

Stable defects of minimizers of constrained variational principles

by

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ABSTRACT. — We prove energy and density bounds for minimizers of certain constrained variational problems, and we deduce limitations for their topological degree.

RÉSUMÉ. — On démontre des estimations *a priori* sur l'énergie et la densité des solutions de certains problèmes variationnels contraints, et l'on en déduit des bornes pour leur degré topologique.

1. INTRODUCTION

The objective of this note is to establish energy and density bounds for minimizers of certain constrained variational problems. As a consequence, we are able to provide limitations for the topological degree of such mappings. The prototype of these questions arose in our study of liquid crystals [HKL₁].

Let $W(A, u)$ be a smooth function of 3×3 matrices A and three-vectors u for which

$$\Lambda^{-1} |\nabla u|^2 \leq W(\nabla u, u) \leq \Lambda |\nabla u|^2 \quad \text{for some } \Lambda > 0, \tag{1.1}$$

whenever $u \in H^1(\Omega; \mathbb{S}^2)$, where $\Omega \subset \mathbb{R}^3$ is a bounded domain. Suppose that u is a W -minimizer in the sense that $u \in H^1(\Omega; \mathbb{S}^2)$ satisfies

$$\int_{\Omega} W(\nabla u, u) \, dx \leq \int_{\Omega} W(\nabla v, v) \, dx \tag{1.2}$$

when $v = u$ on $\partial\Omega$ for $v \in H^1(\Omega; \mathbb{S}^2)$

A conclusion of [HKL₁] in the case of a liquid crystal integrand is that u is Hölder continuous, in fact smooth, in a neighborhood of any point $a \in \Omega$ where the normalized energy

$$\mathbb{E}_{r,a}(u) = \frac{1}{8\pi r} \int_{\mathbb{B}_r(a)} |\nabla u|^2 \, dx, \quad r > 0,$$

is sufficiently small and, in particular, in a neighborhood of any point a where

$$\lim_{r \rightarrow 0} \mathbb{E}_{r,a}(u) = 0.$$

In this case

$$\begin{aligned} W(\nabla v, v) &= \frac{1}{2} \kappa_1 (\operatorname{div} v)^2 + \frac{1}{2} \kappa_2 (v \cdot \operatorname{curl} v)^2 \\ &\quad + \frac{1}{2} \kappa_3 |v \wedge \operatorname{curl} v|^2 + \frac{1}{2} \alpha [\operatorname{tr}(\nabla v)^2 - (\operatorname{div} v)^2] \end{aligned} \tag{1.3}$$

where each $\kappa_i > 0$ and we may, by [HKL₁], 1.2, choose $\alpha = \min\{\kappa_1, \kappa_2, \kappa_3\}$ without loss of generality. The argument leading to

partial regularity extends to general smooth W satisfying (1.1) without serious alteration provided that the blow-up functional [HKL₁], 2.2, [Lu] is elliptic.

Thus, the set of singularities of u in Ω is precisely

$$Z_u = \{a \in \Omega; \Theta(a) > 0\}$$

where

$$\Theta(a) = \limsup_{r \downarrow 0} \frac{1}{8\pi r} \int_{B_r(a)} |\nabla u|^2 dx.$$

Since $u \in H^1(\Omega; \mathbb{S}^2)$ it is immediate that Z_u has one dimensional Hausdorff measure $\mathcal{H}^1(Z_u) = 0$.

The first conclusion of the present note is an *energy density bound*: *If u is a W -minimizer, then*

$$\Theta(a) \leq M \text{ where } M \text{ depends only on } \Lambda.$$

A second conclusion is an *interior energy bound*:

If u is a W -minimizer and K is a compact subset of Ω , then

$$\int_K |\nabla u|^2 dx \leq C \text{ where } C \text{ depends only on } K \text{ and } \Lambda.$$

Thus the set of W -minimizers is bounded in $H^1_{loc}(\Omega)$. We shall, by imposing a convexity condition on W , prove the stronger statement that the set of W -minimizers is compact $H^1_{loc}(\Omega)$ (in the topology induced by the norm). More precisely, if (u_j) is a sequence of W -minimizers with, say,

$$\|u_j\|_{H^1(\Omega)} \leq \text{Const.},$$

then there is a $u \in H^1(\Omega; \mathbb{S}^2)$ and a subsequence $(u_{j'})$ such that

- (i) u is a W -minimizer and
- (ii) $\|u_{j'} - u\|_{L^2(K)} + \|\nabla(u_{j'} - u)\|_{L^2(K)} \rightarrow 0$ as $j' \rightarrow \infty$.

This is directly analogous to the well known Montel space property of bounded harmonic functions. The arguments here, in contrast to those of [SU], 4.6 and [HL₁], 6.4 do not make use of the regularity theory.

To give some perspective to the assertion about the density, note that a special case of W is

$$W(\nabla u) = \frac{1}{2} |\nabla u|^2, \tag{1.4}$$

the integrand of a harmonic mapping of $\Omega \subset \mathbb{R}^3$ into \mathbb{S}^2 . A W -minimizer u is then a solution of the system

$$\Delta u + |\nabla u|^2 u = 0 \quad \text{in } \Omega. \quad (1.5)$$

When $\Omega = \mathbb{B} = \{|x| < 1\}$, a large class of examples of solutions of (1.5) is given by the homogeneous extensions of conformal or anticonformal mappings of \mathbb{S}^2 onto itself, that is, by rational functions of z or \bar{z} . So if

$$\Pi: \mathbb{S}^2 \rightarrow \mathbb{C}$$

denotes stereographic projection and $f(z) = p(z)/q(z)$ is a rational function, $\gcd(p, q) = 1$, then

$$n_f(x) = \Pi^{-1} f[\Pi(x/|x|)] \quad (1.6)$$

is a solution of (1.3). Moreover, for such an n_f ,

$$\Theta(0) = \max(\deg p, \deg q).$$

Our density bound illustrates that not all such f give rise to minima. Moreover, numerical experiments of M. Luskin *et al.* [CHKLL] indicate that $\Theta(a) = 2$ probably does not occur. H. Brezis, J. P. Coron, and E. Lieb [BCL₁], [BCL₂] have shown that when $\Theta(a) \neq 0$, then

$$\Theta(a) = 1$$

and that any nonconstant homogeneous-degree-0 minimizer must be in the form

$$Q\left(\frac{x}{|x|}\right)$$

for some rotation Q of \mathbb{R}^3 . Combined with the interior regularity theory of [SU] and the asymptotic decay estimate of [S], the latter result implies that near a point a with $\Theta(a) \neq 0$, the minimizer u behaves like

$$u(x) = Q\left(\frac{x-a}{|x-a|}\right) + \text{higher order terms}$$

for some rotation Q . In this harmonic mapping case, the energy bounds of the present paper lead to further results on the stability, the number, and the location of singularities [AL], [HL₂].

The bounds established here apply to a minimizer $u \in H^1(\Omega; \mathbb{S}^2)$ of any functional satisfying the uniform growth condition (1.1). From our arguments, we deduce in paragraph 4 two other consequences:

$\nabla u \in L^q_{loc}(\Omega)$ for some $q > 2$ depending only on Λ , and

$\mathcal{H}^{3-q}(Z_u) = 0$; hence, the Hausdorff dimension of Z_u is < 1 .

In paragraph 6 we discuss how these results further generalize. They hold, roughly, for any mapping from a smooth compact Riemannian manifold with boundary to a smooth compact simply-connected Riemannian manifold that is a quasi-minimizer of some integrand satisfying uniform growth conditions. Without the simple connectivity hypothesis on the target, there may be no interior energy bound as shown by harmonic maps into the circle. In (1.1) one may also replace 2 by a number $p > 1$ provided one insists that the target manifold be simply $[p]$ -1 connected [HL₁], § 6. For $p \neq 2$, minimizers are in general only $\mathcal{C}^{1,\alpha}$ regular at their points of continuity (see [HL₁], § 3, [Lu]).

For a fixed compact domain Ω and compact manifold N , the results here indicate similarities between the family of *energy-minimizing* (not just energy-stationary) mappings from Ω to N and a uniformly bounded family of harmonic functions on the disk. The results are also somewhat analogous with the universal density and mass bounds of [Mo].

In an earlier work [HKL₂] we described how some of these ideas may be used to study an experiment of Williams, Pieranski and Cladis [WPC]. Discussion of the static theory of liquid crystals may be found in [BC], [E] and [L]. Analytical questions which arise are discussed in [HK₁], [HK₂] and [HKLu].

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2. DENSITY BOUND

Our proof of the density bound is based on the following consequence of [HKL], 2.3. For the reader's convenience and for other applications we give a proof in the appendix. Note that this lemma differs from the corresponding estimate of R. Schoen and K. Uhlenbeck [SU], 4.3 in that there is in [HL₁], 6.2 no "smallness" assumption applied to the right hand side of the inequality.

2.1. LEMMA. — *If $u \in H^1(\Omega; \mathbb{S}^2)$ and $a \in \Omega$, then for almost every positive $r < \text{dist}(a, \partial\Omega)$, there is a function $w \in H^1[\mathbb{B}_r(a); \mathbb{S}^2]$ such that*

$$w = u \quad \text{on } \partial\mathbb{B}_r(a)$$

and

$$\int_{\mathbb{B}_r(a)} |\nabla w|^2 dx \leq C \left(\int_{\partial\mathbb{B}_r(a)} |\nabla_{\text{tan}} u|^2 dS \cdot \int_{\partial\mathbb{B}_r(a)} |u - \xi|^2 dS \right)^{1/2}, \quad (2.1)$$

where $\xi \in \mathbb{R}^3$ is arbitrary and C is an absolute constant.

A proof of the lemma is given in the appendix. In particular, assume that u satisfies (1.2) in a domain Ω . Then u satisfies (1.2) in $\mathbb{B}_\rho(a)$ for almost every $\rho > 0$ sufficiently small so

$$\begin{aligned} \Lambda^{-1} \int_{\mathbb{B}_\rho} |\nabla u|^2 dx &\leq \int_{\mathbb{B}_\rho} W(\nabla u, u) dx \\ &\leq \int_{\mathbb{B}_r} W(\nabla w, w) dx \\ &\leq \Lambda \int_{\mathbb{B}_r} |\nabla w|^2 dx \\ &\leq \Lambda C \left(\int_{\partial\mathbb{B}_r} |\nabla_{\text{tan}} u|^2 dS \cdot \int_{\partial\mathbb{B}_r} |u - \xi|^2 dS \right)^{1/2}, \end{aligned}$$

or

$$\begin{aligned} \int_{\mathbb{B}_\rho(a)} |\nabla u|^2 dx \\ \leq \Lambda^2 C \left(\int_{\partial\mathbb{B}_r(a)} |\nabla_{\text{tan}} u|^2 dS \cdot \int_{\partial\mathbb{B}_r(a)} |u - \xi|^2 dS \right)^{1/2} \end{aligned} \quad (2.2)$$

2.2 COROLLARY. — *If $u \in H^1(\Omega, \mathbb{S}^2)$ is a W -minimizer and $\mathbb{B}_r(a) \subset \Omega$, then*

$$\mathbb{E}_{(1/2)r, a}(u) \leq \gamma [\mathbb{E}_{r, a}(u)]^{1/2}$$

where $\gamma = 6\pi^{1/2}\Lambda^2 C$ depends only on Λ .

Proof. — By Fubini's Theorem,

$$\int_{\partial\mathbb{B}_s(a)} |\nabla_{\text{tan}} u|^2 dS \leq 2r^{-1} \int_{\mathbb{B}_r(a)} |\nabla u|^2 dx = 2\mathbb{E}_{r, a}(u) \quad (2.3)$$

for some s with $\frac{1}{2}r < s < r$. Choosing $w \in H^1[\mathbb{B}_s(a); \mathbb{S}^2]$ as in Lemma 2.1 with $\xi=0$ and r replaced by s , we conclude from (2.2), and the equality $|u|=1$, that

$$\begin{aligned} \Lambda^{-1} \left(\frac{1}{2}r \right) E_{(1/2)r, a}(u) &= \Lambda^{-1} \int_{\mathbb{B}_{r/2}(a)} |\nabla u|^2 dx \\ &\leq \Lambda^{-1} \int_{\mathbb{B}_s(a)} |\nabla u|^2 dx \\ &\leq \Lambda C \left(\int_{\partial \mathbb{B}_s(a)} |\nabla_{\tan} u|^2 dS \cdot \int_{\partial \mathbb{B}_s(a)} |u|^2 dS \right)^{1/2} \\ &\leq \Lambda C [2 E_{r, a}(u) \cdot 4 \pi s^2]^{1/2} \\ &\leq 6 \pi^{1/2} \Lambda C \left(\frac{1}{2}r \right) [E_{r, a}(u)]^{1/2}. \quad \square \end{aligned}$$

2.3. THEOREM. — If $u \in H^1(\Omega; \mathbb{S}^2)$ is a W -minimizer and $a \in \Omega$, then

$$\Theta(a) = \limsup_{r \downarrow 0} \frac{1}{8 \pi r} \int_{\mathbb{B}_r(a)} |\nabla u|^2 dx \leq M$$

where $M = 2\gamma^2$ depends only on Λ .

Proof. — With $R = \text{dist}(a, \partial\Omega)$, we apply 2.2 iteratively with $r=R, \frac{1}{2}R, \frac{1}{4}R, \dots$ to find that, for each $j \in \{0, 1, 2, \dots\}$,

$$E_{2^{-j}R, a}(u) \leq \beta^{1 + \dots + (1/2)^j} [E_{R, a}(u)]^{(1/2)^j}.$$

Letting $j \rightarrow \infty$, we see that

$$\limsup_{j \rightarrow \infty} E_{2^{-j}R, a}(u) \leq \gamma^2 \cdot 1.$$

Finally, we may for any $0 < r < R$, choose $j \in \{0, 1, 2, \dots\}$ so that $2^{-j-1}R \leq r < 2^{-j}R$; hence,

$$E_{r, a}(u) \leq 2 E_{2^{-j}R, a}(u). \quad \square$$

Let us consider briefly another application of Lemma 2.1. Let $f(z)$ be a given polynomial of degree m and for $\varepsilon > 0$ consider the harmonic

mapping $n_{\varepsilon f}(x)$ of the unit ball \mathbb{B} into \mathbb{S}^2 defined via homogeneous extension as given in (1.6). With $n_{\varepsilon f}(x)$ as boundary values on $\partial\mathbb{B}$, let u_ε denote a minimizer of the Dirichlet integral, that is,

$$\int_{\mathbb{B}} |\nabla u_\varepsilon|^2 dx = \inf_{\mathcal{A}} \int_{\mathbb{B}} |\nabla v|^2 dx,$$

$$\mathcal{A} = \{v \in H^1(\mathbb{B}; \mathbb{S}^2) : v = n_{\varepsilon f} \text{ on } \partial\mathbb{B}\}.$$

Although $n_{\varepsilon f}$ is a solution of the equilibrium equations (1.5) it need not equal u_ε . Indeed, $\varepsilon f(z) \rightarrow 0$ pointwise in the complex plane, from which it immediately follows that

$$n_{\varepsilon f} \rightarrow -e_3 \quad (= \text{the south pole}) \text{ in } L^2(\mathbb{S}^2).$$

Moreover, $\nabla_{\tan} u_\varepsilon = \nabla_{\tan} n_{\varepsilon f}$, so

$$\int_{\partial\mathbb{B}} |\nabla_{\tan} u_\varepsilon|^2 dS = \int_{\partial\mathbb{B}} |\nabla_{\tan} n_{\varepsilon f}|^2 dS = 8\pi m.$$

Consequently,

$$\int_{\mathbb{B}} |\nabla u_\varepsilon|^2 dx \leq C \left(8\pi m \int_{\partial\mathbb{B}} |u_\varepsilon + e_3|^2 dS \right)^{1/2} \rightarrow 0$$

as $\varepsilon \rightarrow 0$. It is immediate that $n_{\varepsilon f}$ is not a minimizer for ε sufficiently small, even when $m = 1$.

3. ENERGY BOUND

The proof of the energy bound is also based on Lemma 2.1.

3.1. THEOREM. — *For any compact subset K of Ω there is a constant C_K depending only on Ω , K , and Λ so that*

$$\int_K |\nabla u|^2 dx \leq C_K$$

for any W -minimizer $u \in H^1(\Omega, \mathbb{S}^2)$.

Proof. — By compactness of K , it suffices to prove such a bound with $\Omega = \mathbb{B}$ and $K = \overline{\mathbb{B}}_{1-\delta}$ for a fixed positive $\delta < 1$. For $0 < \rho < 1$, let

$$\mathbb{D}(\rho) = \int_{\mathbb{B}_\rho} |\nabla u|^2 dx,$$

and note that \mathbb{D} is monotone increasing with

$$\mathbb{D}'(\rho) = \int_{\partial\mathbb{B}_\rho} |\nabla u|^2 dS \quad \text{for almost all } 0 < \rho < 1.$$

By the W -minimality of u and 2.1 with $\xi=0$, we find that for almost all $0 < \rho < 1$,

$$\mathbb{D}(\rho) \leq \alpha \rho [\mathbb{D}'(\rho)]^{1/2} \quad \text{or} \quad \alpha/\rho^2 \leq \mathbb{D}'(\rho)/\mathbb{D}(\rho)^2$$

where $\alpha = 2\pi^{1/2} C$. Integrating this inequality from $1-\delta$ to 1 gives

$$\alpha\delta/(1-\delta) \leq 1/\mathbb{D}(1-\delta) \quad \text{or} \quad \mathbb{D}(1-\delta) \leq \alpha^{-1}[(1-\delta)/\delta]. \quad \square$$

4. HIGHER INTEGRABILITY

We illustrate here how our estimate 2.1 may be used to ascertain higher integrability of the gradient of a minimizer. The tool for this is the Reverse Hölder inequality, cf. Gehring [Ge], Meyers and Elcrat [ME], and Giaquinta and Modica [GM], or Giaquinta's book [Gi], Chapter V.

4.1. THEOREM. — *There is a $q > 2$ depending only on Λ so that any W -minimizer u belongs to $H_{loc}^{1,q}(\Omega)$. Moreover, for any ball $B \subset\subset \Omega$,*

$$\|u\|_{H^{1,q}(B)} \leq C_1 \|u\|_{H^1(B)} \leq C_2$$

where C_1 and C_2 depend only on B and Λ .

Proof. — Suppose $B_{2r} = \mathbb{B}_{2r}(a) \subset\subset \Omega$. From 2.1, for any $\varepsilon > 0$,

$$\int_{B_\rho} |\nabla u|^2 dx \leq \varepsilon \int_{\partial B_\rho} |\nabla u|^2 dS + c \varepsilon^{-1} \int_{\partial B_\rho} |u - \xi|^2 dS$$

for $0 < \rho \leq 2r$, where c depends only on Λ . Integrating this from r to $2r$ shows that

$$\int_{B_r} |\nabla u|^2 dx \leq \varepsilon r^{-1} \int_{B_{2r}} |\nabla u|^2 dx + c_0 (\varepsilon r)^{-1} \int_{B_{2r}} |u - \xi|^2 dx.$$

Set $\varepsilon = \delta r$ and choose

$$\xi = \bar{u} = |B_{2r}|^{-1} \int_{B_{2r}} u dx.$$

Note that by Sobolev's inequality,

$$\int_{B_{2r}} |u - \bar{u}|^2 dx \leq c_1 \left(\int_{B_{2r}} |\nabla u|^p dx \right)^{2/p},$$

where $p = 2n/2 + n = 6/5 < 2$ and c_1 is an absolute constant. Hence

$$\int_{B_r} |\nabla u|^2 dx \leq \delta \int_{B_{2r}} |\nabla u|^2 dx + c_0 c_1 (\delta r^2)^{-1} \left(\int_{B_{2r}} |\nabla u|^p dx \right)^{2/p}.$$

We now divide this inequality by $|B_r|$. We then obtain, for some constant c_2 depending only on Λ and δ that

$$\int_{B_r} |\nabla u|^2 dx \leq 8\delta \int_{B_{2r}} |\nabla u|^2 dx + c_2 \left(\int_{B_{2r}} |\nabla u|^p dx \right)^{2/p}.$$

Fixing $\delta < 1/8$, we may now apply this reverse-Hölder inequality as in the references cited above to conclude the existence of a $q > 2$ such that

$$\nabla u \in L^q_{\text{loc}}(\Omega).$$

Moreover, for c_3, c_4 depending on r and Λ ,

$$\begin{aligned} \|\nabla u\|_{L^q(B_r)} &\leq c_3 (\|\nabla u\|_{L^2(B_{2r})} + \|\nabla u\|_{L^p(B_{2r})}) \\ &\leq c_4 \|\nabla u\|_{L^2(B_{2r})}, \end{aligned}$$

since $p < 2$. The conclusion now follows using 3.1. \square

4.2. COROLLARY. — *The singular set $Z_u = \{a \in \Omega : \Theta(a) > 0\}$ of the minimizer u has $\mathcal{H}^{3-q}(Z_u) = 0$. In particular, the Hausdorff dimension of Z_u is strictly less than 1.*

Proof. — For $\mathbb{B}_r(a) \subset \Omega$, we infer from Hölder’s inequality that

$$\left(\int_{\mathbb{B}_r(a)} |\nabla u|^2 dx \right)^{1/2} \leq \left(\int_{\mathbb{B}_r(a)} |\nabla u|^q dx \right)^{1/q} |\mathbb{B}_r|^{(q-2)/2q}.$$

Squaring and multiplying by $r^{-1} = r^{2(q-3)/q} \cdot r^{-3(q-2)/q}$, we see that

$$\mathbb{E}_{r,a}(u) \leq \left(r^{q-3} \int_{\mathbb{B}_r(a)} |\nabla u|^q dx \right)^{2/q} |\mathbb{B}|^{(q-2)/q},$$

hence,

$$Z_u \subset \left\{ a \in \Omega : \limsup_{r \downarrow 0} r^{q-3} \int_{\mathbb{B}_r(a)} |\nabla u|^q dx > 0 \right\},$$

and the corollary follows from 4.1 and a covering argument [G], §IV, 2.2. \square

4.3. REMARK. — Corollary 4.2 (with a possibly different $q > 2$) may also be derived directly from 2.3, the regularity lemma [HKL], 2.5, and the “Work-raccoon theorem” [W], 5.1 in the manner of [W], 5.2.

5. THE COMPACTNESS OF MINIMIZERS

By imposing a convexity condition on W we shall show that a bounded set of minimizers has some compactness properties. For ease of exposition, we shall assume that $\Omega = \mathbb{B}$, a unit ball.

This is an opportunity to distinguish between a local W -minimizer and a W -minimizer. To this point, the functions we have called W -minimizers need only satisfy (1.2) in some subdomain Ω of their domain of definition in order that the conclusions of the previous results hold in that subdomain. In particular, a function which satisfies (1.2) in some neighborhood of every point of its domain of definition must have $\Theta \leq M$ throughout. However a function may have the property that (1.2) is satisfied in some neighborhood of every point without minimizing the functional in the entire domain. A simple example of this is given by the mapping $u: \mathbb{B}_r \rightarrow \mathbb{S}^2$ defined by

$$u(x) = (\cos \theta(x), \sin \theta(x), 0), \quad x \in \mathbb{B}_r,$$

where θ is an ordinary harmonic function in \mathbb{R}^3 , which fails to minimize the Dirichlet integral (1.4) for large enough r among all functions $v: \mathbb{B}_r \rightarrow \mathbb{S}^2$ with $v=u$ on $\partial\mathbb{B}_r$. In order that the conclusions about compactness be valid, all the mappings in question must be W -minimizers in the same domain.

5. 1. PROPOSITION. — *Suppose that the functional*

$$\int_{\Omega} W(\nabla v, v) dx$$

is lower semicontinuous with respect to weak convergence in $H^1(\mathbf{B}; \mathbb{S}^2)$. If (u_j) is a sequence of W -minimizers and if

$$u_j \rightarrow u \quad \text{in } H_{\text{loc}}^1(\mathbf{B}) \text{ weakly,}$$

then u is a W -minimizer.

The lower semicontinuity hypothesis is implied by the convexity condition (5.9) discussed below [M]. Also the weak convergence hypothesis always holds for some subsequence by Theorem 3.1.

Proof. — By rescaling slightly, we may assume that $(u_j) \subset H_{\text{loc}}^1(\mathbf{B}_{1+\rho}; \mathbb{S}^2)$, $\rho > 0$, is the sequence of minimizers so that $u_j, u \in H^1(\mathbf{B}; \mathbb{S}^2)$ and $u_j|_{\mathbf{B}}$ are W -minimizers. Assume that

$$w \in H^1(\mathbf{B}; \mathbb{S}^2) \quad \text{with } w=u \text{ on } \partial\mathbf{B}.$$

Given $\delta > 0$, choose $\eta \in H^{1,\infty}(\mathbf{B})$ so that

$$\eta = 1 \quad \text{on } \mathbf{B}_{1-\delta}, \quad \eta = 0 \quad \text{on } \partial\mathbf{B},$$

and

$$|\nabla \eta| \leq 1/\delta \quad \text{on } \mathbf{A}_{\delta} \alpha = \mathbf{B} - \mathbf{B}_{1-\delta}.$$

Let us set

$$v_j = (1 - \eta)u_j + \eta w \quad \text{in } \mathbf{B}$$

so that

$$|v_j| = |w| = 1 \quad \text{on } \partial\mathbf{B}_{1-\delta} \quad \text{and} \quad |v_j| = |u_j| = 1 \quad \text{on } \partial\mathbf{B}.$$

Applying our Extension Lemma A.1 in the appendix, with Ω chosen to be the region $A_\delta = B \sim B_{1-\delta}$, we may find a function $w_j \in H^1(A_\delta; \mathbb{S}^2)$ such that

$$\int_{A_\delta} |\nabla w_j|^2 dx \leq C \int_{A_\delta} |\nabla v_j|^2 dx, \tag{5.2}$$

$$w_j = v_j = w \quad \text{on } \partial B_{1-\delta}$$

and

$$w_j = v_j = u_j \quad \text{on } \partial B.$$

Extending w_j to all of B by letting

$$w_j = w \quad \text{on } B_{1-\delta},$$

we see that w_j is then an admissible variation with boundary data u_j , hence

$$\int_B W(\nabla u_j, u_j) dx \leq \int_B W(\nabla w_j, w_j) dx.$$

From this inequality and the assumed lower semicontinuity, we see that given $\varepsilon > 0$, for j sufficiently large,

$$\int_B W(\nabla u, u) dx - \frac{1}{2} \varepsilon \leq \int_B W(\nabla w_j, w_j) dx \tag{5.3}$$

for all sufficiently large j . Expanding the right hand side and employing (5.2), we have that

$$\begin{aligned} \int_B W(\nabla w_j, w_j) dx &\leq \int_{B_{1-\delta}} W(\nabla w_j, w_j) dx + \int_{A_\delta} W(\nabla w_j, w_j) dx \\ &\leq \int_{B_{1-\delta}} W(\nabla v, v) dx + \Lambda \int_{A_\delta} |\nabla w_j|^2 dx \\ &\leq \int_B W(\nabla v, v) dx + C \Lambda \int_{A_\delta} |\nabla v_j|^2 dx. \end{aligned} \tag{5.4}$$

For the latter integral we observe that

$$\begin{aligned} \int_{A_\delta} |\nabla v_j|^2 dx &\leq \int_{A_\delta} |\nabla(u_j - v)|^2 dx + \int_{A_\delta} |(u_j - v) \otimes \nabla \eta|^2 dx \\ &\leq 2 \int_{A_\delta} |\nabla u_j|^2 dx + 2 \int_{A_\delta} |\nabla v|^2 dx \\ &\quad + \delta^{-2} \int_{A_\delta} |u_j - v|^2 dx. \end{aligned} \quad (5.5)$$

Next we observe that by a version of Poincaré's inequality, there is a constant c_B so that for all sufficiently small δ ,

$$\begin{aligned} \int_{A_\delta} |\zeta|^2 dx &\leq c_B \delta^2 \int_{A_\delta} |\nabla \zeta|^2 dx \\ &\text{for all } \zeta \in H^1(A_\delta) \text{ with } \zeta = 0 \text{ on } \partial B. \end{aligned} \quad (5.6)$$

Applying (5.6) with $\zeta = u - v$ now gives

$$\begin{aligned} \delta^{-2} \int_{A_\delta} |u_j - v|^2 dx &\leq \delta^{-2} \int_{A_\delta} |u_j - u|^2 dx + \delta^{-2} \int_{A_\delta} |u - v|^2 dx \\ &\leq \delta^{-2} \int_{A_\delta} |u_j - u|^2 dx + c_B \int_{A_\delta} |\nabla(u - v)|^2 dx. \end{aligned} \quad (5.7)$$

Finally, according to Hölder's inequality and Theorem 4.1,

$$\begin{aligned} \int_{A_\delta} |\nabla u_j|^2 dx &\leq \left(\int_{A_\delta} |\nabla u_j|^q dx \right)^{2/q} |A_\delta|^{1-2/q} \\ &\leq \left(\int_{\Omega_0} |\nabla u_j|^q dx \right)^{2/q} |A_\delta|^{1-2/q} \\ &\leq C_2^2 |A_\delta|^{1-2/q}. \end{aligned}$$

Consequently by (5.5) and (5.7)

$$\begin{aligned} \int_{A_\delta} |\nabla v_j|^2 dx &\leq 2C_2^2 |A_\delta|^{1-2/q} + 2 \int_{A_\delta} |\nabla v|^2 dx \\ &\quad + \delta^{-2} \int_{A_\delta} |u_j - u|^2 dx + c_B \int_{A_\delta} |\nabla(u - v)|^2 dx. \end{aligned}$$

Now permit $j \rightarrow \infty$. Since $u_j \rightarrow u$ in $L^2(B)$, the third term vanishes so that

$$\begin{aligned} \limsup_{j \rightarrow \infty} \int_{A_\delta} |\nabla v_j|^2 dx &\leq 2 C_2^2 |A_\delta|^{1-2/q} + 2 \int_{A_\delta} |\nabla v|^2 dx + c_B \int_{A_\delta} |\nabla(u-v)|^2 dx. \end{aligned}$$

Choosing δ small enough to make the right-hand side less than $\frac{1}{2}(CA)^{-1} \epsilon$, we see that (5.3) and (5.4) now imply that

$$\int_B W(\nabla u, u) dx \leq \int_B W(\nabla w, w) dx + \epsilon. \quad \square$$

We shall now impose a few assumptions about W . Suppose

$$|W_A(A, u)| \leq \text{const}(|A| + 1) \quad \text{for } |u| = 1, \tag{5.8}$$

and

there is a $\lambda > 0$ such that for all $A, u, |u| = 1$,

$$W_{AA}(A, u) B \cdot B \geq \lambda |B|^2 \quad \text{for all } 3 \times 3 \text{ matrices } B. \tag{5.9}$$

It follows from (5.9) that for any (A, u) and (B, v) with $|u| = |v| = 1$,

$$\begin{aligned} W(B, v) - W(A, u) &\geq W_A(A, v) \cdot (B - A) \\ &\quad + \lambda W |B - A|^2 + W(A, v) - W(A, u). \end{aligned} \tag{5.10}$$

It ought to be noted that the liquid crystal integrand (1.3) may be written

$$W(A, u) = \frac{1}{2} \alpha |A|^2 + V(A, u)$$

where $V(A, u)$ is a nonnegative quadratic form in A which is smooth in u , cf. [HKL₁], [HK₁].

5.2. LEMMA. — *If*

$$v_j \rightarrow v \quad \text{weakly in } H^1(\Omega; S^2),$$

and if

$$\int_{\Omega} \mathbf{W}(\nabla v_j, v_j) dx \rightarrow \int_{\Omega} \mathbf{W}(\nabla v, v) dx, \quad (5.11)$$

then a subsequence of the (v_j) converges strongly in $H^1(\Omega; \mathbb{S}^2)$.

Proof. — First select a subsequence of the (v_j) so that, after changing notations,

$$v_j \rightarrow v \quad \text{pointwise a. e. in } \Omega.$$

Integrating (5.10) gives that

$$\begin{aligned} \lambda \int_{\Omega} |\nabla(v_j - v)|^2 dx &\leq \int_{\Omega} [\mathbf{W}(\nabla v_j, v_j) - \mathbf{W}(\nabla v, v)] dx \\ &\quad - \int_{\Omega} \mathbf{W}_A(\nabla v, v_j) \cdot (\nabla v_j - \nabla v) dx \\ &\quad - \int_{\Omega} [\mathbf{W}(\nabla v, v_j) - \mathbf{W}(\nabla v, v)] dx, \quad (5.12) \end{aligned}$$

and it suffices to show that the right-hand side of (5.12) approaches zero.

The first term on the right-hand side of (5.12) approaches zero by our hypothesis (5.11).

Note that

$$\mathbf{W}_A(\nabla v, v_j) \rightarrow \mathbf{W}_A(\nabla v, v) \quad \text{pointwise a. e.}$$

and

$$|\mathbf{W}_A(\nabla v, v_j)| \leq \text{const}[|\nabla v| + 1].$$

Thus

$$\mathbf{W}_A(\nabla v, v_j) \rightarrow \mathbf{W}_A(\nabla v, v) \quad (\text{strongly}) \text{ in } L^2(\Omega).$$

In as much as

$$\nabla v_j \rightarrow \nabla v \quad \text{weakly in } L^2(\Omega),$$

the second term on the right-hand side of (5.12) also converges to zero.

Finally the third term goes to zero because

$$\mathbf{W}(\nabla v, v_j) \rightarrow \mathbf{W}(\nabla v, v) \quad \text{pointwise a. e.}$$

and

$$|W(\nabla v, v_j)| \leq \text{const} [|\nabla v|^2 + 1] \in L^1(\Omega). \quad \square$$

5.3. THEOREM. — Suppose that W satisfies (5.8) and (5.9). For any sequence of W -minimizers, there is a subsequence (u_j) and a $u \in H^1_{\text{loc}}(\mathbb{B}; \mathbb{S}^2)$ such that

$$u_j \rightarrow u \quad \text{in } H^1_{\text{loc}}(\mathbb{B}; \mathbb{S}^2)$$

and

u is a W -minimizer for any subdomain $\Omega \Subset \mathbb{B}$.

Proof. — By Theorem 3.1, a subsequence (u_j) is weakly convergent in $H^1_{\text{loc}}(\mathbb{B}; \mathbb{S}^2)$ to a function $u \in H^1_{\text{loc}}(\mathbb{B}; \mathbb{S}^2)$. Moreover, u is W -minimizing by Proposition 5.1 because (5.9) implies the weak lower semicontinuity of the functional

$$\int_{\Omega} W(\nabla v, v) dx.$$

To prove the strong convergence in $H^1_{\text{loc}}(\mathbb{B}; \mathbb{S}^2)$ it now suffices by this lower semicontinuity and Lemma 5.2 to show that

$$\int_{\Omega} W(\nabla u, u) dx \geq \limsup_{j \rightarrow \infty} \int_{\Omega} W(\nabla u_j, u_j) dx \quad (5.13)$$

for any smooth domain $\Omega \Subset \mathbb{B}$. This we will establish by an argument similar to the proof of Proposition 5.1. In the argument we may rescale slightly and take $\Omega = \mathbb{B}$. For $\mathbb{B}_{1-\delta}$ and η as before, we here let

$$v_j = (1 - \eta)u_j + \eta u \quad \text{in } \mathbb{B}.$$

We again apply Extension Lemma A.1 to the function v_j restricted to the region $A_{\delta} = \mathbb{B} \sim \mathbb{B}_{1-\delta}$ to obtain $w_j \in H^1(A_{\delta}; \mathbb{S}^2)$ satisfying (5.2). Extending w_j to all of $\mathbb{B}_{1-\delta}$ by letting $w_j = v$ on $\mathbb{B}_{1-\delta}$, we infer from the minimality of $u_j | \mathbb{B}$ that

$$\begin{aligned} \int_{\mathbb{B}} W(\nabla u_j, u_j) dx &\leq \int_{\mathbb{B}} W(\nabla w_j, w_j) dx \\ &= \int_{\mathbb{B}_{1-\delta}} W(\nabla u, u) dx + \int_{A_{\delta}} W(\nabla w_j, w_j) dx \\ &\leq \int_{\mathbb{B}} W(\nabla u, u) dx + C \Lambda \int_{A_{\delta}} |\nabla v_j|^2 dx. \quad (5.14) \end{aligned}$$

We estimate the last term as before. Namely, using the Reverse Hölder inequality 4.1 as before and realizing that the limit u is also a minimizer,

$$\begin{aligned} \int_{A_\delta} |\nabla v_j|^2 dx &\leq \int_{A_\delta} |\nabla(u_j - u)|^2 dx + \int_{A_\delta} |(u_j - u) \otimes \nabla \eta|^2 dx \\ &\leq 2 \int_{A_\delta} |\nabla u_j|^2 dx + 2 \int_{A_\delta} |\nabla u|^2 dx + \delta^{-2} \int_{A_\delta} |u_j - u|^2 dx \\ &\leq 4 C_2^2 |A_\delta|^{1-2/q} + \delta^{-2} \int_{A_\delta} |u_j - u|^2 dx. \end{aligned}$$

Thus

$$\limsup_{j \rightarrow \infty} \int_{A_\delta} |\nabla v_j|^2 dx \leq 4 C_2^2 |A_\delta|^{1-2/q}.$$

Combining this with (5.14) gives (5.15) and completes the proof. \square

6. GENERALIZATION

We make the following assumptions:

(a) *Domain:* We wish to let the domain be an arbitrary smooth compact Riemannian manifold M with boundary. However, since the results will be only local bounds with constants depending on M and since the functionals considered in (b) are general enough to include the affect of an arbitrary smooth metric on M , we will assume, without losing generality, that *the domain is an open subset Ω of \mathbb{R}^n with the ordinary Euclidean metric.*

(b) *Functional:* $\mathcal{A}(u) = \int_{\Omega} A(x, u, \nabla u) dx$ where A is a measurable function satisfying the uniform growth conditions

$$\Lambda^{-1} |z|^{p-\mu} \leq A(x, y, z) \leq \Lambda |z|^{p+\mu}$$

for some constants $\Lambda \geq 1$, $\mu \geq 0$ and $p > 1$. For further assumptions on the functional which lead to partial regularity, see e. g. [EG], [FH], [G], [GG], [GM], [Lu]. It might be noted, in addition, that under appropriate

assumptions, minimizers of unconstrained problems are actually fairly smooth [CE].

(c) *Target:* N is a smooth compact, simply $[p]-1$ connected (i. e. $\pi_0(N) = \pi_1(N) = \dots = \pi_{[p]-1}(N) = 0$) Riemannian manifold without boundary. Via an isometric embedding, we view N as a Riemannian submanifold of \mathbb{R}^k .

(d) *Quasi-minimizer:* $u \in L^{1,p}(\Omega, N)$ [HL₁], §1 and, for some constant $Q \geq 1$,

$$\int_{\mathbb{B}_r(a)} A(x, u, \nabla u) dx \leq Q \int_{\mathbb{B}_r(a)} A(x, w, \nabla w) dx$$

whenever $\mathbb{B}_r(a) \subset \Omega$, $w \in L^{1,p}[\mathbb{B}_r(a), N]$, and $w|_{\partial\mathbb{B}_r(a)} \equiv u|_{\partial\mathbb{B}_r(a)}$. This includes minimizers for certain vector-valued obstacle problems. See [G], p. 252. Partial regularity results for higher dimensional smooth obstacle mapping problems have been obtained in [DF] and [F]. These issues become significantly more complicated when the system of equilibrium equations is far from diagonal, cf. e. g. [K].

Under these assumptions, we now use the notations

$$\mathbb{D}(r) = \int_{\mathbb{B}_r(a)} |\nabla u|^p dx \quad \text{and} \quad \mathbb{E}_{r,a}(u) = r^{p-n} \int_{\mathbb{B}_r(a)} |\nabla u|^p dx.$$

Note that for $p \geq n$, $\lim_{r \downarrow 0} \mathbb{E}_{r,a}(u) = 0$ for all $a \in \Omega$ by the absolute continuity of $|\nabla u|^p dx$.

By the topological assumption in (c), [HL₁], 6.2 provides a suitable replacement for 2.1. The conclusion now involves an additive inequality

$$\begin{aligned} r^{p-n} \int_{\mathbb{B}_r} |\nabla w|^p dx \\ \leq \lambda r^{p-n+1} \int_{\partial\mathbb{B}_r} |\nabla_{\tan} u|^p dS + C \lambda^{-p} r^{1-n} \int_{\partial\mathbb{B}_r} |u - \xi|^p dS \end{aligned} \quad (6.1)$$

that is valid for all $\lambda > 0$.

In generalizing 3.1 we find that

$$\begin{aligned} \mathbb{D}(r) \leq \lambda r \mathbb{D}'(r) \\ + \kappa \Lambda^{2p+2} Q^p \lambda^{-p} s^{1-n} \int_{\partial\mathbb{B}_s(a)} |u - \xi|^p dS + \kappa \Lambda \mu Q r^p \end{aligned}$$

for some constant κ depending only on $n, k,$ and p . Moreover,

$$\mathbb{D}(r) \leq \alpha [\max \{r^{1/p} \mathbb{D}'(r), r^{1+p}\}]^{p/(1+p)}$$

where $\alpha = 2 + 2\kappa \Lambda^{2p+2} Q^p (n + \mu) |\mathbb{B}|$. The second conclusion follows from the first by taking $\lambda = [r^p \mathcal{D}'(r)]^{-1/(1+p)}$. One verifies the first by applying (6.1) with λ replaced by $\lambda/(\Lambda^2 Q)$ to obtain a suitable comparison function w with $w|_{\partial\mathbb{B}_r(a)} \equiv u|_{\partial\mathbb{B}_r(a)}$; hence,

$$\int_{\mathbb{B}_r(a)} A(x, u, \nabla u) dx \leq Q \int_{\mathbb{B}_r(a)} A(u, w, \nabla w) dx.$$

Similarly in generalizing Corollary 2.2, we find that

$$\begin{aligned} \mathbb{E}_{(1/2)r, a}(u) &\leq \lambda \mathbb{E}_{r, a}(u) \\ &\quad + \kappa \Lambda^{2p+2} Q^p \lambda^{-p} s^{1-n} \int_{\partial\mathbb{B}_s(a)} |u - \xi|^p dS + \kappa \Lambda \mu Q r^p \end{aligned}$$

for some s with $\frac{1}{2}r < s < r$ and constant κ depending only on n, k and p .

Moreover,

$$\mathbb{E}_{(1/2)r, a}(u) \leq \beta [\max \{\mathbb{E}_{r, a}(u), r^{1+p}\}]^{p/(1+p)}$$

where $\beta = 2 + 2\kappa \Lambda^{2p+2} Q^p (n + \mu) |\mathbb{B}|$.

The new version of Theorem 3.1 should be:

For any compact subset K of Ω there is a constant C depending only on $\Omega, K, N, \Lambda, \mu, Q,$ and p so that

$$\int_K |\nabla u|^p dx \leq C$$

for any W -minimizer u .

In modifying the proof of 3.1, we now find that

$$[\mathbb{D}(r)]^{-(p+1)/p} \mathbb{D}'(r) \geq \alpha^{-(p+1)/p} r^{-1/p} \geq \alpha^{-(p+1)/p}$$

for almost all $0 < r < 1$; hence,

$$\mathbb{D}(1 - \delta) \leq (p - 1)^p \alpha^{p+1} \delta^{-p}.$$

Next we note that trivially, for $0 < r \leq 2\beta$,

$$\left(\frac{1}{2}r\right)^{1+p} \leq \beta [\max\{E_{r,a}(u), r^{1+p}\}]^{p/(1+p)},$$

and we may modify the proof of 2.2 by iterating using the quantity $\max\{E_{r,a}(u), r^{1+p}\}$ in place of $E_{r,a}(u)$. We now obtain the density bound

$$\Theta(a) = \limsup_{r \downarrow 0} E_{r,a}(u) = \limsup_{r \downarrow 0} \max\{E_{r,a}(u), r^{1+p}\} \leq 2^{p-n} \beta^{1+p}.$$

We also obtain the reverse-Hölder inequality

$$\begin{aligned} \left[\int_{\mathbb{B}_{r/2}(a)} |\nabla u|^p dx \right]^{1/p} &\leq \lambda \left[\int_{\mathbb{B}_r(a)} |\nabla u|^p dx \right]^{1/p} \\ &\quad + \gamma \Lambda^{2+(2/p)} Q \lambda^{-p} \left[\int_{\mathbb{B}_r(a)} |\nabla u|^m dx \right]^{1/m} + \alpha (\Lambda \mu Q)^{1/p} r \end{aligned}$$

where γ now depends on n, k and p . Since $1 < m < p$ we infer from [G], §V,1.1 that

the gradient of the minimizer u belongs to $L^q_{loc}(\Omega)$ for some $q > p$ depending only on n, k, p, Λ, μ and Q .

Using this, Hölder's inequality and [G], §IV,2.2 as before, we deduce that

the set $Z_u = \{a \in \Omega : \Theta(a) > 0\}$ has \mathcal{H}^{n-q} measure zero, and hence Hausdorff dimension strictly less than $n-p$. \square

Finally, we take note of the necessity of the simply connected hypothesis in (c). For example, the function

$$u_\lambda(x) = (\cos \lambda x_1, \sin \lambda x_1)$$

is the unique energy minimizing (even stationary) harmonic map from \mathbb{B} to \mathbb{S}^1 having boundary values $(\cos \lambda x_1, \sin \lambda x_1)$. But

$$\int_{\mathbb{B}_r} |\nabla u_\lambda|^2 dx \rightarrow \infty \quad \text{as } \lambda \rightarrow \infty \text{ for any } r < 1,$$

ruling out the possibility of an interior energy bound.

APPENDIX

Here we give succinct proofs of the extension lemmas which are used in this paper.

A. 1. EXTENSION LEMMA. — Let Ω be \mathbb{B}_1 , the unit ball, or the annulus $\mathbb{B} \sim \mathbb{B}_s$ for some $s, \frac{1}{2} < s < 1$. For any $v \in H^1(\Omega; \mathbb{R}^3)$ with $|v|=1$ on $\partial\Omega$, there exists a function $w \in H^1(\Omega; \mathbb{S}^2)$ such that

$$w = v \text{ on } \partial\Omega$$

and

$$\int_{\Omega} |\nabla w|^2 dx \leq C \int_{\Omega} |\nabla v|^2 dx$$

for an absolute constant C , independent of Ω .

Proof. — For $|a| < 1$ consider the function

$$w_a(x) = [v(x) - a] / |v(x) - a|, \quad x \in \Omega,$$

whose gradient (with respect to x) is

$$\nabla w_a = |v - a|^{-1} \nabla v - |v - a|^{-3} (v - a) \otimes (v - a) \nabla v.$$

Thus

$$|\nabla w_a| \leq 2^{1/2} |v - a|^{-1} |\nabla v| \text{ in } \Omega,$$

and hence

$$\int_{\mathbb{B}_{1/2}} |\nabla w_a|^2 da \leq 2 \int_{\mathbb{B}_{1/2}} |v - a|^{-2} |\nabla v|^2 da < +\infty \text{ a. e. in } \Omega.$$

Indeed, elementary considerations show that

$$\int_{\mathbb{B}_{1/2}} |v - a|^{-2} da \leq K < \infty$$

independent of $v \in \mathbb{R}^3$.

Integrating over Ω , we obtain

$$\int_{\mathbb{B}_{1/2}} \int_{\Omega} |\nabla w_a|^2 da \leq \int_{\Omega} \int_{\mathbb{B}_{1/2}} |\nabla w_a|^2 da \leq 2K \int_{\Omega} |\nabla v|^2 dx.$$

Hence there is an a_0 with $|a_0| \leq \frac{1}{2}$, such that

$$\int_{\Omega} |\nabla w_{a_0}|^2 da \leq 8K |\mathbb{B}_1| \int_{\Omega} |\nabla v|^2 dx.$$

Now observe that on $\partial\Omega$,

$$w_a(x) = [v(x) - a] / |v(x) - a| \quad \text{with } |v| = 1.$$

Let

$$\Pi_a(\xi) = |\xi - a|^{-1} (\xi - a) \quad \text{for } \xi \in \mathbb{S}^2 \quad \text{and } |a| \leq \frac{1}{2}.$$

This is a bilipshitz homeomorphism of \mathbb{S}^2 onto itself. Indeed,

$$\Pi_a^{-1}(\eta) = a + [(a \cdot \eta)^2 + (1 - |a|^2)]^{1/2} \eta$$

with

$$|\nabla \Pi_a^{-1}(\eta)| \leq L$$

uniformly independent of a with $|a| \leq \frac{1}{2}$. Thus we may choose

$$w = \Pi_{a_0}^{-1} \circ w_{a_0}$$

and the constant

$$C = 8K |\mathbb{B}_1| L^2. \quad \square$$

Proof of Lemma 2.1. — By Fubini's theorem,

$$\int_{\partial \mathbb{B}_r(a)} |\nabla u|^2 dS < \infty$$

for almost every positive $r < \text{dist}(a, \partial\Omega)$. For any such r , we abbreviate $B = \mathbb{B}_r(a)$ and let v be the harmonic extension of u to B and determine w by Lemma A.1 with $\Omega = B$. Thus

$$\Delta v = 0 \quad \text{in } B,$$

$$v = u \quad \text{on } \partial B.$$

It is well known and easy to check that v satisfies

$$0 \leq \int_{\mathbf{B}} |\nabla v|^2 dx = r \int_{\partial \mathbf{B}} |\nabla_{\text{tan}} v|^2 dS - r \int_{\partial \mathbf{B}} v_p^2 dS,$$

so

$$\begin{aligned} \int_{\partial \mathbf{B}} v_p^2 dS &\leq \int_{\partial \mathbf{B}} |\nabla_{\text{tan}} v|^2 dS \\ &= \int_{\partial \mathbf{B}} |\nabla_{\text{tan}} u|^2 dS < \infty. \end{aligned}$$

Now by Lemma A.1, for any $\xi \in \mathbb{R}^3$,

$$\begin{aligned} \int_{\mathbf{B}} |\nabla w|^2 dx &\leq C \int_{\mathbf{B}} |\nabla v|^2 dx = C \int_{\partial \mathbf{B}} v_p \cdot (v - \xi) dS \\ &\leq C \left(\int_{\partial \mathbf{B}} v_p^2 dS \int_{\partial \mathbf{B}} |v - \xi|^2 dS \right)^{1/2} \\ &\leq C \left(\int_{\partial \mathbf{B}} |\nabla_{\text{tan}} u|^2 dS \int_{\partial \mathbf{B}} |u - \xi|^2 dS \right)^{1/2}. \quad \square \end{aligned}$$

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