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Finite dimensional behavior for weakly damped driven Schrödinger equations

by

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ABSTRACT. — We study the long time behavior of nonlinear Schrödinger equations with a zero order dissipation when they are driven by an external force. We show that this behavior is described by an attractor which captures all the trajectories. One of our main results concerns the estimate of the uniform Lyapunov exponents on this attractor, which allows us to prove its finite dimensional character.

RÉSUMÉ. — Nous étudions le comportement asymptotique, lorsque le temps tend vers l'infini, des solutions des équations de Schrödinger non linéaires, avec dissipation d'ordre zéro et en présence d'une force extérieure. Nous montrons que ce comportement est décrit par un attracteur qui capture toutes les trajectoires. Un de nos résultats principaux concerne l'estimation des exposants de Lyapunov uniformes sur cet attracteur.

Celle-ci nous permet d'établir, en particulier, que cet ensemble est de dimension finie.

Mots clés : Attracteurs, Schrödinger, dimension.

Classification A.M.S. : 35 Q 20, 76 A 99; 78 A 60.

0. INTRODUCTION

Our aim in this work is to obtain some information on the long time behavior (i. e. as $t \rightarrow \infty$) of the solutions to the nonlinear Schrödinger equation

$$i\frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial x^2}+g(|u|^2)u+i\gamma u=f. \tag{0.1}$$

Recall that, thanks to inverse scattering theory, much is known on the conservative case:

$$i\frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial x^2}+|u|^2 u=0, \tag{0.2}$$

where $x \in \mathbb{R}$ [18]. In that case the long time behavior of the solutions depends actually on an infinite number of degrees of freedom (the components of the initial data in the solitons basis) and (0.2) enjoys the properties of infinite dimensional hamiltonian systems. In (0.1) we take into account the effect of a zero order dissipation ($\gamma>0$) and of an external excitation f. It has been observed numerically and using physical arguments that concerning the long time behavior of (0.1),

(i) chaotic attractors exist;

(ii) a finite dimension "space" confine the attractors,

we refer e. g. to [1], [14], this last reference gives the derivation of (0.1) in plasma physics. Thus the dissipation term drastically changes the long time behavior. In this work we will give a contribution to each of the two points above. We are going to show that the long time behavior of solutions to (0.1), with appropriate boundary conditions, is described by a compact attractor. Moreover this attractor will be finite dimensional as shown by the estimates of the uniform Lyapunov exponents on it. This kind of results is well-known for parabolic dissipative equations (*see* [16] for an extensive review on the subject) and has been recently extended to nonlinear waves equations [7]. *See* also [6] concerning the case of strong dissipation in (0.1). From our point of view, it was not obvious that the weak dissipation mechanism in (0.1) would be sufficient to produce a finite dimensional behavior. Moreover a straight application of the methods of [3] (which were done successfully in many situations [16], [7]) does not lead here to the finite dimensionnality of the compact attractors. And, indeed, in our study of (0.1), we have introduced what we think to as a new ingredient which generalizes the method of [3]. Although presented on the particular problem that we consider, we believe to its generality (other applications will reported elsewhere, in particular to weakly damped Korteweg-de Vries equations [20]). *See* the introduction to the Section 3 for more details.

We consider the equation (0.1) for x varying in a finite interval $[0, L]$, $0<L<\infty$ and $t\in\mathbb{R}$. The boundary conditions are either of the Dirichlet type

$$(0.3)_D \qquad u(0,t)=u(L,t)=0, \qquad t\in\mathbb{R}$$

or of the Neumann type

$$(0.3)_N \qquad \frac{\partial u}{\partial x}(0,t)=\frac{\partial u}{\partial x}(L,t)=0, \qquad t\in\mathbb{R}$$

or finally periodic boundary conditions i.e. $u(x,t)$ is defined for $x\in\mathbb{R}$, $t\in\mathbb{R}$ and

$$(0.3)_P \qquad u(x,t)=u(x+L,t), \qquad \forall x\in\mathbb{R}, \quad \forall t\in\mathbb{R}.$$

It is well-known that the Cauchy problem on $\mathbb{R}$ for (0.1) with $\gamma=0$, $f=0$ and g suitably chosen, does not lead to global in time existence results ([10], *see* also [17] for the case $\gamma\neq 0$, $f=0$). We are going to impose some growth condition on the smooth (C^∞) function g which is defined on $\mathbb{R}_+=[0,\infty[$ with values in $\mathbb{R}$. More precisely we assume that

$$(0.4) \qquad \lim_{s\to+\infty}\frac{G_+(x)}{s^3}=0,$$

$$(0.5) \qquad \text{there exists } \omega>0 \text{ such that } \limsup_{s\to+\infty}\frac{h(s)-\omega G(s)}{s^3}\leqq 0,$$

where

$$(0.6) \qquad h(s)=sg(s), \qquad G(s)=\int_0^s g(\sigma)\,d\sigma$$

and

$$G_+(s)=\operatorname{Max}(G(s),0), \qquad G_-(s)=\operatorname{Max}(-G(s),0).$$

We observe that (0.4)-(0.5) are in a certain sense complementary to the conditions in [10], [17] that lead to the blow up in finite time result. Also the classical function $g(\sigma)=\sigma$ occurring in (0.2) satisfy (0.4)-(0.5) and more generally when

$$g(|u|^2)u=|u|^{2\delta}u, \quad \text{i.e. } g(\sigma)=\sigma^\delta, \qquad \delta>0,$$

(0.4)-(0.5) reduce to $\delta<2$. In fact, if there exists $\varepsilon>0$ and C such that

$$|g(\sigma)|\leqq C(1+\sigma)^{2-\varepsilon}, \qquad \forall\sigma\geqq 0,$$

(0.4) and (0.5) always hold.

Due to the physical origin of the problem the force f is frequently a time periodic function and very often

$$f(x,t)=f_0(x)\,e^{i\omega_0 t} \tag{0.7}$$

where $\omega_0\in\mathbb{R}$. Changing $u(x,t)$ into $u(x,t)\,e^{i\omega_0 t}$ in (0.1) leads then to

$$i\frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial x^2}+(g(|u|^2)-\omega_0)\,u+i\gamma u=f_0$$

which is autonomous, and amounts to change $g(\sigma)$ into $g(\sigma)-\omega_0$. We note that (0.4) and (0.5) are invariant by this transformation. Therefore we will mainly consider the long time behavior of (0.1) when it is autonomous (*see* also Section 2.2.2 for general time-periodic forces).

The plan of this work is as follows. In the first Section we derive some time estimates on the solutions of (0.1), uniform with respect to various norms. In the second Section we consider the autonomous case and show that the long time behavior is described by a compact attractor. Finally the third section contains the result on its (finite) dimension.

We have only considered the one dimensional case (i.e. $x\in\mathbb{R}$) for two reasons. First it is a physically relevant case. Second extensions to higher dimensions, which are possible, mainly differ from the cases that we study here, by technicalities including various Sobolev imbeddings that could, as we think, hidden the (hoped) readability of the paper. However in a subsequent work we will consider 2D and 3D cases together with explicit lower and upper bounds on the dimension of the universal attractors which follows from the methods of this paper. The main results of this work has been announced in [19].

CONTENTS

1. TIME UNIFORM ESTIMATES

In this Section we first derive some *a priori* estimates on the solutions to the nonlinear Schrödinger equations

$$i\frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial x^2}+g(|u|^2)u+i\gamma u=f,$$

where u satisfy one of the boundary conditions (0.3). As already said, we impose the two following conditions on g

$$\lim_{s\to+\infty}\frac{G_+(s)}{s^3}=0,$$

there exists $\omega>0$ such that

$$\limsup_{s\to+\infty}\frac{h(s)-\omega G(s)}{s^3}\leqq 0$$

where h and G are related to g in (0.6). Then we recall how these estimates lead by classical techniques to a well-set Cauchy problem in an appropriate space (energy space). We return to the differential inequalities satisfied by the solutions, and deduce the existence of bounded absorbing sets in the H^1-norm. Finally we extend the results to the H^2-norm.

1.1. *A priori* estimates

This paragraph has two goals. First we derive the *a priori* estimates that lead to global (in time) existence of solutions to the previous nonlinear Schrödinger equations. Second we introduce some notations (functional spaces, norms, ...) that will be used in the paper.

(*a*) *Evolution of certain scalar quantities*

We multiply (0.1) by $\bar{u}$ and integrate on]0, L[, it follows that

$$(1.1)\quad i\int_0^L u_t\bar{u}\,dx+\int_0^L u_{xx}\bar{u}\,dx+\int_0^L |u|^2 g(|u|^2)\,dx + i\gamma\int_0^L |u|^2\,dx=\int_0^L f\bar{u}\,dx.$$

Since u satisfies one of the conditions (0.3) we have

$$\int_0^L u_{xx}\bar{u}\,dx=-\int_0^L |u_x|^2\,dx$$

and the imaginary and real parts of (1.1) read

$$(1.2)\qquad \frac{1}{2}\frac{d}{dt}\int_0^L |u|^2\,dx+\gamma\int_0^L |u|^2\,dx=\mathrm{Im}\int_0^L f\bar{u}\,dx,$$

$$(1.3)\qquad \mathrm{Im}\int_0^L u\bar{u}_t\,dx-\int_0^L |u_x|^2\,dx+\int_0^L h(|u|^2)\,dx=\mathrm{Re}\int_0^L f\bar{u}\,dx,$$

where h is given in (0.6). Next we multiply (0.1) by $\bar{u}_t$ and integrate the real part of the resulting identity on]0, L[:

$$(1.4)\quad \mathrm{Re}\int_0^L u_{xx}\bar{u}_t\,dx+\mathrm{Re}\int_0^L g(|u|^2)u\bar{u}_t\,dx-\gamma\,\mathrm{Im}\int_0^L u\bar{u}_t\,dx=\mathrm{Re}\int_0^L f\bar{u}_t\,dx.$$

Since, thanks to (0.3),

$$\int_0^L u_{xx}\,\bar{u}_t\,dx = -\int_0^L u_x\,\bar{u}_{xt}\,dx,$$

we deduce from (1.4) that [*see* (0.6) for the expression of G]

$$(1.5)\quad \frac{1}{2}\frac{d}{dt}\int_0^L \{-|u_x|^2 + G(|u|^2)\}\,dx - \gamma\,\mathrm{Im}\int_0^L u\,\bar{u}_t\,dx = \mathrm{Re}\int_0^L f\bar{u}_t\,dx.$$

We multiply (1.3) by γ and add the resulting identity to (1.5), we find:

$$(1.6)\quad \frac{1}{2}\frac{d}{dt}\int_0^L \{|u_x|^2 - G(|u|^2) + 2\,\mathrm{Re}(f\bar{u})\}\,dx$$

$$\gamma\int_0^L \{|u_x|^2 - h(|u|^2) + \mathrm{Re}(f\bar{u})\}\,dx = \mathrm{Re}\int_0^L f_t\,\bar{u}\,dx.$$

Denoting by $(\ ,\ )_0$ and $|\ \ |_0$ the scalar product and norms on $L^2(0, L)$:

$$(1.7)\qquad (u, v)_0 = \int_0^L u(x)\,\bar{v}(x)\,dx, \qquad |v|_0 = \{(v, v)_0\}^{1/2},$$

we rewrite (1.2) as

$$(1.8)\qquad \frac{1}{2}\frac{d}{dt}|u|_0^2 + \gamma|u|_0^2 = \mathrm{Im}(f, u)_0.$$

Concerning (1.6) we introduce the two functionals

$$(1.9)\qquad \varphi(v) = |v_x|_0^2 + 2\,\mathrm{Re}(f, v)_0 - \int_0^L G(|v|^2)\,dx,$$

$$(1.10)\qquad \psi(v) = |v_x|_0^2 + \mathrm{Re}(f, v)_0 - \int_0^L h(|v|^2)\,dx,$$

so that (1.6) reads

$$(1.11)\qquad \frac{1}{2}\frac{d}{dt}\varphi(u) + \gamma\,\psi(u) = \mathrm{Re}(f_t, u)_0.$$

(b) Consequences of the hypotheses on g

We derive from (0.4)-(0.5) two inequalities which are useful in the study of φ and ψ.

LEMMA 1.1. — *Under the hypothesis* (0.4), *for every* $\varepsilon>0$ *there exists a constant* c'_ε *which only depends on g and* ε *such that for every v*

$$(1.12)\qquad |v_x|_0^2-\int_0^L G_+(|v|^2)\,dx\geqq(1-8\,\varepsilon\,|v|_0^4)\,|v_x|_0^2-\frac{2\,\varepsilon}{L^2}|v|_0^6-LC'_\varepsilon.$$

LEMMA 1.2. — *Under the hypothesis* (0.5), *for every* $\varepsilon>0$, *there exists a constant* C''_ε *which only depends on g and* ε *such that for every v*

$$(1.13)\qquad \int_0^L\{h(|v|^2)-\omega G(|v|^2)\}\,dx\leqq 8\,\varepsilon\,|v|_0^4\,|v_x|_0^2+\frac{2\,\varepsilon}{L^2}|v|_0^6+LC''_\varepsilon.$$

The proof of these Lemmas rely on the following inequality

$$(1.14)\qquad \operatorname*{Sup}_{0\leqq x\leqq L}|v(x)|^2\leqq|v|_0\left(2\,|v|_1+\frac{1}{L}|v|_0\right)$$

which is obtained by integrating on $]x,y[\,\subset\,]0,L[$ the relation $(|v|^2)_x=2\,\mathrm{Re}\,(v\bar{v}_x)$.

Proof of (1.12). — According to (0.4), for every $\varepsilon>0$, there exists $C'_\varepsilon\geqq 0$ such that

$$(1.15)\qquad G_+(s)\leqq\varepsilon\,s^3+C'_\varepsilon,\qquad \forall\,s\geqq 0,$$

hence

$$(1.16)\qquad |v_x|_0^2-\int_0^L G_+(|v|^2)\,dx\geqq|v_x|_0^2-\varepsilon\int_0^L|v|^6\,dx-LC'_\varepsilon.$$

Now thanks to (1.14),

$$(1.17)\qquad \int_0^L|v|^6\,dx\leqq|v|_0^2\operatorname*{Sup}_{0\leqq x\leqq L}|v(x)|^4\leqq|v|_0^4\left(8\,|v_x|_0^2+\frac{2}{L^2}|v|_0^2\right),$$

therefore combining (1.16) and (1.17), we obtain (1.12). $\square$

Proof of (1.13). — By (0.5), for every $\varepsilon>0$, there exists $C''_\varepsilon \geqq 0$ such that

$$h(s)-\omega G(s) \leqq \varepsilon s^3 + C''_\varepsilon, \qquad \forall s \geqq 0. \tag{1.18}$$

We take $s=|v|^2$ and integrate this relation on $]0, L[$, using (1.17) we find (1.13). □

(*c*) A priori *estimates*

We denote by $L^2(I)$, I an interval of $\mathbb{R}$, the space of measurable complex functions on I whose modulus is square integrable on I. For $m \in \mathbb{N}^*$, $H^m(I)$ denotes the subspace of $L^2(I)$ of functions whose distribution derivatives of order $\leqq m$ are in $L^2(I)$. When E is a Banach space, $L^p(I. E)$, $1 \leqq p \leqq \infty$, denotes as usual the space of measurable functions on I whose norm in E belongs to $L^p(I)$ [$L^\infty(I)$ is the space of essentially bounded functions on I while $L^p(I)$, $1 \leqq p \leqq \infty$, is the space of (class of) functions whose *p*th power is integrable on I]. We also denote by $\mathscr{C}(I, E)$ the set of continuous functions from I into E. And finally $L^p_{loc}(I, E)$ [resp. $H^m_{loc}(I)$] denotes the space of functions which are locally in $L^p(I, E)$ [resp. $H^m(I)$].

It is clear on (1.8) that if $f \in L^\infty(\mathbb{R}_+; L^2(0, L))$, and $u_0 \in L^2(0, L)$ then $u \in L^\infty(\mathbb{R}_+; L^2(0, L))$. Indeed denoting

$$|f|_{0,\infty} = \underset{t \geqq 0}{\text{ess sup}} |f(t)|_0 \tag{1.19}$$

we deduce thanks to the Cauchy-Schwartz inequality

$$\frac{1}{2}\frac{d}{dt}|u|_0^2 + \gamma |u|_0^2 \leqq \frac{\gamma |u|_0^2}{2} + \frac{|f|_{0,\infty}^2}{2\gamma},$$

hence

$$\frac{d}{dt}|u|_0^2 + \gamma |u|_0^2 \leqq \frac{|f|_{0,\infty}^2}{\gamma}. \tag{1.20}$$

After integration, it follows that

$$|u(t)|_0^2 \leqq |u(0)|_0^2 e^{-\gamma t} + \frac{|f|_{0,\infty}^2}{\gamma^2}(1-e^{-\gamma t}). \tag{1.21}$$

Now thanks to this bound it is clear on (1.12) that for ε sufficiently small, $\varphi(u)$ is coercive [*see* (1.24)]. For we are going to derive an *a priori* estimate on u, solution of (0.1), in $L^\infty(\mathbb{R}_+; H^1(0, L))$:

PROPOSITION 1.1. — *Let u be a regular solution of* (0.1), (0.3), *under the hypotheses* (0.4)-(0.5), *there exists a constant* φ_∞ [see (1.33)] *such that*

$$\varphi(u(t)) \leqq \varphi(u(0)) e^{-\gamma\omega t} + \varphi_\infty (1 - e^{-\gamma\omega t}), \qquad \forall t \geqq 0, \tag{1.22}$$

where in (1.33) *we take*

$$e_\infty^2 \equiv \operatorname*{Sup}_{t \geqq 0} |u(t)|_0^2 \leqq \operatorname{Max}\left(|u(0)|_0^2, \frac{|f|_{0,\infty}^2}{\gamma^2}\right), \tag{1.23}$$

and ε_0 *in* (1.31) *is chosen in order that*

$$\varphi(u(t)) \geqq \frac{1}{4} |u_x|_0^2 - \frac{2\varepsilon_0}{L^2} e_\infty^6 - L C'_{\varepsilon_0} - 2|f|_{0,\infty} e_\infty, \quad \forall t \geqq 0. \tag{1.24}$$

Proof. — We first observe that when (0.4)-(0.5) hold we can always suppose that $0 < \omega \leqq 1$:

$$\exists \omega \in]0, 1], \quad \limsup_{s \to +\infty} \frac{h(s) - \omega G_+(s)}{s^3} \leqq 0. \tag{1.25}$$

Indeed, let $n \in \mathbb{N}$ be such that $n < \omega \leqq n+1$, we write

$$h(s) - (\omega - n) G(s) = h(s) - \omega G(s) + n G_+(s) - n G_-(s),$$

therefore

$$h(s) - (\omega - n) G(s) \leqq h(s) - \omega G(s) + n G_+(s)$$

and thanks to (0.4) and (0.5) we find (0.5) with ω replaced by $\omega - n$ and this shows (1.25).

From this observation and (1.13) we deduce that, for every $\varepsilon > 0$, there exists C''_ε such that for every v

$$\omega\varphi(v) - \psi(v) \leqq (\omega - 1) |v_x|_0^2 + 2\omega \operatorname{Re}(f, u)_0 - \operatorname{Re}(f, u)_0 + 8\varepsilon |v|_0^4 |v_x|_0^2 + \frac{2\varepsilon}{L^2} |v|_0^6 + L C''_\varepsilon,$$

and with (1.23), applications of Cauchy-Schwarz inequalities and $0<\omega\leqq 1$ we deduce

(1.26)
$$\omega\varphi(u)-\psi(u)\leqq 3|f|_{0,\infty}e_\infty+8\varepsilon e_\infty^4|u_x|_0^2+\frac{2\varepsilon}{L^2}e_\infty^6+L\,C_\varepsilon'', \qquad \forall t\geqq 0.$$

Then we rewrite (1.11) as

$$\frac{1}{2}\frac{d\varphi(u)}{dt}+\gamma\omega\varphi(u)=\gamma(\omega\varphi(u)-\psi(u))+\operatorname{Re}(f_t, u)_0. \tag{1.27}$$

First we majorize the term $\operatorname{Re}(f_t, u)_0$ as follows (1)

$$|\operatorname{Re}(f_t, u)_0|\leqq|f_t|_0|u|_0\leqq|f_t|_{0,\infty}(|u|_0^2+L^2|u_x|_0^2)^{1/2}$$

$$|\operatorname{Re}(f_t, u)_0|\leqq|f_t|_{0,\infty}e_\infty+\frac{\omega\gamma}{4}|u_x|_0^2+\frac{L^2|f_t|_{0,\infty}^2}{\omega\gamma}. \tag{1.28}$$

Second we choose ε. For we minorize $\varphi(u)$, using (1.9), (1.12):

$$\varphi(u)\geqq(1-8\varepsilon|u|_0^4)|u_x|_0^2-\frac{2\varepsilon}{L^2}|u|_0^6-LC_\varepsilon'+2\operatorname{Re}(f, u)_0. \tag{1.29}$$

Hence by (1.23) and (1.26) $(0<\omega\leqq 1)$

$$\omega\varphi(u)-\psi(u)+\frac{\omega}{4}|u_x|_0^2 \leqq\frac{\omega}{2}\varphi(u)+\frac{3\varepsilon_0 e_\infty^6}{L^2}+4|f|_{0,\infty}e_\infty+L(C_{\varepsilon_0}'+C_{\varepsilon_0}''), \tag{1.30}$$

where $\varepsilon_0>0$ is chosen such that

$$\frac{\omega}{4}-8\left(1+\frac{\omega}{2}\right)\varepsilon_0 e_\infty^4=0. \tag{1.31}$$

Now with this choice of ε, (1.29) implies (1.24).

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(1) This estimate is artificial when $f_t\in L^\infty(\mathbb{R}_+;\ L^2(0, L))$, but we shall make later a weaker hypothesis on f_t that leads to an equality similar to (1.28). *See* (1.41).

Using (1.27), (1.28) and (1.30) we find

$$(1.32)\quad \frac{d\varphi(u)}{dt}+\gamma\omega\,\varphi(u)\leqq\frac{6\,\varepsilon_0\,e_\infty^6\,\gamma}{L^2}+(8\,\gamma\,|f|_{0,\,\infty}+2\,|f_t|_{0,\,\infty})\,e_\infty + \frac{2\,L^2}{\omega\gamma}|f_t|_{0,\,\infty}^2+2\,L\,\gamma(C'_{\varepsilon_0}+C''_{\varepsilon_0}),$$

which shows (1.22) when we set

$$(1.33)\qquad \varphi_\infty\equiv\frac{1}{\gamma\omega}\ \text{[the right hand side of (1.32)]}.\quad\square$$

1.2. Remarks on the Cauchy problem

(a) *Functional setting*

We introduce on the basic Hilbert space $H=L^2(0, L)$ the unbounded linear operator A,

$$(1.34)\qquad A\,v=-v_{xx},$$

with domain ([2])

$$(1.35)\qquad \begin{array}{ll} H^2(0,L)\cap H_0^1(0,L) & \text{in the case of b. c. } (0.3)_D,\\ \{v\in H^2(0,L),\ v_x(0)=v_x(L)=0\} & \text{in the case of b. c. } (0.3)_N,\\ \{v\in H^2_{loc}(\mathbb{R}),\ v(x+L)=v(x),\ \forall\, x\in\mathbb{R}\} & \text{in the case of b. c. } (0.3)_P. \end{array}$$

For every $r>0$, the operator $A+r$ is an isomorphism from D (A) (equipped with the graph norm) onto H. Since the imbedding of D(A) into H is compact, $(A+r)^{-1}$ is a compact operator on H. It is self adjoint, therefore there exists a Hilbert basis of H made of eigenvectors of A. We denote by $\{\lambda_i\}_{i=0}^{\infty}$ the nondecreasing sequence of eigenvalues counting multiplicities

$$0\leqq\lambda_1\leqq\lambda_2\leqq\ldots\leqq\lambda_j\to+\infty\quad \text{as}\, j\to\infty$$

the corresponding orthonormalized eigenvectors are denoted by $\{w_j\}_{i=0}^{\infty}$. These eigenelements are explicitely known here but we do not need for

([1]) As usual $H_0^1(0, L)=\{v\in H^1(0, L),\ v(0)=v(L)=0)$.

the moment their actual values. The powers A^s, $s \in \mathbb{R}$, are well defined with domain $D(A^s)$. For example $V = D(A^{1/2})$ (which is of interest in what follows) is also, according to the boundary condition,

$$V = \begin{cases} H_0^1(0, L) & \text{for } (0.3)_D, \\ H^1(0, L) & \text{for } (0.3)_N, \\ \{v \in H^1_{loc}(\mathbb{R}), v(x+L) = v(x), \forall x \in \mathbb{R}\} & \text{for } (0.3)_P. \end{cases}$$

We have $V' = D(A^{-1/2})$ (after identification of H and its antidual H').

(b) *Functional formulation of the equations*

We assume that f is given such that

$$(1.36) \qquad f \in L^\infty_{loc}(\mathbb{R}, H), \qquad f_t \in L^\infty_{loc}(\mathbb{R}, V'),$$

and our aim is to find a function

$$(1.37) \qquad u \in \mathscr{C}(\mathbb{R}, V)$$

which satisfy

$$(1.38) \qquad iu_t - A\,u + g(|u|^2)\,u + i\gamma u = f,$$

$$(1.39) \qquad u(0) = u_0$$

where (1.38) is understood in the distribution-sense on $\mathbb{R}$ (with values in V') and u_0 is a given element in V.

Remark 1.1. – Since $H^1(0, L)$ is continuously imbedded in $\mathscr{C}([0, L])$, the function $x \to g(|v(x)|^2)\,v(x)$ is continuous on [0, L], therefore it belongs to V' (even here to V) when $v \in V$. In (1.38) we have denoted by the same symbol the operator $v \to g(|v|^2)\,v$ and the corresponding complex function, but this should not produce any confusion. □

(c) *Existence and uniqueness of solutions*

It is well known (*see* e. g. I. Segal [15]) that the Cauchy problem (1.37)-(1.39) possesses a unique solution for $t \in [0, T_*[$ with the classical alternative: either $T_* = +\infty$ or $\limsup_{t \to T_*} |u(t)|_V = \infty$. But according to Proposition 1.1 (which follows from our hypotheses on g), we have seen that the latter case does not occur. More precisely we have the

THEOREM 1.1. — *We assume that g satisfy the hypotheses* (0.4) *and* (0.5). *For every* $u_0 \in V$ *and f satisfying* (1.36), *the problem* (1.37)-(1.39) *possesses a unique solution and for every* $t \in \mathbb{R}$,

(1.40) *the mapping* $u_0 \to u(t)$ *is continuous on* V.

Moreover, if we have

$$f \in L^\infty(\mathbb{R}_+; H), \qquad f_t \in L^\infty(\mathbb{R}_+; V'),$$

then

$$u \in L^\infty(\mathbb{R}_+; V).$$

We postpone the proof of the time uniform estimates, in the last part of this result, to the next Section (Proposition 1.2). Concerning the first part, we briefly mention the main steps of a proof of these classical results based on the techniques of J. L. Lions [11] rather than that of [15]. We first construct finite dimensional approximations (Galerkin approximations) of (1.38)-(1.39) based on the spaces spanned by the $\{w_j\}_{j=0}^m$, $m=0, 1, \ldots$, that we denote by $\{u_m(t)\}_{m\in\mathbb{N}}$. Thanks to (1.8), which is now rigourous with u_m instead of u, it follows that the functions $u_m(t)$ are defined for $t \in \mathbb{R}$. Indeed reversing time t amounts to change the sign of γ. Since this sigh is not important for local in time estimates we deduce the result. Then using (1.11) with u_m instead of u we deduce for $t \in [0, T]$, T being fixed, $0 < T < \infty$, an estimate on the H^1-norm of $u_m(t)$ which is independent of m. Let us simply notice that since we have only supposed that $f_t \in L^\infty_{loc}(\mathbb{R}, V')$, we replace (1.28) by

$$|\operatorname{Re}\langle f_t, u\rangle| \leq |f_t|_* (|u|_0^2 + L^2 |u_x|_0^2)^{1/2} \tag{1.41}$$

where $\langle\ ,\ \rangle$ denotes the pairing between V and V′ and $|.|_*$ the dual-norm of the following norm on V:

$$|v|_V = (|v|_0^2 + L^2 |v_x|_0^2)^{1/2}. \tag{1.42}$$

Then we pass to the limit $m \to \infty$ in the equation satisfied by u_m using standard techniques (in particular a compactness lemma of [11]). This produces a solution $u \in L^\infty_{loc}(\mathbb{R}, V)$ to (1.37)-(1.39), which is weakly continuous from $\mathbb{R}$ into V. The strong continuity in (1.37) is obtained by writing (1.38) as a linear Schrödinger equation

$$iu_t - \mathrm{A}\, u = \tilde{f} = f - i\gamma u - g(|u|^2)\, u$$

and using a result of [12]. Finally uniqueness and continuous dependence (1.40) are proved as follows. We denote by u_1 and u_2 two solutions of (1.37)-(1.38). We set $w = u_1 - u_2$, then

$$iw_t - \mathrm{A}\, w + i\gamma w = g(|u_2|^2)\, u_2 - g(|u_1|^2)\, u_1. \tag{1.43}$$

Now since $u_i \in \mathscr{C}(\mathbb{R}, \mathrm{V})$, $iw_t - \mathrm{A}\, w \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}, \mathrm{V})$ and one can show (using the technics of [12]) that since $w \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}, \mathrm{V})$,

$$\frac{1}{2}\frac{d}{dt}|w_x|_0^2 = \mathrm{Im}\langle iw_t - \mathrm{A}\, w, \mathrm{A}\, w\rangle \tag{1.44}$$

and

$$\frac{1}{2}\frac{d}{dt}|w|_0^2 = \mathrm{Im}\langle iw_t - \mathrm{A}\, w, w\rangle \tag{1.45}$$

in the distribution sense on $\mathbb{R}$. Then (1.43), (1.44) and (1.45) allow one to derive a standard Gronwall inequality on the norm $|w|_{\mathrm{V}}$ which shows (1.40) and uniqueness when $w(0) = 0$. □

Remark 1.2. – When u_0 and f are more regular, the solution u is more regular too. *See* Section 1.4. □

1.3. Existence of bounded absorbing sets in the H^1-norm

In this paragraph we show the time uniform estimate stated in the last part of Theorem 1.1. We assume that in addition to (1.36)

$$f \in \mathrm{L}^\infty(\mathbb{R}_+; \mathrm{H}), \qquad f_t \in \mathrm{L}^\infty(\mathbb{R}_+; \mathrm{V}'). \tag{1.46}$$

According to (1.22)-(1.24) it is clear that the solution u of Theorem 1.1 belongs to $\mathrm{L}^\infty(\mathbb{R}_+; \mathrm{V})$. More precisely we have

PROPOSITION 1.2. – *We assume that* (0.4)-(0.5) *holds. There exists a constant* $\rho_{\infty,1}$ [see (1.57)] *such that for every* $\mathrm{R} > 0$ *and for every* $u_0 \in \mathrm{V}$ *with*

$$|u_0|_0^2 + \mathrm{L}^2 |u_{0x}|_0^2 \leq \mathrm{R}^2, \tag{1.47}$$

there exists $T_1(R)>0$ *such that the solution of* (1.37)-(1.39) *satisfy*

$$|u(t)|_0^2+L^2|u_x(t)|_0^2\leqq(\rho_{\infty,1})^2,\qquad \forall t\geqq T_1(R). \tag{1.48}$$

Proof. — We first note that it follows from (1.21) and (1.47):

$$|u(t)|_0^2\leqq R^2e^{-\gamma t}+\frac{|f|_{0,\infty}^2}{\gamma^2}(1-e^{-\gamma t})$$

so that

$$|u(t)|_0^2\leqq\frac{2|f|_{0,\infty}^2}{\gamma^2},\qquad t\geqq T_0(R) \tag{1.49}$$

for $T_0(R)=\dfrac{2}{\gamma}\operatorname{Log}(|f|_{0,\infty}/\gamma R)$.

We only consider $t\geqq T_0(R)$ so that (1.26) holds with e_∞^2 replaced by $\dfrac{2|f|_{0,\infty}^2}{\gamma^2}$. Then we choose ε_1 solution to the analogue of (1.31) i. e.

$$\frac{\omega}{4}-8\left(1+\frac{\omega}{2}\right)\frac{4\varepsilon_1|f|_{0,\infty}^2}{\gamma^4}=0, \tag{1.50}$$

so that ε_1 does not depend on R. Using (1.41) instead of (1.28), we find in place of (1.32)

$$\begin{aligned}\frac{d\varphi(u)}{dt}+\gamma\omega\varphi(u)\leqq{}&\frac{48\varepsilon_1|f|_{0,\infty}^6}{\gamma^6L^2}+16|f|_{0,\infty}^2\\&+\frac{4|f|_{0,\infty}}{\gamma}|f_t|_{*,\infty}+\frac{2L^2|f_t|_{*,\infty}^2}{\omega\gamma}+2L\gamma(C'_{\varepsilon_1}+C''_{\varepsilon_1}),\end{aligned} \tag{1.51}$$

where

$$|f_t|_{*,\infty}^2=\operatorname*{ess\,sup}_{t\geqq0}|f_t(t)|_*. \tag{1.52}$$

It follows from (1.51) (we denote by $\tilde{\varphi}_\infty$ its right hand side) that for $t\geqq T_0(R)$

$$\varphi(u(t))\leqq\varphi(u(T_0(R)))\,e^{-\gamma\omega(t-T_0(R))}+\frac{\tilde{\varphi}_\infty}{\gamma\omega}. \tag{1.53}$$

We choose $T(R) \geqq T_0(R)$ such that

$$(1.54) \qquad \varphi(u(T_0(R))\, e^{-\gamma\omega(T(R)-T_0(R))} \leqq \frac{\varphi_\infty}{\gamma\omega},$$

holds for every u_0 satisfying (1.47). This is possible since we know from (1.22) and (1.24) that $\varphi(u(T_0(R)))$ is bounded by a quantity that only depends on R and the data of the problem. Then according to (1.53)-(1.54),

$$(1.55) \qquad \varphi(u(t)) \leqq \frac{2\tilde{\varphi}_\infty}{\gamma\omega}, \qquad t \geqq T(R).$$

Since (1.24) is valid for $t \geqq T(R) \geqq T_0(R)$, with ε_0 replaced by ε_1 given in (1.50), and e_∞ replaced by $\frac{2|f|_{0,\infty}^2}{\gamma^2}$; we deduce from (1.55) that

$$(1.56) \qquad |u_x|_0^2 \leqq \frac{8\tilde{\varphi}_\infty}{\gamma\omega} + \frac{64}{\gamma^6}|f|_{0,\infty}^6 + 4\,L\,C'_{\varepsilon_1} + \frac{8\sqrt{2}|f|_{0,\infty}^2}{\gamma}, \quad t \geqq T(R).$$

Combining (1.49) and (1.56), we obtain (1.48) with

$$(1.57)\, (\rho_{\infty,1})^2 = \frac{2|f|_{0,\infty}^2}{\gamma^2} + L^2. \quad \text{[the right hand side of (1.56)].} \quad \square$$

1.4. Existence of bounded absorbing sets in the H^2 norm

Our aim is to show an analogue of Proposition 1.2 with respect to the H^2-norm. We strengthen the hypothesis (1.36) as

$$(1.58) \qquad f \in L^\infty_{loc}(\mathbb{R}, H), \qquad f_t \in L^\infty_{loc}(\mathbb{R}, H),$$

and we claim that when $u_0 \in D(A)$, the solution u obtained in Theorem 1.1 satisfies

$$(1.59) \qquad u \in L^\infty_{loc}(\mathbb{R}, D(A)), \qquad u_t \in L^\infty_{loc}(\mathbb{R}, H),$$

the mapping (1.40) being continuous in D(A). Indeed we set $\eta = u_t$, and differentiate (1.38) with respect to t, we find

$$(1.60) \quad i\eta_t - A\eta + \{g(|u|^2) + g'(|u|^2)|u|^2\}\eta + g'(|u|^2)u^2\bar{\eta} + i\gamma\eta = f_t.$$

According to (1.38), we have

$$\eta(0)=u_t(0)=-i\mathrm{A}\,u_0+ig(|u|^2)\,u_0-\gamma\,u_0-if(0)\in\mathrm{H} \tag{1.61}$$

since $u_0\in\mathrm{D}(\mathrm{A})$ and $f\in\mathscr{C}(\mathbb{R};\mathrm{H})$ [a consequence of (1.58)]. Hence taking the imaginary part of the scalar product in H ([3]) of (1.60) with η, we find

$$\frac{1}{2}\frac{d}{dt}|\eta|_0^2+\gamma|\eta|_0^2+\operatorname{Im}(g'(|u|^2)\,u^2,\eta^2)_0=\operatorname{Im}(f_t,\eta)_0. \tag{1.62}$$

Using the fact that for every finite $\mathrm{T}>0$,

$$\sup_{\substack{0\leq x\leq \mathrm{L}\\ |t|\leq \mathrm{T}}}|u(x,t)|<\infty, \tag{1.63}$$

We deduce from (1.58), (1.61) and (1.62) by using Gronwall inequalities that

$$u_t=\eta\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R},\mathrm{H}). \tag{1.64}$$

Returning to (1.38) we have

$$\mathrm{A}\,u=iu_t+g(|u|^2)\,u+i\gamma\,u-f \tag{1.65}$$

and thanks to (1.58), (1.64) and the fact already known that $u\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R},\mathrm{V})$, we deduce (1.59). Then the continuity of the mapping (1.40) in D(A) follows as in the Theorem 1.1.

Remark 1.3. — Using the technics of [8], it is possible to derive further regularity results. □

When we supplement (1.58) with

$$f\in\mathrm{L}^\infty(\mathbb{R}_+;\mathrm{H}) \quad \text{and} \quad f_t\in\mathrm{L}^\infty(\mathbb{R}_+;\mathrm{H}), \tag{1.66}$$

we have the

PROPOSITION 1.3. — *We assume that* (0.4)-(0.5) *holds together with* (1.66). *There exists a constant* $\rho_{\infty,2}$ *such that for every* $\mathrm{R}>0$ *and for every*

([3]) Here again these formal manipulations are rigorous on the Galerkin approximations and the estimates are obtained at the limit $m\to\infty$.

$u_0 \in D(A)$ *with*

$$(1.67) \qquad |u_0|_0^2 + L^2 |u_{0x}|_0^2 + L^4 |u_{0\,xx}|_0^2 \leqq R^2.$$

there exists $T_2(R)$ *such that the solution of* (1.37)-(1.39) *satisfy*

$$(1.68) \qquad |u(t)|_0^2 + L^2 |u_x(t)|_0^2 + L^4 |u_{xx}(t)|_0^2 \leqq (\rho_{\infty,\,2})^2, \quad \forall t \geqq T_2.$$

Proof. — Thanks to Proposition 1.2, we already know that (1.48) holds. We have to estimate $|u_{xx}(t)|_0^2$ for large t. For we take the real part of the scalar product of (1.38) with $A u_t + \gamma A u$, it follows that

$$(1.69) \quad -\text{Re}(A u, A u_t + \gamma A u)_0 + \text{Re}(g(|u|^2) u - f, A u_t + \gamma A u)_0 = 0$$

We first notice that since $g(|u|^2) u \in V$,

$$(1.70) \quad \text{Re}(g(|u|^2) u A u)_0 = \int_0^L \{ g(|u|^2) |u_x|^2 + g'(|u|^2) \text{Re}(|u_x|^2 \bar{u} + u \bar{u}_x^2) \} \, dx$$

therefore using L^∞ estimates on the terms involving u and L^1 on that involving $|u_x|^2$ we deduce from (1.48) that

$$(1.71) \qquad |\text{Re}(g(|u|^2 u), A u)_0| \leqq \rho'_{\infty,\,1}, \qquad t \geqq T_1(R)$$

where $\rho'_{\infty,\,1}$ depends only on $\rho_{\infty,\,1}$.

It remains to study the term

$$(1.72) \qquad \text{Re}(g(|u|^2) u, A u_t) = \text{Re} \int_0^L (g(|u|^2) u)_x \bar{u}_{xt} \, dx.$$

It is equal to

$$\int_0^L g(|u|^2) \text{Re}(u_x \bar{u}_{xt}) \, dx + \int_0^L g'(|u|^2) \text{Re}(u \bar{u}_x) \text{Re}(u \bar{u}_{xt}) \, dx$$

that we rewrite as

$$(1.73) \quad \frac{1}{2} \frac{d}{dt} \int_0^L \{ g(|u|^2 |u_x|^2 + 2 g'(|u|^2) \text{Re}(u \bar{u}_x)^2 \} \, dx - R(u)$$

where

$$(1.74)\quad R(u)=\int_0^L \{g'(|u|^2)(|u_x|^2\,\mathrm{Re}(u\bar{u}_t) \\ +2\,\mathrm{Re}(u\bar{u}_x)\,\mathrm{Re}(u_t\bar{u}_x+2g''(|u|^2\,\mathrm{Re}(u\bar{u}_t)\,\mathrm{Re}(u\bar{u}_x)\}.$$

Now according to (1.37) we have

$$(1.75)\qquad u_t=-i\,A\,u+h$$

where thanks to (1.48)

$$(1.76)\qquad |h(t)|_0\leqq \rho''_{\infty,1},\qquad t\geqq T_1(R).$$

We replace (1.75) in (1.74), then using the L^∞ estimate on u, (1.14) and (1.48) we deduce that

$$(1.77)\qquad |R(u)|\leqq \rho^{(3)}_{\infty,1}\left\{\int_0^L |u_x|^2\,|A\,u|\,dx+\int_0^L |u_x|^2\,|h|\,dx\right\}.$$

We apply again (1.14) with $v=u_x$: for every $t\geqq 0$,

$$(1.78)\qquad \sup_{0\leqq x\leqq L}|u_x(x,t)|^2\leqq |u_x(t)|_0\left(2\,|A\,u|_0+\frac{1}{L}|u_x(t)|_0\right),$$

and then the right hand side of (1.77) is bounded by

$$\rho^{(3)}_{\infty,1}\,|u_x|_{L^\infty}\,|u_x|_0\,(|A\,u|_0+|h|_0).$$

Thanks to (1.76) and (1.78), we have for $t\geqq T_1(R)$,

$$(1.79)\qquad |R(u)|\leqq \rho^{(4)}_{\infty,1}(1+|A\,u|_0^{3/2}),$$

hence returning to (1.69), with (1.71) to (1.73), we have

$$(1.80)\quad \frac{1}{2}\frac{d}{dt}\varphi_1(u)+\gamma\psi_1(u)\leqq \rho'_{\infty,1}+(f_t,A\,u)_0+\rho^{(4)}_{\infty,1}(1+|A\,u|_0^{3/2})$$

for $t\geqq T_1(R)$ and where

$$(1.81)\quad \varphi_1(u)=|A\,u|_0^2+2(f,A\,u)_0 \\ -\int_0^L \{g(|u|^2)\,|u_x|^2+2g'(|u|^2)\,\mathrm{Re}(u\bar{u}_x)^2\}\,dx,$$

$$(1.82)\qquad \psi_1(u)=|Au|_0^2+(f, Au)_0.$$

Now using again (1.48) and the Cauchy-Schwarz inequality we infer that, for $t \geqq T_1(R)$,

$$(1.83)\qquad |Au|_0^2 \leqq 2\varphi_1(u)+4|f|_{0,\infty}^2+\rho_{\infty,1}^{(5)},$$

$$(1.84)\qquad \varphi_1(u)-\psi_1(u) \leqq |f|_{0,\infty}|Au|_0+\rho_{\infty,1}^{(6)}.$$

We deduce then form (1.80),

$$(1.85)\qquad \frac{d}{dt}\varphi_1(u)+\gamma\varphi_1(u) \leqq \frac{\gamma}{2}\varphi_1(u)+\rho_{\infty,1}^{(7)}$$

where we have used the following Young inequalities

$$(|f_t|_{0,\infty}+\gamma|f|_{0,\infty})|Au|_0 \leqq \frac{\gamma}{4}|Au|_0^2+\rho_{\infty,1}^{(8)},$$

$$\rho_{\infty,1}^{(4)}|Au|_0^{3/2} \leqq \frac{\gamma}{4}|Au|_0^2+\rho_{\infty,1}^{(9)},$$

and (1.83).

It follows from (1.85) that

$$(1.86)\qquad \varphi_1(u(t)) \leqq \varphi_1(u(T_1(R))\exp-\frac{\gamma}{2}(t-T_1(R))+\frac{2}{\gamma}\rho_{\infty,1}^{(7)}$$

which shows that for $t \geqq T_2(R)$,

$$(1.87)\qquad \varphi_1(u(t)) \leqq \frac{4}{\gamma}\rho_{\infty,1}^{(7)}.$$

We finish the proof of (1.69) using (1.83) and (1.87) concerning the bound on $|u_{xx}(t)|_0$, and (1.48) for the remaining terms. □

2. THE NONLINEAR GROUP AND THE LONG TIME BEHAVIOR

In this Section we mainly consider the case where the external force f is time independent. We show that the long time behavior of $S(t)$ in $D(A)$ is characterized by an attractor. Then we give some remarks concerning the behavior of $S(t)$ in V and on the time periodic forced case.

2.1. The universal attractor

(a) The nonlinear group

Let f be given in H, according to Theorem 1.1, the equation (1.38) with $f(t)\equiv f$, $\forall t\in\mathbb{R}$, processes a unique solution when we prescribe $u(0)=u_0$, $u_0\in V$. We set

$$S(t)\,u_0=u(t),\qquad \forall t\in\mathbb{R}, \tag{2.1}$$

which defines according to (1.40) a family of continuous mappings on V. Since (1.38) is autonomous, $(S(t))_{t\in\mathbb{R}}$ form a group acting on V. Moreover, as discussed in Section 1.4, $S(t)$ is continuous on D(A). Let us now interpret the uniform boundedness properties which result from Propositions 1.2 and 1.3.

(b) Bounded absorbing sets

We recall that a bounded set B_a in V [resp. D(A)] is absorbing in V [resp. D(A)], if for every bounded set B in V [resp. D(A)] there exists $T(B)\in\mathbb{R}$ such that

$$S(t)\,B\subset B_a,\qquad \forall t\geqq T(B). \tag{2.2}$$

With this definition it is clear that we can deduce from Propositions 1.2 and 1.3 the

COROLLARY 2.1. — *The set*

$$B_1=\{v\in V,\,|v|_0^2+L^2|v_x|_0^2\leqq(\rho_{\infty,1})^2\} \tag{2.3}$$

is a bounded absorbing set for $\{S(t)\}$ *in* V. *While*

$$B_2=\{v\in D(A),\,|v|_0^2+L^2|v_x|_0^2+L^4|v_{xx}|_0^2\leqq(\rho_{\infty,2})^2\} \tag{2.4}$$

is a bounded absorbing set for $\{S(t)\}$ *in* D(A).

(c) Omega limit sets

Thanks to (2.2), as far as the long time behavior of $S(t)$ is concerned, we can restrict ourselves to initial data which are in B_a. Therefore we introduce the following ω-limit set

PROPOSITION 2.1. — *The set*

$$\omega(B_2) = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} S(t) B_2} \tag{2.5}$$

where the closures are taken with respect to the weak topology of D(A), *is included in* B_2 *and non empty. It is invariant by* S(*t*) *i. e.*

$$S(t)\,\omega(B_2) = \omega(B_2), \qquad \forall t \in \mathbb{R}. \tag{2.6}$$

Remarks 2.1. — 1. If we take B a bounded set in D(A), we can also define ω(B) by (2.5) and when $B \neq \varnothing$, ω(B) is also included in B_2, non empty and invariant. According to (2.2) it is clear that $\omega(B) \subset \omega(B_2)$, and this shows that $\omega(B_2)$ is maximal in a certain sense, *see* also the Theorem 2.1.

2. It is also possible to define $\omega(B_1)$ in the weak topology of V, *see* next paragraph. □

The *proof of Proposition* 2.1 is easy using the following facts

(2.7) S (*t*) is weakly continuous on D(A),

(2.8) a point *b* belongs to ω(B) if and only there exists two sequences $t_n \in \mathbb{R}$, $b_n \in B$ such that $t_n \to \infty$ and $S(t_n)\,b_n$ weakly converges in D (A) as *n* goes to ∞.

In order to prove (2.7), we must show that if a sequence v_n converges to *v* weakly in D(A), $S(t)\,v_n$ converges to $S(t)\,v$ in the same topology. By the compactness of the injection of D(A) into V, we know that v_n converges strongly to *v* in V and since S(*t*) is continuous on V, $S(t)\ v_n$ converges strongly to *v* in V. On the other hand, v_n is bounded in D(A) so that $S(t)\,v_n$ in also bounded in D(A), and we can extract a sequence $S(t)\,v_{n'}$ which weakly converges to some *w* in D(A). As before $S(t)\,v_n$ converges to *w* in the strong topology of V and $w = S(t)\,v$. This shows that the whole sequence $S(t)\,v_n$ weakly converges to $S(t)\,v$ in D(A). Q.E.D. Concerning (2.8), we know that the weak topology of D(A) is metrizable on B_2 (or more generally on bounded sets in D(A), we have implicitely used this fact before) and we denote by d^w an associated distance. Thanks to (2.2) the convergence property in (2.8) can be phrased as: there exists $w \in D(A)$ such that $d^w(S(t_n)\,b_n, w)$, goes to 0 as $n \to \infty$ and (2.8) becomes the classical definition of accumulation points in a metric space. □

(d) The universal attractor

As already noticed, due to the fact that B_2 is a bounded absorbing set in $D(A)$, the set $\omega(B_2)$ is maximal in the sense that

THEOREM 2.1. — *The set*

$$\mathscr{A} = \omega(B_2) \tag{2.9}$$

satisfies

(2.10) $\mathscr{A}$ *is bounded and weakly closed in* $D(A)$,

$$S(t)\mathscr{A} = \mathscr{A}, \qquad \forall t \in \mathbb{R}, \tag{2.11}$$

(2.12) *for every bounded set* B *in* $D(A)$,

$$\lim_{t \to +\infty} d^w(S(t)B, \mathscr{A}) = 0.$$

Moreover it is the maximal set (in the sense of inclusion) that satisfies (2.10), (2.11) *and* (2.12). *It is connected in the weak topology of* $D(A)$.

By the compact imbedding of $D(A)$ into $D(A^s)$, $s<1$, and the continuous imbedding of $D(A^s)$ into $H^{2s}(0, L)$ we know that $D(A)$ is compactly imbedded in $H^{2s}(0, L)$ for every s, $s<1$. Then it follows that the convergence in (2.10) holds with respect to the strong (norm) topology of $H^{2s}(0, L)$ for such s. In particular ($s=1/2$).

COROLLARY 2.2. — *For every bounded set* B *in* $D(A)$, *the set* $S(t)B$ *converges to* $\mathscr{A}$ *with respect to the* V*-norm.*

Remarks 2.2. — 1. The uniqueness of a set satisfying (2.8)-(2.11) is obvious. We shall term this set as the *universal attractor.*

2. Recall that $d^w(X, Y) = \operatorname{Sup}_{x \in X} \operatorname{Inf}_{y \in Y} d^w(x, y)$ and (2.12) is not ambiguous since $\mathscr{A} \subset B_2$ (Proposition 2.1) and thanks to Corollary 2.1, $S(t)B \subset B_2$ for $t \geqq T(B)$. □

Proof. — The properties (2.10) and (2.11) follow from Proposition 2.1. Concerning (2.12), we argue by contradiction and assume that there exist two sequences $t_n \in \mathbb{R}$ and $b_n \in B$ [thanks to Corollary 2.1 we can assume that $S(t_n) b_n \in B_2$] such that $t_n \to \infty$ and

$$d^w(S(t_n) b_n, \mathscr{A}) \geqq \varepsilon_0 > 0, \tag{2.13}$$

for some ε_0 independent of n. Since $S(t_n) b_n$ is bounded in $D(A)$ we can assume that there exists $b \in B_2$ such that

$$d^w(S(t_n) b_n, b) \to 0 \quad \text{as } n \to \infty.$$

Writting $S(t_n) b_n = S(t_n/2)\ (S(t_n/2) b_n)$ we see that thanks to (2.8), $b \in \omega(B_2) = \mathscr{A}$ and this contradicts (2.13). The maximality asserted in the Theorem is clear. The connexity follows from the observation that B_2 is connected (since it is convex), compact in the weak topology and the following abstract result (which is easy to prove).

LEMMA 2.1. — *Let $(\mathscr{E}, d)$ be a metric space and $\mathscr{F}$ a non empty compact connected set. Let also denote by $(\Sigma(t))_{t \geqq 0}$ a semi group on $\mathscr{E}$ that satisfy*

(i) *for every $t \in \mathbb{R}_+$, $\Sigma(t)$ is continuous on $\mathscr{E}$;*

(ii) *for every $e \in \mathscr{E}$, $t \to \Sigma(t)\, e$ is continuous from $\mathbb{R}_+$ into $\mathscr{E}$;*

(iii) *there exists a compact set K and $t_0 \in \mathbb{R}_+$ such that*

$$\Sigma(t)\mathscr{F} \subset K, \qquad \forall t \geqq t_0.$$

Then the omega limit set of $\mathscr{F}$ under $\Sigma(t)$:

$$\omega(\mathscr{F}) = \bigcap_{s \geqq 0} \overline{\bigcup_{t \geqq s} \Sigma(t)\mathscr{F}}$$

is a connected and compact non empty set. □

We apply this lemma with $\mathscr{E} = B_2$, $d = d^w$, $\Sigma(t) = S(t)$, $\mathscr{F} = K = B_2$, $t_0 = T(B_2)$, the property (i) follows from (2.7), (ii) follows in the same manner. □

2.2. Remarks and complements

In this paragraph we briefly address two different questions: the behavior of $S(t)$ in V and the case where the function f depends periodically on t.

2.2.1. *Behavior of* $S(t)$ *in* V

According to Corollary 2.1, the set B_1 is a bounded absorbing set in V, therefore the behavior of $S(t)$ in V (as $t \to \infty$) is related to its action on B_1. We introduce

$$\mathscr{A}^* = \bigcap_{s>0} \overline{\bigcup_{t \geq s} S(t) B_1} \tag{2.14}$$

where this time the closures are taken with respect to the weak topology of V. In the previous analysis i. e. in the proof of Proposition 2.1, we have used that $S(t)$ is weakly continuous in the relevant topology, i. e. (2.7), and this was a consequence of the fact that $S(t)$ was continuous on V. Here we do not know whether or not $S(t)$ is even defined on a subspace of V which is compactly imbedded in V. For instance it is not known whether or not $S(t)u_0$ has a sense for $u_0 \in H$ (as it is the case for the linear equation). However we are going to derive a property of $S(t)$ that shows its weak continuity on V:

PROPOSITION 2.2. — *For every* $t \in \mathbb{R}$, *the mapping* $S(t)$ *is continuous on bounded sets of* V *for the topology of the norm in* H.

And then

COROLLARY 2.3. — *The set* $\mathscr{A}^*$ *defined in* (2.12) *is the universal weak attractor for* $S(t)$ *in* V.

First, we notice that the weak continuity of $S(t)$ in V follows readily from Proposition 2.2 [as in the proof of (2.7)]. Then by mimicking the proof of Proposition 2.1 and Theorem 2.1, we obtain Corollary 2.2. Concerning the *proof of Proposition* 2.2 we return to (1.43), with (1.45). Hence

$$\frac{1}{2}\frac{d}{dt}|w|_0^2 + \gamma |w|_0^2 = \operatorname{Im} \int_0^L (g(|u|^2)u_2 - g(|u|^2)u_1)(\bar{u}_2 - \bar{u}_1)\,dx \tag{2.15}$$

where $u_i = S(t)u_{0i}$, $w = u_2 - u_1$. When u_{01} and u_{02} belong to a fixed bounded set in V, $|u_{0i}| \leqq R$, we know that for every T, $0 < T < \infty$, the number

$$\operatorname{Sup} |u_i(x, t)|$$

where the supremum is taken on $i = 1, 2$, $x \in [0, L]$, $|t| \leqq T$, $|u_{01}| \leqq R$ is finite. Hence there exists a constant $C(T, R)$ such that the right hand side of (2.15) is bounded in absolute value by $C(T, R)|w|^2$. This shows Proposition 2.2 thanks to the classical Gronwall lemma. □

Remark 2.3. — Since $\mathscr{A}$ is bounded in D(A), it is bounded in V and by the invariance property (2.9), we have

$$\mathscr{A} \subset \mathscr{A}^*. \tag{2.16}$$

On the contrary if we knew (as in [7] for instance) that $\mathscr{A}^*$ is bounded in D(A) then the opposite inclusion would hold. □

2.2.2. *The time-periodic case*

We have already noticed in the Introduction that the special case

$$f(t)=f_0\, e^{i\,\omega_0\, t}, \qquad f_0 \in \mathrm{H}, \quad \omega_0 \in \mathbb{R} \tag{2.17}$$

is contained in the previous analysis. Concerning the more general case where f satisfy (1.67) and there exists $\mathrm{T}>0$ such that

$$f(t+\mathrm{T})=f(t), \qquad \forall\, t \in \mathbb{R}, \tag{2.18}$$

we consider the family of applications on V, $s \in \mathbb{R}$, $t \in \mathbb{R}$,

$$\mathrm{S}(t, s): u_0 \to u(t), \quad \text{solution of (1.38) with } u(s)=u_0.$$

It is clear that

$$\mathrm{S}(t, s)\,\mathrm{S}(s, \sigma)=\mathrm{S}(t, \sigma), \qquad \forall\, t, s, \sigma \in \mathbb{R},$$

and thanks to (2.18),

$$\mathrm{S}(t+\mathrm{T}, \sigma+\mathrm{T})=\mathrm{S}(t, \sigma), \qquad \forall\, t, \sigma \in \mathbb{R}.$$

It follows then that for every $s \in [0, \mathrm{T}[$, the family $\{\, \mathrm{S}(s+m\,\mathrm{T}, s)\,\}_{m \in \mathbb{Z}}$ forms a discrete group. All the results of Section 2.1 can be transposed in this case (*see* [9] for the details). For instance, the set

$$\mathscr{A}_s = \bigcap_{n\in\mathbb{N}} \overline{\bigcup_{m\geq n} S(s+m\,T, s)\,B_2}$$

is the universal attractor for $\{S(s+m\,T, s)\}_{m\in\mathbb{Z}}$ in $D(A)$ and moreover

$$\mathscr{A}_{s_1} = S(s_1, s_2)\,\mathscr{A}_{s_2}, \qquad \forall\, s_i \in \mathbb{R}. \quad \square$$

3. DIMENSION OF INVARIANT SETS

In this Section we prove that the universal attractor in $D(A)$, $\mathscr{A}$, has finite fractal dimension [*see* (3.48) for the definition]. And this clearly implies that it is the case for every invariant set which is bounded in $D(A)$. First we show that when m is sufficiently large, the differential of the semi-group $S(t)$ contracts the m-dimensional volumes in V [*see* (3.20)]. Second, we apply a general result of Constantin, Foias and Temam [3] that leads to the result on the dimension. The proof of Theorem 3.1, which is a result on the evolution of m-dimensional volumes in V, contains a generalization of a method presented in [3]. Indeed, instead of considering the classical norm on this space, (1.42), we introduce a family of quadratic functional $q_\mu(t;\,.)$ that depend on t through a solution $u(t) = S(t)\,u_0$ which leads to the desired result. This method, which seems fairly general, could be adapted to cases where the *a priori* estimates on the solutions are obtained using nonlinear multipliers.

3.1. The linearized equation

Let u_0 be given in V, we denote by $v(t)$ the solution to the nonautonomous $\mathbb{R}$-linear equation

$$(3.1) \qquad i v_t - A\,v + [g(|u|^2) + |u|^2 g'(|u|^2)]\,v + g'(|u|^2)\,u^2\,\bar{v} + i\gamma\,v = 0,$$

$$(3.2) \qquad v(0) = v_0,$$

where $u(t) = S(t)\,u_0$. Since we know that $u \in \mathscr{C}(\mathbb{R}, V)$, it is a simple matter to check that (3.1)-(3.2) possesses a unique solution with

$$(3.3) \qquad v \in \mathscr{C}(\mathbb{R}, V).$$

This equation, which is formally obtained by differentiation of (1.38) with respect to u, gives the value of the differential of $S(t)$ at u_0. More precisely, we consider (1.38) as a system of two equations on $\mathbb{R}$ for the real and imaginary parts of u, then the mapping $DS(t)u_0$ defined by

$$(3.4) \qquad (DS(t)u_0).v_0 = v(t), \qquad v_0 \in V$$

is the $\mathbb{R}$-differential of $S(t)$ at u_0. We are obliged to deal with $\mathbb{R}$-linear operator since (3.1)-(3.2) is not $\mathbb{C}$-linear [the nonlinear term $g(|u|^2)u$ is not holomorphic in general]. In the sequel the identification $E_{\mathbb{C}} = E^2_{\mathbb{R}}$, $E = H, V, \ldots$ is understood. We have the following result that makes precise the fact that $DS(t)u_0$ is the differential of $S(t)$ at u_0:

PROPOSITION 3.1. – *Let* R, R_1 *and* T *be three positive numbers. There exists a constant* $C = C(R, R_1, T)$ *such that for every* u_0, v_0, t *with* $|u_0|_V \leqq R_1$, $|v_0|_V \leqq R$, $|t| \leqq T$, *we have*

$$(3.5) \qquad |S(t)(u_0+v_0) - S(t)u_0 - (DS(t)u_0)v_0|_V \leqq C|v_0|^2_V.$$

The proof of this result, which is lengthy but classic, is left to the reader. It should be noticed that (3.5) shows that $S(t)$ is uniformly differentiable on bounded sets of V, an important property as far as the proof of finite dimensionality is concerned. □

An energy equality. We set

$$(3.6) \qquad w(x,t) = v(x,t)e^{\gamma t},$$

and when v is solution to (3.1)-(3.2), we have

$$(3.7) \qquad iw_t - Aw + (g|u|^2) + |u|^2 g'(|u|^2))w + g'(|u|^2)u\bar{w} = 0,$$

$$(3.8) \qquad w(0) = v_0.$$

We take the real part of the scalar product of (3.7) with $-\bar{w}_t$:

$$(3.9) \quad \frac{1}{2}\frac{d}{dt}|w_x|_0^2 - \mathrm{Re}\int_0^L \{g(|u|^2)w\bar{w}_t + 2g'(|u|^2)\mathrm{Re}(u\bar{w})u\bar{w}_t)\,dx = 0$$

and this leads to set

$$(3.10) \quad \Phi(t,w) = \int_0^L \{|w_x|^2 - g(|u|^2)|w|^2 - 2g'(|u|^2)\mathrm{Re}(u\bar{w})^2\}\,dx;$$

so that (3.9) reads

$$\frac{d}{dt}\Phi(t, w) = r(t, w) \tag{3.11}$$

where

$$r(t, w) = -\int_0^L \left\{ |w|^2 \frac{\partial}{\partial t}\{g(|u|^2)\} + 2\,\mathrm{Re}(u\bar{w})^2 \times \frac{\partial}{\partial t}\{g'(|u|^2)\} + 4g'(|u|^2)\,\mathrm{Re}(u\bar{w})\,\mathrm{Re}\left(\frac{\partial u}{\partial t} w\right)\right\} dx. \tag{3.12}$$

On the other hand the imaginary part of the scalar product of (3.7) with $\bar{w}$ reads

$$\frac{1}{2}\frac{d}{dt}|w|_0^2 + 2\int_0^L g'(|u|^2)\,\mathrm{Re}(u\bar{w})\,\mathrm{Im}(u\bar{w})\,dx = 0. \tag{3.13}$$

If we set for $\mu \in \mathbb{R}$,

$$q_\mu(t, w) = \Phi(t, w) + \mu |w|_0^2, \tag{3.14}$$

and

$$r_\mu(t, w) = r(t, w) - 4\mu \int_0^L g'(|u|^2)\,\mathrm{Re}(u\bar{w})\,\mathrm{Im}(u\bar{w})\,dx, \tag{3.15}$$

we deduce from (3.11) and (3.13) the following family of scalar evolution equations:

$$\frac{d}{dt}\{q_\mu(t, w(t))\} = r_\mu(t, w(t)), \qquad \forall \mu \in \mathbb{R}. \tag{3.16}$$

When $u(x, t)$ belongs to a fixed compact set in $\mathbb{C}$, it is clear by (3.10), (3.14) that we can choose μ so that $w \to q_\mu(t, w)$ is a norm on V, equivalent to the usual one. This is the reason why we have introduced such a parameter μ.

3.2. Evolution of the m-dimensional volumes in V

In this paragraph we study how the operators $\mathrm{DS}(t)u_0$ transform the m-dimensional volumes in V. We take m elements $v_0^1, \ldots, v_0^m$ in V and study the evolution of the quantities

$$|v^1(t) \wedge \ldots \wedge v^m(t)|_V^2 = \det_{1 \leq i, j \leq m} (v^i(t), v^j(t))_V \tag{3.17}$$

where $v^i(t) = (DS(t)u_0)v_0^i$, and we have set

$$(v, w)_V = \operatorname{Re} \int_0^L \{v\bar{w} + L^2 v_x \bar{w}_x\} dx. \tag{3.18}$$

The Gram determinant (3.17) represents the square of $m!$-times the volume of the m-dimensional polyhedron defined by the vectors $v^1(t), \ldots, v^m(t)$. We are going to show that for sufficiently large m this determinant decays exponentially as $t \to +\infty$. More precisely we consider an invariant set X which is bounded in D(A):

$$S(t)X = X, \quad \forall t \in \mathbb{R}; \qquad X \text{ bounded in } D(A). \tag{3.19}$$

We have

THEOREM 3.1. — *Let* X *be an invariant set which is bounded in* D(A). *There exists two constants* C_1 *and* C_2 *such that for every* $u_0 \in X$, $m \geqq 1$ *and* $t \geqq 0$, *the* $v^i(t) = (DS(t)u_0)v_0^i$ *satisfy*

$$|v^1(t) \wedge \ldots \wedge v^m(t)|_V \leqq |v_0^1(t) \wedge \ldots \wedge v_0^m(t)|_V C_1^m \exp(C_2 - \gamma m)t, \quad \forall v_0^1 \in V. \tag{3.20}$$

Proof. — We first make the transform (3.6) so that

$$|v^1(t) \wedge \ldots \wedge v^m(t)|_V^2 = e^{-2\gamma mt} G_m(t) \tag{3.21}$$

where

$$G_m(t) = |w^1(t) \wedge \ldots \wedge w^m(t)|_V^2. \tag{3.22}$$

We are going to use instead of $|.|_V^2$, the (time dependent) quadratic form $q_\mu(t, .)$ with appropriate μ.

(a) *Choice of* μ

According to (3.19), the number

$$|X|_\infty = \sup_{v \in X} \sup_{x \in [0, L]} |v(x)|, \tag{3.23}$$

is finite since the norm of v in $D(A)$ majorize the L^∞-norm of v on $[0, L]$. We take

$$\mu = \mu(X) = \frac{1}{L^2} + \sup_{0 \leqq \sigma \leqq |X|_\infty^2} \{ |g(\sigma)| + 2\sigma |g'(\sigma)| \} \tag{3.24}$$

then $q_\mu(t, .)$ defined in (3.14) satisfy

$$\begin{cases} \mu \{ |w|_0^2 + L^2 |w_x|_0^2 \} \leqq q_\mu(t, w) \leqq L^{-2} \{ |w|_0^2 + L^2 |w_x|_0^2 \}, \\ \qquad \forall w \in V. \end{cases} \tag{3.25}$$

Hence, for fixed t, $\{ q_\mu(t, .) \}^{1/2}$ is a norm on V, which is equivalent to the $|.|_V$ norm (recall that $V_{\mathbb{C}} \simeq V_{\mathbb{R}} \times V_{\mathbb{R}}$). On the other hand, we claim that there exists a constant $C_3 = C_3(X)$ such that

$$|r_\mu(t, w)| \leqq C_3 |w|_0^{3/2} (|w|_0^2 + L^2 |w_x|_0^2)^{1/4}, \qquad \forall w \in V. \tag{3.26}$$

Indeed, let us first bound $r(t, w)$ given in (3.12). According to (1.38) we have

$$i u_t = A u - g(|u|^2) u - i \gamma u + f,$$

and since $u(t) \in X$, $\forall t \in \mathbb{R}$ and X is bounded in $D(A)$ by hypothesis we deduce that there exists a constant $C_4 = C_4(X)$ such that

$$|u_t|_0 \leqq C_4, \qquad \forall t \in \mathbb{R}. \tag{3.27}$$

Then using (3.23), which shows that $|u(x, t)| \leqq |X|_\infty$, $\forall x, t$, we deduce from (3.12) that

$$\begin{aligned} |r(t, w)| &\leqq C_5 \int_0^L |w|^2 |u_t| \, dx \\ &\leqq C_4 C_5 \left\{ \int_0^L |w|^4 \, dx \right\}^{1/2} \\ &\leqq [\text{by } (1.14)] \end{aligned}$$

$$|r(t, w)| \leqq C_4 C_5 |w|_0^{3/2} \left(2 |w_x|_0 + \frac{1}{L} |w|_0 \right)^{1/2}. \tag{3.28}$$

Now according to (3.15),

$$|r_\mu(t, w)| \leqq |r(t, w)| + 4\mu C_6 \int_0^L |w|^2 dx,$$

and (3.26) follows with (3.28).

(b) A first estimation on $G_m(t)$

We introduce the $\mathbb{R}$-bilinear forms on $V_{\mathbb{R}} \times V_{\mathbb{R}}$

$$(3.29) \quad \varphi(t; \eta, \zeta) = \operatorname{Re} \int_0^L \{ \eta_x \bar{\zeta}_x + \mu\eta\bar{\zeta} - g(|u|^2)\eta\bar{\zeta} \\ - 2g'(|w|^2)\operatorname{Re}(u\bar{\eta})\operatorname{Re}(u\bar{\zeta}) + \mu\eta\bar{\zeta} \} dx$$

which depend on t through $u = u(t)$. The quadratic form which is associated to $\varphi(t; .,.)$ is $q_\mu(t,.)$, hence according to (3.25), $\varphi(t; .,.)$ is a scalar product on V which is continuous and coercive. We introduce the Gram determinants

$$(3.30) \qquad H_m(t) = \det_{1 \leqq i, j \leqq m} \varphi(t; w^i(t), w^j(t)).$$

We relate the $H_m(t)$ to the $G_m(t)$ by the

LEMMA 3.1. — *With the previous notations and thanks to* (3.25) *we have*

$$(3.31) \qquad \mu^m G_m(t) \leqq H_m(t) \leqq L^{-2m} G_m(t), \qquad \forall t \in \mathbb{R}.$$

We detail the proof of this result since it contains another interpretation of the Gram determinant that is very useful for later purposes. We recall [4] that the Gram determinant of m vectors $\xi^1, \ldots, \xi^m$ in a Hilbert space $\mathscr{H}$ with scalar product $\psi(.,.)$ is also the determinant of the quadratic form on $\mathbb{R}^m$

$$(x_1, \ldots, x_m) \to \psi\left\{ \sum_{j=1}^m x_j \xi^j, \sum_{j=1}^m x_j \xi^j \right\}.$$

Since this determinant is also equal to the product of the m eigenvalues of this quadratic form, we have

$$(3.32) \qquad \det_{a \leqq i, j \leqq m} \psi(\xi^i, \xi^j) = \prod_{l=1}^{m} \operatorname*{Max}_{\substack{G \subset \mathbb{R}^m \\ \dim G = l}} \operatorname*{Min}_{\substack{\sum_{i=1}^{m} x_i^2 = 1 \\ x \in G}} \psi(x_j \xi^j, x_j \xi^j),$$

thanks to the classical min-max principle. It is clear on (3.32) that if ψ_1 is another scalar product on $\mathscr{H}$ which is continuous and coercive, i. e.

$$(3.33) \qquad \alpha\psi(\xi, \xi) \leqq \psi_1(\xi, \xi) \leqq \beta\psi(\xi, \xi), \qquad \forall \xi \in \mathscr{H},$$

then

$$(3.34) \quad \alpha^m \det_{1 \leqq i, j \leqq m} \psi(\xi^i, \xi^j) \leqq \det_{1 \leqq i, j \leqq m} \psi_1(\xi^i, \xi^j) \leqq \beta^m \det_{1 \leqq i, j \leqq m} \psi(\xi^i, \xi^j).$$

The proof of (3.31) is done when we notice that, for fixed t, the scalar products on $\mathscr{H} = V$ that appears in (3.22) and (3.30) satisfy according to (3.25), the relation (3.33) with $\alpha = \mu$ and $\beta = L^{-2}$. □

(c) *An estimate on* $H_m(t)$

We are going to derive from (3.16) the time derivative of $H_m(t)$ given in (3.30). According to the classical rule of differentiation of a determinant,

$$(3.35) \qquad \frac{dH_m(t)}{dt} = \sum_{l=1}^{m} \det_{1 \leqq i, j \leqq m} \varphi(t; w^i(t), w^j(t))_l$$

where we have set

$$(3.36) \quad \varphi(t; w^i(t), w^j(t))_l = (1 - \delta_{jl})\, \varphi(t; w^i(t), w^j(t)) + \delta_{jl} \frac{d}{dt} \{ \varphi(t; w^i(t), w^j(t)) \}$$

and δ_{jl} is the usual Kronecker symbol. Now thanks to (3.16) and the formula

$$4\,\varphi(t; w^i(t), w^j(t)) = q_\mu(t, w^i(t) + w^j(t)) - q_\mu(t, w^i(t) - w^j(t))$$

we deduce that

$$\frac{d}{dt}\{\varphi(t; w^i(t), w^j(t))\} = \rho(t; w^i(t), w^j(t)) \tag{3.37}$$

where

$$4\rho(t; \eta, \zeta) = r_\mu(t; \eta+\zeta) - r_\mu(t, \eta-\zeta) \tag{3.38}$$

is the (time dependent) symetric bilinear forms associated to $r_\mu(t,.)$. We claim that according to (3.37), (3.35) yields

$$\frac{dH_m(t)}{dt} = H_m(t) \sum_{l=1}^{m} \underset{\substack{F \subset \mathbb{R}^m \\ \dim F = l}}{\text{Max}} \ \underset{\substack{x \in F \\ x \neq 0}}{\text{Min}} \frac{r_\mu\left(t, \sum_{j=1}^{m} x_j w^j(t)\right)}{q_\mu\left(t, \sum_{j=1}^{m} x_j w^j(t)\right)}. \tag{3.39}$$

We assume that this formula holds for the moment and we finish the

(*d*) *Proof of* (3.20)

We first note that if $|v_0^1 \wedge \ldots \wedge v_0^m|_V = 0$ then the v_0^i are linearly dependent, so are the $v^i(t)$ and (3.20) is obvious. If $|v_0^1 \wedge \ldots \wedge v_0^m|_V \neq 0$, we know by a continuity argument that $|v^1 \wedge \ldots \wedge v^m|_V \neq 0$ for small t, and then $\{w^1(t), \ldots, w^m(t)\}$ are independent too. Hence for $G \subset \mathbb{R}^m$, $x \in G \setminus \{0\}$, $q_\mu\left(t, \sum_{j=1}^{m} x_j w^j(t)\right) \neq 0$ and according to (3.25) and (3.26) we have

$$\frac{r_\mu\left(t, \sum_{j=1}^{m} x_j w^j(t)\right)}{q_\mu\left(t, \sum_{j=1}^{m} x_j w^j(t)\right)} \leq \frac{C_3}{\mu} \frac{\left|\sum_{j=1}^{m} x_j w^j(t)\right|_0^{3/2}}{\left|\sum_{j=1}^{m} x_j w^j(t)\right|_V^{3/2}}. \tag{3.40}$$

Since $H_m(t) \geqq 0$, we infer from (3.39) that

$$\frac{dH_m(t)}{dt} \leqq \frac{C_3 H_m(t)}{\mu} \sum_{l=1}^{m} \underset{\substack{G \subset \mathbb{R}^m \\ \dim G = l}}{\text{Max}} \ \underset{\substack{x \neq 0 \\ x \in G}}{\text{Min}} \frac{\left| \sum_{j=1}^{m} x_j w^j(t) \right|_0^{3/2}}{\left| \sum_{j=1}^{m} x_j w^j(t) \right|_V^{3/2}}. \tag{3.41}$$

Now we notice that when $F \subset \mathbb{R}^m$, dim $F = l$ is given; for $x \in F$, the $\sum_{j=1}^{m} x_j w^j(t)$ span $F(t)$ an l-dimensional subspace of V so that

$$\underset{\substack{x \in F \\ x \neq 0}}{\text{Min}} \frac{\left| \sum_{j=1}^{m} x_j w^j(t) \right|_0^{3/2}}{\left| \sum_{j=1}^{m} x_j w^j(t) \right|_V^{3/2}} \leqq \underset{\substack{\mathcal{F} \subset V \\ \dim \mathcal{F} = l}}{\text{Max}} \ \underset{\substack{\xi \in \mathcal{F} \\ \xi \neq 0}}{\text{Min}} \frac{|\xi|_0^{3/2}}{|\xi|_V^{3/2}} \tag{3.42}$$

Now the right hand side of (3.42) is explicitly known since

$$\underset{\substack{\mathcal{F} \subset V \\ \dim \mathcal{F} = l}}{\text{Max}} \ \underset{\substack{\xi \in \mathcal{F} \\ \xi \neq 0}}{\text{Min}} \frac{|\xi|_0^2}{|\xi|_0^2 + L^2 |\xi_x|_0^2} = \frac{1}{1 + L^2 \lambda_l}. \tag{3.43}$$

We then conclude from (3.41)-(3.43) that

$$\frac{dH_m(t)}{dt} \leqq \frac{C_3}{\mu} \left(\sum_{l=1}^{m} \frac{1}{(1 + L^2 \lambda_l)^{3/4}} \right) H_m(t).$$

Since $\lambda_l \sim C_0 L^{-2} l^2$ as $l \to \infty$, $\sum_{l=1}^{m} \frac{1}{(1 + L^2 \lambda_l)^{3/4}} < \infty$ and there exists C_2 such that

$$\frac{dH_m(t)}{dt} \leqq 2 C_2 H_m(t); \tag{3.44}$$

which shows that

$$H_m(t) \leqq e^{2 C_2 t} H_m(0), \qquad \forall t \geqq 0.$$

Combining (3.21), (3.22), (3.31) and this last inequality we obtain (3.20) with $C_1 = 1/(L\sqrt{\mu})$.

(*e*) *Proof of* (3.39)

As before we can assume that the $\{w^j(t)\}_{j=1}^m$ are independent. Then we use the following Lemma

LEMMA 3.2. — *We consider two bilinear and symetric forms on* $\mathbb{R}^m$, ψ *and* ψ_2 *and we assume that* ψ *is definite and positive. Then denoting by* $\{\kappa_i\}_{i=1}^m$ *the ordered eigenvalues of* ψ_2 *with respect to* ψ, *i. e.*

$$\kappa_l = \underset{\substack{F \subset \mathbb{R}^m \\ \dim F = l}}{\text{Max}} \ \underset{\substack{x \in F \\ x \neq 0}}{\text{Min}} \frac{\psi_2(x, x)}{\psi(x, x)}, \tag{3.45}$$

for every family $\{\psi^1, \ldots, \psi^m\}$ *in* $\mathbb{R}^m$, *we have*

$$\sum_{l=1}^m \det_{1 \leq i, j \leq m} \psi(\xi^j, \xi^j)_l = \left(\sum_{l=1}^m \kappa_l\right) \det_{1 \leq i, j \leq m} \psi(\xi^i, \xi^j) \tag{3.46}$$

where

$$\psi(\xi^i, \xi^j)_l = (1 - \delta_{jl}) \psi(\xi^i, \xi^j) + \delta_{jl} \psi_2(\xi^i, \xi^j) \quad \square \tag{3.47}$$

This lemma is very similar to [3], Lemma 4.1, the present formulation is more adapted to the proof of (3.29). Let us give now a proof which is slightly different from that of [3]. We denote by $\{\eta^1, \ldots, \eta^m\}$ a basis of $\mathbb{R}^m$ where both ψ and ψ_2 are diagonal, i. e.

$$\psi(\eta^i, \eta^j) = \delta_{ij}, \qquad \psi_2(\eta^i, \eta^j) = \kappa_i \delta_{ij}.$$

We decompose the ξ^i in that basis: $\xi^i = \sum_j P_{ij} \eta^j$, then

$$\psi(\xi^i, \xi^j)_l = \sum_a P_{ia} P_{ja} ((1 - \delta_{jl}) + \kappa_a \delta_{jl})$$
$$= \sum_a P_{ia} Q^l_{aj}$$

where $Q^l_{aj} = ((1 - \delta_{jl}) + \kappa_a \delta_{jl}) P_{ja}$.

We deduce that the left hand side of (3.46) is equal to

$$\sum_{l=1}^m \det(PQ^l) = \det P \sum_{l=1}^m \det Q^l.$$

Now we notice that the application $P \rightarrow \sum_{l=1}^{m} \det Q^l$ is an m-multilinear alternating map as a function of the rows of P. Therefore it is proportional to det P. Taking P=I leads to

$$\sum_{l=1}^{m} \det Q^l = (\det P) \sum_{l=1}^{m} \det_{a,j} \{ ((1-\delta_{jl}) + \kappa_a \delta_{jl}) \delta_{ja} \}$$
$$= \det P \sum_{l=1}^{m} \kappa_l.$$

and this shows (3.46). □

Now (3.39) follows from this Lemma when we notice that in the right hand side of (3.35), t is simply a parameter so that we can take for each fixed t

$$\psi(x, y) = \varphi(t; x_j w^j(t), y_j w^j(t)),$$
$$\psi_2(x, y) = \rho(t; x_j w^j(t), y_j w^j(t)),$$

and apply Lemma 3.2. □

3.3. The result on the dimension

We recall that the fractal dimension of a metric space $\mathscr{E}$ is defined by the following limit [13]

(3.48) $$d_F(\mathscr{E}) = \limsup_{\varepsilon \to 0^+} \frac{\operatorname{Log} N_\varepsilon(\mathscr{E})}{\operatorname{Log}(1/\varepsilon)}$$

where $N_\varepsilon(\mathscr{E})$ denotes the minimal number of balls of radius ε which are necessary to cover $\mathscr{E}$. We always have $d_H(\mathscr{E}) \leq d_F(\mathscr{E})$, where d_H denotes the Hausdorff dimension of $\mathscr{E}$. We recall that the converse may not be; it is even possible that $d_F(\mathscr{E}) = \infty$ while $d_H(\mathscr{E}) = 0$.

We are going to deduce from Theorem 3.1 that the universal attractor $\mathscr{A}$ of Theorem 2.1 has finite fractal dimension in V. We first recall an abstract result of P. Constantin, C. Foias and R. Temam [3] that generalizes one of A. Douady and J. Oesterlé [5]. We are given a nonlinear mapping S on a compact subset X of a Hilbert space $\mathscr{H}$. And it is assumed that

$$SX = X, \tag{3.49}$$

and for every $u \in X$, there exists a linear operator $L(u) \in \mathcal{L}(\mathcal{H}, \mathcal{H})$ such that

$$\lim_{\varepsilon \to 0} \ \operatorname*{Sup}_{\substack{u, v \in X \\ 0 < |u - v|_{\mathcal{H}} < \varepsilon}} \frac{|Sv - Su - L(u)(v-u)|_{\mathcal{H}}}{|v-u|_{\mathcal{H}}} = 0, \tag{3.50}$$

and

$$\operatorname*{Sup}_{u \in X} |L(u)|_{\mathcal{L}(\mathcal{H})} < \infty. \tag{3.51}$$

Finally we denote by $\omega_m(L)$ the norm of the mth exterior product of L in $\Lambda^m \mathcal{H}$ (*see* [2]):

$$\omega_m(L) = \operatorname{Sup}\big(\det_{1 \leqq i, j \leqq m} (L\xi^i, L\xi^j)_{\mathcal{H}}\big)^{1/2} \tag{3.52}$$

where the supremum is taken on all the $\{\xi^i\}_{i=1}^m$ with $\det_{1 \leqq i, j \leqq m} (\xi^i, \xi^j)_{\mathcal{H}} \leqq 1$.

We set

$$\bar{\omega}_m = \operatorname*{Sup}_{x \in X} \omega_m(L(u)), \tag{3.53}$$

and define the uniform Lyapunov exponents on X by

$$\mu_1 = \operatorname{Log} \bar{\omega}_1, \qquad \mu_j = \operatorname{Log} \bar{\omega}_j - \operatorname{Log} \bar{\omega}_{j-1}, \quad j \geqq 2. \tag{3.54}$$

We can state according to [3] the

THEOREM. — *Under the hypotheses* (3.49) *to* (3.51), *if there exists* $m \geqq 0$ *such that*

$$\mu_1 + \ldots + \mu_{m+1} < 0 \tag{3.55}$$

then

$$d_H(X) \leqq m+1, \tag{3.56}$$

and

$$d_F(X) \leqq (m+1) \operatorname*{Max}_{1 \leqq l \leqq m} \left(1 + \frac{|\mu_1 + \ldots + \mu_l|}{|\mu_1 + \ldots + \mu_{m+1}|}\right). \ \square \tag{3.57}$$

In fact is assumed in [3] that the operators $L(u)$ are compact but this hypothesis is not necessary as shown in [7], and this will be used in the application below.

We can prove the

THEOREM 3.2. – *The universal attractor $\mathscr{A}$ of Theorem 2.1 has finite fractal and Hausdorff dimension in* V.

Proof. – We fix some positive $t_0>0$ and consider the mapping $S=S(nt_0)$ where $n\geqq 1$ will be chosen later on. We take $\mathscr{H}=V$, $X=\mathscr{A}$ which is compact in V by (2.10) and the compact injection of D(A) into V. According to Proposition 3.1, $L(u_0)=DS(nt_0).u_0$ satisfy (3.50), (3.51) and according to Theorem 3.1 and (3.52),

$$\omega_m(L(u_0))\leqq C_1^m \exp(C_2-\gamma m)\, nt_0.$$

Since the right hand side is independent of $u_0\in\mathscr{A}$, we deduce that

$$\bar{\omega}_m\leqq C_1^m \exp(C_2-\gamma m)\, nt_0. \tag{3.58}$$

We fix $m\in\mathbb{N}$ with $\gamma m>C_2$, then there exists n_0 such that for $S=S(n_0 t_0)$

$$\bar{\omega}_m<1 \tag{3.59}$$

or equivalently (3.55). Hence (3.56) and (3.57) hold and Theorem 3.2 is proved. □

Remark 3.1. – Theorem 3.1 is still valid [with γ replaced by $\gamma/2$ in (3.20)] when we consider an invariant set X which is only bounded in V because in that case (3.26) can be replaced by

$$|r_\mu(t,w)|\leqq C_4\,|w|_0^{1/2}(|w|_0^2+L^2\,|w_x|_0^2)^{3/4}.$$

Hence we can take $X=\mathscr{A}^*$ given by Corollary 2.2. The properties (3.49) to (3.51) hold also in this case with $\mathscr{H}=V$ but we do not know whether or not $\mathscr{A}^*$ is compact in V and therefore we cannot apply the result on the dimensions. □

Remark 3.2. – In the time periodic forced case (Section 2.2.2), Theorem 3.1 is again valid and Theorem 3.2 shows that all the $\mathscr{A}_s$ are finite dimensional.

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