Ann. Inst. Henri Poincaré,
Vol. 3, n° 5, 1986, p. 391-409. *Analyse non linéaire*

On the definition and the lower semicontinuity of certain quasiconvex integrals

by

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ABSTRACT. — Let us consider vector-valued functions $u:\Omega \to \mathbb{R}^N$, defined in an open bounded set $\Omega \subset \mathbb{R}^n$. Let $f(x, \xi)$ be a continuous function in $\Omega \times \mathbb{R}^{nN}$, quasiconvex with respect on ξ, that satisfies, for some $p \leqq q$, the growth conditions $c_1 |\xi|^p \leqq f(x, \xi) \leqq c_2(1 + |\xi|^q)$.

The integral $\mathrm{I}(u) = \int_\Omega f(x, \mathrm{D}u(x))dx$ is well defined if $u \in \mathrm{H}^{1,q}(\Omega; \mathbb{R}^N)$. We extend the integral $\mathrm{I}(u)$ to functions $u \in \mathrm{H}^{1,p}(\Omega; \mathbb{R}^N)$, and we study its lower semicontinuity in the weak topology of $\mathrm{H}^{1,p}(\Omega; \mathbb{R}^N)$, in order to obtain existence of minima.

RÉSUMÉ. — Soit $\Omega \subset \mathbb{R}^n$ un ouvert borné et soit $u:\Omega \to \mathbb{R}^N$. Soit $f(x, \xi)$ une fonction continue sur $\Omega \times \mathbb{R}^{nN}$, quasi-convexe en ξ et qui satisfait à la condition $c_1 |\xi|^p \leqq f(x, \xi) \leqq c_2(1 + |\xi|^q)$ avec $p \leqq q$.

L'intégrale $\mathrm{I}(u) = \int_\Omega f(x, \mathrm{D}u(x))dx$ est bien définie si $u \in \mathrm{H}^{1,q}(\Omega; \mathbb{R}^N)$. On étend l'intégrale $\mathrm{I}(u)$ aux fonctions $u \in \mathrm{H}^{1,p}(\Omega; \mathbb{R}^N)$ et on étudie la semi-continuité dans la topologie faible de $\mathrm{H}^{1,p}(\Omega; \mathbb{R}^N)$ pour obtenir l'existence de minimum.

Liste de mots-clés : Lower semicontinuity, quasiconvexity.
Classification AMS : 49 A 50.

Annales de l'Institut Henri Poincaré - Analyse non linéaire - 0294-1449
Vol. 3/86/05/391/19/$ 3,90/

1. INTRODUCTION

In this paper we study the definition, the lower semicontinuity, and the existence of minima of some quasiconvex integrals of the calculus of variations. To introduce our results, first we describe a situation studied by Ball [3] [4], of interest in nonlinear elasticity.

Ball considers a deformation of an elastic body that occupies a bounded domain $\Omega \subset \mathbb{R}^n$ ($n \geq 2$). If $u : \Omega \to \mathbb{R}^n$ is the displacement, and if Du is the $n \times n$ matrix of the deformation gradient, then the total energy can be represented by an integral of the type

$$\text{(1.1)} \qquad I(u) = \int_\Omega f(x, Du(x))dx .$$

The energy function $f(x, \xi)$, defined for $x \in \Omega$ and $\xi \in \mathbb{R}^{n \times n}$, is *quasiconvex* with respect to ξ in Morrey's sense [23]; that is, for every vector-valued function $\phi \in C_0^1(\Omega ; \mathbb{R}^n)$:

$$\text{(1.2)} \qquad \int_\Omega f(x, \xi + D\phi(y))dy \geq |\Omega| f(x, \xi) .$$

One of the simplest, but typical, examples considered by Ball ([4], section 7), is given by a function f of the type:

$$\text{(1.3)} \qquad f(\xi) = g(\xi) + h(\det \xi) ,$$

where $\det \xi$ is the determinant of the $n \times n$ matrix ξ, and g, h are nonnegative convex functions, that satisfy the growth conditions:

$$\text{(1.4)} \qquad g(\xi) \geq c_1 |\xi|^p ; \qquad \lim_{t \to +\infty} h(t) = +\infty .$$

The constant c_1 is greater than zero, and the exponent p satisfies the inequalities $1 < p < n$. In particular, the condition $p \leq n$ is necessary to study the existence of equilibrium solutions with *cavities*, i. e. minima of the integral (1.1) that are discontinuous at one point where a cavity forms; in fact, every u with finite energy belongs to the Sobolev space $H^{1,p}(\Omega, \mathbb{R}^n)$, and thus it is a continuous function if $p > n$.

Ball assumes also that g is singular, in the sense that $g = +\infty$ at some finite ξ, and that $h(t) \to +\infty$ as $t \to 0^+$. These assumptions, very natural for applications to nonlinear elasticity, are not relevant from the point of view of lower semicontinuity of the integral (1.1). In fact, we can approximate the convex functions g and h, considered by Ball, by increasing sequences of convex functions g_k and h_k, each of them being finite everywhere. If the integral corresponding to g_k, h_k is lower semicontinuous with respect to a fixed convergence, then also the integral corresponding to g, h will be lower semicontinuous with respect to that convergence, since it turns out to be the supremum of a sequence of lower semicontinuous functionals.

We can choose the growth at ∞ of each g_k and h_k so that they satisfy the conditions (we do not denote the dependence on k):

$$(1.5)\qquad c_1 |\xi|^p \leq g(\xi) \leq c_2(1 + |\xi|^p), \quad c_3 |t| \leq h(t) \leq c_4(1 + |t|).$$

If $p \leq n$, since $|\det \xi| \leq c_5(1 + |\xi|^n)$, we obtain that the function f in (1.3) satisfies:

$$(1.6)\qquad c_1 |\xi|^p \leq f \leq c_6(1 + |\xi|^n).$$

Thus it is clear the interest to study the lower semicontinuity of integral (1.1) under the assumption (1.7) that follows.

Let $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$. We consider a continuous function $f(x, \xi)$ defined for $x \in \Omega$ and $\xi \in \mathbb{R}^{nN}$, that satisfies

$$(1.7)\qquad c_1 |\xi|^p \leq f(x, \xi) \leq c_7(1 + |\xi|^q), \qquad \forall x \in \Omega, \quad \forall \xi \in \mathbb{R}^{nN},$$

where c_1, c_7 are positive constants, and $1 < p \leq q$. Under the coercivity on the left-hand side of (1.7), in order to obtain existence of minima it is natural to study the lower semicontinuity of integral (1.1) with respect to the weak convergence in the Sobolev space $H^{1,p}(\Omega; \mathbb{R}^N)$. However, there is an other interesting problem to consider, before to study semicontinuity: How to define integral (1.1) for every $u \in H^{1,p}(\Omega; \mathbb{R}^N)$? More precisely, integral (1.1) is well defined if u is a smooth vector function, say $u \in C^1(\Omega; \mathbb{R}^N)$. By the right-hand side of (1.7), the integral (1.1) is continuous in the strong topology of $H^{1,q}(\Omega; \mathbb{R}^N)$; thus it is well defined for $u \in H^{1,q}(\Omega; \mathbb{R}^N)$. It remains undetermined the meaning of the integral (1.1) if $u \in H^{1,p}(\Omega; \mathbb{R}^N)$, but $u \notin H^{1,q}(\Omega; \mathbb{R}^N)$. To explain this point, we study in our context a case already considered by Ball [3] [4].

Let us discuss the meaning to give, for $n = N > p$, to the integral:

$$(1.8)\qquad \int_{|x|<1} \{ |Du|^p + |\det Du| \} \, dx, \qquad \text{for} \quad u(x) = \frac{x}{|x|}.$$

Like in [3] [4], it is useful to consider also functions of the form $u(x) = xv(|x|)/|x|$. If we denote by $u \equiv (u^i)$, $x \equiv (x_\alpha)$, we have:

$$(1.9)\qquad u(x) = v(|x|)\frac{x}{|x|}, \quad \frac{\partial u^i}{\partial x_\alpha} = v' \frac{x_i x_\alpha}{|x|^2} + v \frac{\delta_{i\alpha}|x| - x_i x_\alpha / |x|}{|x|^2}.$$

It is easy to see that $|Du(x)|^2$ and $\det Du(x)$ are radially symmetric functions. Thus we can compute them for $x \equiv (|x|, 0, \ldots, 0)$. For such an x, we obtain

$$(1.10)\qquad \frac{\partial u^1}{\partial x_1} = v'; \quad \frac{\partial u^i}{\partial x_i} = \frac{v}{|x|}, \qquad i = 2, \ldots, n; \quad \frac{\partial u^i}{\partial x_\alpha} = 0, \quad i \neq \alpha.$$

Thus, for u given by (1.9) we have

$$(1.11)\quad |Du|^2 = (v')^2 + (n-1)\left(\frac{v}{|x|}\right)^2; \quad \det Du = v'\left(\frac{v}{|x|}\right)^{n-1}, \quad \forall x \neq 0.$$

First, let us procede formally to compute the integral (1.8). By using (1.11) with $v = 1$, since $\det Du = 0$ a. e., we obtain

$$(1.12)\qquad \int_{|x|<1} |Du|^p dx = \int_{|x|<1} \left(\frac{n-1}{|x|}\right)^p dx = \omega_n \frac{(n-1)^p}{n-p}.$$

We have denoted by ω_n the measure of the $(n-1)$ dimensional sphere $\{|x| = 1\}$. In particular, as well known, we can see from (1.12) that $u(x) = x/|x|$ belongs to $H^{1,p}(\Omega; \mathbb{R}^n)$ for every $p < n$.

On the other hand, we can define the integral (1.8) as the limit of values of the integral $I(u_k)$, where u_k is a sequence of smooth functions that converges to u strongly in $H^{1,p}(\Omega; \mathbb{R}^n)$, and satisfies:

$$(1.13)\qquad u_k(x) = v_k(|x|)\frac{x}{|x|}, \qquad v_k(0) = 0, \quad v_k(1) \to 1, \quad v_k' \geq 0.$$

This situation happens if we define, for example, $u_k = u * \alpha_k$, where $\alpha(x)$ is a radially symmetric mollifier, and, as usual, $\alpha_k(x) = k^n \alpha(kx)$. Two other examples are obtained: the first for $v_k(r) = r/(r + 1/k)$; the second for $v_k(r) = kr$ if $0 \leq r \leq 1/k$, and $v_k(r) = 1$ if $r > 1/k$.

Since u_k converges to u in $H^{1,p}(\Omega; \mathbb{R}^n)$, by (1.11) we obtain

$$\begin{aligned}(1.14)\quad \lim_{k\to+\infty} I(u_k) &= \int_{|x|<1} |Du|^p dx + \lim_{k\to+\infty} \int_{|x|<1} |\det Du_k|\, dx \\ &= \int_{|x|<1} |Du|^p dx + \lim_{k\to+\infty} \omega_n \int_0^1 v_k'\left(\frac{v_k}{r}\right)^{n-1} r^{n-1} dr \\ &= \int_{|x|<1} |Du|^p dx + \frac{\omega_n}{n} \lim_{k\to+\infty} \left[v_k^n(1) - v_k^n(0)\right] = \omega_n \frac{(n-1)^p}{n-p} + \frac{\omega_n}{n}.\end{aligned}$$

Thus the methods for computing integral (1.8) turns out to be different in the two cases (1.12) and (1.14). In this paper we will follow the second method. In fact we will show in section 5 that, if p is not much smaller than n, then the value (1.14) is the correct one for the integral (1.8) in the sense of the next definition. We follow a very classical method that was introduced by Lebesgue in his thesis *[17]* to define the area of a surface, by mean of the elementary area of approximating polyhedra.

DEFINITION. — *For every* $u \in C^1(\Omega; \mathbb{R}^N)$ *let* $I(u)$ *be the integral (1.1) (the integral* $I(u)$ *is well defined also if* $u \notin C^1(\overline{\Omega}; \mathbb{R}^N)$, *since the integrand is nonnegative). For* $u \in H^{1,p}(\Omega; \mathbb{R}^N)$ *we define*

$$(1.15)\qquad F(u) = \inf\{\liminf_k I(u_k)\}$$

for all sequences u_k that converge to u in the weak topology of $H^{1,p}(\Omega;\mathbb{R}^N)$, *and such that* $u_k \in C^1(\Omega;\mathbb{R}^N)$ *for every k.*

Let us recall that, after Lebesgue, the scheme of the above definition has been used by many authors, for integrals of area type. For example we quote the researches by De Giorgi, Giusti, Miranda (see e. g. [*14*] [*22*]), Serrin [*24*], Morrey ([*23*], definition 9.1.4), and more recently [*12*] [*7*] [*11*].

By the very definition, F is lower semicontinuous in the weak topology of $H^{1,p}(\Omega;\mathbb{R}^N)$; in fact F is the maximum functional not greater than I, and lower semicontinuous with respect to the weak convergence in $H^{1,p}(\Omega;\mathbb{R}^N)$. The definition of F is well motivated if F is an extension of the integral I, i. e. if $F(u) = I(u)$ for every $u \in C^1(\Omega;\mathbb{R}^N)$. This fact happens if and only if:

$$I(u) \leq \liminf_k I(u_k), \tag{1.16}$$

for every $u, u_k \in C^1(\Omega;\mathbb{R}^N)$, such that u_k converges to u in the weak topology of $H^{1,p}(\Omega;\mathbb{R}^N)$.

We succeeded in proving the semicontinuity result in (1.16) only if p is not much smaller than q, precisely if $p > qn/(n+1)$. The integrand f is assumed to be a general quasiconvex function, and not only polyconvex (we recall the definition in section 3), that satisfies the structure condition:

$$f(x, t\xi) \leq c_8(1 + f(x, \xi)), \qquad \forall t \in [0, 1], \tag{1.17}$$

and for every $x \in \Omega$, $\xi \in \mathbb{R}^{nN}$. In some sense, (1.17) is an intermediate condition in between the quasiconvex and the polyconvex case. In fact we will show in section 3 that, in the polyconvex case, (1.17) is a consequence of the other assumptions.

In section 2 we prove the semicontinuity result (1.16) in the general quasiconvex case. In section 3 we study semicontinuity in the polyconvex case. In section 4 we prove existence of minima in the Sobolev class $H^{1,p}(\Omega;\mathbb{R}^N)$ of functions with prescribed values at the boundary of $\partial\Omega$. Finally, in section 5 we compute the integral in (1.8) according to the definition (1.15).

2. THE GENERAL QUASICONVEX CASE

Let Ω be a bounded open set of $\mathbb{R}^n$. Let $f(x, \xi)$ be a *continuous* function for every $x \in \Omega$, and every $n \times N$ matrix ξ. In this section we assume that f is *quasiconvex* with respect to ξ in Morrey's sense [*23*], i. e. for every $x \in \Omega$ and $\xi \in \mathbb{R}^{nN}$:

$$\int_\Omega f(x, \xi + D\phi(y))dy \geq |\Omega| f(x, \xi), \qquad \forall \phi \in C_0^1(\Omega;\mathbb{R}^N). \tag{2.1}$$

Moreover we assume that f satisfies the conditions:

$$(2.2) \qquad 0 \leq f(x, \xi) \leq c_7(1 + |\xi|^q);$$

$$(2.3) \qquad f(x, t\xi) \leq c_8(1 + f(x, \xi)), \qquad \forall t \in [0, 1];$$

for every $x \in \Omega$ and $\xi \in \mathbb{R}^{nN}$, where $q \geq 1$ and $c_7, c_8 \geq 0$. Finally we assume that there exists a modulus of continuity $\lambda(t)$ (i. e. $\lambda(t)$ is a nonnegative increasing function that goes to zero as $t \to 0^+$) with the property that, for every compact subset Ω_0 of Ω, there exists $x_0 \in \Omega_0$ such that

$$(2.4) \qquad f(x_0, \xi) \leq f(x, \xi) + \lambda(|x - x_0|)[1 + f(x, \xi)],$$

for every $x \in \Omega_0$ and $\xi \in \mathbb{R}^{nN}$.

Condition (2.4) is a generalization of *conditions of type* I *or type* II by Serrin (see [*24*] or [*23*], p. 96-97). It is satisfied for example if $f(x, \xi)$ has the form:

$$(2.5) \qquad f(x, \xi) = a(x)F(\xi),$$

if $a(x)$ is a continuous function greater or equal than zero, by taking as x_0 a minimum point of $a(x)$ in Ω_0.

The following semicontinuity result holds:

THEOREM 2.1. — *Let $f(x, \xi)$ be a continuous function satisfying (2.1), (2.2), (2.3) and (2.4). Then we have*

$$(2.6) \qquad \int_\Omega f(x, Du)dx \leq \liminf_k \int_\Omega f(x, Du_k)dx,$$

for every $u, u_k \in C^1(\Omega; \mathbb{R}^N)$, such that u_k converges to u in the weak topology of $H^{1,p}(\Omega; \mathbb{R}^N)$, for p (≥ 1) strictly greater than $qn/(n + 1)$.

We obtain the proof of theorem 2.1 through some lemmas.

LEMMA 2.2. — *Let $f(x, \xi)$ be a quasiconvex function satisfying the growth condition (2.2). If $q - 1 \leq p \leq q$, then there exists a constant c_9 such that for every $w_1, w_2 \in L^{p/(p-q+1)}(\Omega; \mathbb{R}^{nN})$, it results:*

$$(2.7) \qquad \int_\Omega |f(x, w_1) - f(x, w_2)| dx \leq c_9 \| 1 + |w_1| + |w_2| \|_{L^p}^{q-1} \cdot \| w_1 - w_2 \|_{L^{p/(p-q+1)}}.$$

Proof. — We proved in section 2 of [*19*] that, by the quasiconvexity and the growth assumption of f, there exists a constant c_9 such that

$$(2.8) \qquad | f(x, \xi) - f(x, \eta) | \leq c_9(1 + |\xi| + |\eta|)^{q-1} |\xi - \eta|, \quad \forall \xi, \eta \in \mathbb{R}^{nN}.$$

The constant c_9 depends only on the constant c_7, and thus it is independent of $x \in \Omega$.

To estimate the left side of (2.7) we use inequality (2.8) and Hölder's inequality with exponents $p/(q-1)$ and $p/(p-q+1)$.

LEMMA 2.3. — *Theorem 2.1 holds if $f = f(\xi)$ is independent of x and if u is affine, i.e. if Du is constant in Ω.*

Proof. — Let us assume that $Du(x) = \xi \in \mathbb{R}^{nN}$ for every $x \in \Omega$, and that u_k converges to u in the weak topology of $H^{1,p}(\Omega; \mathbb{R}^N)$. Like in De Giorgi [*9*] (see also [*19*]), let Ω_0 be a fixed open set compactly contained in Ω. Let $R = \operatorname{dist}(\overline{\Omega}_0, \partial\Omega)/2$, and let v be a positive integer. For $i = 1, 2, \ldots, v$ let us define

$$\Omega_i = \left\{ x \in \Omega : \operatorname{dist}(x, \Omega_0) < \frac{i}{v} R \right\}. \tag{2.9}$$

For $i = 1, 2, \ldots, v$ we choose scalar functions $\phi_i \in C_0^1(\Omega_i)$ such that

$$0 \leq \phi_i \leq 1\,; \quad \phi_i = \begin{cases} 1 & \text{in} \quad \Omega_{i-1} \\ 0 & \text{in} \quad \Omega - \Omega_i \end{cases}; \quad |D\phi_i| \leq \frac{v+1}{R}. \tag{2.10}$$

Let us consider functions $v_{ki} \in H^{1,p}(\Omega; \mathbb{R}^N)$ defined by $v_{ki} = (1-\phi_i)u + \phi_i u_k$. For every k and i the function $v_{ki} - u$ has its support contained in Ω. Thus, by the quasiconvexity assumption, we have

$$\begin{aligned} \int_\Omega f(Du)dx = f(\xi)|\Omega| &\leq \int_\Omega f(Dv_{ki})dx \\ &= \int_{\Omega - \Omega_i} f(Du)dx + \int_{\Omega_i - \Omega_{i-1}} f(Dv_{ki})dx + \int_{\Omega_{i-1}} f(Du_k)dx. \end{aligned} \tag{2.11}$$

Let us sum up with respect to $i = 1, 2, \ldots, v$:

$$\begin{aligned} v \int_\Omega & f(Du)dx \\ &\leq v \int_{\Omega - \Omega_0} f(Du)dx + \sum_{i=1}^{v} \int_{\Omega_i - \Omega_{i-1}} f(Dv_{ki})dx + v \int_\Omega f(Du_k)dx. \end{aligned} \tag{2.12}$$

Now we estimate the second addendum in the right side. We have $Dv_{ki} = (1-\phi_i)Du + \phi_i Du_k + D\phi_i(u_k - u)$. In order to apply lemma 2.2, we define w_1 and w_2 by:

$$w_1 = Dv_{ki}, \quad w_2 = \phi_i Du_k, \qquad \text{in} \quad \Omega_i - \Omega_{i-1}. \tag{2.13}$$

Thus $w_1,\ w_2 \in L^\infty(\Omega_\nu;\mathbb{R}^{nN})$. By Lemma 2.2 we obtain

$$(2.14)\quad \sum_{i=1}^{\nu}\int_{\Omega_i-\Omega_{i-1}}|f(Dv_{ki})-f(\phi_i Du_k)|\,dx=\int_{\Omega_\nu-\Omega_0}|f(w_1)-f(w_2)|\,dx$$

$$\leq c_9\|1+|w_1|+|w_2|\|_{L^p}^{q-1}\left(\sum_{i=1}^{\nu}\int_{\Omega_i-\Omega_{i-1}}|(1-\phi)Du+D\phi_i(u_k-u)|^{\frac{p}{p-q+1}}dx\right)^{\frac{p-q+1}{p}}$$

$$\leq c_9\left\|1+|\xi|+2|Du_k|+\frac{\nu+1}{R}|u_k-u|\right\|_{L^p}^{q-1}$$

$$\cdot\left(|\xi|\,|\Omega-\Omega_0|^{(p-q+1)/p}+\frac{\nu+1}{R}\|u_k-u\|_{L^{p/(p-q+1)}(\Omega_\nu)}\right).$$

Since $p > qn/(n+1)$, we have also $np/(n-p) > p/(p-q+1)$. Thus, as $k \to +\infty$, u_k converges to u in the strong topology of $L^{p/(p-q+1)}(\Omega_\nu)$. Therefore since Du_k is bounded in L^p, there exists a real number c_{10}, independent of k and ν, such that:

$$(2.15)\qquad \limsup_k \sum_{i=1}^{\nu}\int_{\Omega_i-\Omega_{i-1}}|f(Dv_{ki})-f(\phi_i Du_k)|\,dx \leq c_{10}.$$

By the structure assumption (2.3) we obtain also

$$(2.16)\qquad \liminf_k \sum_{i=1}^{\nu}\int_{\Omega_i-\Omega_{i-1}} f(Dv_{ki})dx$$

$$\leq c_{10}+\liminf_k \sum_{i=1}^{\nu}\int_{\Omega_i-\Omega_{i-1}} f(\phi_i Du_k)dx$$

$$\leq c_{11}+c_8\liminf_k\int_\Omega f(Du_k)dx.$$

From (2.12), (2.16) it follows that

$$(2.17)\quad \int_\Omega f(Du)dx\leq\int_{\Omega-\Omega_0} f(Du)dx+\frac{c_{11}}{\nu}+\frac{\nu+c_8}{\nu}\liminf_k\int_\Omega f(Du_k)dx.$$

We obtain the result as $\nu \to +\infty$ and $\Omega_0 \to \Omega$.

LEMMA 2.4. — *Theorem 2.1 holds if $f = f(\xi)$ is independent of x.*

Proof. — Let $u \in C^1(\Omega;\mathbb{R}^N)$. Let Ω_0 be an open set compactly contained in Ω. Of course $u \in C^1(\overline{\Omega}_0, \mathbb{R}^N)$. For every positive integer ν, let us consider a subdivision of Ω_0 into open sets Ω_i such that

$$(2.18)\quad \Omega_i\cap\Omega_j=\emptyset,\ \text{if}\ i\neq j;\quad \sum_i|\Omega_i|=|\Omega_0|;\ \text{diameter}\ (\Omega_0)<\frac{1}{\nu},\ \forall i.$$

For every i, we define a vector $\xi_i \in \mathbb{R}^{nN}$ by

$$\xi_i = \frac{1}{|\Omega_i|}\int_{\Omega_i} Du(x)dx\,. \tag{2.19}$$

Since Du is uniformly continuous in Ω_0, for every $\varepsilon > 0$ there exists ν_0 such that

$$\sup_i \sup\{\,|Du(x) - \xi_i| : x \in \Omega_i\,\} < \varepsilon, \quad \forall \nu > \nu_0\,. \tag{2.20}$$

Let u_k be a sequence in $C^1(\Omega\,;\mathbb{R}^N)$ that converges to u in the weak topology of $H^{1,p}(\Omega\,;\mathbb{R}^N)$. For every i, we define in Ω_i the sequence

$$v_{ki}(x) = u_k(x) - u(x) + \langle \xi_i, x \rangle, \qquad x \in \Omega_i\,. \tag{2.21}$$

As $k \to +\infty$, v_{ki} converges to $v_i(x) = \langle \xi_i, x \rangle$ in the weak topology of $H^{1,p}(\Omega_i, \mathbb{R}^N)$. Thus, by lemma 2.3, we have:

$$\liminf_k \int_{\Omega_i} f(Dv_{ki})dx \geq \int_{\Omega_i} f(\xi_i)dx, \qquad \forall i\,. \tag{2.22}$$

We apply lemma 2.2 on the domain Ω_0, with $w_1 = Du_k$ and w_2 defined in each Ω_i by $w_2 = Du_k - Du + \xi_i$. By (2.20), there exists a constant c_{12} such that

$$\sum_i \int_{\Omega_i} |f(Du_k) - f(Dv_{ki})|\,dx < \varepsilon c_{12}, \qquad \forall \nu > \nu_0\,. \tag{2.23}$$

For a similar reason we have

$$\sum_i \int_{\Omega_i} |f(Du) - f(\xi_i)|\,dx < \varepsilon c_{12}, \qquad \forall \nu > \nu_0\,. \tag{2.24}$$

From (2.22), (2.23), (2.24) we obtain

$$\begin{aligned}\liminf_k \int_{\Omega} f(Du_k)dx &\geq \liminf_k \int_{\Omega_0} f(Du_k)dx \\ &\geq \liminf_k \sum_i \int_{\Omega_i} f(Dv_{ki})dx - \varepsilon c_{12} \\ &\geq \sum_i \int_{\Omega_i} f(\xi_i)dx - \varepsilon c_{12} \geq \int_{\Omega_0} f(Du)dx - 2\varepsilon c_{12}.\end{aligned} \tag{2.25}$$

We obtain the result as $\nu \to +\infty$, $\varepsilon \to 0$, $\Omega_0 \to \Omega$.

Proof of theorem 2.1. — Let Ω_0 be an open set compactly contained in Ω. For every positive integer ν let us consider a subdivision of Ω_0 into open sets Ω_i, like in (2.18). In particular the diameter of each Ω_i is less than $1/\nu$. We define $f_\nu(x, \xi)$ by using assumption (2.4):

$$\text{if} \quad x \in \Omega_i \quad \text{then} \quad f_\nu(x, \xi) = f(x_i, \xi), \tag{2.26}$$

where x_i is the point in $\bar{\Omega}_i$ for which we have

$$f(x_i, \xi) \leq \left[1 + \lambda\left(\frac{1}{\nu}\right)\right] f(x, \xi) + \lambda\left(\frac{1}{\nu}\right) \tag{2.27}$$

for every $x \in \Omega_i$.

Then, if u_k converges to u in the weak topology of $H^{1,p}(\Omega ; \mathbb{R}^N)$, by the previous lemma 2.4 we obtain

$$\begin{aligned} \left[1 + \lambda\left(\frac{1}{\nu}\right)\right] \liminf_k \int_\Omega f(x, Du_k)dx + \lambda\left(\frac{1}{\nu}\right) \\ \geq \liminf_k \int_{\Omega_0} f_\nu(x, Du_k)dx \geq \sum_i \liminf_k \int_{\Omega_i} f(x_i, Du_k)dx \\ \geq \sum_i \int_{\Omega_i} f(x_i, Du)dx = \int_{\Omega_0} f_\nu(x, Du)dx. \end{aligned} \tag{2.28}$$

Since $f(x, \xi)$ is uniformly continuous with respect to $x \in \Omega_0$, f_ν converges to f as $\nu \to +\infty$. Moreover $f_\nu(x, \xi)$ is bounded in terms of $f(x, \xi)$, by (2.27). Thus, we can go to the limit as $\nu \to +\infty$, by the dominated convergence theorem. Finally we obtain the result as $\Omega_0 \to \Omega$.

REMARK 2.5. — It is clear by the given proof that theorem 2.1 holds if we assume, more generally, that c_7, c_8 in (2.2), (2.3) are continuous functions of $x \in \Omega$ (possibly unbounded at the boundary of Ω). On the contrary it is not known if it is possible to extend theorem 2.1 to Carathédory functions $f(x, \xi)$, or to integrals whose integrand depends explicitly on u, other than Du. About this point, see the examples in section 6 of [*19*].

3. THE POLYCONVEX CASE

We say that a function $f(x, \xi)$ is *polyconvex* with respect to ξ in Ball's sense [*3*] if there exists a function $g(x, \eta)$, convex with respect to η, such that

$$f(x, \xi) = g(x, \xi, \det_1 \xi, \det_2 \xi, \ldots), \tag{3.1}$$

where each $\det_i \xi$ is a subdeterminant (or adjoint) of the $n \times N$ matrix ξ.

It is possible to verify that every polyconvex function is quasiconvex, according to definition (2.1).

In order to deduce from theorem 2.1 a semicontinuity result in the polyconvex case, let us discuss the structure condition (2.3). It is easy to handle the particular case of a function f defined, for $n = N$, by

$$f(\xi) = g(\det \xi), \tag{3.2}$$

where g is a convex function. In fact, for $t \in [0, 1]$, we have

$$f(t\xi) = g(t^n \det \xi) \leq (1 - t^n)g(0) + t^n g(\det \xi) \leq f(0) + f(\xi). \tag{3.3}$$

Thus a function given by (3.2) satisfies (2.3), if $f(0)$ is finite. Of course also any convex function of ξ, finite at $\xi = 0$, satisfies (2.3). We will use the following result:

LEMMA 3.1. — *Let $f(x, \xi) = g(x, \det_1 \xi, \dots)$ be a polyconvex function such that*

$$0 \leq g(x, \eta) \leq c_{13}(1 + |\eta|), \tag{3.4}$$

for some constant c_{13}. If $p \geq \min\{n - 1, N - 1\}$, then for every $\varepsilon \in (0, 1]$ the function

$$f_\varepsilon(x, \xi) = f(x, \xi) + \varepsilon |\xi|^p \tag{3.5}$$

satisfies the structure inequality (2.3).

Proof. — We prove the theorem in the case $n \geq N$; otherwise it is sufficient to interchange the role of n and N. Let us consider the $n \times N$ matrix ξ as a vector $\xi \equiv (\xi^i)$ for $\xi^i \in \mathbb{R}^n$, $i = 1, 2, \dots, N$. The function f_ε in (3.5) is convex with respect to each ξ^i. Thus, for every $t \in [0, 1]$, we have

$$\begin{aligned} f_\varepsilon(x, t\xi) &= f_\varepsilon(x, t\xi^1, t\xi^2, \dots, t\xi^N) \\ &\leq f_\varepsilon(x, 0, t\xi^2, \dots, t\xi^N) + f_\varepsilon(x, \xi^1, t\xi^2, \dots, t\xi^N) \\ &\leq f_\varepsilon(x, 0, 0, \dots, t\xi^N) + f_\varepsilon(x, 0, \xi^2, \dots, t\xi^N) \\ &+ f_\varepsilon(x, \xi^1, 0, \dots, t\xi^N) + f_\varepsilon(x, \xi^1, \xi^2, \dots, t\xi^N) \\ &\leq \dots \leq f_\varepsilon(x, 0, 0, \dots, 0) + \dots + f_\varepsilon(x, \xi^1, \xi^2, \dots, \xi^N). \end{aligned} \tag{3.6}$$

Other than by $f_\varepsilon(x, 0)$ and by $f_\varepsilon(x, \xi)$, we have estimated the left side of (3.6) by some intermediate addenda $f(x, \bar\xi)$ computed for vectors $\bar\xi \equiv (\bar\xi^i)$ with at least one component equal to zero. For these intermediate addenda we have

$$\begin{aligned} f_\varepsilon(x, \bar\xi) &= g(x, \bar\xi, \det_1 \bar\xi, \dots) + \varepsilon |\bar\xi|^p \\ &\leq c_{13}(1 + |\bar\xi| + |\det_1 \bar\xi| + \dots) + |\bar\xi|^p \\ &\leq c_{14}(1 + |\bar\xi|^{N-1}) + |\bar\xi|^p \leq c_{15}(1 + |\bar\xi|^p). \end{aligned} \tag{3.7}$$

We have used the fact that all the determinants of order N are equal to zero, when computed at the vectors $\bar{\xi} \equiv (\bar{\xi}^i)$ with some null components. By (3.6), (3.7) we obtain the result with a constant c_{16} that depends on ε:

$$(3.8)\quad \begin{aligned} f_\varepsilon(x, t\xi) &\leq f_\varepsilon(x, 0) + f_\varepsilon(x, \xi) + (2^N - 2)c_{15}(1 + |\xi|^p) \\ &\leq c_{13} + f_\varepsilon(x, \xi) + (2^N - 2)c_{15}(1 + f_\varepsilon(x, \xi)/\varepsilon) \leq c_{16}(1 + f_\varepsilon(x, \xi)). \end{aligned}$$

Now we deduce from theorem 2.1 a semicontinuity result for polyconvex functions $f(x, \xi) = g(x, \xi, \det_1 \xi, \ldots)$.

Let us assume that $g : \mathbb{R}^n \times \mathbb{R}^m \to [0, +\infty]$ satisfies:

(3.9) *The set where g is finite is independent of x, it is not empty and it is open in $\mathbb{R}^m$; g is continuous with respect to $x \in \Omega$ and it is convex and lower semicontinuous with respect to $\eta \in \mathbb{R}^m$.*

THEOREM 3.2. — *Let f be a nonnegative polyconvex function, and let us assume that the corresponding g satisfies (3.9). Let us assume also that there exists a positive constant c_{17} such that $g(x, \eta) \geq c_{17} |\eta|$. Then we have*

$$(3.10)\quad \int_\Omega f(x, Du)dx \leq \liminf_k \int_\Omega f(x, Du_k)dx,$$

for every $u, u_k \in C^1(\Omega; \mathbb{R}^N)$, such that u_k converges to u in the weak topology of $H^{1,p}(\Omega; \mathbb{R}^N)$, for $p > \min\{n; N\} \cdot n/(n+1)$.

Proof. — It is sufficient to give the proof also assuming that $p \leqslant \min\{n; N\}$. Like in [8] we can approximate g by an increasing sequence of continuous functions g_h of the form

$$(3.11)\quad g_h(x, \eta) = \max\{(a_j(x), \eta) + b_j(x) : j \leq h\}$$

with $a_j \in C_0^0(\Omega; \mathbb{R}^m)$ $b_j \in C_0^0(\Omega)$. By changing g_h with $\max\{g_h(x, \eta); c_{17} |\eta|\}$ (we still denote by g_h this function) we have

$$(3.12)\quad c_{17} |\eta| \leq g_h(x, \eta) \leq c_h(1 + |\eta|).$$

Since the functions a_j, b_j are continuous, for every h there exists a modulus of continuity $\lambda_h(t)$ such that

$$(3.13)\quad |g_h(x, \eta) - g_h(x_0, \eta)| \leq \lambda_h(|x - x_0|)(1 + |\eta|),$$

for every $x, x_0 \in \Omega$ and $\eta \in \mathbb{R}^m$. For every $\varepsilon \in (0, 1]$, the function

$$(3.14)\quad f_h(x, \xi) + \varepsilon |\xi|^p = g_h(x, \xi, \det_1 \xi, \ldots) + \varepsilon |\xi|^p$$

satisfies the assumptions of theorem 2.1. In fact it satisfies the continuity condition (2.4) with modulus of continuity λ_h/c_{17}; it is a quasiconvex function; it satisfies the growth condition (2.2) with $q = \min\{n; N\}$. Moreover by lemma 3.1, it satisfies the structure condition (2.3). By the semicontinuity result of theorem 2.1, if we denote by c_{18} an upper bound for the $H^{1,p}(\Omega;\mathbb{R}^N)$-norm of u_k, we obtain

$$(3.15)\quad \liminf_k \int_\Omega f(x, Du_k)dx \geq \liminf_k \int_\Omega f_h(x, Du_k)dx$$
$$\geq \liminf_k \int_\Omega \{ f_h(x, Du_k) + \varepsilon | Du_k |^p \} dx - \varepsilon c_{18}$$
$$\geq \int_\Omega f_h(x, Du)dx + \varepsilon \int_\Omega | Du |^p dx - \varepsilon c_{18}.$$

We obtain the semicontinuity result as $\varepsilon \to 0$ and $h \to +\infty$.

REMARK 3.3. — We have proved in this section that the semicontinuity theorem 2.1 is an extension to the general quasiconvex case of a semicontinuity result stated in section 5 of [*19*] (see the following section 6). We used in [*19*] a different method. The only point in common in the two proofs is the Rellich-Kondrachov imbedding theorem. Essentially this fact determine the lower bound for p. We use here the imbedding theorem in the proof of lemma 2.3 for the quasiconvex case, while we used it in [*19*] to obtain continuity in the sense of distributions of all the subdeterminants of the matrix Du.

4. EXISTENCE OF MINIMA

In this section we apply the results of the previous sections to define the integral out of C^1, and to obtain existence of minima. Let us assume:

(4.1) $f(x, \xi)$ *is continuous in* x, *and it is quasiconvex in* ξ;

(4.2) $$c_1 | \xi |^p \leq f(x, \xi) \leq c_7(1 + | \xi |^q);$$

(4.3) $$f(x, t\xi) \leq c_8(1 + f(x, \xi)), \quad \forall t \in [0, 1];$$

(4.4) *same as in (2.4)*.

We assume also that $1 < p \leq q$, and $p > qn/(n+1)$.

For every $u \in C^1(\Omega;\mathbb{R}^N)$ we denote by $I(u)$ the integral in (1.1). Moreover for every $u \in H^{1,p}(\Omega;\mathbb{R}^N)$ we define $F(u)$ as in (1.15). By the semicontinuity result of section 2 if follows:

THEOREM 4.1. — *Under assumptions (4.1), (4.2), (4.3), (4.4), for every* $u \in C^1(\Omega;\mathbb{R}^N)$ *we have* $F(u) = I(u)$. *Moreover, for every* $u_0 \in H^{1,p}(\Omega;\mathbb{R}^N)$ *such that* $F(u_0) < +\infty$, *the functional* F *has a minimum on the class* $H_0^{1,p}(\Omega;\mathbb{R}^N) + u_0$.

Now we consider a polyconvex function f, and we assume that the corresponding g satisfies (3.9) and the coercivity condition

$$(4.5)\quad f(x,\xi)=g(x,\xi,\det_1\xi,\det_2\xi,\dots)\geq c_{19}\Big(|\xi|^p+\sum_i|\det_i\xi|\Big),$$

for $p>\min\{n,N\}\cdot n/(n+1)$. Like before, we denote by $I(u)$ the integral in (1.1) for $u\in C^1(\Omega;\mathbb{R}^N)$.

We define also $F(u)$ as in (1.15) for $u\in H^{1,p}(\Omega;\mathbb{R}^N)$. We obtain the following:

THEOREM 4.2. — *Under assumptions (3.9), (4.5), for every $u\in C^1(\Omega;\mathbb{R}^N)$ we have $F(u)=I(u)$. Moreover, for every $u_0\in H^{1,p}(\Omega;\mathbb{R}^N)$ such that $F(u_0)<+\infty$, the functional F has a minimum on the Sobolev class $H_0^{1,p}(\Omega;\mathbb{R}^N)+u_0$.*

5. THE RADIALLY SYMMETRIC CASE

Let $n=N$ and let us consider the integral

$$(5.1)\quad \int_{|x|<1}\{|Du|^p+|\det Du|\}\,dx,\qquad \text{for}\quad u(x)=v(|x|)\frac{x}{|x|},$$

where $v:[0,1]\to[v(0),v(1)]$ is a nonnegative increasing function. The integral in (5.1) is well defined if $u\in C^1(\Omega;\mathbb{R}^n)$. Let us define, for every $u\in H^{1,p}(\Omega;\mathbb{R}^n)$:

$$(5.2)\quad F(u)=\inf\Big\{\liminf_k\int_{|x|<1}\{|Du_k|^p+|\det Du_k|\}\,dx\Big\},$$

for all sequences $u_k\in C^1(\Omega;\mathbb{R}^n)$ that converge to u in the weak topology of $H^{1,p}(\Omega;\mathbb{R}^n)$.

If $p>n^2/(n+1)$, then we will prove that

$$(5.3)\quad F(u)=\int_{|x|<1}|Du|^p dx+\frac{\omega_n}{n}v(1)^n;$$

as usual, ω_n/n is the measure of the unit ball in $\mathbb{R}^n$.

Notice that this value for $F(u)$, when $v\equiv 1$, is the same as in (1.14).

The stated result follows by a lemma, whose proof, based on a method by De Giorgi [9], is the same as the proof of lemma 2.3. Again we denote by $I(u)$ the integral (1.1).

LEMMA 5.1. — *Let $p, q\geq 1$ such that $p>qn/(n+1)$. Let $f(x,\xi)$ be a continuous quasiconvex function satisfying (2.2), (2.3). Let Ω_0 be an* *open set compactly contained in* Ω. *If* $u \in H^{1,p}(\Omega; \mathbb{R}^N)$ *is such that* $u \in C^1(\overline{\Omega} - \Omega_0; \mathbb{R}^N)$, *then:*

$$(5.4)\quad \inf\{\liminf_k I(u_k) : u_k \in C^1(\Omega; \mathbb{R}^N),\ u_k \rightharpoonup u \text{ in } H^{1,p}(\Omega; \mathbb{R}^N)\}$$
$$= \inf\{\liminf_k I(u_k) : u_k \in C^1(\Omega; \mathbb{R}^N),\ u_k - u \rightharpoonup 0 \text{ in } H_0^{1,p}(\Omega; \mathbb{R}^N)\}.$$

Let us go back to (5.1), (5.2). Since $p > n^2(n+1)$, we can apply lemma 5.1. Thus, in the definition (5.2) of $F(u)$, we can take sequences $u_k \in C^1(\Omega; \mathbb{R}^n)$ that converge to u in the weak topology of $H^{1,p}(\Omega; \mathbb{R}^n)$, and such that $u_k = u$ on the boundary $\{|x| = 1\}$. Thus we have $u_k(x) = v(1)x$ for $|x| = 1$. We use the inequality of quasiconvexity:

$$(5.5)\quad \int_{|x|<1} |\det Du_k|\, dx \geq |\{|x| < 1\}|\, |\det D(v(1)x)| = \frac{\omega_n}{n} v(1)^n.$$

Therefore, by the semicontinuity of the $H^{1,p}$-norm, we have

$$(5.6)\qquad F(u) \leq \int_{|x|<1} |Du|^p dx + \frac{\omega_n}{n} v(1)^n.$$

The opposite inequality follows similarly to the computation in (1.13), (1.14) in the introduction.

REMARK 5.2. — The definition of the integral (1.8) adopted here is different from the definition adepted by Ball and Murat in [*5*]; in fact they use the computation in (1.12). Ball and Murat showed that, by taking (1.12) as the value for the integral (1.8), then one obtain a functional that is not lower semicontinuous in the weak topology of $H^{1,p}(\Omega; \mathbb{R}^n)$, whatever is $p < n$.

REMARK 5.3. — If one compute the integral (5.1) without taking into account the singularity of the determinant of the gradient at $x = 0$, then one obtain

$$(5.7)\qquad \int_{|x|<1} |Du|^p dx + \frac{\omega_n}{n}[v(1)^n - v(0)^n].$$

Thus the difference $(\omega_n/n)v(0)^n$ in between the two values is a measure concentrated at $x = 0$. The quantity $(\omega_n/n)v(0)^n$ is also the Lebesgue measure of the cavity that forms around the origin.

6. ADDENDUM TO SECTION 5 OF THE PAPER [*19*]

The statement of lemma 5.3 in Marcellini [*19*] is wrong. It has been quoted in a wrong way a right result by De Giorgi [*8*]. More precisely, it is not true that, for every k, the function $g_k(x, \eta)$ is uniformly continuous in $\Omega \times \mathbb{R}^m$. In fact, each function g_k is defined, like in (3.11), as the maximum of a finite number of functions that are linear with respect to η and uniformly continuous with respect to x.

Thus $g_k(x, \eta)$ is uniformly continuous separately for $x \in \Omega$ and $\eta \in \mathbb{R}^m$, but not with respect to $(x, \eta) \in \Omega \times \mathbb{R}^m$.

For this reason lemma 5.3 and the consequences 5.4, 5.5, 5.6, 5.7 of [*19*] have not been proved. Here we indicate (we hope in a right way!) how to modify the approach of section 5 of [*19*]. We note explicitely that sections 1, 2, 3, 4 and 6 of [*19*] do not need to be modified.

First of all the results of section 5 of [*19*] are true if the function $g(x, \eta)$ is independent of x. In fact, in this case we can choose g_k essentially (see the details below) independent of x, and thus we can operate with a sequence of uniformly continuous functions. In this particular case we obtain the following semicontinuity result:

THEOREM 6.1. — *Let $g : \mathbb{R}^m \to [0, +\infty]$ be a convex and lower semicontinuous function, not identically $+\infty$. Let v_h and v be functions of $L^1_{loc}(\Omega; \mathbb{R}^m)$, and assume that v_h converges to v in the sense of distributions, i. e.:*

$$\lim_h \int_\Omega (v_h, \phi)dx = \int_\Omega (v, \phi)dx, \qquad \forall \phi \in C_0^\infty(\Omega; \mathbb{R}^m). \tag{6.1}$$

Then we have

$$\liminf_h \int_\Omega g(v_h(x))dx \geq \int_\Omega g(v(x))dx. \tag{6.2}$$

Theorem 6.1 will be consequence of the following lemma.

LEMMA 6.2. — *There exists an increasing sequence $g_k(\eta)$ that pointwise converge, as $k \to +\infty$, to $g(\eta)$. For every k, g_k is a function of class C^∞, it is convex, it is Lipschitz-continuous in $\mathbb{R}^m$. Moreover $g_k(\eta) \geq -1$.*

Proof. — The convex lower semicontinuous function $g(\eta)$ is the supremum of a numerable family of linear functions $l_j(\eta)$ such that $l_j \leq g$.

Let us assume that $l_1(\eta)$ is identically equal to zero. As usual, we define

$$m_k(\eta) = \max \{ l_j(\eta) : j \leq k \}. \tag{6.3}$$

We have an increasing sequence of nonnegative functions. Each m_k is convex and Lipschitz-continuous in $\mathbb{R}^m$. Starting from this sequence m_k, by regularization like in lemma 5.4 of [*19*], we can define a new increasing sequence of functions $g_k(\eta)$ that satisfies the statement of lemma 6.2.

Proof of theorem 6.1. — Let us define for every k

$$(6.4)\qquad \begin{cases} \Omega_k = \{ x \in \Omega : \operatorname{dist}(x, \partial\Omega) > 1/k \} \\ \phi_k \in C_0^\infty(\Omega_k) \\ 0 \le \phi_k \le 1 \quad \text{in } \Omega \\ \phi_k = 1 \quad \text{in } \Omega_{k-1} . \end{cases}$$

Let us denote by f_k the sequence of the previous lemma that converges to g. We define:

$$(6.5)\qquad g_k(x, \eta) = \phi_k(x)(f_k(\eta) + 1) - 1 .$$

For every k the derivative $D_\eta g_k = \phi_k D_\eta f_k$ is C^∞, it is bounded in $\Omega \times \mathbb{R}^m$ and it is equal to zero if $\operatorname{dist}(x, \partial\Omega) \le 1/k$. Moreover the sequence $g_k(x, \eta)$ in increasing and pointwise converges to $g(\eta)$ as $k \to +\infty$.

Let α be a mollifier, i. e. $\alpha \in C_0^\infty(\mathbb{R}^n)$, $\int \alpha dx = 1$, $\alpha \ge 0$, $\alpha = 0$ if $|x| \ge 1/2$. As usual we set $\alpha_\varepsilon(x) = \varepsilon^{-n}\alpha(x/\varepsilon)$. For every k, if $\varepsilon < 1/k$, we define in Ω_k

$$(6.6)\qquad v_\varepsilon(x) = v * \alpha_\varepsilon(x) = \int_\Omega v(x - y)\alpha_\varepsilon(y)dy, \qquad x \in \Omega_k .$$

By the convexity of g_k with respect to η, similarly to Serrin [*24*], we have

$$(6.7)\qquad g_k(x, v_n) \ge g_k(x, v_\varepsilon) + (D_\eta g_k(x, v_\varepsilon), v_h - v_\varepsilon) .$$

Note that $g_k(x, v_\varepsilon)$ and $D_\eta g_k(x, v_\varepsilon)$ are well defined in Ω_k and can be extended to Ω with values respectively -1 and 0.

Since $D_\eta g_k(x, v_\varepsilon) \in C_0^\infty(\Omega ; \mathbb{R}^m)$, by (6.1) and (6.7) we have

$$(6.8)\qquad \liminf_h \int_\Omega g(v_h)dx \ge \liminf_h \int_\Omega g_k(x, v_h)dx \ge \int_\Omega g_k(x, v_\varepsilon)dx + \int_\Omega (D_\eta g_k(x, v_\varepsilon), v - v_\varepsilon)dx .$$

We go to the limit as $\varepsilon \to 0$: in the first term of the right side we use Fatou's lemma, and in the second term the fact that $D_\eta g_k$ is bounded in $\Omega \times \mathbb{R}^m$ independently of ε. Finally we obtain the result as $k \to +\infty$, by the monotone convergence theorem.

Now let us turn our attention to a function $g(x, \eta)$ defined for $x \in \Omega$ and $\eta \in \mathbb{R}^m$, with values in $[0, +\infty]$, satisfying (3.9).

THEOREM 6.3. — *Let $g(x, \eta)$ satisfy (3.9). Let v_h, v be functions of*

$L^1_{loc}(\Omega; \mathbb{R}^m)$ *and assume that* v_h *converges to* v *in the sense of distributions. Then we have*

$$\liminf_h \int_\Omega g(x, v_h(x))dx \geq \int_\Omega g(x, v(x))dx, \tag{6.9}$$

if at least one of the following assumptions is satisfied:

i) the continuity condition (2.4) holds for the function g;

ii) $g(x, \eta) \geq c_{20} | \eta |$, *for some positive constant* c_{20};

iii) the sequence v_h *is bounded in* $L^1_{loc}(\Omega; \mathbb{R}^m)$.

Proof. — If *i)* holds, then we can proceed like in the proof of theorem 2.1, using the fact that our statement holds, as proved in theorem 6.1, for integrands g independent of x. In particular, like in (2.28), we obtain

$$\left[1 + \lambda\left(\frac{1}{v}\right)\right] \liminf_h \int_\Omega g(x, v_h)dx + \lambda\left(\frac{1}{v}\right) \geq \int_{\Omega_0} g_v(x, v)dx, \tag{6.10}$$

where g_v is defined similarly to f_v in (2.26). We use Fatou's lemma to go to the limit as $v \to +\infty$. Finally we go to the limit as $\Omega_0 \to \Omega$.

Now let us assume that *ii)* holds. Like in [*8*] we can approximate g by an increasing sequence of continuous functions g_k of the form (3.11). Proceeding like in section 3, we can see that g_k satisfies the continuity condition (2.4) of the previous part *i)*. Therefore the lower semicontinuity result holds for g_k and, by approximation, holds for g too.

Finally let us assume that *iii)* holds. We apply the case *ii)* to the integrand $g(x, \eta) + \varepsilon | \eta |$. Let Ω_0 be an open set compactly contained in Ω. If c_{21} is a bound for the L^1-norm of v_h on the set Ω_0, we have

$$\begin{aligned} \liminf_h \int_\Omega g(x, v_h)dx &\geq -\varepsilon c_{21} + \liminf_h \int_{\Omega_0} \{ g(x, v_h) + \varepsilon | v_h | \} dx \\ &\geq -\varepsilon c_{21} + \int_{\Omega_0} \{ g(x, v) + \varepsilon | v | \} dx. \end{aligned} \tag{6.11}$$

We obtain the result as $\varepsilon \to 0$ and $\Omega_0 \to \Omega$.

Corollaries 5.6 and 5.7 of [*19*] hold for an integrand $g(x, \eta)$ that satisfies the assumptions of the previous theorem 6.3. In particular we have a different proof of theorem 3.2 of section 3.

ACKNOWLEDGEMENT

The author wish to thank Stefan Hildebrandt for his advices in the manuscript and for his hospitality at the Mathematic Institute at the University of Bonn.

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(Manuscrit reçu le 15 avril 1985)
(révisé le 14 janvier 1986)