H : y(t) \neq 0, \forall t\}$.

We denote by $\|u\|_1^2 = \int |\dot{u}|^2 + \int |u|^2$ (4) the norm in H.

Define $f_T : H \to \mathbb{R} \cup \{+\infty\}$ by setting

$$f_T(u) = \frac{1}{2}\int |\dot{u}(t)|^2\, dt - T^2 \int V(u(t))\, dt.$$

(4) From now on we will assume that each integral is taken from 0 to 1.

[where $V(0) = -\infty$].

Then $f_T \in C^1(\Lambda; \mathbb{R})$ and, if $u \in \Lambda$ and $f'_T(u) = 0$ then $y(t) = u(t/T)$ is a non-collision solution of (P_T).

THEOREM 1. — *Suppose* $V \in C^1(\mathbb{R}^N \setminus \{0\}; \mathbb{R})$ *satisfies* (V1), (V2) *and* (V3). *Then* $\exists T^*$ *such that* $\forall T \geqq T^*$ *problem* (P) *has at least one non-collision solution* x *such that* $\{x(t)\} \not\subset \Gamma$.

2. ESTIMATES FOR THE MINIMUM OF f_T ON COLLISION ORBITS

It is easy to show that it exists a function $\psi : \mathbb{R}^+ \to \mathbb{R}$ of class C^1 such that: (i) $\psi(s) \to -\infty$ as $s \to 0^+$; (*ii*) *max* $\psi = b$; (iii) $V(x) \leqq \psi(|x|)$, $\forall x \in \mathbb{R}^N \setminus \{0\}$; (iv) ψ is not decreasing. Let $g_T : H^1(0, 1; \mathbb{R}^+) \to \mathbb{R}$ be defined by

$$g_T(r) = \frac{1}{2}\int |r'(t)|^2\, dt - T^2 \int \psi(r(t))\, dt.$$

Consider now $u \in H$. Setting $r(t) = |u(t)|$, one has $r \in H^1(0, 1; \mathbb{R}^+)$ and

$$\int |r'|^2 = \int |\langle \dot{u}, u/|u| \rangle|^2 \leqq \int |\dot{u}|^2.$$

Then

$$f_T(u) \geqq \frac{1}{2}\int |r'|^2 - T^2 \int V(u) \geqq \frac{1}{2}\int |r'|^2 - T^2 \int \psi(r) = g_T(r).$$

Moreover, if $u \in H \setminus \Lambda$ there exists a $\theta \in [0, 1[$ such that $|u(t+\theta)| \in H^1_Q(0, 1; \mathbb{R}^+)$. Hence

LEMMA 2. — $m_T = \inf\{f_T(u) : u \in H \setminus \Lambda\} \geqq \inf\{g_T(r) : r \in H^1_0(0, 1; \mathbb{R}^+)\}$. □

LEMMA 3. — g_T *attains its minimum on* $H^1_0(0, 1; \mathbb{R}^+)$.

Proof. — Trivial since g_T is coercive and weakly lower semi continuous. □

LEMMA 4. — *$\exists c, \tau>0$ such that $\forall \mathrm{T}>\tau$*

$$\min\{g_{\mathrm{T}}(r): r\in \mathrm{H}_0^1(0,\,1;\,\mathbb{R}^+)\}\geqq c\mathrm{T}-b\mathrm{T}^2.$$

Proof. — Let r_{T} be such that $g_{\mathrm{T}}(r_{\mathrm{T}})=\min\{g_{\mathrm{T}}(r): r\in \mathrm{H}_0^1(0,\,1;\,\mathbb{R}^+)\}$. Then set

$$\mathrm{T}^2\,\mathrm{E}_{\mathrm{T}}\equiv\frac{1}{2}|r_{\mathrm{T}}'(\mathrm{t})|^2+\mathrm{T}^2\,\psi(r_{\mathrm{T}}(t)).$$

From the conservation of energy it follows that $\mathrm{T}^2\,\mathrm{E}_{\mathrm{T}}$ is a constant of the motion. Fix now T_0 and correspondingly $r_0=r_{\mathrm{T}_0}$ and $\mathrm{E}_0=\mathrm{E}_{\mathrm{T}_0}$. We claim that $\mathrm{E}_0<b$. In fact, since $\exists t_0$ such that $r'(t_0)=0$ [we recall that $r_0(0)=r_0(\mathrm{T})=0$], then $\mathrm{T}_0^2\,\mathrm{E}_0=\mathrm{T}_0^2\,\psi(r_0(t_0))\leqq\mathrm{T}_0^2\,b$, hence $\mathrm{E}_0\leqq\mathrm{b}$. If $\mathrm{E}_0=b$, then $\psi'(r_0(t_0))=0$ and $r_0(t)=r_0(t_0)>0$, $\forall t$, contradiction which proves the claim.

Take now any $\mathrm{T}>\mathrm{T}_0$. Distinguish between: (i) $\mathrm{E}_{\mathrm{T}}\leqq\mathrm{E}_0$ and (ii) $\mathrm{E}_{\mathrm{T}}>\mathrm{E}_0$. If (i) holds, from $\mathrm{T}^2\,\psi(r_{\mathrm{T}})\leqq\mathrm{T}^2\,\mathrm{E}_{\mathrm{T}}$ and $\mathrm{E}_{\mathrm{T}}\leqq\mathrm{E}_0$ it follows that

$$g_{\mathrm{T}}(r_{\mathrm{T}})\geqq-\mathrm{T}^2\int\psi(r_{\mathrm{T}})\geqq-\mathrm{T}^2\,\mathrm{E}_0. \tag{2}$$

Suppose now that (ii) holds. Let t_{T} be such that $r_{\mathrm{T}}'(t_{\mathrm{T}})=0$ and $r_{\mathrm{T}}'(t)>0$ $\forall t\in[0,\,t_{\mathrm{T}}[$. Set $\rho_{\mathrm{T}}=r_{\mathrm{T}}(t_{\mathrm{T}})$. From $\mathrm{E}_{\mathrm{T}}>\mathrm{E}_0$ and the monotonicity of ψ it follows that $\rho_{\mathrm{T}}\geqq\rho_0\equiv r_0(t_0)$. Since $r_{\mathrm{T}}'(t)>0$, $\forall t\in[0,\,t_{\mathrm{T}}[$, we can solve $r_{\mathrm{T}}(t)=\rho$ in $[0,\,t_{\mathrm{T}}[$ to get $t=\tau_{\mathrm{T}}(\rho)$ such that $r_{\mathrm{T}}(\tau_{\mathrm{T}}(\rho))=\rho$, $\forall\rho\in[0,\,\rho_{\mathrm{T}}[$. From the conservation of energy we get, since $\mathrm{E}_{\mathrm{T}}>\mathrm{E}_0$

$$(2\mathrm{T}^2)^{-1}\,r_{\mathrm{T}}'(\tau_{\mathrm{T}}(\rho))^2>(2\mathrm{T}_0^2)^{-1}\,r_0'(\tau_0(\rho))^2,\qquad\forall\rho\in[0,\,\rho_0[.$$

We can now evaluate

$$\begin{aligned}\frac{1}{2}\int_0^1|r_{\mathrm{T}}'(t)|^2\,dt&\geqq\frac{1}{2}\int_0^{t_{\mathrm{T}}}|r_{\mathrm{T}}'(t)|^2\,dt\\ =\frac{1}{2}\int_0^{\rho_{\mathrm{T}}}r_{\mathrm{T}}'(\tau_{\mathrm{T}}(\rho))\,d\rho&\geqq\frac{1}{2}\,\frac{\mathrm{T}}{\mathrm{T}_0}\int_0^{\rho_0}r_0'(\tau_0(\rho))\,d\rho\end{aligned} \tag{3}$$

Set

$$c=\frac{1}{2\,\mathrm{T}_0}\int_0^{\rho_0}r_0'(\tau_0(\rho))\,d\rho.$$

Then $c>0$ and from (3) and $\psi(r)\leqq b$ it follows

$$g_T(r_T)\geqq c\,T-b\,T^2, \qquad \forall\, T>T_0 \tag{4}$$

(2) and $E_0<b$ jointly with (4) prove the Lemma. □

3. PROOF OF THE THEOREM

We start by showing

LEMMA 5. — $\forall\,\varepsilon>0$, f_T satisfies the PS *condition in the set*

$$\{x\in\Lambda : f_T(u)\leqq\alpha_T-\varepsilon\},$$

where $\alpha_T=\min\{m_T, {}_{-\beta T}{}^2\}$.

Proof. — Let $(u_n)\subset\Lambda$ be such that

$$f_T(u_n)\leqq\alpha_T-\varepsilon, \qquad f'_T(u_n)\to 0.$$

Then $\frac{1}{2}\int|u'_n|^2\leqq$Const., hence, setting $w_n=u_n-\int u_n$, $w_n\to w$ in $C^0(S^1;\mathbb{R}^N)$.

Suppose, by contradiction, that $\xi_n=\int u_n\to+\infty$. Then $|u_n(t)|\to+\infty$ uniformly and using (V3)

$$f_T(u_n)\geqq-\beta\,T^2-\varepsilon/2 \qquad \text{for} \quad n \text{ sufficiently large},$$

contradiction which proves the boundedness of $\|u_n\|_1$. We immediately deduce that $u_n\to u$ strongly in $C^0(S^1;\mathbb{R}^N)$ and weakly in $H^1(S^1;\mathbb{R}^N)$. Moreover, from the weakly lower semi-continuity of f_T, we deduce

$$f_T(u)\leqq\liminf_{n\to+\infty} f_T(u_n)\leqq\alpha_T-\varepsilon<m_T,$$

hence $u\in\Lambda$. Usual arguments then prove that $u_n\to u$ in H^1. □

Proof of Theorem 1. — Consider the set of functions
$$\Sigma=\{x\in H \text{ such that } x(t)=\xi\cos(2\pi t)+\eta\sin(2\pi t)+x_0,$$
$$\xi,\eta,x_0\in\mathbb{R}^N,\ |\xi|=|\eta|\leqq 1,$$
$$\langle\xi,\eta\rangle=\langle\xi,x_0\rangle=\langle\eta,x_0\rangle=0,\ |x_0|^2=1-|\xi|^2-|\eta|^2\}.$$

Then, $\forall x \in \Sigma$, $|x(t)| = 1$, $\forall t$.

If $\Phi : S^{N-1} \to \Gamma$ is the radial projection [which is a diffeomorphism by (V2)], set

$$\Sigma' = \{ \Phi(x) \text{ such that } x \in \Sigma \}.$$

We have that:

$$\begin{aligned} f_T(u) &= \frac{1}{2} \int |\Phi'(u(t))|^2 |\dot{u}(t)|^2 - T^2 b \\ &\leqq c_1 \int |\dot{u}(t)|^2 - T^2 b \qquad (c_1 = \sup_{x \in S^{N-1}} |\Phi'(x)|^2) \\ &\leqq 4\pi^2 c_1 - T^2 b. \end{aligned} \tag{5}$$

By Lemmas 2, 3 and 4

$$m_T \geqq c\,T - b\,T^2, \qquad \forall T > \tau. \tag{6}$$

From (5) and (6) it follows: $\exists \tau_1$ such that

$$f_T(\Sigma') < m_T, \qquad \forall T \geqq \tau_1. \tag{7}$$

Moreover, since $\beta < b$, one has

$$f_T(\Sigma') < -\beta T^2,$$

and hence

$$f_T(\Sigma') < \alpha_T = \min \{ m_T, -\beta T^2 \}. \tag{8}$$

One can now proceed as in the proof of the theorem by Lyusternik and Fet on the existence of one closed geodesic on a compact Riemannian manifold (*see* [9], Theorem A.1.5). In fact, letting $\varepsilon > 0$ be such that $f_T(\Sigma') < \alpha_T - \varepsilon$, we can work in the set $\{ u: f_T(u) \leqq \alpha_T - \varepsilon \}$, where the PS condition holds according to Lemma 5. Since the minimum on such a set is achieved on $\{ x \in \mathbb{R}^N : V(x) = b \}$, set which is homeomorphic (through Φ) to S^{N-1}, the existence of a critical point u such that $-T^2 b < f_T(u) < \alpha_T$ follows. Lastly, if such a critical point is such that $u(t) \in \Gamma$, $\forall t$, then for the corresponding solution $y(t) = u(t/T)$ one would find $y(t) = y_0$. Hence one finds $f_T(u) = -T^2 b$, a contradiction. This completes the proof. □

4. FINAL REMARKS

PROPOSITION 6. — *Let* $N=2$ *and* $V \in C^1(\mathbb{R}^N \setminus \{0\}, \mathbb{R})$ *satisfy* (V1-2). *Then*

(i) $\exists T^*$ *such that* $\forall T > T^*$, (P_T) *has a non constant solution* $y(t)$ *which is a non-collision orbit;*

(ii) *if* Ω *is convex, then* $y(t) \in \Omega$, $\forall t$.

Proof. — The only point where (V3) has been used is in proving Lemma 5. If $N=2$, this can be avoided using Lemmas 2-4 jointly with the arguments of [6]. We will be sketchy here. Let $\Lambda_0 = \{u \in \Lambda: u \text{ is non-contractible to a constant in } \Lambda\}$. It is possible to show that f_T is (bounded from below on H and) coercive on Λ_0. Since $\Sigma \subset \Lambda_0$, (7) implies that $\inf\{f_T(u): u \in \Lambda_0\} < m_T$ for T large. Then it follows that $\exists u_0 \in \Lambda_0: f_T(u_0) = \min\{f_T(u): u \in \Lambda_0\}$. This proves (i).

As for (ii), consider

$$U(x) = \begin{cases} V(x), & \forall x \in \Omega \\ b, & \forall x \in \mathbb{R}^2 \setminus \Omega. \end{cases}$$

U is of class C^1. Applying (i) above we find a T-periodic solution of

$$-\ddot{y} = U'(y)$$

with $y \in \Lambda_0$. It follows easily that such a solution must be contained in Ω for every t (in fact, if it hits the boundary it must be a straight line in the past or in the future, so that it cannot be periodic). □

Remark 7. — By a suitable modification of Lemma 5, it would be possible to show that Proposition 6 (ii) holds even if $N > 2$.

PROPOSITION 8. — *Let the assumptions of Theorem* 5 *be satisfied. Then for every compact set* $K \subset \Omega$, $\exists T_0$: $\forall T \geqq T_0$, (P_T) *has a solution* y_T *with* $\{y_T(t)\} \not\subset K$.

Proof. — From the proof of Theorem 5 we know that for T large enough f_T has a critical point u_T such that

$$f_T(u_T) < 4\pi^2 c_1 - b T^2. \tag{9}$$

If there is a compact set $K \subset \Omega$ such that $u_T(t) \in K$, $\forall t$, one would have

$$f_T(u_T) \geqq -T^2 \int V(u_T) \geqq -T^2 \max_K V.$$

Since $\max_K V < b$, this is in contradiction with (9) for T large. □

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