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On a non linear partial differential equation having natural growth terms and unbounded solution

by

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ABSTRACT. — We prove the existence of a solution of the nonlinear elliptic equation: $A(u)+g(x, u, Du)=h(x)$, where A is a Leray-Lions operator from $W_0^{1,p}(\Omega)$ into $W^{-1,p'}(\Omega)$ and g is a nonlinear term with "natural" growth with respect to Du [i.e. such that $|g(x, u, \xi)| \leqq b(|u|)(|\xi|^p+c(x))$], satisfying the sign condition $g(x, u, \xi)u \geqq 0$ but no growth condition with respect to u. Here h belongs to $W^{-1,p'}(\Omega)$, thus the solution u of the problem does not in general be more smooth than $W_0^{1,p}(\Omega)$. The existence of a solution is also proved for the corresponding obstacle problem.

RÉSUMÉ. — Nous démontrons l'existence d'une solution du problème elliptique non linéaire $A(u)+g(x, u, Du)=h(x)$, où A est un opérateur de Leray-Lions de $W_0^{1,p}(\Omega)$ à valeurs dans $W^{-1,p'}(\Omega)$ et où g est un

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terme non linéaire à croissance «naturelle» en Du [i.e. tel que $|g(x, u, \xi)| \leqq b(|u|)(|\xi|^p + c(x))$], qui satisfait la condition de signe $g(x, u, \xi)u \geqq 0$ mais dont la croissance en u n'est pas limitée. Le second membre h appartient à $W^{-1, p'}(\Omega)$, et la solution u du problème n'est donc pas, en général, plus régulière que $W_0^{1, p}(\Omega)$. Nous démontrons également l'existence d'une solution pour l'inéquation variationnelle avec obstacle associée à ce problème.

INTRODUCTION

In this paper we prove the existence of solutions of non linear elliptic equations of the type

$$A(u) + g(x, u, Du) = h,$$

where A is a Leray-Lions operator from $W_0^{1, p}(\Omega)$ into $W^{-1, p'}(\Omega)$, $h \in W^{-1, p'}(\Omega)$, and g is a non linear lower order term having natural growth (of order p) with respect to $|Du|$. With respect to $|u|$, we do not assume any growth restrictions, but we assume the "sign-condition"

$$g(x, u, \xi)u \geqq 0.$$

It will turn out that, for any solution u, $g(x, u, Du)$ will be in $L^1(\Omega)$, but, for each $v \in W_0^{1, p}(\Omega)$, $g(x, v, Dv)$ can be very odd, and does not necessarily belong to $W^{-1, p'}(\Omega)$.

In the present paper, the main features are the "sign-condition" and the non smoothness of the right hand side h.

Let us point out that, if h is sufficiently smooth, existence results of bounded solutions have been recently obtained in [1], [3], [2], [8], [9], [10], [19]. But, in general, it is well known that, even for the corresponding linear equation, one cannot expect an L^∞ solution: the solutions can be only in $W_0^{1, p}(\Omega)$. They are bounded if $h \in W^{-1, q}(\Omega)$, $q > N/(p-1)$, *see* [5].

When g does not depend on Du, existence results for this type of problems have been proved in [15], [20], [11], [4]. When g depends on Du

with natural growth, the case where A is linear has been solved in [7], [6]. This result was generalized to non linear A's in [13], [16]. The proofs of the four last paper are based on the almost everywhere convergence of the gradients, a result which is due to J. Frehse [14] in the non-linear case.

In the present paper we present a proof which proceeds from different ideas. We consider u_ε defined by

$$u_\varepsilon \in W_0^{1,p}(\Omega): \quad A(u_\varepsilon) + \frac{g(x, u_\varepsilon, Du_\varepsilon)}{1+\varepsilon\,|g(x, u_\varepsilon, Du_\varepsilon)|} = h.$$

Because of the "sign-condition" it is easy to obtain a $W_0^{1,p}(\Omega)$-estimate on u_ε. Extracting a subsequence, u_ε tends to u in $W_0^{1,p}(\Omega)$ weakly. The problem will be solved whenever the convergence will be proved to be *strong* in $W_0^{1,p}(\Omega)$. We obtain this result proving that the positive part u_ε^+ of u_ε strongly converges to u^+ (and that the similar property holds for u_ε^-). The proof consists in two steps. In the first one, we prove that the "exeeding" part of u_ε^+, defined as $(u_\varepsilon^+ - u_k^+)^+$ (where u_k^+ is the truncation of u^+ at level k) is controlled in $W_0^{1,p}(\Omega)$ in terms of $(u^+ - u_k^+)$. The second step is to prove that the "bounded" part $(u_\varepsilon^+ - u_k^+)^-$ of u_ε^+ strongly converges to zero in $W_0^{1,p}(\Omega)$. For this we use the technique of multiplying by a non linear test function $\varphi((u_\varepsilon^+ - u_k^+)^-)$ introduced in [8], [9], [10]; this is allowed because $0 \leqq (u_\varepsilon^+ - u_k^+)^- \leqq k$.

Finally we give also an existence result for the corresponding obstacle problem.

1. STATEMENT OF THE PROBLEM

1.1. Assumptions

Let Ω be a bounded open set of $\mathbb{R}^N$. Let $1 < p < +\infty$ be fixed and A be a non linear operator from $W_0^{1,p}(\Omega)$ into its dual $W^{-1,p'}(\Omega)\left(\frac{1}{p}+\frac{1}{p'}=1\right)$ defined by

$$A(u) = -\operatorname{div} a(x, u, Du),$$

where $a(x, s, \xi)$ is a Carathéodory function $a: \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ such that

$$\left.\begin{array}{c} |a(x, s, \xi)| \leqq \beta[|s|^{p-1}+|\xi|^{p-1}+k(x)] \\ {[a(x, s, \xi)-a(x, s, \eta)].(\xi-\eta)>0, \quad \forall \xi \neq \eta} \\ a(x, s, \xi).\xi \geqq \alpha|\xi|^p, \end{array}\right\} \tag{1.1}$$

where $k(x) \in L^{p'}(\Omega)$, $k \geqq 0$, $\beta > 0$, $\alpha > 0$.

Let $g(x, s, \xi)$ be a Carathéodory function such that

$$\left.\begin{array}{c} g(x, s, \xi) s \geqq 0 \\ |g(x, s, \xi)| \leqq b(|s|)(|\xi|^p+c(x)), \end{array}\right\} \tag{1.2}$$

where b is a continuous and increasing function with (finite) values on $\mathbb{R}^+$, and $c \in L^1(\Omega)$, $c \geqq 0$.

We consider

$$h \in W^{-1, p'}(\Omega). \tag{1.3}$$

1.2. The main result

Consider the non linear elliptic problem with Dirichlet boundary conditions

$$\left.\begin{array}{c} A u+g(x, u, Du)=h \quad \text{in } \mathscr{D}'(\Omega) \\ u \in W_0^{1, p}(\Omega), \quad g(x, u, Du) \in L^1(\Omega), \quad u g(x, u, Du) \in L^1(\Omega) \end{array}\right\} \tag{1.4}$$

Our objective is to prove the following

THEOREM 1.1. — *Under the assumptions* (1.1), (1.2), (1.3) *there exists a solution of* (1.4).

Before giving the proof of the theorem, let us emphasize that the main difficulty stems from the fact that u is unbounded. Since h is only in $W^{-1, p'}(\Omega)$, it is impossible to expect the existence of an L^∞ solution (*see* [5]).

Usually boundedness plays an important role in the study of equations of the type (1.4). Indeed, to overcome the difficulties due to the quadratic growth of the non linear term, non linear test functions (with respect to the solution) are used in [2], [8], [9], [10] and it is important to know beforehand that such exponentials remain bounded. We shall see that it is possible to avoid this assertion, in the present framework.

2. PROOF OF THEOREM 1.1

2.1. An approximation scheme

Let us define

$$g_\varepsilon(x, s, \xi) = \frac{g(x, s, \xi)}{1+\varepsilon\,|g(x, s, \xi)|}, \tag{2.1}$$

and let us consider the equation

$$\left.\begin{array}{r} \mathrm{A}(u_\varepsilon)+g_\varepsilon(x, u_\varepsilon, \mathrm{D}u_\varepsilon)=h \\ u_\varepsilon \in \mathrm{W}_0^{1,p}(\Omega) \end{array}\right\} \tag{2.2}$$

which has a solution by the classical result of J. Leray and J. L. Lions [17].

Multiplying (2.2) by u_ε and using (1.2) we get

$$\langle \mathrm{A}(u_\varepsilon), u_\varepsilon \rangle \leqq \langle h, u_\varepsilon \rangle$$

hence

$$\alpha \| u_\varepsilon \|^2 \leqq \| h \| . \| u_\varepsilon \|$$

from which wet get

$$\| u_\varepsilon \|_{\mathrm{W}_0^{1,p}} \leqq c_1. \tag{2.3}$$

Hence we can extract a subsequence, still denoted by u_ε, with

$$u_\varepsilon \rightharpoonup u \quad \text{in } \mathrm{W}_0^{1,p}(\Omega) \text{ weakly and a. e.} \tag{2.4}$$

2.2. Convergence of the positive part of u_ε

Our objective in this paragraph is to prove that

$$u_\varepsilon^+ \to u^+ \quad \text{in } \mathrm{W}_0^{1,p}(\Omega) \text{ strongly.} \tag{2.5}$$

Let k be a positive constant. Let us define

$$u_k^+ = u^+ \wedge k.$$

In a first stage we shall fix k, and use the notation

$$z_\varepsilon = u_\varepsilon^+ - u_k^+. \tag{2.6}$$

First step: the behaviour of z_ε^+

Note that $z_\varepsilon \in W_0^{1,p}(\Omega)$, therefore $z_\varepsilon^+ \in W_0^{1,p}(\Omega)$. Multiplying (2.2) by z_ε^+ yields

$$\langle A\, u_\varepsilon, z_\varepsilon^+ \rangle + \int_\Omega g_\varepsilon(x, u_\varepsilon, Du_\varepsilon)\, z_\varepsilon^+ \, dx = \langle h, z_\varepsilon^+ \rangle. \tag{2.7}$$

Note that where $z_\varepsilon^+ > 0$, $u_\varepsilon^+ > 0$ hence $u_\varepsilon > 0$ and from (1.2) $g_\varepsilon(x, u_\varepsilon, Du_\varepsilon) \geqq 0$. Therefore we deduce from (2.7)

$$\langle A(u_\varepsilon), z_\varepsilon^+ \rangle \leqq \langle h, z_\varepsilon^+ \rangle.$$

Hence

$$\int_\Omega a(x, u_\varepsilon, Du_\varepsilon)\, Dz_\varepsilon^+ \, dx \leqq \langle h, z_\varepsilon^+ \rangle.$$

Since $u_\varepsilon = u_\varepsilon^+$ on the set $\{x \in \Omega,\ z_\varepsilon^+(x) > 0\}$, we can also write

$$\int_\Omega a(x, u_\varepsilon, Du_\varepsilon^+)\, Dz_\varepsilon^+ \, dx \leqq \langle h, z_\varepsilon^+ \rangle,$$

which implies

$$\int_\Omega [a(x, u_\varepsilon, Du_\varepsilon^+) - a(x, u_\varepsilon, Du_k^+)]\, D(u_\varepsilon^+ - u_k^+)^+ \, dx$$

$$\leqq -\int_\Omega a(x, u_\varepsilon, Du_k^+)\, D(u_\varepsilon^+ - u_k^+)^+ \, dx + \langle h, z_\varepsilon^+ \rangle \tag{2.8}$$

As $\varepsilon \to 0$, we have

$$z_\varepsilon^+ \to (u^+ - u_k^+)^+ \quad \text{a.e.}$$

But z_ε^+ is bounded in $W_0^{1,p}(\Omega)$, hence also

$$z_\varepsilon^+ \rightharpoonup (u^+ - u_k^+)^+ \quad \text{in } W_0^{1,p}(\Omega) \text{ weakly, as } \varepsilon \to 0 \ (k \text{ fixed}).$$

Define

$$R_k = -\int_\Omega a(x, u, Du_k^+) D(u^+ - u_k^+)^+ \, dx + \langle h, (u^+ - u_k^+)^+ \rangle. \tag{2.9}$$

Since we have

$$(u^+ - u_k^+)^+ \to 0 \quad \text{in } W_0^{1,p}(\Omega), \text{ as } k \to +\infty,$$

we obtain that

$$R_k \to 0, \text{ as } k \to +\infty;$$

passing to the limit in ε (for fixed k) in (2.8) yields

$$\overline{\lim_{\varepsilon \to 0}} \int_\Omega [a(x, u_\varepsilon, Du_\varepsilon^+) - a(x, u_\varepsilon, Du_k^+)] D(u_\varepsilon^+ - u_k^+)^+ \, dx \leqq R_k. \tag{2.10}$$

Second step: behaviour of z_ε^-

We shall use as a test function in (2.2) the function

$$v_\varepsilon = \varphi_\lambda(z_\varepsilon^-), \qquad \varphi_\lambda(s) = s\, e^{\lambda s^2},$$

where λ will be chosen later. Note that

$$0 \leqq z_\varepsilon^- \leqq k, \tag{2.11}$$

hence $z_\varepsilon^- \in L^\infty(\Omega)$ and, since $z_\varepsilon \in W_0^{1,p}(\Omega)$, clearly $v_\varepsilon \in W_0^{1,p}(\Omega)$. Therefore v_ε is an admissible test function for (2.2). We deduce

$$\int_\Omega a(x, u_\varepsilon, Du_\varepsilon) Dz_\varepsilon^- \varphi_\lambda'(z_\varepsilon^-) \, dx + \int_\Omega g_\varepsilon(x, u_\varepsilon, Du_\varepsilon) \varphi_\lambda(z_\varepsilon^-) \, dx = \langle h, \varphi_\lambda(z_\varepsilon^-) \rangle. \tag{2.12}$$

Define

$$E_\varepsilon = \{x \in \Omega : u_\varepsilon^+(x) \leqq u_k^+(x)\}, \qquad F_\varepsilon = \{x \in \Omega : 0 \leqq u_\varepsilon(x) \leqq u_k^+(x)\}.$$

We have

$$\int_{\Omega} g_\varepsilon(x, u_\varepsilon, Du_\varepsilon)\, \varphi_\lambda(z_\varepsilon^-)\, dx = \int_{E_\varepsilon} g_\varepsilon(x, u_\varepsilon, Du_\varepsilon)\, \varphi_\lambda(z_\varepsilon^-)\, dx. \qquad (2.13)$$

Note that $\varphi_\lambda(z_\varepsilon^-) \geqq 0$ and that $g_\varepsilon(x, u_\varepsilon, Du_\varepsilon) \leqq 0$ whenever $u_\varepsilon \leqq 0$. Using then (1.2) and (1.1) we have

$$\begin{aligned}\int_{E_\varepsilon} g_\varepsilon(x, u_\varepsilon, Du_\varepsilon)\, \varphi_\lambda(z_\varepsilon^-)\, dx &\leqq \int_{F_\varepsilon} g_\varepsilon(x, u_\varepsilon, Du_\varepsilon)\, \varphi_\lambda(z_\varepsilon^-)\, dx \\ &\leqq \int_{F_\varepsilon} b(|u_\varepsilon|)\,[|Du_\varepsilon|^p + c(x)]\, \varphi_\lambda(z_\varepsilon^-)\, dx \\ &\leqq b(k) \int_{F_\varepsilon} [|Du_\varepsilon|^p + c(x)]\, \varphi_\lambda(z_\varepsilon^-)\, dx \\ \leqq \frac{b(k)}{\alpha} \int_{F_\varepsilon} a(x, u_\varepsilon, Du_\varepsilon)\, Du_\varepsilon\, \varphi_\lambda(z_\varepsilon^-)\, dx &+ b(k) \int_{\Omega} c(x)\, \varphi_\lambda(z_\varepsilon^-)\, dx. \qquad (2.14)\end{aligned}$$

Since $u_\varepsilon \leqq 0$ implies $z_\varepsilon^- = u_k^+$, we obtain

$$\begin{aligned}\int_{\Omega} -a(x, u_\varepsilon, Du_\varepsilon^+)\, Dz_\varepsilon^-\, \varphi_\lambda'(z_\varepsilon^-)\, dx &= \int_{\Omega} -a(x, u_\varepsilon, Du_\varepsilon)\, Dz_\varepsilon^-\, \varphi_\lambda'(z_\varepsilon^-)\, dx \\ &+ \int_{\Omega} [a(x, u_\varepsilon, Du_\varepsilon) - a(x, u_\varepsilon, Du_\varepsilon^+)]\, Du_k^+\, \varphi_\lambda'(u_k^+)\, dx.\end{aligned}$$

Moreover, using (2.12), (2.13) and (2.14), we obtain

$$\begin{aligned}\int_{\Omega} &-[a(x, u_\varepsilon, Du_\varepsilon^+) - a(x, u_\varepsilon, Du_k^+)]\, D(u_\varepsilon^+ - u_k^+)^-\, \varphi_\lambda'(z_\varepsilon^-)\, dx \\ &= \int_{\Omega} [a(x, u_\varepsilon, Du_\varepsilon) - a(x, u_\varepsilon, Du_\varepsilon^+)]\, Du_k^+\, \varphi_\lambda'(u_k^+)\, dx + \langle -h, \varphi_\lambda(z_\varepsilon^-) \rangle \\ &+ \int_{\Omega} g_\varepsilon(x, u_\varepsilon, Du_\varepsilon)\, \varphi_\lambda(z_\varepsilon^-)\, dx + \int_{\Omega} a(x, u_\varepsilon, Du_k^+)\, Dz_\varepsilon^-\, \varphi_\lambda'(z_\varepsilon^-)\, dx \\ &\leqq \int_{\Omega} [a(x, u_\varepsilon, Du_\varepsilon) - a(x, u_\varepsilon, Du_\varepsilon^+)]\, Du_k^+\, \varphi_\lambda'(u_k^+)\, dx + \langle -h, \varphi_\lambda(z_\varepsilon^-) \rangle +\end{aligned}$$

$$+\int_\Omega a(x, u_\varepsilon, Du_k^+)\, Dz_\varepsilon^-\, \varphi'_\lambda(z_\varepsilon^-)\, dx$$

$$+\frac{b(k)}{\alpha}\int_{F_\varepsilon} a(x, u_\varepsilon, Du_\varepsilon)\, Du_\varepsilon\, \varphi_\lambda(z_\varepsilon^-)\, dx$$

$$+b(k)\int_\Omega c(x)\, \varphi_\lambda(z_\varepsilon^-)\, dx$$

$$=\int_\Omega [a(x, u_\varepsilon, Du_\varepsilon)-a(x, u_\varepsilon, Du_\varepsilon^+)]\, Du_k^+\, \varphi'_\lambda(u_k^+)\, dx+\langle -h, \varphi_\lambda(z_\varepsilon^-)\rangle$$

$$+\int_\Omega a(x, u_\varepsilon, Du_k^+)\, Dz_\varepsilon^-\, \varphi'_\lambda(z_\varepsilon^-)\, dx$$

$$+\frac{b(k)}{\alpha}\int_\Omega [a(x, u_\varepsilon, Du_\varepsilon^+)-a(x, u_\varepsilon, Du_k^+)]\, D(u_\varepsilon^+-u_k^+)\, \varphi_\lambda(z_\varepsilon^-)\, dx$$

$$+\frac{b(k)}{\alpha}\int_\Omega a(x, u_\varepsilon, Du_\varepsilon^+)\, Du_k^+\, \varphi_\lambda(z_\varepsilon^-)\, dx$$

$$+\frac{b(k)}{\alpha}\int_\Omega a(x, u_\varepsilon, Du_k^+)\, D(u_\varepsilon^+-u_k^+)\, \varphi_\lambda(z_\varepsilon^-)\, dx$$

$$+b(k)\int_\Omega c(x)\, \varphi_\lambda(z_\varepsilon^-)\, dx. \quad (2.15)$$

Now we choose $\lambda=\dfrac{b(k)^2}{4\alpha^2}$. So $\varphi'_\lambda-\dfrac{b(k)}{\alpha}\varphi_\lambda \geqq \dfrac{1}{2}$ and we deduce

$$-\frac{1}{2}\int_\Omega [a(x, u_\varepsilon, Du_\varepsilon^+)-a(x, u_\varepsilon, Du_k^+)]\, D(u_\varepsilon^+-u_k^+)^-\, dx$$

$$\leqq \int_\Omega [a(x, u_\varepsilon, Du_\varepsilon)-a(x, u_\varepsilon, Du_\varepsilon^+)]\, Du_k^+\, \varphi'_\lambda(u_k^+)\, dx+\langle -h, \varphi_\lambda(z_\varepsilon^-)\rangle$$

$$+\int_\Omega a(x, u_\varepsilon, Du_k^+)\, Dz_\varepsilon^-\, \varphi'_\lambda(z_\varepsilon^-)\, dx$$

$$+\frac{b(k)}{\alpha}\int_\Omega a(x, u_\varepsilon, Du_\varepsilon^+)\, Du_k^+\, \varphi_\lambda(z_\varepsilon^-)\, dx+\cdots$$

$$+\frac{b(k)}{\alpha}\int_\Omega a(x, u_\varepsilon, Du_k^+)\,D(u_\varepsilon^+ - u_k^+)\,\varphi_\lambda(z_\varepsilon^-)\,dx$$

$$+b(k)\int_\Omega c(x)\,\varphi_\lambda(z_\varepsilon^-)\,dx. \quad (2.16)$$

Extracting a subsequence such that

$$a(x, u_\varepsilon, Du_\varepsilon) \rightharpoonup \gamma \quad \text{in } (L^{p'}(\Omega))^N \text{ weakly}$$
$$a(x, u_\varepsilon, Du_\varepsilon^+) \rightharpoonup \sigma \quad \text{in } (L^{p'}(\Omega))^N \text{ weakly}$$

(which is still possible), and using Lebesgue's dominated convergence Theorem, it is easy to pass to the limit in ε (for k fixed) in the right hand side of (2.16). The limit is

$$\int_\Omega [\gamma(x)-\sigma(x)]\,Du_k^+\,\varphi_\lambda'(u_k^+)\,dx + \langle -h, \varphi_\lambda((u^+ - u_k^+)^-)\rangle$$

$$+\int_\Omega a(x, u, Du_k^+)\,D(u^+ - u_k^+)^-\,\varphi_\lambda'((u^+ - u_k^+)^-)\,dx$$

$$+\frac{b(k)}{\alpha}\int_\Omega \sigma(x)\,Du_k^+\,\varphi_\lambda((u^+ - u_k^+)^-)\,dx$$

$$+\frac{b(k)}{\alpha}\int_\Omega a(x, u, Du_k^+\,D(u^+ - u_k^+)\,\varphi_\lambda((u^+ - u_k^+)^-)\,dx$$

$$+b(k)\int_\Omega c(x)\,\varphi_\lambda((u^+ - u_k^+)^-)\,dx = \int_\Omega [\gamma(x)-\sigma(x)]\,Du_k^+\,\varphi_\lambda'(u_k^+)\,dx,$$

since $(u^+ - u_k^+)^- = 0$ and $\varphi_\lambda(0) = 0$; moreover

$$[a(x, u_\varepsilon, Du_\varepsilon) - a(x, u_\varepsilon, Du_\varepsilon^+)]\,(u_\varepsilon)_k^+ = 0 \quad \text{a. e.}$$

which implies $[\gamma(x)-\sigma(x)]\,u_k^+ = 0$; so the last term is zero too. Thus passing to the limit in ε for k fixed in (2.16) gives

$$\overline{\lim_{\varepsilon\to 0}} - \int_\Omega [a(x, u_\varepsilon, Du_\varepsilon^+) - a(x, u_\varepsilon, Du_k^+)]\,D(u_\varepsilon^+ - u_k^+)^-\,dx \leqq 0. \quad (2.17)$$

Third step: conclusion

From (2.10) and (2.17) we deduce

$$\overline{\lim_{\varepsilon \to 0}} \int_\Omega [a(x, u_\varepsilon, Du_\varepsilon^+) - a(x, u_\varepsilon, Du^+)] D(u_\varepsilon^+ - u^+) \, dx$$

$$\leq \overline{\lim_{\varepsilon \to 0}} \int_\Omega [a(x, u_\varepsilon, Du_\varepsilon^+) - a(x, u_\varepsilon, Du_k^+)] D(u_\varepsilon^+ - u_k^+)^+ \, dx$$

$$+ \overline{\lim_{\varepsilon \to 0}} - \int_\Omega [a(x, u_\varepsilon, Du_\varepsilon^+) - a(x, u_\varepsilon, Du_k^+)] D(u_\varepsilon^+ - u_k^+)^- \, dx$$

$$+ \overline{\lim_{\varepsilon \to 0}} \int_\Omega [a(x, u_\varepsilon, Du_\varepsilon^+) - a(x, u_\varepsilon, Du_k^+)] D(u_k^+ - u^+) \, dx$$

$$+ \overline{\lim_{\varepsilon \to 0}} \int_\Omega [a(x, u_\varepsilon, Du_k^+) - a(x, u_\varepsilon, Du^+)] D(u_\varepsilon^+ - u^+) \, dx$$

$$\leq R_k + \int_\Omega [\sigma(x) - a(x, u, Du_k^+)] D(u_k^+ - u^+) \, dx. \quad (2.18)$$

Letting k tend to $+\infty$, the right hand side tends to zero. By a variation of a result of Leray-Lions (for the proof *see* e. g. [12], [10]), this implies

$$u_\varepsilon^+ \to u^+ \quad \text{in } W_0^{1,p}(\Omega) \text{ strongly.} \quad (2.19)$$

2.3. Convergence of the negative part of u_ε

Similarly to the preceding Section, we want to prove now that

$$u_\varepsilon^- \to u^- \quad \text{in } W_0^{1,p}(\Omega) \text{ strongly.} \quad (2.20)$$

Define

$$u_k^- = u^- \wedge k, y_\varepsilon = u_\varepsilon^- - u_k^-.$$

Multiplying (2.2) by y_ε^+, we get

$$\int_\Omega a(x, u_\varepsilon, Du_\varepsilon) Dy_\varepsilon^+ \, dx + \int_\Omega g_\varepsilon(x, u_\varepsilon, Du_\varepsilon) y_\varepsilon^+ \, dx = \langle h, y_\varepsilon^+ \rangle.$$

But

$$g_\varepsilon(x, u_\varepsilon, Du_\varepsilon) y_\varepsilon^+ \leq 0 \quad \text{a. e.}$$

therefore we obtain as in (2.10) that for k fixed:

$$\overline{\lim_{\varepsilon \to 0}} \int_\Omega -[a(x, u_\varepsilon, -Du_\varepsilon^-)-a(x, u_\varepsilon, -Du_k^-)] D(u_\varepsilon^- - u_k^-)^+ \, dx \leqq Q_k, \tag{2.21}$$

where $Q_k \to 0$, if $k \to +\infty$.

The next step is to study the behaviour of $(u_\varepsilon^- - u_k^-)^-$. Considering again as test function

$$v_\varepsilon = \varphi_\lambda(y_\varepsilon^-),$$

we deduce as in (2.17) that

$$\overline{\lim_{\varepsilon \to 0}} \int_\Omega [a(x, u_\varepsilon, -Du_\varepsilon^-)-a(x, u_\varepsilon, -Du_k^-)] D(u_\varepsilon^- - u_k^-)^- \, dx \leqq 0. \tag{2.22}$$

Combining (2.21) and (2.22) we deduce, as in (2.18), that

$$u_\varepsilon^- \to u^- \quad \text{in } W_0^{1,p}(\Omega) \text{ strongly.}$$

2.4. Convergence

From (2.19) and (2.20) we deduce that for a subsequence

$$u_\varepsilon \to u \quad \text{in } W_0^{1,p}(\Omega) \text{ and a.e.} \tag{2.23}$$

$$Du_\varepsilon \to Du \quad \text{a.e.} \tag{2.24}$$

Since g is continuous in the two last arguments we have

$$\left.\begin{aligned} g_\varepsilon(x, u_\varepsilon, Du_\varepsilon) &\to g(x, u, Du) \quad \text{a.e.} \\ g_\varepsilon(x, u_\varepsilon, Du_\varepsilon)\, u_\varepsilon &\to g(x, u, Du)\, u \quad \text{a.e.} \end{aligned}\right\} \tag{2.25}$$

From (2.2) and (2.3) we infer that

$$0 \leqq \int_\Omega g_\varepsilon(x, u_\varepsilon, Du_\varepsilon)\, u_\varepsilon \, dx \leqq c_2. \tag{2.26}$$

For any measurable subset E of Ω and any $m > 0$, we have

$$\int_{E} |g_\varepsilon(x, u_\varepsilon, Du_\varepsilon)|\,dx = \int_{E \cap X_m^\varepsilon} |g_\varepsilon(x, u_\varepsilon, Du_\varepsilon)|\,dx + \int_{E \cap Y_m^\varepsilon} |g_\varepsilon(x, u_\varepsilon, Du_\varepsilon)|\,dx,$$

where

$$X_m^\varepsilon = \{x \in \Omega : |u_\varepsilon(x)| \leqq m\}$$
$$Y_m^\varepsilon = \{x \in \Omega : |u_\varepsilon(x)| > m\}.$$

So

$$\int_{E} |g_\varepsilon(x, u_\varepsilon, Du_\varepsilon)|\,dx \leqq \int_{E \cap X_m^\varepsilon} |g_\varepsilon(x, u_\varepsilon, Du_\varepsilon)|\,dx + \frac{1}{m}\int_{\Omega} g_\varepsilon(x, u_\varepsilon, Du_\varepsilon)\,u_\varepsilon\,dx$$
$$\leqq b(m)\int_{E} (|Du_\varepsilon|^p + c(x))\,dx + c_2\frac{1}{m}. \quad (2.27)$$

Since the sequence Du_ε strongly converges in $(L^p(\Omega))^N$, (2.27) implies the equi-integrability of $g_\varepsilon(x, u_\varepsilon, Du_\varepsilon)$. Now (2.25) and Vitali's Theorem yield:

$$g_\varepsilon(x, u_\varepsilon, Du_\varepsilon) \to g(x, u, Du) \quad \text{strongly in } L^1(\Omega). \quad (2.28)$$

Because of (2.23) and (2.28) it is easy to pass to the limit in

$$\langle A(u_\varepsilon), v\rangle + \int_{\Omega} g_\varepsilon(x, u_\varepsilon, Du_\varepsilon)\,v\,dx = \langle h, v\rangle$$

to obtain

$$\left.\begin{array}{l} \langle A(u), v\rangle + \int_{\Omega} g(x, u, Du)\,v\,dx = \langle h, v\rangle \\ \text{for any } v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega). \end{array}\right\} \quad (2.29)$$

Moreover since $g_\varepsilon(x, u_\varepsilon, Du_\varepsilon)\,u_\varepsilon \geqq 0$ a. e. it follows from (2.25), (2.26) and Fatou's Lemma that

$$g(x, u, Du)\,u \in L^1(\Omega). \quad (2.30)$$

Thus Theorem 1.1 is proved.

Finally let us note that

$$\langle A(u), u \rangle + \int_\Omega g(x, u, Du)\, u\, dx = \langle h, u \rangle; \tag{2.31}$$

indeed put $v = u_k$ in (2.29) where u_k is the truncation of u. We have

$$\langle A(u) - h, u_k \rangle \to \langle A u - h, u \rangle$$

and

$$g(x, u, Du)\, u_k \to g(x, u, Du)\, u \quad \text{in } L^1(\Omega),$$

by Lebesgue's dominated convergence Theorem, since

$$|g(x, u, Du)\, u_k| \leqq |g(x, u, Du)| . |u| \in L^1(\Omega),$$

by (2.30), and

$$g(x, u, Du)\, u_k \to g(x, u, Du)\, u \quad \text{a. e.}$$

3. VARIATIONAL INEQUALITIES

In this Section we extend our main result (Theorem 1.1) to variational inequalities. Let ψ be a measurable function with values in $\overline{\mathbb{R}}$ such that $\psi^+ \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$; note that this implies:

$$\{v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) : v \geqq \psi \text{ a. e.}\} \neq \varnothing. \tag{3.1}$$

We consider the problem

$$\left.\begin{array}{c} \langle A(u), v-u \rangle + \displaystyle\int_\Omega g(x, u, Du)(v-u)\, dx \geqq \langle h, v-u \rangle \\ \forall v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \qquad v \geqq \psi \quad \text{a. e.} \\ u \in W_0^{1,p}(\Omega), \qquad u \geqq \psi \quad \text{a. e.} \\ g(x, u, Du) \in L^1(\Omega), \qquad u\, g(x, u, Du) \in L^1(\Omega) \end{array}\right\} \tag{3.2}$$

We have the following result

THEOREM 3.1. — *Under the assumptions* (1.1), (1.2), (1.3) *and* (3.1), *there exists a solution of* (3.2).

Proof. — We follow the developments made in Section 2. We just emphasize the necessary changes. We start with the approximate problem

$$\begin{gathered}\langle A(u_\varepsilon), v-u_\varepsilon\rangle + \int_\Omega g_\varepsilon(x, u_\varepsilon, Du_\varepsilon)(v-u_\varepsilon)\,dx \geqq \langle h, v-u_\varepsilon\rangle \\ \forall v\in W_0^{1,p}(\Omega), \qquad v\geqq\psi \quad \text{a. e.} \\ u\in W_0^{1,p}(\Omega), \qquad u\geqq\psi \quad \text{a. e.}\end{gathered} \tag{3.3}$$

using the same approximation g_ε of g as in the case of the equation [*see* (2.1)]. For the existence of a solution of (3.3), *see* [18].

Using $v=\psi^+$ as test function in (3.3), we easily deduce that u_ε remains in a bounded set of $W_0^{1,p}(\Omega)$ and that $\int_\Omega g_\varepsilon(x, u_\varepsilon, Du_\varepsilon)\,u_\varepsilon\,dx$ remains bounded. Actually this is the only point in the present proof where the hypothesis $\psi^+\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$; is used; with some technicalities it is possible to obtain the same estimates just using (3.1) as hypothesis.

Pick now a subsequence such that

$$u_\varepsilon \rightharpoonup u \quad \text{in } W_0^{1,p}(\Omega) \text{ weakly and a. e.}$$

Note that $u\geqq\psi$ a. e.

We first study the convergence of the positive part u_ε^+. Define again

$$z_\varepsilon = u_\varepsilon^+ - u_k^+$$

and consider the test function

$$v = u_\varepsilon - (u_\varepsilon^+ - u_k^+)^+.$$

Since k will tend to $+\infty$, we may without loss of generality assume that

$$k\geqq\psi \quad \text{a. e.}$$

which is possible since ψ is bounded above. By this choice of k, the above test function is admissible. We recover immediately the inequality (2.10).

We then prove (2.17). We consider for that the test function

$$v_\varepsilon = u_\varepsilon + \varphi_\lambda(z_\varepsilon^-)$$

which is clearly admissible. Using this test function we recover (2.15) with the first = replaced by $\leqq$ and (2.17) follows.

Therefore (2.19) is proved as in the case of the equation.

We turn now to the convergence of the negative part. As in Section 2.3 we define

$$y_\varepsilon = u_\varepsilon^- - u_k^-.$$

We begin proving (2.21). We can use

$$v_\varepsilon = u_\varepsilon + y_\varepsilon^+$$

as a test function in (3.3) because it is clearly admissible. We easily deduce

$$\int_\Omega a(x, u_\varepsilon, Du_\varepsilon)\, Dy_\varepsilon^+ \, dx \geqq \langle h, y_\varepsilon^+ \rangle$$

from which (2.21) follows as in the case of the equation.

The final step is to recover (2.22). Our test function will now be

$$v_\varepsilon = u_\varepsilon - \delta_\varepsilon \varphi_\lambda (y_\varepsilon^-)$$

where δ_ε is a positive constant such that $\delta_\varepsilon E_\varepsilon(x) \leqq 1$, where $E_\varepsilon = \exp \lambda (y_\varepsilon^-)^2$; such a constant exists since $0 \leqq y_\varepsilon^- \leqq k$. Let us check that this test function is admissible. Consider a point where $\psi \geqq 0$, then $u_\varepsilon \geqq 0$, $u \geqq 0$, $y_\varepsilon = 0$, and $v_\varepsilon = u_\varepsilon$. Consider next a point where $\psi < 0$, $u \geqq 0$, then $y_\varepsilon = u_\varepsilon^-$, $y_\varepsilon^- = 0$, and again $v_\varepsilon = u_\varepsilon$. Assume from now on $\psi < 0$, $u < 0$; note that $u^- \leqq -\psi$, $u_k^- \leqq -\psi$. Suppose that $u_\varepsilon \geqq 0$, then $y_\varepsilon = -u_k^-$, $y_\varepsilon^- = u_k^-$, $v_\varepsilon = u_\varepsilon - \delta_\varepsilon E_\varepsilon u_k^-$. But $\delta_\varepsilon E_\varepsilon u_k^- \leqq u_k^- \leqq -\psi$, thus $v_\varepsilon \geqq -\delta_\varepsilon E_\varepsilon u_k^- \geqq \psi$. Suppose finally $u_\varepsilon < 0$ and $u_\varepsilon^- < u_k^-$ (otherwise $y_\varepsilon^- = 0$), then

$$v_\varepsilon = -u_\varepsilon^- - \delta_\varepsilon E_\varepsilon (u_k^- - u_\varepsilon^-) = -(1 - \delta_\varepsilon E_\varepsilon)\, u_\varepsilon^- - \delta_\varepsilon E_\varepsilon u_k^-.$$

Noting that $u_\varepsilon^- \leqq -\psi$, $u_k^- \leqq -\psi$ we deduce again $v_\varepsilon \geqq \psi$.

With this choice, we derive (2.22) and (2.20) follows as in the case of the equation.

From the above arguments we can assert that

$$\left.\begin{aligned} u_\varepsilon \to u \quad & \text{in } W_0^{1,p}(\Omega) \text{ and a. e.} \\ Du_\varepsilon \to Du \quad & \text{a. e.} \end{aligned}\right\} \tag{3.4}$$

By the same argument as in Section 2.4, we deduce again that

$$g_\varepsilon(x, u_\varepsilon, Du_\varepsilon) \to g(x, u, Du) \quad \text{in } L^1(\Omega),$$

and by Fatou's Lemma that $g(x, u, Du)\, u \in L^1(\Omega)$ with

$$\int_\Omega g(x, u, Du)\, u\, dx \leqq \underline{\lim} \int_\Omega g_\varepsilon(x, u_\varepsilon, Du_\varepsilon)\, u_\varepsilon\, dx.$$

By passing to the limit in (3.3) with $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, one recovers immediately (3.2), which completes the proof of Theorem 3.1.

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