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On embedded minimal disks in convex bodies

by

M. GRÜTER and J. JOST
Centre for Mathematical Analysis
Hanna Neumann Building, Australian National University
GPO Box 4, Canberra ACT 2601, Australia

ABSTRACT. — If $A \subset \mathbb{R}^3$ is a convex body we prove the existence of an embedded minimal disk $M \subset A$ meeting ∂A orthogonally.

RÉSUMÉ. — Si $A \subset \mathbb{R}^3$ est un ensemble convexe, nous prouvons l'existence d'une sous-variété minimale $M \subset A$ du type disque, intersectant ∂A orthogonalement.

INTRODUCTION

Let A be a bounded open strictly convex subset of $\mathbb{R}^3$ with boundary ∂A of class C^4.

In the present paper, we consider the free boundary value problem for minimal surfaces in A. This means that we seek a minimal surface $M \subset \overline{A}$ whose interior is contained in A and whose boundary ∂M is contained in ∂A which is stationary (for the area integral) with respect to all variations preserving the inclusion $\partial M \subset \partial A$. This implies in particular that M has to meet ∂A orthogonally.

Our result is

THEOREM. — *There exists an embedded minimal disk* M *in* $\overline{A}$ *solving the free boundary value problem.*

Liste de mots-clés : Minimal surfaces, free boundary problems, geometric measure theory, minimaxing procedure.

Classification A. M. S. : 49 F 10, F 20, F 22, 53 A 10, 58 E 12.

Our proof has several ingredients:

a) The minimaxing methods of Pitts [P] are used to connect two distinct points on ∂A by a sequence of disks meeting ∂A transversally. We obtain a minimaxing varifold which has a certain almost minimizing property in the sense of Pitts [P] and Simon-Smith [SS].

b) The methods for minimizing among embedded surfaces of Almgren-Simon [AS] and Meeks-Simon-Yau [MSY] are used for local replacement arguments.

c) We use the (easy) extension to free boundaries of the curvature estimates for stable minimal surfaces of Schoen-Simon [SRS] for some compactness arguments.

d) The regularity at the free boundary depends on the companion paper [GJ], where Allard's regularity theorems for stationary varifolds [A1, A2] are extended to solutions of free boundary value problems.

e) Finally, Simon-Smith [SS] showed that any (regular) metric on S^3 admits a minimal embedded two dimensional sphere. Besides using many of their arguments in *a*) and *b*), we shall make use of their paper in an essential way to show that the almost minimizing varifold produced in *a*) and shown to be an embedded minimal surface in *b*), *d*) is actually simply connected, i. e. a disk or a collection of disks.

We remark that our arguments easily generalize to the case where the ambient space is replaced by a three-dimensional Riemmanian manifold of class C^5 and A is a strictly convex ball in this manifold provided there are no minimal embedded spheres in A. We did not include the details, because it was already demonstrated in [P] and [MSY] how to generalize such arguments to manifolds, and also because the present paper is already long enough.

A corresponding parametric problem was recently treated by Struwe [St], using a method of Sacks-Uhlenbeck [SU]. He showed that given an embedded surface S in $\mathbb{R}^3$ of class C^4, diffeomorphic to the standard sphere, there exists a parametric minimal surface $f : D \to \mathbb{R}^3$, where D is the unit disk with $f(\partial D) \subset S$ and meeting S orthogonally. It is not clear, however, whether his solution is embedded (at least if S is strictly convex) or at least immersed. He does not assume that S is convex, but in the general case his solution cannot be confined to lie in the interior of S. For these reasons, we believe that our result captures the physical and geometric essence of the problem better than his.

Finally, it was shown by Smyth [Sm] that if T is a tetrahedron in $\mathbb{R}^3$ (i. e. having a boundary formed by four planar pieces) then there exist three embedded minimal disks meeting T orthogonally. The rather explicit boundary in his problem made it possible to apply arguments of a much more elementary nature than ours.

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§1. THE EXISTENCE OF AN ALMOST MINIMIZING VARIFOLD

Terminology.

A is a bounded open strictly convex subset of $\mathbb{R}^3$, $\partial A \in C^4$, $U \subset \mathbb{R}^3$ open

$$I(U, A) := \{ \psi = \{ \psi_t \}, \psi_t : \mathbb{R}^3 \to \mathbb{R}^3 \text{ isotopy of class } C^1 \ (t \in [0,1]),$$
$$\psi_t \llcorner \mathbb{R}^3 \sim K = \text{id, for some } K \subset\subset U, \psi_t(\partial A) \subset \partial A, \psi_0 = \text{id} \}$$

$$D_r := \{ x \in \mathbb{R}^2 : |x| \leq r \}$$

$$\mathcal{M} := \{ \phi(D_1), \phi : D_1 \to \overline{A} \text{ injective, of class } C^2 \text{ and of maximal rank}$$
$$\phi(D_1) \text{ meets } \partial A \text{ transversally} \}$$

$$\tilde{\mathcal{M}} := \{ N = \overline{\text{int}_M \gamma}, M \in \mathcal{M}, \gamma \text{ piecewise } C^2 \text{ Jordan arc on } M \}$$

$$|\Sigma| := \text{area}(\Sigma).$$

For $\Sigma \in \mathcal{M}$, we define

$$I(\Sigma, U, A, \alpha) := \{ \psi = \{ \psi_t \}_{0 \leq t \leq 1} \in I(U, A) : |\psi_t(\Sigma)| \leq |\Sigma| + \alpha \text{ for all } t \in [0,1] \}.$$

Furthermore, for $\varepsilon > 0$, $\alpha > 0$,

$$S(U, A, \varepsilon, \alpha) := \{ \Sigma \in \mathcal{M} : \text{if } \psi \in I(U, A) \text{ with } |\psi_t(\Sigma)| \leq |\Sigma| + \alpha \text{ for all } t \in [0,1], \text{ then } |\psi_1(\Sigma)| \geq |\Sigma| - \varepsilon \}.$$

Let

$$\mathcal{U} := \left\{ (U_1, U_2) : U_i \subset \mathbb{R}^3 \text{ open}, \quad U_i \cap A \neq 0, \quad \text{vol}(U_i \cap A) < \frac{1}{4} \text{vol } A \right.$$
$$\text{for } i = 1, 2 \quad \text{dist}(U_1, U_2) \geq \min(\text{diam } U_1, \text{diam } U_2) \}.$$

Furthermore, if $U \subset \mathbb{R}^3$ is open, $\sigma > 0$,

$$U^\sigma := \{ x \in U : \text{dist}(x, \partial U) > \sigma \}$$
$$\mathcal{U}^\sigma := \{ (U_1^\sigma, U_2^\sigma) : (U_1, U_2) \in \mathcal{U} \}$$

For V, $W \in V_k(\mathbb{R}^n)$:

$$\underline{\underline{F}}(V, W) = \sup \{ |V(f) - W(f)| : f \in C^{0,1}(G_k(\mathbb{R}^n)), |f| \leq 1, \text{Lip}(f) \leq 1 \}.$$

DEFINITION. — *A varifold* $V \in V_2(A)$ $(:= \{ W \in V_2(\mathbb{R}^3) : \text{spt} \, \|W\| \subset \overline{A} \})$, $V \neq 0$, *is called uniformly almost minimizing among disks relative to* $\mathcal{U}^\sigma$, *if for each* $\varepsilon > 0$ *there is* $\alpha > 0$ *and* $\Sigma \in \mathcal{M}$ *with* $\underline{\underline{F}}(V, \underline{v}(\Sigma)) < \varepsilon$ *and* $\Sigma \in S(U_i^\sigma, A, \varepsilon, \alpha)$ *for at least one* $i \in \{1, 2\}$ *for each* $(U_1^\sigma, U_2^\sigma) \in \mathcal{U}^\sigma$. *We also say, that (for this i)* V *is almost minimizing among disks on* U_i^σ.

Note that for $\sigma_1 < \sigma_2$ the property of uniformly almost minimizing among disks relative to $\mathcal{U}^{\sigma_1}$ implies the same property relative to $\mathcal{U}^{\sigma_2}$.

We also say that $V \in V_2(A)$ is almost minimizing among disks in U (U an open set) if for each $\varepsilon > 0$ there is $\alpha > 0$ and $\Sigma \in \mathcal{M}$ with $\underline{\underline{F}}(V, \underline{\underline{v}}(\Sigma)) < \varepsilon$ and $\Sigma \in S(U, A, \varepsilon, \alpha)$.

In this paragraph, we use the methods of [P, § 4] together with their modifications by [SS] in order to obtain the existence of a varifold which is uniformly almost minimizing among disks.

Let $\Lambda \in \mathcal{M}$.

We consider the set of maps $P(\Lambda)$

$$\phi_t : \bar{A} \to \bar{A} \qquad 0 \leq t \leq 1$$

with

(1) $\phi_0(\bar{A}) = z_0, \quad \phi_1(\bar{A}) = z_1, \quad z_0, z_1 \in \partial A$

(2) $\phi : [0, 1] \times \bar{A} \to \bar{A}, \quad (t, x) \to \phi_t(x)$ is C^1 for $t \in (0, 1)$

(3) ϕ_t is a diffeomorphism of $\bar{A}$ for each $t \in (0, 1)$

(4) $A_1^1 = \bar{A}$, where we have two families (A_t^i), $0 \leq t \leq 1$, $i = 1, 2$, of closed subsets of $\bar{A}$,

uniquely defined via

$$A_0^1 = \{ \tau_0 \}$$

$$A_t^1 \cup A_t^2 = \bar{A}, \qquad A_t^1 \cap A_t^2 = \phi_t(\Lambda)$$

$$\partial A_t^i \cap A = \phi_t(\Lambda) \cap A \qquad (i = 1, 2)$$

and

$$t \to A_t^i \quad \text{is continuous}.$$

Put

$$M := \inf_{P(\Lambda)} \sup_{0 \leq t \leq 1} | \phi_t(\Lambda) |$$

$$\underline{\underline{C}}(\Lambda) := \{ V \in V_2(\bar{A}) : V = \lim_{k \to \infty} \underline{\underline{v}}(\phi_{t_k}^k(\Lambda)), \text{ where } (\phi_{t_k}^k(\Lambda))_{k \in \mathbb{N}} \text{ is a sequence with}$$
$$\phi^k \in P(\Lambda), \quad t_k \in [0, 1], \quad \lim_{k \to \infty} |\phi_{t_k}^k(\Lambda)| = M = \lim_{k \to \infty} \big(\sup_{0 \leq t \leq 1} |\phi_t^k(\Lambda)| \big) \}$$

$\underline{\underline{C}}(\Lambda)$ is the set of critical varifolds.

It follows from the isoperimetric inequality that $M > 0$. Actually

$$M \geq \frac{1}{2} h_A \operatorname{vol}(A)$$

with

$$h_A = \inf \left\{ \frac{| \Sigma |}{\min(\operatorname{vol} S_1, \operatorname{vol} S_2)} : \Sigma \in \mathcal{M}, \quad S_1 \cup S_2 = \bar{A}, \right.$$
$$\left. S_1 \cap S_2 = \Sigma, \quad \partial S_1 \cap A = \partial S_2 \cap A = \Sigma \cap A \right\}.$$

LEMME 1. — *There exists* $V \in \underline{\underline{C}}(\Lambda)$ *which is uniformly almost minimizing among disks relative to* $\mathcal{U}^\sigma$ *for each* $\sigma > 0$.

Proof (1). — Since $\underline{\underline{C}}$ is compact relative to the topology defined by $\underline{\underline{F}}$, and since the almost minimizing property considered here is preserved under limits in $V_2(\overline{A})$, it suffices to show that for each $\sigma > 0$, there is $V_\sigma \in \underline{\underline{C}}$ which is uniformly almost minimizing among disks relative to $\mathcal{U}^\sigma$.

We suppose that this is false, i. e. that for some $\sigma > 0$ no $V \in \underline{\underline{C}}$ has the required property. Then, for each $v \in \underline{\underline{C}}$, there exists $\varepsilon_V > 0$ with the property that for each $\alpha > 0$ and $\Sigma \in \mathcal{M}$ with $\underline{\underline{F}}(\underline{\underline{v}}(\Sigma), V) < \varepsilon_v$, there is $(U_1(\Sigma, \alpha, V), U_2(\Sigma, \alpha, V)) \in \mathcal{U}$ for which Σ is neither in $S(U_1^\sigma(\Sigma, \alpha, V))$ nor in $S(U_2^\sigma(\Sigma, \alpha, V))$. This means that there exist isotopies

$$\psi^i \in I(U_i^\sigma, A)$$

with

$$(5) \qquad |\psi_t^i(\Sigma)| \leq |\Sigma| + \alpha$$

$$(6) \qquad |\psi_1^i(\Sigma)| < |\Sigma| - \varepsilon_V$$

for $i = 1, 2$, $0 \leq t \leq 1$.

Let $N_\sigma(U) := \{ W \in V_2(A) : \underline{\underline{F}}(V, W) < \sigma \}$.

Since $\underline{\underline{C}}$ is compact relative to the topology defined by $\underline{\underline{F}}$,

$$(7) \qquad \underline{\underline{C}} \subset \bigcup_{j=1}^{n_0} N_{\varepsilon_{V_j}}(V_j)$$

for suitable $V_1, \ldots, V_{n_0} \in \underline{\underline{C}}$.

Let
$$\varepsilon_1 = \min_{1 \leq j \leq n_0} \varepsilon_{V_j}.$$

Using again a compactness argument, one easily sees that there is some $\varepsilon_2 > 0$ with the property that if $\{ \phi_t \} \in P(\Lambda)$ with

$$(8) \qquad \sup_{0 \leq t \leq 1} |\phi_t(\Lambda)| < M + \varepsilon_2$$

and if for some $t_0 \in [0, 1]$

$$(9) \qquad |\phi_{t_0}(\Lambda)| > M - \varepsilon_2$$

then for some $j \in \{ 1, \ldots, n \}$

$$(10) \qquad \underline{\underline{v}}(\phi_{t_0}(\Lambda)) \in N_{\varepsilon_{V_j}}(V_j).$$

Let Θ be a finite covering of $\overline{A}$ by balls of radius $\sigma/4$. Then there exists a finite partition of unity $\{ \theta_l, l = 1, \ldots, L \}$ subordinate to Θ with

$$(11) \qquad \sup_{x \in \overline{A}} \sum_{l=1}^{L} |D\theta_l(x)| \leq \frac{c}{\sigma}, \qquad \text{where } c \text{ is an absolute constant}.$$

(1) We shall largely follow [SS].

Let

(12) $$\varepsilon = \min\left(\frac{1}{4}, \varepsilon_1, \varepsilon_2, \frac{1}{2}\mathrm{M}, \frac{1}{40(\mathrm{M}+1)}\right).$$

Let (ϕ_t) be a path in $\mathrm{P}(\Lambda)$ with

(13) $$\sup_{0 \le t \le 1} |\phi_t(\Lambda)| \le \mathrm{M} + \frac{1}{4}\varepsilon .$$

We want to modify (ϕ_t), using (5) and (6), to obtain a new path with

$$\sup_{0 \le t \le 1} |\tilde{\phi}(\Lambda)| < \mathrm{M}$$

and hence the desired contradiction.

Using (1) and (2), for some $\delta_0 > 0$

(14) $$|\phi_t(\Lambda)| < \mathrm{M} - \varepsilon \quad \text{if} \quad 0 \le t \le \delta_0 \quad \text{or} \quad 1 - \delta_0 \le t \le 1 .$$

Let

$$\kappa := \max_{\substack{0 \le t \le 1 \\ x \in \bar{\mathrm{A}}}} \left\| \frac{\partial \phi}{\partial t}(t, x) \right\| .$$

We choose $\delta > 0$ having the following four properties

(15) $$\delta \le \varepsilon$$

(16) $$\kappa \frac{c}{\sigma} \delta \le \varepsilon^2 \qquad \text{(cf. (11))},$$

if $t, t' \in [\delta_0, 1 - \delta_0]$, $|t - t'| < \delta$, then

(17) $$|\phi_t \circ \phi_{t'}^{-1}(x) - x| \le \varepsilon^2 \sigma \qquad \text{for all} \quad x \in \bar{\mathrm{A}}$$

and

(18) $$\|(\mathrm{D}\phi_t) \circ \phi_{t'}^{-1}\| \le 1 + \varepsilon^2 .$$

We note that by (18), since $\varepsilon \le 1$, for $t, t' \in [\delta_0, 1 - \delta_0]$, $|t - t'| < \delta$

(19) $$|\phi_t(\Lambda)| < (1 + 3\varepsilon^2)|\phi_{t'}(\Lambda)| .$$

We choose a partition

$$\delta_0 = t_0 < t_1 \ldots < t_n = 1 - \delta_0$$

of $[\delta_0, 1 - \delta_0]$ with $|t_j - t_{j-1}| < \delta$ $(j = 1, \ldots, n)$.

$$\mathscr{J} := \{ t_j : |\phi_{t_j}(\Lambda)| > \mathrm{M} - \varepsilon \} .$$

By (14), $t_0, t_n \notin \mathscr{J}$.

Since $\varepsilon \le \varepsilon_2$, (8) and (9) are satisfied for $t_j \in \mathscr{J}$, and thus, by (10), there is $k_j \in \{1, \ldots, n_0\}$ with

$$\underline{\underline{v}}(\phi_{t_j}(\Lambda)) \in \mathrm{N}_{\varepsilon v_{k_j}}(\mathrm{V}_{k_j}) .$$

Putting $\alpha = \varepsilon^2$ in (5), for each j with $t_j \in \mathscr{J}$ there is $(U_{1j}, U_{2j}) \in \mathscr{U}$ and isotopies $\psi^{ij} \in I(U_{ij}, A)$ with

(20) $$|\psi_t^{ij}(\phi_{t_j}(\Lambda))| \leq |\phi_{t_j}(\Lambda)| + \varepsilon^2$$

and

(21) $$|\psi_1^{ij}(\phi_{t_j}(\Lambda))| < |\phi_{t_j}(\Lambda)| - \varepsilon$$

for $i = 1, 2,\ 0 \leq t \leq 1$.

Let

$$\mathscr{J} = \bigcup_{k=1}^{n(\mathscr{J})} C_k,$$

where each C_k is of the form

$$C_k = \{ t_{j_k+i}, i = 0, 1, \ldots, r_k \} \subset \mathscr{J}, t_{j_k-1} \notin \mathscr{J}, t_{j_k+r_k+1} \notin \mathscr{J}.$$

For given $k \in \{1, \ldots, n(\mathscr{J})\}$ we perform the modification of ϕ_t in the interval $t_{j_k-1} \leq t \leq t_{j_k+r_k+1}$. For simplicity of notation, we shall suppress the subscript k in the sequel. We thus want to construct $(\tilde{\phi}_t)$ with

(22) $$|\tilde{\phi}_t(\Lambda)| \leq M - \frac{1}{2}\varepsilon \quad \text{if} \quad t_{j-1} \leq t \leq t_{j+r+1}$$

and $$\tilde{\phi}_{t_{j-1}} = \phi_{t_{j-1}}, \qquad \tilde{\phi}_{t_{j+r+1}} = \phi_{t_{j+r+1}}.$$

First of all

(23) $$\tilde{\phi}_t = \phi_{2t-t_{j-1}} \quad \text{for} \quad t_{j-1} \leq t \leq \frac{1}{2}(t_{j-1} + t_j).$$

Since $t_{j-1} \notin \mathscr{J}$, (19) implies, in case $t_{j-1} \leq t \leq \frac{1}{2}(t_{j-1} + t_j)$,

(24) $$|\phi_{2t-t_j}(\Lambda)| < M - \frac{3}{4}\varepsilon.$$

Next, let $i_0 \in \{1, 2\}$ and

(25) $$\tilde{\phi}_t = \psi^{i_0 j}_{\tau_1(t)} \circ \phi_{t_j}$$

with $\tau_1(t) = \dfrac{2t - t_j - t_{j-1}}{t_j - t_{j-1}}$ for $\frac{1}{2}(t_{j-1} + t_j) \leq t \leq t_j$.

Using (20) and (24) for $t = \frac{1}{2}(t_{j-1} + t_j)$, (22) also holds for $\frac{1}{2}(t_{j-1} + t_j) \leq t \leq t_j$.

Now suppose inductively that $\tilde{\phi}_t$ has been defined for $t_j \leq t \leq t_l\ j < l < j + r$ with

(26) $$|\tilde{\phi}_t(\Lambda)| \leq M - \frac{1}{2}\varepsilon \quad \text{for} \quad t_j \leq t \leq t_l$$

and

$$\tilde{\phi}_{t_l} = \psi_1^{i_0 l} \circ \phi_{t_l} \qquad \text{for} \qquad i_0 = 1 \text{ or } 2\,. \tag{27}$$

We then want to construct $\tilde{\phi}_t$ for $t_l \leq t \leq t_{l+1}$, with (26) and (27) holding with $l + 1$ instead of l.

Since $\operatorname{dist}(\mathrm{U}_1, \mathrm{U}_2) \geq \min(\operatorname{diam} \mathrm{U}_1, \operatorname{diam} \mathrm{U}_2)$ for $(\mathrm{U}_1, \mathrm{U}_2) \in \mathscr{U}$, we can find U_{il}, $\mathrm{U}_{i',l+1}$ with

$$\mathrm{U}_{il} \cap \mathrm{U}_{i',l+1} = \varnothing\,. \tag{28}$$

Let
$$0 < s_1 < s_2 < \frac{1}{4}(t_{l+1} - t_l)\,.$$

If $i = i_0$ in (28) and (27), we put

$$\tilde{\phi}_t = \psi_1^{il} \circ \phi_{t_l} \qquad \text{for} \qquad t_l \leq t \leq t_l + s_2\,. \tag{29 a}$$

Then, by (26) and (27), (22) is satisfied for $t_l \leq t \leq t_l + s_2$. If $i \neq i_0$, we put

$$\begin{aligned} \tilde{\phi}_t &= \psi^{il}_{(t-t_l)/s_1} \circ \psi_1^{i_0 l} \circ \phi_{t_l} && \text{for} \quad t_l \leq t \leq t_l + s_1 \\ &= \psi_1^{il} \circ \psi^{i_0 l}_{(t_l+s_2-t)/(s_2-s_1)} \circ \phi_{t_l} && \text{for} \quad t_l + s_1 \leq t \leq t_l + s_2\,. \end{aligned} \tag{29 b}$$

Since in this case

$$\operatorname{supp} \psi^{il} \cap \operatorname{supp} \psi^{i_0 l} = \varnothing\,,$$

where $\operatorname{supp} \psi^{il} = \{ x : \psi^{il}(x) \neq x \}$, (20) and (21) again imply (22) for this interval. Moreover

$$\tilde{\phi}_{t_l+s_2} = \psi_1^{il} \circ \phi_{t_l}\,. \tag{30}$$

We now recall the partition of unity satisfying (11). We put

$$\zeta_{ij} := 1 - \sum_l \theta_l \delta(j, l)$$

where

$$\delta(j, l) = \begin{cases} 1 & \text{if} \quad \operatorname{spt} \theta_l \subset \mathrm{W}_l \in \Theta, \quad \mathrm{W}_l \cap \mathrm{U}_{ij}^\sigma \neq \varnothing \\ 0 & \text{otherwise} \end{cases}$$

Thus

$$\begin{aligned} &\zeta_{ij} \,|\, \mathrm{U}_{ij}^\sigma = 0, \quad \zeta_{ij} \,|\, \overline{\mathrm{A}} \backslash \mathrm{U}_{ij}^{\sigma/2} = 1 \\ &|\, \mathrm{D}\zeta_{ij} \,| \leq \frac{c}{\sigma}. \end{aligned} \tag{31}$$

We choose $0 < s_2 < s_3 < s_4 < s_5 < t_{l+1} - t_l$. We put

$$\beta_t^1(y) := \phi_{t_l + \zeta_{il}(y)\tau_2(t)} \circ \phi_{t_l}^{-1} \circ \psi_1^{il}(y)$$

with $\quad \tau_2(t) = (t_{l+1} - t_l)(t - t_l - s_2)/(s_3 - s_2) \quad$ for $\quad t_l + s_2 \leq t \leq t_l + s_3$

$$\beta_t^2(y) := \phi_{t_l + \zeta_{il}(y)(t_{l+1} - t_l)} \circ \phi_{t_l}^{-1} \circ \psi^{il}_{\tau_3(t)}(y)$$

with $\tau_3(t) = (t_l + s_5 - t)/(s_5 - s_4)$ for $t_l + s_4 \leq t \leq t_l + s_5$

$$\beta_t^3(y) := \phi_{\tau_4(t)} \circ \phi_{t_l}^{-1}(y)$$

with $$\tau_4(t) = \frac{t(t_{l+1} - t_l - s_5) - s_5(t_{l+1} - t) + \zeta_{il}(y)(t_{l+1} - t_l)(t_{l+1} - t)}{t_{l+1} - t_l - s_5}$$

for $$t_l + s_5 \leq t \leq t_{l+1}$$

and

$$\begin{aligned}\tilde{\phi}_t &= \beta_t^1 \circ \phi_{t_l} \quad \text{for} \quad t_l + s_2 \leq t \leq t_l + s_3 \\ &= \psi^{i',l+1}_{(t-t_l-s_3)/(s_4-s_3)} \circ \beta^1_{t_l+s_3} \circ \phi_{t_l} \quad \text{for} \quad t_l + s_3 \leq t \leq t_l + s_4 \\ &= \psi_1^{i',l+1} \circ \beta_t^2 \circ \phi_{t_l} \quad \text{for} \quad t_l + s_4 \leq t \leq t_l + s_5 \\ &= \psi_1^{i',l+1} \circ \beta_t^3 \circ \phi_{t_l} \quad \text{for} \quad t_l + s_5 \leq t \leq t_{l+1}.\end{aligned}$$

The idea behind this complicated construction is quite simple. By (28), the supports of ψ^{il} and $\psi^{i',l+1}$ are disjoint. Since by (21); we can decrease the energy by at least ε, performing a modification on U_{il}, we have some freedom for operations on $U_{i',l+1}$, still preserving (22), and vice versa.

We thus want to show that

$$|\phi_{\tilde{t}}(\Lambda)| \leq M - \frac{1}{2}\varepsilon \quad \text{for} \quad t_l + s_2 \leq t \leq t_{l+1}. \tag{32}$$

Since the estimates on the different subintervals are rather similar (and taken from [SS] anyway), we confine ourselves here to carry out only one (typical) example, namely $t_l + s_3 \leq t \leq t_l + s_4$.

We divide

$$\begin{aligned}\tilde{\phi}_t(\Lambda) &= \tilde{\phi}_t(\Lambda \cap \phi^{-1}_{t_{l+1}}(U^{\sigma/2}_{i',l+1})) \\ &\quad \cup (\tilde{\phi}_t(\Lambda \backslash \phi^{-1}_{t_{l+1}}(U^{\sigma/2}_{i',l+1})) \cap U^{\sigma}_{il}) \cup (\tilde{\phi}_t(\Lambda \backslash \phi^{-1}_{t_{l+1}}(U^{\sigma/2}_{i',l+1})) \backslash U^{\sigma}_{il}) \\ &=: \mathrm{I} \cup \mathrm{II} \cup \mathrm{III}\end{aligned}$$

By (17)

$$\mathrm{I} \subset U^{(\frac{1}{2}-\varepsilon^2)\sigma}_{i',l+1} \tag{33}$$

$$\mathrm{II} \cup \mathrm{III} \subset \bar{A} \backslash U^{(\frac{1}{2}+\varepsilon^2)\sigma}_{i',l+1}. \tag{34}$$

Since $\zeta_i = 1$ on $U^{(\frac{1}{2}+\varepsilon^2)\sigma}_{i',l+1}$, noting (28)

$$\mathrm{I} = \psi^{i',l+1}_{(t-t_l-s_3)/(s_4-s_3)} \circ \phi_{t_{l+1}}(\Lambda \cap \phi^{-1}_{t_{l+1}}(U^{\sigma/2}_{i',l+1})).$$

Thus, by (20)

$$|\mathrm{I}| \leq |\phi_{t_{l+1}}(\Lambda) \cap U^{\sigma/2}_{i',l+1}| + \varepsilon^2. \tag{35}$$

Next

$$\begin{aligned}\mathrm{II} &= \beta^1_{t_l+s_3}(\phi_{t_l}(\Lambda \backslash \phi^{-1}_{t_{l+1}}(U^{\sigma/2}_{i',l+1})) \cap U^{\sigma}_{il}) \\ &= \psi^{il}_1(\phi_{t_l}(\Lambda) \backslash \phi_{t_l} \circ \phi^{-1}_{t_{l+1}}(U^{\sigma/2}_{i',l+1})) \cap U^{\sigma}_{il},\end{aligned}$$

since $\zeta_{il} = 0$ on U^{σ}_{il}.

With (21)

$$
\begin{aligned}
|\,\mathrm{II}\,| &\leq \left|(\phi_{t_l}(\Lambda)\backslash\phi_{t_l}\circ\phi_{t_{l+1}}^{-1}(\mathrm{U}_{i',l+1}^{\sigma/2}))\cap \mathrm{U}_{il}^{\sigma}\right| - \varepsilon \\
&= \left|(\phi_{t_l}\circ\phi_{t_{l+1}}^{-1})\circ\phi_{t_{l+1}}(\Lambda\backslash\phi_{t_{l+1}}^{-1}(\mathrm{U}_{i',l+1}^{\sigma/2}))\cap \mathrm{U}_{il}^{\sigma}\right| - \varepsilon \\
&\leq (1+3\varepsilon^2)\left|(\phi_{t_{l+1}}(\Lambda\backslash\phi_{t_{l+1}}^{-1}(\mathrm{U}_{i',l+1}^{\sigma/2})))\cap\phi_{t_{l+1}}\circ\phi_{t_l}^{-1}(\mathrm{U}_{il}^{\sigma})\right| - \varepsilon \qquad \text{by (19)} \\
&\leq \left|(\phi_{t_{l+1}}(\Lambda)\backslash\mathrm{U}_{i',l+1}^{\sigma/2})\cap\phi_{t_{l+1}}\circ\phi_{t}^{-1}(\mathrm{U}_{il}^{\sigma})\right| + 3\varepsilon^2\left(\mathrm{M}+\frac{1}{4}\varepsilon\right) - \varepsilon \qquad \text{by (13)}
\end{aligned}
$$

Finally

$$
\mathrm{III} = \beta_{t_l+s_3}^{1}(\phi_{t_l}(\Lambda\backslash\phi_{t_{l+1}}^{-1}(\mathrm{U}_{i',l+1}^{\sigma/2}))\backslash\mathrm{U}_{il}^{\sigma}).
$$

Thus putting $\mathrm{R} = (\phi_{t_l}(\Lambda)\ \phi_{t_l}\circ\phi_{t_{l+1}}^{-1}(\mathrm{U}_{i',l+1}^{\sigma/2}))\backslash\mathrm{U}_{il}^{\sigma}$

$$
(37) \qquad |\,\mathrm{III}\,| \leq \int_{\mathrm{R}} \|\,\mathrm{D}\beta_{t_l+s_3}^{1}(y)\,\|^2 d\mathscr{H}^2(y).
$$

Now, on R

$$
\begin{aligned}
\mathrm{D}\beta_{t_l+s_3}^{1}(y) = \left.\frac{\partial\phi_\tau}{\partial\tau}\right|\tau = t_l + s_3\circ\phi_{t_l}^{-1}(y)\,\cdot \\
\cdot\langle\,\mathrm{D}\zeta_{il}(y),\cdot\,\rangle(t_{l+1}-t_l) + \mathrm{D}\phi_{t_l+s_3}(\phi_{t_l}^{-1}(y))\cdot\mathrm{D}\phi_{t_l}^{-1}(y).
\end{aligned}
$$

Therefore, noting (31),(18), $|\,t_{l+1}-t_l\,| \leq \delta$

$$
(38) \qquad
\begin{aligned}
\|\,\mathrm{D}\beta_{t_l+s_3}^{1}(y)\,\| &\leq \kappa\cdot\frac{c}{\sigma}\delta + (1+\varepsilon^2) \\
&\leq 1 + 2\varepsilon^2 \qquad \text{by (16)}
\end{aligned}
$$

(37) and (38) yield

$$
\begin{aligned}
(39) \qquad |\,\mathrm{III}\,| &\leq (1+2\varepsilon^2)\,|\,\phi_{t_l}(\Lambda\backslash\phi_{t_{l+1}}^{-1}(\mathrm{U}_{i',l+1}^{\sigma/2}))\backslash\mathrm{U}_{il}^{\sigma}\,| \\
&\leq (1+2\varepsilon^2)(1+3\varepsilon^2)\left|(\phi_{t_{l+1}}(\Lambda\backslash\mathrm{U}_{i',l+1}^{\sigma/2})\backslash\phi_{t_{l+1}}\circ\phi_{t_l}^{-1}(\mathrm{U}_{il}^{\sigma})\right|
\end{aligned}
$$

By (35), (36), (39) and (13)

$$
\begin{aligned}
|\,\tilde{\phi}_t(\Lambda)\,| &\leq |\,\phi_{t_{l+1}}(\Lambda)\,| - \varepsilon + \varepsilon^2(1+10\mathrm{M}) \\
&\leq |\,\phi_{t_{l+1}}(\Lambda)\,| - \frac{3}{4}\varepsilon \qquad \text{by choice of } \varepsilon \text{ (cf. (12))} \\
&\leq \mathrm{M} - \frac{1}{2}\varepsilon \qquad \text{by (13)}
\end{aligned}
$$

for

$$
t_l + s_3 \leq t \leq t_l + s_4.
$$

Handling the other subintervals in a similar way, we obtain (32). Moreover, by construction

$$
(40) \qquad \phi_{t_{l+1}} = \psi_1^{i',l+1}\circ\phi_{t_{l+1}}.
$$

This holds for $l = j-1,\ldots,j+r-1$. By induction likewise

$$
(41) \qquad |\,\tilde{\phi}_t(\Lambda)\,| \leq \mathrm{M} - \frac{1}{2}\varepsilon \qquad \text{for} \quad t_{j-1} \leq t \leq t_{j+r}.
$$

We finally put

$$\tilde{\phi}_t = \psi^{i',j+r}_{\tau_5(t)} \circ \phi_{t_{j+r}}$$

with $$\tau_5(t) = (t_{j+r+1} + t_{j+r} - 2t)/(t_{j+r+1} - t_{j+r})$$

for $$t_{j+r} \leq t \leq \frac{1}{2}(t_{j+r} + t_{j+r+1})$$

and

$$\tilde{\phi}_t = \phi_{2t-t_{j+r+1}} \qquad \frac{1}{2}(t_{j+r} + t_{j+r+1}) \leq t \leq t_{j+r+1}\,.$$

By the same argument as for $t_{j-1} \leq t \leq t_j$, we see that (41) continues to hold for $t_{j+r} \leq t \leq t_{j+r+1}$.

Putting

$$\tilde{\phi}_t = \phi_t \qquad \text{for} \quad t \notin \bigcup_{k=1}^{n(\mathscr{J})} [j_k - 1, j_k + r_k + 1]$$

(19) then implies that (41) also holds for those t. Hence $\tilde{\phi}_t$ is defined on $[0, 1]$ and satisfies

$$|\tilde{\phi}_t(\Lambda)| \leq \mathrm{M} - \frac{1}{2}\varepsilon \qquad \text{for} \quad 0 \leq t \leq 1\,. \tag{42}$$

Since by definition of $\mathscr{U}$, we only performed modification on sets with volume less than a quarter of the volume of $\overline{\mathrm{A}}$, we see that $(\tilde{\phi}_t)$ also satisfies condition (4) (observe that we made sure that (40) holds at every step of the construction, so $(\tilde{\phi}_t)$ never deviated enough from (ϕ_t) to become a path with $\mathrm{A}_1^1 = \{ z_1 \}$.) We then smooth out $(\tilde{\phi}_t)$ to get a C^1-path $(\tilde{\tilde{\phi}}_t)$ with

$$|\tilde{\tilde{\phi}}_t(\Lambda)| \leq \mathrm{M} - \frac{1}{4}\varepsilon \qquad \text{for} \quad 0 \leq t \leq 1\,.$$

Thus, we have obtained the desired contradiction finishing the proof.

q. e. d.

COROLLARY 1. — a) *There is at most one point $x \in \overline{\mathrm{A}}$ at which* V *is not almost minimizing among disks.*

b) *For each $x \in \overline{\mathrm{A}}$ there is some $r = r(x) > 0$ with the property that* V *is almost minimizing among disks in* $\mathrm{B}(x, r) \setminus \{ x \}$.

Proof. — *a*) If the almost minimizing property would fail at x_1 and x_2 $(x_1 \neq x_2)$, we take r_1, r_2 with

$$r_1 + r_2 \leq \frac{1}{2} | x_1 - x_2 |$$

$$\operatorname{vol}(\mathrm{B}(x_i, r_i)) < \frac{1}{4} \operatorname{vol}(\overline{\mathrm{A}})$$

in Lemma 1 to obtain a contradiction.

b) If V is not almost minimizing in $\overset{\circ}{B}(x, r) =: U_1$ for all $r \leq r_0(x)$, then it is almost minimizing in $U_2 := \overset{\circ}{B}(x, r_0) \setminus B(x, 4r)$ for all $r \leq \frac{1}{4} r_0$ by Lemma 1, hence in $\overset{\circ}{B}(x, r_0) \setminus \{ x \}$. q. e. d.

§ 2. MINIMIZING SEQUENCES OF SURFACES AT FREE BOUNDARIES

We extend the methods of [AS] and [MSY] to free boundary value problems using the regularity theorem of [GJ].

Let $U \subset \mathbb{R}^3$ be open, of class C^2, and let $\partial U \cap A$ be simply connected.

Let $M \in \mathcal{M}$ intersect ∂U transversally and

$$\partial M \cap A \cap U = \emptyset .$$

Let Λ be a component of $M \setminus (M \cap U)$ with

$$\partial M \cap A \cap \Lambda = \emptyset .$$

Then there exist $F \subset \partial U \cap A$ and $C \subset \overline{A} \setminus U$ with

$$\partial F \cap A = \Lambda \cap \partial U \cap A \tag{1}$$

$$\partial C \cap A = \Lambda \cup F \tag{2}$$

$$\operatorname{vol} C \leq c_0 \mathcal{H}^2(\Lambda \cup F)^{3/2} . \tag{3}$$

The constant c_0 depends only on ∂A. This easily follows from the isoperimetric inequality.

We also define for U as above and $t \geq 0$,

if $\pi : \mathbb{R}^3 \setminus U \to \partial U$ denotes the nearest point projection,

$$R(t) := \{ x \in \mathbb{R}^3 \setminus U : \operatorname{dist}(x, \partial U) = t, \pi(x) \in \partial U \cap A \}$$

$$U'(t) := \bigcup_{0 \leq s \leq t} R(s)$$

$$U_t := \{ x \in U : \operatorname{dist}(x, \partial U) \geq t \} .$$

LEMMA 1. — *Let U be an open subset of $\mathbb{R}^3$ with a convex boundary ∂U of class C^2. Suppose on ∂U the following isoperimetric inequality holds:*

If λ is a system of Jordan curves in $\partial U \cap A$ dividing $\partial U \cap A$ into two (not necessarily connected) components E_1, E_2, then

$$\min(\mathcal{H}^2(E_1), \mathcal{H}^2(E_2)) \leq \beta(\text{length } \lambda)^2$$

for some $\beta > 0$.

Suppose

$$T > c_1 \mathcal{H}^2(\partial U \cap A)^{\frac{1}{2}}$$

with

$$c_1 = \max\left(\frac{3}{16} c_0, 2\beta^{\frac{1}{2}}\right)$$

and

$$U'(T) \subset A.$$

Let

$$\theta > 0$$

and let $M \in \mathcal{M}$ *intersect* ∂U *transversally. Suppose* $\partial M \cap A$ *is not contained in any set* C *satisfying* (2), (3) *where* Λ *is a component of* $M \backslash (M \cap U)$ *with* $\partial M \cap A \cap \Lambda = \emptyset$ *and* $\tilde{F} \subset \partial U \cap A$ *satisfies* (1).

Then there exists $\tilde{M} \in \mathcal{M}$ *with*

$$\partial \tilde{M} \cap A = \partial M \cap A$$

$$\tilde{M} \backslash (\tilde{M} \cap U) \subset M \backslash (M \cap U)$$

$$\tilde{M} \cap U_\theta \subset M \cap U_\theta$$

$\tilde{M}$ *intersects* ∂U *transversally*

$$\mathcal{H}^2(\tilde{M}) + \mathcal{H}^2((M \backslash \tilde{M}) \cap U_\theta) \leq \mathcal{H}^2(M)$$

$$\tilde{M} \cap U = \bigcup_{j=1}^{k} N_j \qquad \textit{where} \quad N_j \in \mathcal{M}.$$

If in addition

$$\mathcal{H}^2(M) \leq \mathcal{H}^2(P) + \varepsilon \qquad \textit{for any} \quad P \in \mathcal{M} \quad \textit{with} \quad \partial P \cap A = \partial M \cap A$$

then there $\varepsilon_1, \ldots, \varepsilon_n \geq 0$ *with* $\sum_{j=1}^{k} \varepsilon_j \leq \varepsilon$ *and*

$$\mathcal{H}^2(N_j) \leq \mathcal{H}^2(P_j) + \varepsilon_j \quad \textit{for any} \quad P_j \in \mathcal{M} \quad \textit{with} \quad \partial P_j \cap A = \partial M_j \cap A$$

$(j = 1, \ldots, k)$.

Proof. — We can proceed as in [AS; p. 457 ff.] once we have demonstrated the following claim:

If Λ, F and C are as above (in particular satisfying (1)-(3)) then

$$\mathcal{H}^2(F) < \mathcal{H}^2(\Lambda).$$

We achieve this as follows.

Since ∂U is convex, $\mathcal{H}^2(R(t))$ is monotonically increasing in t, and, by assumption

$$R(t) \subset A \qquad \text{for} \quad 0 \leq t \leq T.$$

If Λ intersects $R(t)$ transversally (which is the case for almost all t by Sard's lemma), it divides $R(t)$ into two (not necessarily connected) sets $F(t)$, $F'(t)$.

We label them in such a way that they depend continuously on t and

$$F(0) = F$$

w. l. o. g.

(4) $$\mathscr{H}^2(\Lambda) < \mathscr{H}^2(\partial U \cap A)$$

and hence

$$\operatorname{vol} C \leq 3c_0 \mathscr{H}^2(\partial U \cap A)^{3/2}.$$

The coarea formula then yields

$$\int_{T/4}^{T/2} \mathscr{H}^2(F(t))dt \leq 3c_0 \mathscr{H}^2(\partial U \cap A)^{3/2}.$$

Hence, by assumption on T, there exists $t_0 \in \left[\frac{T}{4}, \frac{T}{2}\right]$ with

(5) $$\mathscr{H}^2(F(t_0)) \leq \frac{1}{4} \mathscr{H}^2(\partial U \cap A) \leq \frac{1}{2} \mathscr{H}^2(R(t_0)).$$

We put

$$E(t) := \Lambda \cap U'(t).$$

Since ∂U is convex, $\Delta \operatorname{dist}(\cdot, U) \geq 0$ on $\mathbb{R}^3 \backslash U$. Thus, from the divergence theorem, if ν denotes the unit normal vector field of Λ,

$$\mathscr{H}^2(F(t_1)) - \mathscr{H}^2(F(t_2)) \leq \int_{E(t_2)\backslash E(t_1)} |\langle \nu, \operatorname{grad} \operatorname{dist}(\cdot, U) \rangle|$$

for $0 \leq t_1 < t_2 \leq T$.

Therefore

(6) $$\mathscr{H}^2(F(t_1)) - \mathscr{H}^2(F(t_2)) \leq \mathscr{H}^2(\overline{E}(t_2)) - \mathscr{H}^2(E(t_1)),$$

and if $E(t_2) \neq E(t_1)$, we even have strict inequality. In particular the claim follows if $\mathscr{H}^2(F(t_2)) = 0$ for some $t_2 \in \left[\frac{T}{2}, T\right]$, noting $\mathscr{H}^2(\overline{E}(t_2)) \leq \mathscr{H}^2(\Lambda)$. In general, we have at least

(7) $$\mathscr{H}^2(F) - \mathscr{H}^2(\overline{E}(t)) < \mathscr{H}^2(F(t))$$

for $0 < t \leq T$.

(5) implies that we can also assume

(8) $$\mathscr{H}^2(F) \leq \frac{1}{2} \mathscr{H}^2(\partial U \cap A),$$

because, if not, we take $U'(t_0) \cup U$, $F(t_0)$ (note (5)), $A(T) := A \cap U'(T)$, and $\Lambda \cap A(T)$ instead of U, F, A, Λ resp., show that (with the arguments below)

$$\mathscr{H}^2(F(t_0)) < \mathscr{H}^2(\Lambda \cap A(T))$$

and apply the divergence theorem to show that

$$\begin{aligned}\mathcal{H}^2(\mathrm{F}) &< \mathcal{H}^2(\mathrm{F}(t_0)) + \mathcal{H}^2(\overline{\mathrm{E}}(t_0)) \\ &< \mathcal{H}^2(\Lambda),\end{aligned}$$

thus demonstrating the claim.

Therefore, assuming (8), we can assume as well

$$\mathcal{H}^2(\Lambda) \leq \frac{1}{2}\mathcal{H}^2(\partial \mathrm{U} \cap \mathrm{A}).$$

Assuming this and using (5), we obtain from (6) ($t_1 = t$, $t_2 = t_0$)

$$\mathcal{H}^2(\mathrm{F}(t)) \leq \frac{3}{4}\mathcal{H}^2(\partial \mathrm{U} \cap \mathrm{A}) \leq \frac{3}{4}\mathcal{H}^2(\mathrm{R}(t))$$

for $0 \leq t \leq \frac{\mathrm{T}}{4}$.

Hence, from the isoperimetric inequality on $\mathrm{R}(t)$

$$\mathcal{H}^2(\mathrm{F}(t)) \leq 4\beta\,(\text{length}\,(\Lambda \cap \mathrm{R}(t)))^2 = 4\beta\left(\frac{d}{dt}\mathcal{H}^2(\mathrm{E}(t))\right)^2$$

for almost all $t \in \left[0, \frac{\mathrm{T}}{4}\right]$.

With (7)

$$\mathcal{H}^2(\mathrm{F}) - \mathcal{H}^2(\mathrm{E}(t)) \leq 4\beta\left(\frac{d}{dt}(\mathcal{H}^2(\mathrm{F}) - \mathcal{H}^2(\mathrm{E}(t)))\right)^2$$

for almost all $t \in \left[0, \frac{\mathrm{T}}{4}\right]$, and since this expression is monotonically decreasing and $\mathcal{H}^2(\mathrm{E}(0)) = 0$

$$\mathcal{H}^2(\mathrm{F})^{\frac{1}{2}} - \left(\mathcal{H}^2(\mathrm{F}) - \mathcal{H}^2\left(\mathrm{E}\left(\frac{\mathrm{T}}{4}\right)\right)\right)^{\frac{1}{2}} \geq \frac{\mathrm{T}}{4\beta^{\frac{1}{2}}},$$

provided

$$\mathcal{H}^2(\mathrm{F}) > \mathcal{H}^2\left(\mathrm{E}\left(\frac{\mathrm{T}}{4}\right)\right).$$

This implies (noting (8))

$$\mathrm{T} \leq 2\beta^{\frac{1}{2}}\mathcal{H}^2(\partial \mathrm{U} \cap \mathrm{A})^{\frac{1}{2}},$$

contradicting the choice of T. Hence

$$\mathcal{H}^2(\mathrm{F}) \leq \mathcal{H}^2\left(\mathrm{E}\left(\frac{\mathrm{T}}{4}\right)\right).$$

Now either $\mathcal{H}^2\left(F\left(\frac{T}{2}\right)\right) = 0$ which case however was already treated after (6), or

$$\mathcal{H}^2\left(E\left(\frac{T}{4}\right) < \mathcal{H}^2\left(E\left(\frac{T}{2}\right)\right)\right.$$

whence the claim follows again, noting $\mathcal{H}^2\left(E\left(\frac{T}{2}\right)\right) \leq \mathcal{H}^2(\Lambda)$. q. e. d.

LEMMA 2 (Boundary filigree).

Assumptions.

$\{ Y_t \}_{t\in[0,1]}$ *increasing family of convex sets where each* Y_t *satisfies the assumptions of the set* U *of Lemma 1.*

$Y_t = \{ x \in \mathbb{R}^3 : f(x) < t \}$, $t > 0$, *where* $f : \mathbb{R}^3 \to \mathbb{R}^+$ *is* C^2 *on* $\mathbb{R}^3 \sim \overline{Y}_0$, $Df \neq 0$ *on* $Y_1 \sim \overline{Y}_0$,

$$\sup_{\overline{Y}_1 \sim \overline{Y}_0} |Df| \leq c_1$$

$\exists c_2 > 0$, $\forall \Gamma_1$, Γ_1 C^2 *Jordan arc in* ∂Y_t, $\partial\Gamma_1 \subset \partial A$,

Γ_1 *in* ∂Y_t *homotopic with fixed endpoints to an arc* $\Gamma \subset \partial A$

$$\exists E \subset \partial Y_t : \partial E \subset \Gamma_1 \cup (\partial A \cap \partial Y_t)$$

$$\mathcal{H}^2(E) \leq c_2(\mathcal{H}^1(\Gamma_1))^2 \tag{11}$$

$M \subset \overline{U}$, $M \in \mathcal{M}$, $\forall t \in (0, 1) : \partial M \cap A$ *is not contained in any set* C *satisfying* (2), (3)

$\exists \varepsilon > 0 : \forall \psi \in I(U, A)$:

$$\mathcal{H}^2(M) \leq \mathcal{H}^2(\psi_1(M)) + \varepsilon . \tag{12}$$

Conclusion:

If $t_0 = 1 - 2c_1\sqrt{c_2} \cdot \sqrt{\mathcal{H}^2(M \cap Y_1)} > 0$ *then*

$$\mathcal{H}^2(M \cap Y_{t_0}) \leq 2\varepsilon . \tag{13}$$

Proof. — By Sard's Lemma, M intersects ∂Y_t transversally for almost every $t \in (0, 1)$, and by assumption $M \subset \overline{A}$ and M meets ∂A transversally. In particular, $\operatorname{int} M \cap \partial A = \emptyset$. Thus, we can apply Lemma 1 and get $\tilde{M}$ with

$$\begin{aligned} &\mathcal{H}^2(\tilde{M}) \leq \mathcal{H}^2(M) \\ &\tilde{M} \cap \partial Y_t \subset M \cap \partial Y_t \end{aligned} \tag{14}$$

$$\tilde{M} \cap \overline{Y}_t = \bigcup_{j=1}^{k} N_j, \qquad N_j \in \mathcal{M}$$

$$\mathcal{H}^2(N_j) \leq \mathcal{H}^2(N) + \varepsilon_j \qquad \forall N = \psi_1(N_j) \tag{15}$$

(16) $$\sum_{j=1}^{k} \varepsilon_j \leq \varepsilon \qquad \psi \in \mathrm{I}(\mathrm{U}, \mathrm{A})$$

(and not only $\mathrm{N}_j \in \tilde{\mathscr{M}}$ as in [AS], p. 457).

Let $\Gamma_j : \partial \mathrm{N}_j \cap \partial \mathrm{Y}_t$. Take $\mathrm{E} \subset \partial \mathrm{Y}_t$, $\partial \mathrm{E} \subset \Gamma_1 \cup (\partial \mathrm{A} \cap \partial \mathrm{Y}_t)$, for which (11) holds.

By (15)

(17) $$\mathscr{H}^2(\mathrm{N}_j) < \mathscr{H}^2(\mathrm{E}) + \varepsilon_j .$$

From (11) and (17)

$$\mathscr{H}^2(\mathrm{N}_j) \leq c_2(\mathscr{H}^1(\mathrm{N}_j \cap \partial \mathrm{Y}_t))^2 + \varepsilon_j .$$

Using (16)

$$\mathscr{H}^2(\tilde{\mathrm{M}} \cap \mathrm{Y}_t) \leq c_2(\mathscr{H}^1(\tilde{\mathrm{M}} \cap \partial \mathrm{Y}_t))^2 + \varepsilon ,$$

and using (14)

$$\mathscr{H}^2(\tilde{\mathrm{M}} \cap \mathrm{Y}_t) \leq c_2(\mathscr{H}^1(\mathrm{M} \cap \partial \mathrm{Y}_t))^2 + \varepsilon$$

and using (12),

(18) $$\mathscr{H}^2(\mathrm{M} \cap \mathrm{Y}_t) \leq c_2(\mathscr{H}^1(\mathrm{M} \cap \partial \mathrm{Y}_t))^2 + 2\varepsilon$$

w. l. o. g. $$\mathscr{H}^2(\mathrm{M} \cap \mathrm{Y}_1) > 2\varepsilon$$

$$t_1 := \inf \{ t : \mathscr{H}^2(\mathrm{M} \cap \mathrm{Y}_t) > 2\varepsilon \}$$

$$g(t) := \mathscr{H}^2(\mathrm{M} \cap \mathrm{Y}_t) - 2\varepsilon \qquad \text{for} \quad t \in [t_1, 1] .$$

Using the coarea formula, (18) yields

(19) $$g(t) \leq c_1^2 c_2 (g'(t))^2 \qquad \text{a. a. } t \in [t_1, 1] .$$

Since g is increasing, the result easily follows from integrating (19).

THEOREM 1. — *Suppose* $\mathrm{S} \in \mathscr{M}$, (ψ^k) *sequence in* $\mathrm{I}(\mathrm{U}, \mathrm{A})$,

$$\lim_{k \to \infty} \operatorname{area}(\psi_1^k(\mathrm{S})) = \inf \{ \operatorname{area}(\psi_1(\mathrm{S})) : \psi \in \mathrm{I}(\mathrm{U}, \mathrm{A}) \} ,$$

$$\mathrm{W} = \lim_{k \to \infty} \underline{\underline{v}}(\psi_1^k(\mathrm{S})) \in \mathrm{V}_2(\mathbb{R}^3)$$

exists in the varifold sense.

Then V *is an integral varifold, and*

(20) $$\mathrm{W} \llcorner \mathrm{U} \cap \overline{\mathrm{A}} \times \mathrm{G}(3, 2) = n\underline{\underline{v}}(\mathrm{M})$$

where M *is a stable embedded minimal surface in* $\mathrm{U} \cap \overline{\mathrm{A}}$ *with* $\partial \mathrm{M} \cap \mathrm{U} \subset \partial \mathrm{A}$, *and* M *meets* $\partial \mathrm{A}$ *orthogonally.*

Proof. — As in [AS], p. 463, we see using the boundary filigree lemma, that W is stationary, rectifiable and there is some $c > 0$ with

(21) $$\Theta_*^2(\| \mathrm{W} \|, x) \geq c$$

for all $x \in \operatorname{spt} \| \mathrm{W} \| \cap \mathrm{U}$.

Interior regularity of W follows from [AS], §§ 5, 6. Also, W is integral.

Let $x_0 \in \operatorname{spt} \| W \| \cap U \cap \partial A$.

We assume for a moment that W has a varifold tangent C at x_0 with $\operatorname{spt} \| C \|$ contained in a half plane H.

Since W is also stationary w. r. t. variations of its boundary on ∂A, C has to contain the interior normal of ∂A at x_0.

W. l. o. g. $x_0 = 0$, and $(0, 1, 0)$ is normal to H.

Let $\tilde{K}_{\rho,\sigma} := ((D_\rho \sim \partial D_\rho) \times (-\sigma, \sigma)) \cap A$.

By rescaling, we can assume w. l. o. g. $\tilde{K}_{1,1} \subset U$.

Put $N_k := \psi_1^k(S)$.

By definition of C,

$$\mu_{r_k \#} W \to C \tag{22}$$

for some sequence $(r_k) \to \infty$ as $k \to \infty$.

Let $\sigma_0 \in (0, 1)$ be given.

(21) implies that we can find $r \in (r_k)$ with

$$\tilde{K}_{1,1} \cap \operatorname{spt} \| \mu_{r_\#} W \| \subset \tilde{K}_{1,\sigma_{0/2}}. \tag{23}$$

W. l. o. g. also

$$\begin{gathered} \mathcal{H}^2(\operatorname{spt} \| \mu_{r_\#} W \| \cap (\partial D_{\frac{1}{2}} \times \mathbb{R}) = \emptyset \\ \mathcal{H}^2(\operatorname{spt} \| \mu_{r_\#} W \| \cap \partial A_r = \emptyset, \quad \text{where} \quad A_r = \mu_r(A). \end{gathered} \tag{24}$$

By assumption

$$\mathcal{H}^2(\mu_r(N_k)) \leq \mathcal{H}^2(N) + r^2 \varepsilon \tag{25}$$

for all $N \in \mathcal{M}$ with $\partial N \cap A_r = \partial \mu_r(N_k) \cap A_r$.

Let $A_r \Leftrightarrow \mu_r(A)$.

Since $\underline{\underline{v}}(\mu_r(N_k)) \to \mu_{r_\#} W$, (23) and the coarea formula yield for almost all $\sigma \in (\sigma_{0/2}, 1)$ and $k \to \infty$

$$\mathcal{H}^1(\mu_r(N_k) \cap (D_1 \times (\{-\sigma\} \cup \{\sigma\}) \cap A_r)) \to 0.$$

Thus, for sufficiently large k, we can find $\sigma_k \in \left(\frac{3}{4}\sigma_0, \sigma_0\right)$ and $\rho_k \in \left(\frac{3}{4}, 1\right)$ with

$$\mu_r(N_k) \cap ((\partial D_{\rho_k} \times (\{-\sigma_k\} \cup \{\sigma_k\})) \cup (\partial A_r \cap D_{\rho_k} \times (\{-\sigma_k\} \cup \{\sigma_k\}))) = \emptyset \tag{26}$$

Furthermore, by Sard's Lemma, we can assume that $\mu_r(N_k)$ intersects $D_{\rho_k} \times (\{-\sigma_k\} \cup \{\sigma_k\})$ and $\partial D_{\rho_k} \times [-1, 1]$ transversally. Moreover $\mu_r(N_k)$ intersects A_r transversally by assumption.

We now want to apply Theorem 1 of [AS] for $M = \mu_r(N_k)$ and $U = \tilde{K}_{\rho_k,\sigma_k}$ (M, U as in [AS], Theorem 1).

As observed in [AS], p. 475, we don't have to worry about the edges of $\tilde{K}_{\rho_k,\sigma_k}$. Because of (26), anyway only the edge $\partial A_r \cap (\partial D_{\rho_k} \times [-\sigma_k, \sigma_k])$

has to be taken into account. The N_j and P from the statement of Theorem 1 in [AS] are then in $\tilde{\mathcal{M}}$ instead of $\mathcal{M}$.

Anyway, we find integers $0 < R_k^1 \le R_k^2 \le R_k^3$ and disks $P_k^1, \ldots, P_k^{R_k^3}$ with

$$\partial P_k^1, \ldots, \partial P_k{}^{R_k^2} \subset \partial D_{\rho_k} \times (-\sigma_k, \sigma_k) \cup (\partial A \cap (D_{\rho_k} \times (-\sigma_k, \sigma_k))) =: E_k$$
$$\partial P_k^{R_k^2+1}, \ldots, \partial P_k^{R_k^3} \subset D_{\rho_K} \times (\{-\sigma_k\} \cup \{\sigma_k\})$$

and $\partial P_k^1, \ldots, \partial P_k^{R_k}$ are homotopically nontrivial in E_k while $\partial P_k^{R_k^1+1}, \ldots, \partial P_k^{R_k^2}$ bound disks in E_k.

Note that because of (26), the edges

$$\partial D_{\rho_k} \times (\{-\sigma_k\} \cup \{\sigma_k\}) \quad \text{and} \quad \partial A_r \cap (D_{\rho_k} \times (\{-\sigma_k\} \cup \{\sigma_k\}))$$

are not intersected by the ∂P_k^l.

Moreover,

(27 $\quad \mathcal{H}^2(P_k^l) \le \mathcal{H}^2(P) + \varepsilon_{k,l} \qquad \forall P \in \tilde{\mathcal{M}}$ with $\partial P = P_k^l \quad l = 1, \ldots, R_k^3$
(for $l = R_k^2 + 1, \ldots, R_k^3$, we can actually assume $P_k^l, P \in \mathcal{M}$) and

$$\sum_{l=1}^{R_k^3} \varepsilon_{k,l} \le r^2 \varepsilon_k$$

and using (14) and [AW1] 2.6 (2) (*d*),

$$\text{(28)} \qquad \mu_{r\#} W \llcorner \tilde{K}_{\frac{1}{2},1} \times G(3,2) = \lim_{k \to \infty} \sum_{l=1}^{R_k^3} \underline{\underline{v}}(P_k^l \cap \tilde{K}_{\frac{1}{2},1}).$$

Then, first of all, $P_k^{R_k^2+1}, \ldots, P_k^{R_k^3}$ can be discarded as in [AS], p. 465 f., without changing the varifold limit in (28).

We now want to delete $P_k^{R_k^1+1}, \ldots, P_k^{R_k^2}$.

Let $\Delta_{k,l}$ be the intersection of the disk bounded by P_k^l in E_k with $\partial D_{\rho_k} \times (-\sigma_k, \sigma_k)$ $(l = R_k^1 + 1, \ldots, R_k^2)$.

Clearly

$$\mathcal{H}^2(\Delta_{k,l}) \le 2\pi\rho_k\sigma_0 .$$

Choosing $P = \Delta_{k,l}$ in (17) and k sufficiently large, hence

$$\text{(29)} \qquad \mathcal{H}^2(P_k^l) < 2\pi\sigma_0 + \varepsilon_{k,l} \qquad (l = R_k^1 + 1, \ldots, R_k^2).$$

Choosing σ_0 sufficiently small and using the boundary filigree lemma for the family of cylinders (which after suitably rescaling and slightly perturbing satisfy the proper assumptions)

$$Y_t = \{ x = (x_1, x_2, x_3) : \sqrt{x_1^2 + x_2^2} < t\rho_k \}$$
$$f(x) = \frac{1}{\rho_k} \sqrt{x_1^2 + x_2^2}$$
$$c_1 = \rho_k^{-1}$$

we get

$$\mathscr{H}^2(P_k \cap \tilde{K}_{\frac{1}{2},1}) \leq 3\varepsilon_{k,l}, \qquad l = R_k^1 + 1, \ldots, R_k^2$$

and thus also these P_k^l can be discarded without changing the varifold limit in (28).

Thus

$$\mu_{r\#} W \llcorner \tilde{K}_{\frac{1}{2},1} \times G(3,2) = \lim_{k\to\infty} \sum_{l=1}^{R_k^1} \underline{\underline{v}}(P_k \cap \tilde{K}_{\frac{1}{2},1}). \tag{30}$$

For $l = 1, \ldots, R_k^1$, we have

$$\frac{1}{2}\pi\rho^2(1 - \delta(r)) \leq \mathscr{H}^2(P_k^l \cap \tilde{K}_{\rho,1}) \qquad (\rho \in (0, \rho_k]), \tag{31}$$

where $\delta(r) \to 0$ as $r \to \infty$, since $\partial A \in C^2$.

Furthermore, comparing P_k^l with either of the parts into which $\partial P_k^l \cap A_r$ divides $(\partial D_{\rho_k} \times (-\sigma_0, \sigma_0)) \cap A_r$, and using (27)

$$\mathscr{H}^2(P_k^l) \leq \frac{1}{2}\pi\rho_k^2 + \pi\rho_k\sigma_0 + \varepsilon_{k,l}.$$

We now choose k so large that $\varepsilon_k < \pi\sigma_0$ (Note that the choice of σ_0 leading to the deletion of P_k^l for $l = R_k^1 + 1, \ldots, R_k^2$ did not depend on k.

Thus

$$\mathscr{H}^2(P_k^l) \leq \frac{1}{2}\pi\rho_k^2 + 2\pi\sigma_0 \tag{32}$$

(30) and (31) imply that R_k^1 is bounded independent of k. After selection of a subsequence, we find a positive integer n and

$$\rho_k \to \rho_0 \in \left[\frac{3}{4}, 1\right] \qquad \text{as} \quad k \to \infty$$

as for $l = 1, \ldots, n$

$$\underline{\underline{v}}(\pi_{\rho_k^{-1}} P_k) \quad \text{converges to a varifold } W_l$$

with (using (31), (32), (33))

$$\frac{1}{2}\pi\rho^2(1 - \delta(r)) \leq \| W_l \| (\tilde{K}_{\rho,1}) \qquad \text{for each} \quad \rho \in [0, 1] \tag{33}$$

$$\| W_l \| (\tilde{K}_{1,1}) \leq \frac{\pi}{2} + 2\pi\sigma_0\rho_0^{-2} \tag{34}$$

$$\leq \frac{\pi}{2} + 20\sigma_0$$

$$\operatorname{spt} \| W_l \| \subset \tilde{K}_{1,\sigma_0} \tag{35}$$

$$(\mu_{r\rho_0^{-1}} W) \llcorner \tilde{K}_{\frac{1}{2},1} \times G(3,2) = \sum_{l=1}^{n} W_l \llcorner \tilde{K}_{\frac{1}{2},1} \times G(3,2). \tag{36}$$

Since $\tilde{K}_{1-\sigma_0,\sigma_0} \subset U(0,1)$, (33) and (35) imply

$$(37) \qquad \| W_l \| (U(0,1)) \geq \| W_l \| K_{1-\sigma_0,\sigma_0} = \| W_l \| K_{1-\sigma_0,1} \geq \frac{\pi}{2}(1-\delta(r))(1-\sigma_0)^2 .$$

Since $U(0,1) \cap A \subset \tilde{K}_{1,1}$, (34) yields

$$(38) \qquad \| W_l \| U(0,1) \leq \frac{\pi}{2} + 20\sigma_0 .$$

Since we can make σ_0 and $\delta(r)$ as small as we want by choosing r sufficiently large (satisfying (23)), we obtain, using the monotonicity at the free boundary of [GJ]

$$(39) \qquad \Theta(\| W \|, x_0) = \frac{n}{2}.$$

We now apply the first part of the proof of $\left(\mu_{\rho_k^{-1}}(P_k^l)\right)$ instead of (N_k) $(l = 1, \ldots, n)$. This, together with the interior regularity of [AS, § 5] implies that each W_l is a stationary integral varifold with density 1 $\| W_l \|$-almost everywhere. Taking σ_0 in (38) sufficiently small, the free boundary regularity of [GJ] implies

$$W_l \llcorner \tilde{K}_{\frac{1}{2},1} \times G(3,2) = \underline{\underline{v}}(M_l), \qquad l = 1, \ldots, n$$

where M_l is a minimal surface which can be represented as a graph over $D_{\frac{1}{2}} \cap \overline{A}_{r\rho_0^{-1}}$:

$$M_l = \{ (x_1, x_2, x_3) : x_3 = u_l(x_1, x_2), \quad x \in D_{\frac{1}{2}} \cap \overline{A}_{r\rho_0^{-1}} \}$$

By (39) (remembering $x_0 = 0$) and (36),

$$u_l(0) = 0 \qquad (l = 1, \ldots, n).$$

Since for $l, m \in \{ 1, \ldots, n \}$ either $u_l \leq u_m$ or $u_l \geq u_m$ in $D_{\frac{1}{2}} \cap \overline{A}_{r\rho_0^{-1}}$ by construction of W_l, and since we can apply the strong maximum principle to the difference of two solutions of the minimal surface equation also at boundary points, $u_l = u_m$ on $D_{\frac{1}{2}} \cap \overline{A}_{r\rho_0^{-1}}$. Hence

$$\mu_{r\rho_0^{-1}\#} W \llcorner \tilde{K}_{\frac{1}{2},1} \times G(3,2) = n\underline{\underline{v}}(M_1).$$

In order to finish the proof, we have to show that at each $x_0 \in \partial A \cap U \cap \operatorname{spt} \| W \|$, there is a varifold tangent C of V of the form $n\underline{\underline{v}}(H)$ with H a half plane and $n \in \mathbb{N}$.

W. l. o. g. $x_0 = 0$ again.

Let $C \in \operatorname{Var Tan}(W, x_0)$

$$C = \lim_{k \to \infty} \mu_{t_k \#} W \qquad \text{for some sequence} \quad (t_k).$$

We choose a sequence (N_k) in $\mathcal{M}$ with

$$N_k = \mu_{r_k}(\psi_k^1(S))$$
$$\underline{\underline{v}}(N_k) \to C \quad \text{as} \quad k \to \infty$$
$$\mathcal{H}^2(N_k) \le \mathcal{H}^2(N) + \tilde{\varepsilon}_k \quad \forall N = \psi_1(S) \quad \psi \in I(U, A)$$
$$\tilde{\varepsilon}_k \to 0 \quad \text{as} \quad k \to \infty .$$

It follows that C is stationary. We reflect C across Tan (A, x_0) and obtain a stationary $\tilde{C}$ (cf. [GJ; 4.11]). We apply the interior arguments of [AS; § 6] to $\tilde{C}$ and deduce that it is contained in a plane. From the two remaining possibilities, the first one, namely that C is a halfplane (containing the interior normal of A at x_0) was already treated above. Therefore we only have to exclude the second possibility, namely

$$C = \operatorname{Tan}(A, x_0) .$$

Let $B_r = B(x_0, r) \cap A$ and assume w. l. o. g. $B_1 \subset U$. By the usual replacement argument, we can assume that each N_k intersects B_{r_k} in a number of disks $P_k^1, \ldots, P_k^{R_k}$ for a suitable $r_k \in \left(\frac{3}{4}, 1\right]$.

Using the coarea formula, for given $\varepsilon > 0$ we can also assume that for all sufficiently large k

$$\mathcal{H}^1(N_k \cap \partial B_{r_k} \cap \{ x : \operatorname{dist}(x, C) > \varepsilon \}) < \varepsilon .$$

It follows that if one of the disks P_k^i, $i \in \{ 1, \ldots, R_k \}$ has part of its boundary on ∂A, i. e. $P_k^i \cap \partial A \neq \emptyset$, we can replace it by a region $A_k^i \subset \partial B(x_0, r_k) \cap A$ with

$$\mathcal{H}^2(A_k^i) \le c\varepsilon$$

where c is a fixed constant. Lemma 2 (again after rescaling and perturbing so that the proper assumptions are satisfied) implies that those P_k^i do not contribute to the limit and can hence be discarded.

Therefore, we may assume

$$N_k \cap \partial A \cap B_{3/4} = \emptyset .$$

We let

$$A_\varepsilon := \{ x \in A : \operatorname{dist}(x, \operatorname{Tan}(A, x_0)) \ge \varepsilon \} .$$

Since A is strictly convex, we may assume if ε is small

$$A \backslash A_\varepsilon \subset B_{3/4} .$$

By smoothing out the corner $\partial A \cap \{ x : \operatorname{dist}(x, \operatorname{Tan}(A, x_0)) = \varepsilon \}$ we can obtain a convex $\tilde{A}_\varepsilon \subset A$ with boundary of class C^2. We can also assume that $\partial \tilde{A}_\varepsilon$ is intersected transversally by all N_k. Thus, we can apply Thm. 1 of [AS] and produce a minimizing sequence

$$(\tilde{N}_k) \subset \tilde{A}_\varepsilon$$

with $\tilde{N}_k \cap \tilde{A}_{2\varepsilon} \subset N_k \cap \tilde{A}_{2\varepsilon}$ and

$$\mathscr{H}^2(\tilde{N}_k) \leq \mathscr{H}^2(N_k) - \mathscr{H}^2(N_k \sim \tilde{N}_k \cap \tilde{A}_{2\varepsilon}).$$

Therefore, by interior regularity $\underline{\underline{v}}(\tilde{N}_k)$ converges to a stationary varifold $\tilde{W}$ with spt $\| \tilde{W} \| \subset \tilde{A}_\varepsilon$ and

$$W \llcorner A_{2\varepsilon} \times G(3,2) = W \llcorner A_{2\varepsilon} \times G(3,2],$$

and

$$\operatorname{spt} \| \tilde{W} \| = \tilde{M}$$

where $\tilde{M}$ is an embedded minimal surface in the interior of $\tilde{A}_\varepsilon$. On the other hand, by interior regularity as well, also spt $\| W \|$ is represented by an embedded minimal surface M in the interior of A. Of course

$$M \cap \tilde{A}_\varepsilon = \tilde{M} \cap \tilde{A}_\varepsilon .$$

Therefore, by unique continuation, M and $\tilde{M}$ also coincide in the interior of A. It follows

$$\operatorname{spt} \| W \| \cap A \backslash A_\varepsilon = \emptyset$$

and in particular $x_0 \notin \operatorname{spt} \| W \|$ which is a contradiction and excludes the possibility

$$C = \operatorname{Tan}(A, x_0).$$

Since Var Tan $(W, x_0) \neq \emptyset$, this completes the proof.

THEOREM 2. — *Let* U *be an open 3-cell in* $\mathbb{R}^3$, S *an embedded surface in* $\tilde{A}$ *which intersects* ∂U *and* ∂A *transversally. Suppose* $S \cap \partial A$ *is connected.* (ψ^j) *a sequence in* I(U, A) *with*

$$(40) \quad \lim_{j \to \infty} \operatorname{area}(S^j) = \inf \{ \operatorname{area}(\psi_1(S) : \psi \in I(U, A) \} \qquad (S^j := \psi_1^j(S)).$$

Suppose

$$(41) \quad W = \lim_{j \to \infty} \underline{\underline{v}}(S^j)$$

exists in $V_2(\mathbb{R}^3)$.

Then

$$(42) \quad W \llcorner U \cap \overline{A} \times G(2,3) \in IV_2(\mathbb{R}^3)$$

$$(43) \quad \operatorname{spt} \| W \| \cap U \cap \overline{A} = M$$

where M *is a stable embedded min. surface in* $U \cap \overline{A}$ *meeting* ∂A *orthogonally,*

$$(44) \quad \partial M \cap U \cap A = \emptyset$$

*(*M *is not necessarily connected).*

Proof. — Since $S \cap U$ neither is necessarily connected nor a disk we first have to perform some reductions as in [MSY], § 3.

Suppose $\gamma > 0$ is given.
Assume that there is some $\tilde{S} = \psi_1(S)$, $\psi \in I(U, A)$ with

$$\text{area}(\tilde{S}\Delta S) < \gamma$$

and some curve λ on $\tilde{S}$ which (possibly together with a curve in ∂A) bounds a disk Δ in $U \cap A$ with

$$A \cap \Delta \cap \tilde{S} = \lambda$$

and

$$\text{area}\,\Delta < \gamma\,,$$

while none of the two parts into which λ divides its component of $\tilde{S} \cap U$ is a disk.

We then cut $\tilde{S}$ along λ, insert Δ into each part, smooth out the corners and move the two inserted disks a bit apart so that we get an embedded surface S_1 with

$$S_1 \cap \mathbb{R}^3 \setminus U = S \cap \mathbb{R}^3 \setminus U$$

$$\text{area}(S_1 \Delta S) < 3\gamma \tag{45}$$

while $S_1 \cap U$ has one more connected component than $S \cap U$.

We then perform a similar reduction with S_1 and so on until we obtain a surface S_k which allows no further such reduction. We note that the number k of possible such reductions is bounded independent of γ by the number of components of $S \cap U$ and their topological complexity. By (45)

$$\text{area}(S_k \Delta S) < 3k\gamma\,. \tag{46}$$

Hence, we can find as in [MSY, § 3] subsequences q_j and $(\tilde{S}^j)$ with (after selection of a subsequence of (S^j))

$$\text{area}(S^{q_j} \Delta \tilde{S}^j) \leq \frac{3K}{j} \tag{47}$$

where K is independent of j and $\tilde{S}^j$ allows no more such reduction for some fixed $\gamma > 0$.

(47) implies

$$\lim_{j \to \infty} \underline{v}(\tilde{S}^j) = \lim_{j \to \infty} \underline{v}(S^{q_j})\,. \tag{48}$$

Thus, we can assume w. l. o. g. that already our original sequence (S^j) allowed no such reductions for some fixed γ.

The proof is then completed by simple modifications of the arguments of [MSY, Th. 2 and § 5] involving Th. 1 (actually in the present context where the ambient space is $\mathbb{R}^3$ instead of a general three dimensional manifold, the proof can even be simplified compared to [MSY]).

§ 3. CURVATURE ESTIMATES FOR STABLE MINIMAL SURFACES AT FREE BOUNDARIES AND AN ABSTRACT REGULARITY THEOREM

We want to extend the interior curvature estimates for stable minimal hypersurfaces of [SRS] to such hypersurfaces which solve a free boundary problem.

Let S be a hypersurface in $\mathbb{R}^{n+1}$, $0 \in S$, and X be a hypersurface in $B^{n+1}(0, \rho_0)$ for some $\rho_0 > 0$, with $\partial X \cap B^{n+1}(0, \rho_0) = S \cap X \cap B^{n+1}(0, \rho_0)$ and X lies entirely on one side of $S \cap B(0, \rho_0)$. Suppose X is embedded and is stationary and stable w. r. t. the area integrand. This implies in particular that X meets S orthogonally.

Suppose S is of class C^4, and $S \cap B(0, \rho_0)$ is diffeomorphic to the n-dimensional disk.

We now perform a C^4-transformation f of coordinates with the following properties

i) $$f(B^{n+1}(0, \rho_0)) = B^{n+1}(0, \rho_0), \qquad f(0) = 0$$

ii) $$f(S \cap B^{n+1}(0, \rho_0)) = (\{0\} \times \mathbb{R}^n) \cap B^{n+1}(0, \rho_0),$$

i. e. $S \cap B^{n+1}(0, \rho_0)$ is mapped into the hyperplane orthogonal to the first coordinate axis.

iii) The area integrand is transformed into a C^3 integrand F satisfying properties (1.2)-(1.6) of [SRS].

iv) Normal vectors to S are mapped onto normal vectors to $f(S)$.

Let $M = f(X)$. Assume $M \in C^2$.

Let e_i be a moving orthonormal frame on M.

Let ξ be a vector field on $B^{n+1}(0, \rho_0)$ with compact support.

Let ν be the normal vector field, $x \in M$, α, $\beta \in T_xM$, and A the second fundamental form of M, i. e.

$$A(\alpha, \beta) = - \langle D_\beta \nu, \alpha \rangle,$$

where D is covariant differentiation on M.

The first variation of F at M w. r. t. ξ is given by

$$\delta F(M, \xi) = \int_M \operatorname{div}_M \xi \, d\mathcal{H}^n + R(\xi), \tag{1}$$

where

$$|R(\xi)| \leq c_1 \mu_1 \int_M (|\xi| + |x| \, |\nabla \xi|) d\mathcal{H}^n, \tag{2}$$

where $x \in M$, ∇ is the derivative in $\mathbb{R}^{n+1}$, and c_1, μ_1 are the constants of [SRS, (1.9)].

Similarly, the second variation is given by

$$(3)\qquad \delta^2 F(M, \xi) = \int_M \Big(\sum_{i=1}^{n} |(D_{e_i}\xi)^\perp|^2 + (\operatorname{div}_M \xi)^2 - \sum_{i=1}^{n} \langle e_i, D_{e_j}\xi \rangle \langle e_j, D_{e_i}\xi \rangle \Big) d\mathcal{H}^n + \tilde{R}(\xi)$$

with

$$(4)\qquad |\tilde{R}(\xi)| \le c_2 \mu_1 \int_M (\mu_1 |\xi|^2 + |\xi| |\nabla\xi| + |x| |\nabla\xi|^2) d\mathcal{H}^n$$

as in [SRS, (1.10), (1.12)], where $^\perp$ denotes orthogonal projection onto the ν-direction $([\nu(x)] \oplus T_x M = T_x \mathbb{R}^{n+1})$.

We now use a normal vector field $\xi = \zeta\nu$ in (3) to obtain

$$(5)\qquad \delta^2 F(M, \zeta\nu) = \int_M (|\nabla\zeta|^2 - |A|^2\zeta^2 + H^2\zeta^2) d\mathcal{H}^n + \tilde{R}(\zeta\nu)$$

where H is the mean curvature of M, cf. [SRS, (1.14)].

We now assume that M is stationary w. r. t. all variations ξ with $\xi(x) \in \{0\} \times \mathbb{R}^n$ for $x \in \{0\} \times \mathbb{R}^n$, i. e. variations which are tangent to the supporting hyperplane, i. e.

$$(6)\qquad \delta F(M, \xi) = 0 \qquad \text{for such vector fields}.$$

This implies

$$(7)\qquad \nu(x) \in \{0\} \times \mathbb{R}^n \qquad \text{for} \quad x \in \{0\} \times \mathbb{R}^n \cap M.$$

(Note $\{0\} \times \mathbb{R}^n = f(S)$).

Furthermore, we assume that M is also stable w. r. t. such variations, i. e. (using (7))

$$(8)\qquad \delta^2 F(M, \zeta\nu) \ge 0$$

for all compactly supported ζ.

Note that this equivalent with the original assumption that $X = f^{-1}(M)$ was stationary and stable w. r. t. variations which are tangential to S. As in [SRS, (1.17)] we deduce

$$(9)\quad \int_M |A|^2\zeta^2 d\mathcal{H}^n \le \int_M |\nabla\zeta|^2 d\mathcal{H}^n + c_3\mu_1 \int_M \{ \mu_1\zeta^2 + \zeta|\nabla\zeta| + \zeta^2|A| + |x| |\nabla\zeta|^2 + \zeta x |\zeta^2|A|^2 + \mu_1 |x|^2\zeta^2|A|^3 \} d\mathcal{H}^n$$

where c_3 depends only on n, μ_1 is as in [SRS, (1.4)] and hence depends on the C^4-norm of S, and ζ is any Lipschitz function on M vanishing near $M \cap \partial B^{n+1}(0, \rho_0)$.

The crucial step now is to use (9) in order to extend Lemma 1 of [SRS] to the present situation. The constants $c_3, c_4, c_5, \ldots$ in the sequel will depend only on n, μ, $\mu_1\rho_0$ (μ, μ_1 as in [SRS, (1.3)-(1.6)]).

LEMMA 1. — *Let* M *as before be a* C^2-*surface in* $B^{n+1}(0, \rho_0)$ *with* $\partial M \cap B^{n+1}(0, \rho_0) = f(S) \cap M \cap B^{n+1}(0, \rho_0)$ *which is stationary and stable w. r. t.* F.

There exists $\varepsilon_0 > 0$, *depending only on* n, μ, $\mu_1\rho_0$, *with the property that if* $\mu_1\rho \leq \varepsilon_0$, $v_0 \in S^n \cap T_0 f(S)$, ϕ *is a bounded locally Lipschitz function vanishing in a neighbourhood of* $\partial M \cap C(0, \rho)$, *where* $C(0, \rho) = B^n(0, \rho) \times \mathbb{R}$, *then*

$$(10)\quad \int_M |A|^2\phi^2 d\mathcal{H}^n \leq c_3 \int_M (1-v-v_0)^2)|\nabla\phi|^2 d\mathcal{H}^n + c_3\mu_1^2\int_M \phi^2 d\mathcal{H}^n .$$

Remark. — We have tacitly assumed that M is complete. As in [SRS] one can also handle singularities, i. e. points where $\overline{M}$ is not locally an embedded hypersurface as long as the $(n-2)$ dimensional Hausdorff measure of the singular set vanishes.

Proof. — We use $\zeta = \phi\ (1 - v - v_0)^2)^{\frac{1}{2}}$ as a test function in (9). It is standard to estimate

$$|\nabla(1 - (v . v_0)^2)^{\frac{1}{2}}| \leq |A|,$$

and hence ζ is locally Lipschitz.

W. l. o. g. $2\rho \leq \rho_0$.

Then (9) gives (cf. [SRS, (2.1)])

$$(11)\quad \int_M |A|^2(1 - v\cdot v_0)^2)\phi^2 d\mathcal{H}^n$$
$$\leq \int_M \{ c_4(1 - (v\cdot v_0)^2)|\nabla\phi|^2 + 2\phi(1 - (v\cdot v_0)^2)^{\frac{1}{2}}\nabla\phi$$
$$\cdot \nabla(1 - (v\cdot v_0)^2)^{\frac{1}{2}} + \phi^2|\nabla(1 - (v - v_0)^2)^{\frac{1}{2}}|^2 \} d\mathcal{H}^n$$
$$+ c_4\int_M (\mu_1\rho|A|^2 + \mu_1|A| + \mu_1^2)\phi^2 d\mathcal{H}^n .$$

We now choose an orthonormal frame $e_1, \ldots, e_n$ on M with the property that on $S \cap M$, $e_1, \ldots, e_{n-1}$ are tangential to S and e_n is normal. We look at the second term on the right hand side of (11) which equals

$$(12)\quad \frac{1}{2}\int_M \nabla\phi^2 \cdot \nabla(1 - v\cdot v_0)^2)d\mathcal{H}^n$$
$$= \frac{1}{2}\int_M \phi^2\Delta(v\cdot v_0)^2 d\mathcal{H}^n - \int_{S\cap M} \phi^2(v\cdot v_0)\langle \operatorname{grad}(v\cdot v_0), e_n\rangle d\mathcal{H}^{n-1}$$

integrating by parts, since ϕ vanishes near $M \cap \partial C(0, \rho)$.

But

$$\langle \operatorname{grad}(\nu \cdot \nu_0), e_n \rangle = \langle \nu_{e_i}, e_j \rangle \langle e_j, \nu_0 \rangle \langle e_i, e_n \rangle$$

since $\langle \nu_{e_i}, \nu \rangle = 0$, employing the standard summation convention. Now

$$\begin{aligned} \langle e_i, e_n \rangle &= 0 \qquad \text{if} \quad i \neq n, \\ \langle e_n, \nu_0 \rangle &= 0 \qquad \text{by assumption} \\ \langle \nu_{e_n}, e_j \rangle &= 0 \\ = \langle \nu_{e_j}, e_n \rangle &= 0 \qquad \text{for} \quad j \neq n, \end{aligned}$$

since $\nu, e_1, \ldots, e_{n-1}$ are always tangent to the hyperplane $f(\mathrm{S}) = \{0\} \times \mathbb{R}^n$, whereas e_n is normal to it.

Hence

$$\langle \operatorname{grad}(\nu \cdot \nu_0), e_n \rangle = 0,$$

and there is no boundary contribution in (12).

Hence we can calculate as in [SRS, (2.8)]

$$\begin{aligned} (13) \quad & \frac{1}{n}\int_{\mathrm{M}} |\mathrm{A}|^2 \phi^2 d\mathscr{H}^n \\ & \leq c_4 \int_{\mathrm{M}} (1 - (\nu \cdot \nu_0)^2) |\nabla\phi|^2 d\mathscr{H}^n + \int_{\mathrm{M}} \phi^2 \mathrm{H}_{e_i}(e_i \cdot \nu_0)(\nu \cdot \nu_0) d\mathscr{H}^n \\ & + c_4 \int_{\mathrm{M}} (\mu_1 \rho |\mathrm{A}|^2 + \mu_1 |\mathrm{A}| + \mu_1^2)\phi^2 d\mathscr{H}^n + \frac{2}{n}\int_{\mathrm{M}} \phi^2 |\mathrm{A}| \cdot |\mathrm{H}| \, d\mathscr{H}^n \end{aligned}$$

We examine the second term on the right hand side of (13):

$$\begin{aligned} \operatorname{div}_{\mathrm{M}}(\mathrm{H}\phi^2(e_i \cdot \nu_0)(\nu \cdot \nu_0)e_i) &= \mathrm{H}_{e_i}\phi^2(e_i \cdot \nu_0)(\nu \cdot \nu_0) + 2\mathrm{H}\phi \cdot \phi_{e_i}(e_i \cdot \nu_0)(\nu \cdot \nu_0) \\ & \quad - \mathrm{H}\phi^2 \cdot \mathrm{H}(\nu \cdot \nu_0)^2 + \mathrm{H}\phi^2(e_i \cdot \nu_0)h_{ij}(e_j \cdot \nu_0). \end{aligned}$$

If we integrate over M we obtain

$$\int_{\mathrm{M}} \operatorname{div}_{\mathrm{M}}(\mathrm{H}\phi^2(e_i \cdot \nu_0)(\nu \cdot \nu_0)e_i) = \int_{\mathrm{M} \cap \mathrm{S}} \mathrm{H}\phi^2(e_i \cdot \nu_0)(\nu \cdot \nu_0) \langle e_i, e_n \rangle \, d\mathscr{H}^{n-1} = 0,$$

since $(e_n \cdot \nu_0) = 0$, since ν_0 is tangential to $f(\mathrm{S})$. Hence the boundary contribution vanishes again, and we conclude as in [SRS, p. 751]

$$\begin{aligned} \int |\mathrm{A}|^2 \phi^2 d\mathscr{H}^n \leq c_5 & \int_{\mathrm{M}} (1 - (\nu \cdot \nu_0)^2) |\nabla\phi^2| \\ & + c_6 \int_{\mathrm{M}} \{ \mu_1 \rho |\mathrm{A}|^2 + \mu_1 |\mathrm{A}| + \mu_1^2 \} \phi^2 d\mathscr{H}^n, \end{aligned}$$

and if $\mu_1 \rho$ is small, we can absorb the terms with $|\mathrm{A}|$ and $|\mathrm{A}|^2$ into the left hand side. q. e. d.

It is now fairly straightforward to extend Theorems 1-3 of [SRS] to the present context to obtain.

LEMMA 2. — *Suppose* S *is a surface of class* C^4 *in* $\mathbb{R}^3$, $0 \in S$, S *intersecting* $B^3(0, \rho_0)$ *in a disk.*

Suppose M *is a complete surface of class* C^2 *with boundary* $\partial M \cap B^3(0, \rho_0) = S \cap M \cap B^3(0, \rho_0)$ *which is stationary and stable with respect to the area integral and variations tangent to* S.

Suppose

$$\mathscr{H}^2(M \cap B^3(0, \rho_0)) \leq \mu\rho_0^2 .$$

Then there exists $\delta_0 > 0$, *depending only on* μ, $\mu_1\rho_0$ *(if the transformation* f *introduced above leads to an integrand satisfying (1.2)-(1.6) of* [SRS] *with constants* μ, μ_1*) with the property that if* $x \in M \cap B^3\left(0, \frac{1}{4}\rho_0\right)$, $0 < \rho < \frac{1}{4}\rho_0$, M′ *is the connected component of*

$$M \cap C(x, \rho) \, (= \{ y=(y_1, y_2, y_3) \in \mathbb{R}^3 : |x_1 - y_1|^2 + |x_2 - y_2|^2 \leq \rho^2 \}),$$

and

$$\sup_{y \in M'} | y_3 - x_3 | \leq \delta_0 \rho$$

$$\mu_1 \rho \leq \delta_0$$

then $M' \cap C\left(x, \frac{1}{2}\rho\right)$ *consists of a disjoint union of graphs of functions* $u_1 < u_2 < \ldots < u_k$ *defined on* $\left\{(y_1, y_2) : |y_1 - x_1|^2 = |y_2 - x_2|^2 \leq \left(\frac{1}{2}\rho\right)^2\right\}$ *satisfying*

$$\sup_{\substack{\{(y_1, y_2) : |y_1 - x_1|^2 \\ + |y_2 - x_2|^2 \leq (\frac{1}{2}\rho)^2\}}} (| Du_i | + \rho | D^2 u_i |) \leq c_7 \delta_0$$

$(i = 1, \ldots, k)$, *where* c_7 *depends only on* μ, $\mu_1 \rho_0$.

Using the techniques of [SRS] it is not too difficult to show

LEMMA 3. — *Suppose* (S_n) *is a sequence of surfaces in* $\mathbb{R}^3$, *for which* $S_n \cap B^3(0, \rho_0)$ *is a disk with uniformly (i. e. independently of* n*) bounded* C^4*-norm (in the sense that the corresponding transformations* f_n, *mapping* S_n *onto a disk and satisfying* i)-iv) *above, have uniformly bounded* C^4*-norms). Suppose* (M_n) *is a sequence of complete orientable surfaces with boundary in* $B^3(0, \rho_0)$, *with*

$$0 \in M_n \cap S_n$$

$$\partial M_n \cap B^3(0, \rho_0) = S_n \cap M_n \cap B^3(0, \rho_0)$$

which are stationary and stable w. r. t. the area integrand and variations tangent to S_n.

Suppose moreover

$$\limsup_{n\to\infty} \mathscr{H}^2(\mathrm{M}_n \cap \mathrm{B}^3(0, \rho_0)) < \infty .$$

Then after selection of a subsequence, we can find a varifold

$$\mathrm{V} = \lim_{n\to\infty} \left| \mathrm{M}_n \cap \mathrm{B}^3\left(0, \frac{1}{2}\rho_0\right)\right|$$

with

$$\mathrm{spt}\, \| \mathrm{V} \| \cap \mathrm{B}^3\left(0, \frac{1}{2}\rho_0\right) = \mathrm{M} \cap \mathrm{B}^3\left(0, \frac{1}{2}\rho_0\right),$$

where M *is a complete orientable surface with boundary in* $\mathrm{B}^3\left(0, \frac{1}{2}\rho_0\right)$,

$$\mathscr{H}^2\left(\mathrm{M} \cap \mathrm{B}^3\left(0, \frac{1}{2}\rho_0\right)\right) < \infty$$

$$\partial \mathrm{M} \cap \mathrm{B}^3\left(0, \frac{1}{2}\rho_0\right) = \mathrm{S} \cap \mathrm{M} \cap \mathrm{B}^3\left(0, \frac{1}{2}\rho_0\right),$$

where S *is a surface of class* C^4.

LEMMA 4. — *Suppose* S *and* M *are as in Lemma 2, in particular*

$$\mathscr{H}^2(\mathrm{M} \cap \mathrm{B}^3(0, \rho_0)) \leq \mu \rho_0^2 .$$

Then there is some constant c_8, *depending only on* μ, $\mu_1 \rho_0$ *with*

$$\sup_{\mathrm{M}\cap\mathrm{B}^3(0,\rho_{0/2}} |\mathrm{A}| \leq c_8 \rho_0^{-1}$$

where A *is the second fundamental form of* M.

We indicate the modifications of the arguments of [SRS] required to prove the preceding lemmata.

We look at the point $x = 0$ and try to represent a surface M satisfying the assumptions of Theorem 1 [SRS] (in the modified form for our free boundary problem) as a graph over the plane which is orthogonal to the normal vector $\nu(0)$ of M ($\nu(0) = e_{n+1}$ in the notations of [SRS]). Note that $\nu(0)$ is tangent to $f(\mathrm{S})$.

In the definition of the excess E_σ on p. 757, we can allow only vectors v_i which are tangent to the plane $f(\mathrm{S})$, since we had to make that restriction in Lemma 1.

Lemma 1 then has to be applied with $v_0 = \nu(0)$ (p. 753 and p. 763) and with $v_0 = v_i$, where v_i realizes the infimum in the definition of E_σ (cf. p. 760, p. 763).

For the harmonic comparison function v_i on p. 766 we then have to require that the normal derivative vanishes at the free boundary.

(In order to fix the notation, assume

$$f(\mathrm{S}) = \{ x \in \mathbb{R}^3 : x_1 = 0 \}, \qquad \nu(0) = (0,0,1), \qquad \mathrm{M} \subset \{ x_1 \geq 0 \}.$$

Then

$$\begin{aligned} v_i &= v_i(x_1 x_2) && (x_1 \geq 0) \\ \Delta v_i &= 0 && \text{for} \quad x_1 \geq 0, \quad x_1^2 + x_2^2 < (\sigma/2)^2 \\ v_i &= \bar{u}_i && \text{if} \quad x_1^2 + x_2^2 = (\sigma/2)^2 \\ \frac{\partial v_i}{\partial x_1} &= 0 && \text{for} \quad x_1 = 0) \end{aligned}$$

v_i then can be reflected as a harmonic function across $\{ x_1 = 0 \}$, and the estimates (4.8) pertain.

Since the graphs u_i also meet $f(\mathrm{S})$ orthogonally,

$$\frac{\partial u_i}{\partial x_1} = 0 \qquad \text{for} \quad x_1 = 0,$$

and hence as on p. 767 for ζ a compactly on $\Omega^{(\sigma)}$ supported Lipschitz function

$$\int_{\mathrm{B}_{\sigma/2} \cap \{x_1 > 0\}} \mathrm{D}u_i \cdot \mathrm{D}\zeta dx$$
$$= \int_{\mathrm{B}_{\sigma/2} \cap \{x_1 > 0\}} \left(1 - \frac{1}{(1 + |\mathrm{D}u_i|^2)^{\frac{1}{2}}}\right) \mathrm{D}u_i \cdot \mathrm{D}\zeta dx - \int_{\mathrm{B}_{\sigma/2} \cap \{x_1 > 0\}} \zeta(x) \mathrm{H}(x, u_i(x)) dx$$

$(x = (x_1, x_2)$ here).

Finally, we note that the vector

$$\nu_0 = (-\mathrm{D}v_i(0), 1) \cdot (1 + |\mathrm{D}v_i(0)|^2)^{-\frac{1}{2}}$$

on p. 770 again is tangential to $f(\mathrm{S})$ and hence admissible.

Moreover, when one performs blowing ups, then in the limit S becomes a plane, i. e. F becomes the area functional, and we can reflect M, since stationary w. r. t. F, across $\mathrm{S} = f(\mathrm{S})$ to apply interior arguments (cf. 4.11 in [GJ]).

Detailed arguments in a similar situation were carried out in [GJ].

We note that also the arguments of chapter 6 of [P] can be carried over to free boundaries without essential difficulties. The arguments of chapter 5 of [P], which are taken from [SSY], however, are not readily generalizable for several reasons. Therefore, we had to take recourse to [SRS] for the curvature estimates.

Let A be a bounded open strictly convex subset of $\mathbb{R}^3$ with $\partial \mathrm{A} \in \mathrm{C}^4$.

Let $x \in \overline{\mathrm{A}}$, $\sigma, \sigma_1, \sigma_2 > 0$

$$\mathrm{U}(x, \sigma) := \{ \tilde{x} \in \overline{\mathrm{A}} : |\tilde{x} - x| < \sigma \}$$
$$\mathrm{A}(x, \sigma_1, \sigma_2) := \{ \tilde{x} \in \overline{\mathrm{A}} : \sigma_1 < |\tilde{x} - x| < \sigma_2 \}.$$

We fix $x_0 \in \bar{A}$ and $\sigma > 0$ and put

$$U = U(x_0, \sigma) \setminus \{ x_0 \}.$$

Let $\mathscr{A} := \mathscr{A}(x_0, \sigma)$ be a nonempty set of varifolds V in $V_2(\mathbb{R}^3)$ with support contained in $\bar{A}$ which are stationary for the area integral w. r. t. variations tangent to ∂A and which enjoy the following property

$$\text{for all} \quad p \in \bar{A}, \quad 0 < \rho_1 < \rho_2 < \sigma \qquad \text{for which} \quad A(p, \rho_1, \rho_2) \subset \dot{U}$$

there is some $V^* \in \mathscr{A}$ with

i) $V^* \llcorner G_2(\mathbb{R}^3 \sim \bar{A}(p, \rho_1, \rho_2)) = V \llcorner G_2(\mathbb{R}^3 \sim \bar{A}(p, \rho_1, \rho_2))$

ii) $V^* \llcorner G_2(A(p, \rho_1, \rho_2)) \in IV_2(\mathbb{R}^3)$ (i. e. an integral varifold)

and

$$\operatorname{spt} \| V^* \| \cap A(p, \rho_1, \rho_2) = M$$

where M is a not necessarily connected embedded minimal surface with boundary

$$\partial M \cap A(p, \rho_1, \rho_2) = \partial A \cap M \cap A(p, \rho_1, \rho_2)$$

which is stable w. r. t. variations tangent to ∂A.

It is now easy, using the arguments of chapter 7 of [P] (cf. also [SRS], chapter 7 and [SS]) in conjunction with Lemmata 2-4 to prove the following abstract regularity theorem (cf. [SS]).

LEMMA 5. — *Let $x_0 \in \bar{A}$ and $\sigma > 0$ so small that $\partial B(x_0, \sigma) \cap \partial A$ is empty or a circle.*

Let $V \in \mathscr{A}(x_0, \sigma)$.

Then V is regular in $U(x_0, \sigma)$ in the sense that

$$V \llcorner G_2(U(x_0, \sigma)) \in IV_2(\mathbb{R}^3)$$
$$\operatorname{spt} \| V \| \cap U(x_0, \sigma) = M$$

where M is a (not necessarily connected) minimal surface with boundary

$$\partial M \cap U(x_0, \sigma) = M \cap \partial A \cap U(x_0, \sigma)$$

which is stable w. r. t. variations tangent to ∂A. In particular, M meets ∂A orthogonally.

Finally, if V^* is constructed from V as in the definition of $\mathscr{A}$, then $V^* = V$.

The idea of the proof is first to show that by comparison with a suitable sequence of replacements, every tangent cone of V is a plane with integer multiplicity. Then one selects spheres which are intersected transversally by $\operatorname{spt} \| V \|$ (using Sard's Lemma) to make suitable replacements which by definition of $\mathscr{A}$ again lead to stationary surfaces so that one can apply a unique continuation result for elliptic equations, taking the decomposition result of Lemma 2 into account. This gives regularity on annuli $A(p, \rho_1, \rho_2)$ for any $0 < \rho_1 < \rho_2 < \sigma$, and regularity at p then is obtained as in [P, 7.12].

§ 4. REGULARITY OF ALMOST MINIMIZING VARIFOLDS AT FREE BOUNDARIES

Besides considerations of free boundaries, we also use arguments of [SS].

LEMMA 1. — $\Sigma \in \mathcal{M}$, $\alpha > 0$, U *open in* $\mathbb{R}^3$, (ϕ^j) *sequence in* $I(\Sigma, U, A, \alpha)$, $\Sigma^j := \phi_1^j(\Sigma)$

$$\lim_{j\to\infty} \text{area}\,(\Sigma^j) = \inf\{\text{area}\,\psi_1(\Sigma),\ \psi \in I(\Sigma, U, A, \alpha)\}. \tag{1}$$

Then for each $x \in U$, *there exists* $\sigma \in (0, \text{dist}\,(x, \partial U))$ *with*

$$\lim_{j\to\infty} \text{area}\,\Sigma^j = \inf\{\text{area}\,\psi_1(\Sigma) : \psi \in I(B(x,\sigma), A)\}. \tag{2}$$

Proof. — We assume w. l. o. g. $x \in \partial A$ since the interior case is similar and already treated in [SS].

After selection of a subsequence

$$V = \lim_{j\to\infty} \underline{\underline{v}}(\Sigma^j) \quad \text{exists and is stationary}. \tag{3}$$

Given $\sigma_0 \in (0, \text{dist}\,(x, \partial U))$, using (3) and the monotonicity formula at the free boundary of [GJ] for V,

$$\text{area}\,(\Sigma^j \cap B(x,\sigma)) < c_1\sigma_0^2 \tag{4}$$

if j is greater than some $j(\sigma_0)$, where $c_1 = c_1(\kappa, \text{dist}\,(x, \partial U))$, and κ is the curvature of ∂A.

Using the coarea formula, and Sard's Lemma, for each j, we can choose $\sigma \in (\sigma_{0/2}, \sigma_0)$ for which Σ^j intersects $\partial B(x, \sigma)$ transversally and

$$\text{length}\,(\Sigma^j \cap \partial B(x,\sigma)) < 8c_1\sigma. \tag{5}$$

Let now $\psi^j \in I(B(x,\sigma), A)$ be given with

$$\text{area}\,(\psi_1^j(\Sigma^j) \cap B(x,\sigma)) < c_1\sigma_0^2 \le 4c_1\sigma^2. \tag{6}$$

We want to show that there is an isotopy

$$\psi^* \in I(B(x,\sigma), A)$$

with

$$\text{area}\,(\psi_t^*(\Sigma^j) \cap B(x,\sigma)) < c_2\sigma^2 \tag{7}$$

and

$$\psi_1^*(\Sigma j) = \psi_1^j(\Sigma^j) \tag{8}$$

where c_2 depends only on c_1 and κ.

Employing a diffeomorphism which changes areas only by some fixed factor (controlled from above and below by κ and an upper bound for σ_0), we can assume that $\partial A \in B(x, \sigma_0)$ is plane.

Since $\psi^j \in I(B(x, \sigma), A)$ there is some $\sigma_1 \in (0, \sigma)$ with

$$\psi^j \mid \mathbb{R}^3 \setminus B(x, \sigma_1) = id \mid \mathbb{R}^3 \setminus B(x, \sigma_1).$$

By (5), we can find σ_2, σ_3, $\sigma_1 < \sigma_2 < \sigma_3 < \sigma$, with the property that Σ^j intersects $\partial B(x, \tau)$ transversally for all $\tau \in [\sigma_2, \sigma_3]$ and

$$\text{length}\,(\Sigma^j \cap \partial B(x, \tau)) < c_3 \sigma \tag{9}$$

with $c_3 = 16c_1$, if $\sigma_2 \leq \tau \leq \sigma_3$ (σ_2 and σ_3 of course depend on Σ^j).

We introduce polar coordinates $r \in [0, \sigma]$ and $\theta \in S^2$ on $B(x, \sigma)$. Let

$$|d\psi_t^j| \leq K$$

$$0 < \theta < \min\left(1, \frac{1}{K}\right)$$

$$\beta(t, r) := \begin{cases} 1 + t(\theta - 1), & 0 \leq r \leq \sigma_2 \\ 1 + t\dfrac{(\sigma_3 - r)(\theta - 1)}{\sigma_3 - \sigma_2}, & \sigma_2 \leq r \leq \sigma_3 \\ 1, & \sigma_3 \leq r \leq \sigma \end{cases}$$

$$\gamma_t(r, \theta) := (\beta(t, r)r, \theta)$$

$$(\gamma_t \mid \mathbb{R}^3 \setminus B(x, \sigma_3) = \text{id} \mid \mathbb{R}^3 \setminus B(x, \sigma_3))$$

$$\psi_t^* := \begin{cases} \gamma_{3t} & 0 \leq t \leq \dfrac{1}{3} \\ \gamma_1 \circ \psi_{3t-1}^j & \dfrac{1}{3} \leq t \leq \dfrac{2}{3} \\ \gamma_{3-3t} \circ \psi_1^j & \dfrac{2}{3} \leq t \leq 1. \end{cases}$$

Using (9) and $\theta < \dfrac{1}{K}$, it is easy to check that ψ^* (after approximation by differentiable isotopies) satisfies (7) and (8).

We now choose

$$\sigma_0 \leq \sqrt{\frac{\alpha}{c_2}}.$$

Hence from (7), since $\sigma \leq \sigma_0$

$$\text{area}\left(\psi_t^*(\Sigma^j) \cap B(x, \sigma)\right) < c_2 \sigma_0^2 \leq \alpha.$$

So far, we have chosen a different σ, $\sigma_{0/2} < \sigma < \sigma_0$, for each j, but since

$$I(B(x, \tau_1), A) \subset I(B(x, \tau_2), A)$$

if $\tau_1 \leq \tau_2$, (2) holds with $\sigma = \sigma_{0/2}$.

LEMMA 2. — *Let* $U \subset \mathbb{R}^3$ *open* $U \cap A \neq \emptyset$, $\Sigma \in \mathcal{M}$, $\alpha > 0$, (ϕ^i) *sequence in* $I(\Sigma, U, A, \alpha)$ *with, abbreviating* $\Sigma^i = \phi^i_1(\Sigma)$,

$$\lim_{i\to\infty} |\Sigma^i| = \inf\{\,|\psi_1(\Sigma)| : \psi \in I(\Sigma, U, A, \alpha)\,\} \tag{10}$$

and

$$W = \lim_{i\to\infty} \underline{\underline{v}}(\Sigma^i) \tag{11}$$

exists in $V_2(\mathbb{R}^3)$.

Thus $W \llcorner G_2(U \cap \overline{A})$ *is an integral varifold and*

$$\operatorname{spt} \|W\| \cap U \cap \overline{A} = M\,,$$

where M *is a (not necessarily connected) minimal surface in* $U \cap \overline{A}$ *with*

$$\partial M \cap U = M \cap \partial A \cap U$$

which is stable w. r. t. variations tangent to ∂A.

A similar statement holds, if $\tilde{\Sigma}^i$ *is any union of components of* $\Sigma^i \cap U$ *and*

$$\tilde{W} = \lim_{i\to\infty} \underline{\underline{v}}(\tilde{\Sigma}^i) \tag{12}$$

exists in $V_2(U)$.

Proof. — By Lemma 1, for any $x \in U \cap \overline{A}$, there exists $\sigma \in (0, \operatorname{dist}(x, \partial U))$ with

$$\lim_{i\to\infty} |\Sigma^i| = \inf\{\,|\psi_1(\Sigma)| : \psi \in I(B(x,\sigma), A)\,\}\,. \tag{13}$$

By Sard's lemma, we can assume that each Σ^i meets $B(x, \sigma)$ transversally.

We also note that W is stationary in $U \cap \overline{A}$ and stable in $B(x, \sigma) \cap \overline{A}$ by (13), e. g.

We now use the idea of Pitts [P, 3.10, 3.11] to construct suitable regular stationary comparison surfaces, in order to apply Lemma 5 of § 3.

Let $x \in \operatorname{spt} \|W\| \cap U \cap \overline{A}$.

Let (χ^k) be a sequence in $I(B(x, \sigma), A)$ with, putting $\Sigma^{ik} = \chi^k_1(\Sigma^i)$

$$\lim_{k\to\infty} |\Sigma^{ik}| = \inf\{\,|\psi_1(\Sigma^i)| : \psi \in I(B(x,\sigma), A)\,\} \tag{14}$$

and for which

$$V^i := \lim_{k\to\infty} \underline{\underline{v}}(\Sigma^{ik}) \quad \text{exists in } V_2(\mathbb{R}^3)\,.$$

By Theorem 2 of § 2, $V^i \llcorner G_2(B(x, \sigma))$ is an integral varifold with

$$\operatorname{spt} \|V^i\| \cap B(x,\sigma) \cap \overline{A} = M^i \tag{15}$$

where M^i is a stable embedded minimal surface (w. r. t. variations tangent to ∂A),

$$\partial M^i \cap B(x, \sigma) = M^i \cap \partial A \cap B(x, \sigma)\,.$$

Selecting a subsequence, we can assume that

$$W^* = \lim_{i\to\infty} V^i \qquad \text{exists in } V_2(\mathbb{R}^3),$$

and by Lemma 3, § 3, it is an integral varifold with

$$\operatorname{spt} \| W^* \| \cap B(x, \sigma/2) \cap \overline{A} = M^*$$

where M^* is a stable embedded minimal surface with

$$\partial M \cap B(x, \sigma) = M \cap \partial A \cap B(x, \sigma).$$

The argument of Pitts ([P, 3.10, 3.11]) then implies that

$$V^* = \begin{cases} W^* & \text{in} \quad B(x, \sigma) \cap \overline{A} \\ W & \text{in} \quad (U \cap \overline{A}) \setminus B(x, \sigma) \end{cases}$$

is stationary.

Moreover, by the same argument, we can also perform replacements on annuli.

Hence, Lemma 2 follows from Lemma 5, § 3.

THEOREM. — *$x_0 \in \overline{A}$, $\sigma > 0$ so small that $\partial A \cap B(x_0, \sigma)$ is empty or a circle.*

V is almost minimizing (uniformly among disks in the sense of § 1) in $\dot{U} := U(x_0, \sigma) \setminus \{ x_0 \}$.

Then V is an integral varifold, and

$$\operatorname{spt} \| V \| \cap B(x_0, \sigma) \cap \overline{A} = M, \tag{13}$$

where M is a (not necessarily connected) embedded minimal surface with

$$\partial M \cap B(x_0, \sigma) = M \cap \partial A \cap B(x_0, \sigma) \tag{14}$$

which is stable w. r. t. variations tangent to ∂A.

Proof. — By the argument of Pitts [P, 3.3], V is stationary (w. r. t. variations tangent to ∂A in our case).

Let T be any annulus in $B(x_0, \sigma) \setminus \{ x_0 \}$. Then V is almost minimizing (in the above sense) in $T \cap \overline{A}$.

Hence, if $\varepsilon_n \to 0$, there is a sequence $\alpha_n \to 0$ and a sequence of disks $\Sigma^n \in \mathcal{M}$ with

$$\underline{\underline{F}}(V, \underline{\underline{v}}(\Sigma_n)) < \varepsilon_n \tag{15}$$

$$\Sigma_n \in S(T, A, \varepsilon_n, \alpha_n). \tag{16}$$

We choose a sequence $(\psi^{nj})_{j\in\mathbb{N}}$ in $I(\Sigma_n, T, A, \varepsilon_n)$ with, putting $\Sigma_n^i = \psi_1^{ni}(\Sigma_n) \in \mathcal{M}$,

$$\lim_{i\to\infty} | \Sigma_n^i | = \inf \{ | \psi_1(\Sigma_n) | : \psi \in I(\Sigma_n, T, A, \varepsilon_n) \}. \tag{17}$$

After selection of subsequences, we get varifold limits

(18) $$V_n^* = \lim_{i\to\infty} \underline{\underline{v}}(\Sigma_n^i) \qquad (n = 1, 2, \ldots)$$

and

(19) $$V^* = \lim_{n\to\infty} V_n^*$$

Applying Lemma 2 with $U = T$, $W = V_n^* \llcorner G_2(T \cap \overline{A})$, $\Sigma^i = \Sigma_i^n$, we infer that V_n^* is an integral varifold with

$$\operatorname{spt} \| V_n^* \| \cap T \cap \overline{A} = M_n^*$$

where M_n^* satisfies the conclusions of the Theorem.

By Lemma 3 of § 3, the same conclusion holds for V^*.

As in the proof of Lemma 2, we then conclude the desired regularity of V.

§ 5. CONTROL OF THE TOPOLOGICAL TYPE OF THE ALMOST MINIMIZING VARIFOLD

In this paragraph, we rather closely follow the corresponding argument of [SS].

From the preceding paragraphs, we infer that there exists a varifold of the form

(1) $$V = \sum_{j=1}^{N} n_j \underline{\underline{v}}(M_j)$$

where each M_j is an embedded compact minimal surface with boundary $\partial M_j = M_j \cap \partial A$ which intersects ∂A orthogonally, $n_j \in \mathbb{N}$, for $j = 1, \ldots, N$.

(Since A is strictly convex, no interior point of M can touch ∂A in particular.)

$M_i \cap M_k \neq \phi$ for $i \neq k$.

Each M_j is uniformly almost minimizing among disks with respect to some collection of open subsets of A.

We now want to show that each M_j is simply connected, i. e. a disk. W. l. o. g., we shall do this for $M := M_1$.

We again put for $x \in \overline{A}$, $\sigma, \sigma_1, \sigma_2 > 0$

$$U(x, \sigma) := \{ y \in \overline{A} : | x - y | < \sigma \}$$
$$A(x, \sigma_1, \sigma_2) := \{ y \in \overline{A} : \sigma_1 < | x - y | < \sigma_2 \} .$$

Let $\delta > 0$ be so small that for any $x \in \partial A$, $\partial U(x, \delta) \cap \partial A$ is a single circle.

$\underline{\underline{v}}(\mathrm{M})$ then is uniformly almost minimizing among disks w. r. t. the collection of pairs of annuli

$$\left\{\left(\mathrm{A}\left(x, \frac{1}{32}\rho, \frac{1}{8}\rho\right), \mathrm{A}\left(x, \frac{1}{4}\rho, \rho\right)\right) : 0 < \rho < \delta,\ x \in \overline{\mathrm{A}}\right\}.$$

We can also require that $\delta > 0$ is so small that for each $x \in \mathrm{M}$, $\mathrm{B}(x, \delta) \cap \mathrm{M}$ is topologically a disk.

For each $x \in \overline{\mathrm{A}}$, we can select

$$\mathrm{A}\left(x, \frac{1}{4}\rho(x), \rho(x)\right), \qquad \frac{1}{8}\delta \leq \rho(x) \leq \delta$$

with the property that $\underline{\underline{v}}(\mathrm{M})$ is uniformly almost minimizing among disks w. r. t. the collection of these annuli.

The balls $\mathrm{U}(x, \rho(x))$ cover $\overline{\mathrm{A}}$, and hence (cf. [F; 2.8.11, 2.8.13]) we can choose a finite number of points $x_j \in \overline{\mathrm{A}}$, $j = 1, \ldots, \mathrm{J}$ with

$$\overline{\mathrm{A}} = \bigcup_{j=1}^{\mathrm{J}} \mathrm{U}(x_j, \rho(x_j))$$

and

$$\overline{\mathrm{U}}\left(x_i, \frac{1}{5}\rho(x_i)\right) \cap \overline{\mathrm{U}}\left(x_k, \frac{1}{5}\rho(x_k)\right) = \emptyset$$

if $i \neq k$.

If γ_0 is any simple closed curve in M, we can isotope it in M to a curve $\gamma \subset \mathrm{M} \cap \mathrm{A}$ with

$$\gamma \cap \bigcup_{j=1}^{\mathrm{J}} \overline{\mathrm{U}}\left(x_j, \frac{1}{5}\rho(x_j)\right) = \emptyset. \tag{2}$$

We need some notation:

$$\mathrm{T}_\theta := \{ x \in \overline{\mathrm{A}} : \operatorname{dist}(x, \gamma) < \theta \} \qquad \text{for} \quad \theta > 0$$
$$\lambda : \mathrm{T}_\theta \to \gamma \qquad \text{nearest point projection}$$
$$\mathrm{Y}_\sigma := \{ x \in \overline{\mathrm{A}} : \operatorname{dist}(x, \mathrm{M}) < \sigma \} \qquad \text{for} \quad \sigma > 0$$
$$\pi : \mathrm{Y}_\sigma \to \mathrm{M} \qquad \text{nearest point projection}.$$

We choose $\sigma > 0$ so small that

$$0 < \sigma < \frac{1}{2}\min\left(\operatorname{dist}\left(\gamma, \bigcup_{j=1}^{\mathrm{J}} \mathrm{U}\left(x_j, \frac{1}{5}\rho(x_j)\right)\right), \operatorname{dist}(\gamma, \partial\mathrm{A})\right) \tag{3}$$

and that

$$\lambda : \mathrm{T}_{2\sigma} \to \gamma$$

and

$$\pi : \mathrm{Y}_{2\sigma} \to \mathrm{M}$$

are continuous, that for any $y \in T_{2\sigma}$, $\lambda(y)$ and $\pi(y)$ are so close on M that they can be joined by a unique shortest geodesic arc on M, and that T_σ intersects no other connected component of spt $\| V \|$ besides $M = M_1$.

In particular, $T_\sigma \cap \partial A = \emptyset$ and

$$T_\sigma \subset \bigcup_{j=1}^{J} A\left(x_j, \frac{1}{5}\rho(x_j), \rho(x_j)\right). \tag{4}$$

By the almost minimizing property of $\underline{\underline{v}}(M)$ w. r. t. the collection of these annuli, for any sequence (ε_k), $\varepsilon_k \to 0$, there is a sequence (α_k) and sequence $(\Sigma_k) \subset \mathscr{M}$ with

$$\underline{\underline{F}}(\underline{\underline{v}}(M), \underline{\underline{v}}(\Sigma_k)) < \varepsilon_k \tag{5}$$

and

$$\Sigma_k \in S\left(A\left(x_j, \frac{1}{5}\rho(x_j), \rho(x_j)\right), A, \varepsilon_k, \alpha_k\right) \tag{6}$$

$(k \in \mathbb{N}, j = 1, \ldots, J)$.

From (4) and (6), for $x \in \gamma$

$$\Sigma_k \in S(U(x, \sigma), A, \varepsilon_k, \alpha_k). \tag{7}$$

We select $y_1, \ldots, y_m \in \gamma$ and $0 < \tau < \sigma$ with

$$\gamma \subset \bigcup_{i=1}^{m} B(y_i, \tau),$$

where the boundaries $\partial B(y_i, \tau)$ all intersect transversally, and for each $y \in \gamma$

$$B(y, 4\tau) \subset A\left(x_j, \frac{1}{5}\rho(x_j), \rho(x_j)\right), \tag{8}$$

for some $j \in \{1, \ldots, J\}$.

$W := \bigcup_{i=1}^{m} B(y_i, \tau)$. The components $W_1, \ldots, W_q$ of $W \setminus \bigcup_{i=1} \partial B(y_i, \tau)$ are then topological balls. By making a further subdivision, if necessary, we can also assume that $M \cap W_j$ is connected for $j = 1, \ldots, q$ (without increasing $M \cap \cup \partial W_j$).

We choose sequences $(\psi^{kl})_{l \in \mathbb{N}} \subset \bigcup_{j=1}^{q} I(\Sigma_k, W_j, A, \varepsilon_k)$ with, putting $\Sigma_k^l = \psi_1^{kl}(\Sigma_k)$

$$\lim_{l \to \infty} |\Sigma_k^l| = \inf\left\{ |\psi_1(\Sigma_k)| : \psi \subset \bigcup_{j=1}^{q} I(\Sigma_k, W_j, A, \varepsilon_k)\right\}. \tag{9}$$

W. l. o. g.

$$|\psi_1^{kl}(\Sigma_k \cap W_j)| \leq |\Sigma_k \cap W_j| \qquad \text{for each } j, k, l,$$

and hence

$$|\Sigma_k^l| \le |\Sigma_k| \quad \text{for each } k, l. \tag{10}$$

Consequently

$$\Sigma_k^l \cap \partial W_j = \Sigma_k \cap \partial W_j \quad \text{for all } j, k, l \tag{11}$$

and by (7), $\tau < \sigma$, (8) and (10)

$$\Sigma_k^l \in S(U(x, \sigma/2), A, \varepsilon_k, \alpha_k) \quad \text{for} \quad x \in \gamma. \tag{12}$$

After selection of subsequences, we get varifold limits

$$V_k = \lim_{l\to\infty} \underline{\underline{v}}(\Sigma_k^l)$$
$$V^* = \lim_{k\to\infty} V_k.$$

By (8), V^* is almost minimizing in $B(y, 2\tau)$ for $y \in \gamma$, and hence regular there, cf. § 4. Since on the other hand, V and V^* coincide outside W, they have to coincide everywhere, i. e.

$$V = V^*. \tag{13}$$

Furthermore, spt $\| V_k \| \cap W_j$ is an embedded minimal surface (cf. Lemma 2, § 4), and the same holds, if instead of $\Sigma_k^l \cap W_j$, we take any union $\Sigma_{k,j}^l$ of components of $\Sigma_k^l \cap W_j$, for which the varifold limit exists as $l \to \infty$.

Let $\Sigma_{k,j}^{l,1}, \ldots, \Sigma_{k,j}^{l,P}$ be the components of $\Sigma_k^l \cap W_j$.

We can assume

$$V_{k,j}^m = \lim_{l\to\infty} \underline{\underline{v}}(\Sigma_{k,j}^{l,m}) \neq 0 \quad (m = 1, \ldots, P) \quad \text{(just discard the other ones).} \tag{14}$$

We can assume that for each k, we can select $l(k)$ with

$$\underline{\underline{F}}(V, \underline{\underline{v}}(\Sigma_k^{l(k)})) \le \underline{\underline{F}}(V, V_k) + \underline{\underline{F}}(V_k, \underline{\underline{v}}(\Sigma_k^{l(k)})) < \frac{1}{k} \tag{15}$$

$$\underline{\underline{F}}(V_{k,j}^m, \underline{\underline{v}}(\Sigma_{k,j}^{l(k)m})) < \frac{1}{k} \tag{16}$$

$$\lim_{k\to\infty} \left| \left(\Sigma_k^{l(k)} \setminus \bigcup_{m=1}^{P} \Sigma_{k,j}^{l(k),m} \right) \cap W_j \right| = 0 \tag{17}$$

Finally, since spt $\| V_{k,j}^m \|$ is an embedded minimal surface, and

$$\lim_{k\to\infty} V_{k,j}^m = n_m \underline{\underline{v}}(M) \quad \text{for some integer } n_m, \tag{18}$$

we can also w. l. o. g. discard $\Sigma_{k,j}^{l,m}$ with $n_m = 0$ by pushing them into ∂W_j. Then, P is bounded independently of k, and thus, after selection of a subsequence,

$$P = P(j) \tag{19}$$

Put

$$\Lambda_k := \Sigma_k^{l(k)} . \tag{20}$$

Let B be any open topological ball in A for which $B \cap M \cap W_j$ is connected for $j = 1, \ldots, q$ and $B \cap M \cap W_j \neq \emptyset$ for at least one $j \in \{1, \ldots, q\}$.

Let $\tilde{\Lambda}_k$ be any connected component of $B \cap \Lambda_k$ having a varifold limit

$$\lim_{k \to \infty} \underline{\underline{v}}(\tilde{\Lambda}_k) \llcorner G_2(B) \neq 0 \tag{21}$$

(such a component exists by (14)).

By (17) and (21), we can find j_0, m_0 and a compact set $K \subset B \cap W_{j_0}$ with $\Sigma_{k,j_0}^{l(k),m_0} \subset \tilde{\Lambda}_k$ and

$$\liminf_{k \to \infty} | \Sigma_{k,j_0}^{l(k),m_0} \cap K | > 0 . \tag{22}$$

After selection of a subsequence, the varifold limit

$$Z = \lim_{k \to \infty} \underline{\underline{v}}(\tilde{\Lambda}_k)$$

exists.

From (17) since $B \cap M \cap W_j$ is connected

$$Z \llcorner G_2(W_j) = m_j \underline{\underline{v}}(B \cap M \cap W_j) \tag{23}$$

$(j = 1, \ldots, q)$, with $m_j \in \mathbb{N}$, and $m_{j_0} \geq 1$ by (22).

LEMMA 1. — *Let* B *be a topological ball in* A *for which* $B \cap M \cap W_j$ *is (empty or) connected and simply connected for* $j = 1, \ldots, q$ *and* $B \cap M \cap W_j \neq \emptyset$ *for at least one* $j \in \{1, \ldots, q\}$. *Let* $\tilde{\Lambda}_k$ *be a connected component of* $B \cap \Lambda_k$.

If $B \cap M \cap W_j \neq \emptyset$, *we have* $m_j \geq 1$ *in (23).*

Proof. — Otherwise, there exist $j_1 \neq j_2 \in \{1, \ldots, q\}$ with $m_{j_1} = 0$, $m_{j_2} \geq 1$, and so that we can choose some nonempty arc $\beta \subset \partial W_{j_1} \cap \partial W_{j_2} \cap B \cap M$, a point x_0 in the interior of β and $\eta < \tau$ with

$$B(x_0, \eta) \subset B$$
$$M \cap B(x_0, \eta) \cap \partial W_j \subset \beta \qquad (j = 1, \ldots, q) .$$

Put

$$B^+ := B(x_0, \eta) \cap W_{j_1}$$
$$B^- := B(x_0, \eta) \cap W_{j_2} .$$

By assumption

$$\begin{aligned} &Z \llcorner B^+ = 0 \\ &Z \llcorner B^- = m_{j_2} \underline{\underline{v}}(M \cap B^-), \qquad m_{j_2} \geq 1 , \end{aligned} \tag{25}$$

and we want to derive a contradiction from (25).

We can also assume, by possibly decreasing $\eta > 0$, that

$$|M \cap B(x_0, \eta)| \leq \frac{11}{10}\pi\eta^2 .$$

We then use (15) (note that $\Lambda_k = \Sigma_k^{l(k)}$, and $\tilde{\Lambda}_k$ was a connected component of $\Lambda_k \cap B$, $V \cap W_j = n_1 \underline{\underline{v}}(M \cap W_j)$) and the coarea formula, in order to find (for $k \geq 10$, say) $x_1 \in M \cap B^+$ and $\eta_k \in \left[\frac{1}{8}\eta, \frac{1}{4}\eta\right]$ for which $B(x_1, \eta_k) \subset B(x_0, \eta)$, $\partial B(x_1, \eta_k)$ intersects Λ_k transversally (Sard's Lemma) and

$$\text{(26)} \qquad \text{length}\,(\partial B(x_1, \eta_k) \cap \tilde{\Lambda}_k) \leq \frac{3}{4}(2\pi\eta_k)$$

$$\text{length}\,(\partial B(x_1, \eta_k) \cap \Lambda_k \setminus \tilde{\Lambda}_k) \leq \left(n_1 - \frac{1}{4}\right)(2\pi\eta_k) .$$

W. l. o. g.
$$\eta_k \to \eta_* \in \left[\frac{1}{8}\eta, \frac{1}{4}\eta\right] .$$

By (12), $\Lambda_k \in S(B(x_1, \eta_k), A, \varepsilon_k, \alpha_k)$.

Thus, for each k, we can find a sequence $(\phi^{kl})_{l \in \mathbb{N}} \subset I(\Lambda_k, B(x_1, \eta_k), A, \alpha_k)$ with, putting $\Lambda_k^l := \phi_1^{kl}(\Lambda_k)$,

$$\lim_{l \to \infty} |\Lambda_k^l| = \inf \{ |\psi_1(\Lambda_k)| : \psi \in I(\Lambda_k, B(x_1, \eta_k), A, \alpha_k) .$$

By Lemma 2 of § 4,

$$\lim_{l \to \infty} \underline{\underline{v}}(\Lambda_k^l) \llcorner G_2(B(x_1, \eta_k)) = \underline{\underline{v}}(E_k) \llcorner G_2(B(x_1, \eta_k))$$

where E_k is a stable embedded minimal surface in $B(x_1, \eta_k)$ which by the boundary regularity results of [AS] is regular up to its boundary

$$\partial E_k = \Lambda_k \cap \partial B(x_1, \eta_k) .$$

Let E_k^1, E_k^2 be the (unions of) components of E_k with

$$\partial E_k^1 = \Lambda_k \cap \partial B(x_1, \eta_k)$$
$$\partial E_k^2 = (\Lambda_k \setminus \tilde{\Lambda}_k) \cap \partial B(x_1, \eta_k) .$$

By (26) and the isoperimetric inequality

$$|E_k^1| \leq \frac{3}{4}\pi\eta_k^2$$

$$\text{(27)} \qquad |E_k^2| \leq \left(n_1 - \frac{1}{4}\right)\pi\eta_k^2 .$$

As in § 4, we infer that

$$\lim_{k\to\infty} \underline{\underline{v}}(E_k^1) = m_1\underline{\underline{v}}(M \cap B(x_1, \eta_k))$$
$$\lim_{k\to\infty} \underline{\underline{v}}(E_k^2) = m_2\underline{\underline{v}}(M \cap B(x_1, \eta_k))$$

where $m_1, m_2 \in \mathbb{N}$. By (27), since $x_1 \in M$,

$$m_1 = 0$$
$$m_2 \leq n_1 - 1 .$$

This is a contradiction, however, since

$$\lim_{k\to\infty} \underline{\underline{v}}(E_k^1 \cup E_k^2) = \lim_{k\to\infty} \underline{\underline{v}}(\Sigma_k \llcorner B(x_1, \eta_k)) = n_1\underline{\underline{v}}(M \cap B(x_1, \eta_k))$$

using again the unique continuation argument of § 4 for the first equality.

q. e. d.

W. l. o. g.

$$\gamma \cap W_1 \neq \emptyset .$$

Let $$x_1 \neq x_2, \qquad x_1, x_2 \in W_1 \cap M \cap T_\sigma ,$$
$$\lambda(x_1) = \lambda(x_2) = p \in \gamma \cap W_1 .$$

($\lambda : T_\sigma \to \gamma$ was the nearest point projection).

Let $L := \max \{ P(j) : j = 1, \ldots, q \}$.

We cover T_σ by a finite collection $B_1, \ldots, B_r$ of topological balls so that $M \cap B_s \cap W_j$ is connected and simply connected for each $s \in \{1, \ldots, r\}$, $j \in \{1, \ldots, q\}$.

Suppose $B_1 \cap M \neq \emptyset$ and let $\tilde{\Lambda}_k$ be any connected component of $\Lambda_k \cap T_\sigma$ with

$$\lim_{k\to\infty} \underline{\underline{v}}(\tilde{\Lambda}_k) \llcorner G_2(B_1) \neq \emptyset$$

(such $\tilde{\Lambda}_k$ exists by (14)).

If $B_1 \cap W_j \cap M \neq \emptyset$, then

$$\lim \underline{\underline{v}}(\tilde{\Lambda}_k) \llcorner G_2(B_1 \cap W_j) = m_{1j}\underline{\underline{v}}(M \cap B_1 \cap W_j) \tag{28}$$

and $m_{1j} \geq 1$ by Lemma 1.

There exist $i_1 \neq 1$ and j_1 with $B_1 \cap B_{i_1} \cap W_{j_1} \neq \emptyset$. Defining m_{ij} via

$$\lim \underline{\underline{v}}(\tilde{\Lambda}_k) \llcorner G_2(B_i \cap W_j) = m_{ij}\underline{\underline{v}}(M \cap B_i \cap W_j) ,$$

we see

$$m_{i_1 j_1} = m_{1 j_1} \geq 1 \qquad \text{by Lemma 1} .$$

Continuing this way, repeatedly using Lemma 1, we get

$$m_{ij} \geq 1 \tag{29}$$

provided $B_i \cap W_j \cap M \neq \emptyset$. Moreover, if $U \subset W_1 \cap T_\sigma$ is open, $U \cap M \neq \emptyset$, then for large k

$$\tilde{\Lambda}_k \cap \bigcup_{m=1}^{P(1)} \Sigma_{k,1}^{l(k),m} \cap U \neq \emptyset .$$

(This follows from (15) and Lemma 1).

Let $y \in \gamma \cap W_1$. We can find a point $z_1 \in T_\sigma \cap \tilde{\Lambda}_k \cap \bigcup_{m=1}^{P(1)} \Sigma_{k,1}^{l(k),m}$ with

$$| z_1 - y | < \frac{\sigma}{2}.$$

Using (29), we can lift γ inside T_σ to $\tilde{\Lambda}_k$ with starting point z_1, i. e. we find $z_2 \in \tilde{\Lambda}_k \cap T_\sigma \cap \bigcup_{m=1}^{P(1)} \Sigma_{k,1}^{l(k),m}$ and $\gamma_1 \subset \Lambda_k \cap T_\sigma$ with endpoints z_1, z_2 and $| z_2 - y | < \frac{\sigma}{2}$ and

$$\lambda(\gamma_1) = \gamma + \gamma'$$

where γ' is some oriented arc in γ with length at most σ.

Likewise, we find a lift γ_2 with starting point z_2, and if we continue this process L times, we obtain $\bar{\gamma} \subset \tilde{\Lambda}_k \cap T_\sigma$ with

$$\lambda_\#(\bar{\gamma}) = L\gamma + \gamma'' \tag{30}$$

length $(\gamma'') \leq \sigma$, and points $z_1, \ldots, z_{L+1} \in \bar{\gamma} \cap \tilde{\Lambda}_k \cap T_\sigma \cap \bigcup_{m=1}^{P(1)} \Sigma_{k,1}^{l(k),m}$ with $| z_i - y | < \frac{\sigma}{2} (i = 1, \ldots, L + 1)$ (here, $\lambda_\#$ is the induced map on homotopy classes, and $L \cdot \gamma$ is of course multiplication in the fundamental group).

On the other hand, since $P(1) \leq L$ (by choice of L), there must be two different points z_{i_1}, z_{i_2} which are contained in the same $\Sigma_{k,1}^{l(k),m_0}$ for some $m_0 \in \{ 1, \ldots, P(1) \}$, since the number of the points z_i is $L + 1$.

Let γ^* be the subarc of $\bar{\gamma}$ with endpoints z_{i_1} and z_{i_2}. By (30))

$$\lambda_\#(\gamma^*) = m\gamma + \gamma'' , \tag{31}$$

where length $(\gamma'') < \sigma$ again, and $m \geq 1$, i. e. up to a small error, λ gives a nontrivial covering γ by γ^*.

We then close γ^* off in $\Sigma_{k,1}^{l(k),m_0}$ to obtain a closed curve γ_0 in $\Lambda_k \cap T_\sigma$ (note that $\Sigma_{k,1}^{l(k),m_0} \subset W_1 \subset T_\sigma$ by construction) with

$$\lambda_\#(\gamma_0) = m\gamma$$

with $m \geq 1$.

Since Λ_k is an embedded disk, γ_0 bounds an embedded disk D_0 on Λ_k. By an elementary cutting procedure (one can e. g. use the topological version of the argument of [AS, § 3]), we can find an embedded disk $D_1 \subset A$ with

$$\partial D_1 = \gamma_0$$
$$D_1 \subset Y_{2\sigma} (= \{ x \in \overline{A} : \operatorname{dist}(x, M) < 2\sigma \}$$
$$D_1 \cap Y_\sigma \subset D_0 \cap Y_\sigma .$$

Hence γ_0 can be homotoped to a point in D_1. Since $D_1 \subset Y_{2\sigma}$, and $\pi : Y_{2\sigma} \to M$ was continuous by choice of σ, $\pi(\gamma_0)$ is homotopic to a point in M.

Moreover, the choice of σ implies that $\pi(\gamma_0)$ and $\lambda(\gamma_0)$ are homotopic in M.

Hence, by (30), $m\gamma$ is homotopic to a point for some $m \geq 1$.

As M is orientable, this implies that γ itself is homotopic to a point, and hence that M is a disk. (That M is orientable follows, e. g., from the following argument: Topologically, A is half of the 3-sphere S^3, and since M meets ∂A transversally, we can reflect M across ∂A to obtain a closed embedded surface $\tilde{M}$ without boundary in S^3. Thus, M and hence also M is orientable).

This completes the proof of our main theorem and thus also this paper.

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Added in proof. Subsequently, stronger results were obtained by the second author (cf. J. Jost, Existence results for embedded minimal surfaces of controlled topological type I, II, III, *Ann. Sc. Norm. Sup. Pisa*, to appear).