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Infinitely many radial solutions of an elliptic system

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Dedicated to the Memory of Professor Charles Conley

ABSTRACT. — We consider a system of equations of the form $\Delta u + \nabla F(u) = 0$. In this and two subsequent papers we find conditions on $F(u)$ to guarantee that this system has infinitely many radial solutions. We also define a notion of winding number for each radial solution and prove that for each positive integer K there exists a radial solution with winding number K.

RÉSUMÉ. — L'on considère un système d'équations de la forme $u + F(u) = 0$. Dans cet article et dans deux articles à paraître, l'on trouve des conditions sur $F(u)$ qui garantissent que le système a une infinité de solutions radiales. L'on définit également un nombre d'enlacements pour chaque solution radiale et l'on démontre que pour tout entier K non nul il existe une solution radiale dont le nombre d'enlacements est K.

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A. Introduction

In recent years a number of authors have considered the question of the existence of radial solutions of the scalar equation

$$\Delta u + g(u) = 0. \tag{1A.1}$$

Here Δ is the usual Laplace operator and u is a function of $x \in \mathbb{R}^n$, $n > 1$. By a radial solution of (1A.1) we mean a nonconstant, bounded solution of the form $u(x) = U(r)$, $r = \|x\|$. We also assume that each radial solution satisfies $\lim_{r \to \infty} U\, r) = 0$. Two questions related to this problem are

(1) find conditions on $g(u)$ to guarantee that (1A.1) has a positive radial solution;

(2) find conditions on $g(u)$ to guarantee that (1A.1) has infinitely many radial solutions.

There have been two general approaches to answering these questions. The first approach has been to use variational methods. For references to this approach *see* [2], [3], [8]. The second approach for studying the existence of radial solutions of (1.A1) has been topological or phase space techniques. Using these methods, Atkinson and Peletier [1] have proven the existence of a positive solution of (1A.1) under very weak conditions on $g(u)$. Jones and Küpper [7] use topological methods to find conditions on $g(u)$ which guarantee the existence of infinitely many radial solutions of (1A.1). Moreover, they were able to characterize the radial solutions by their nodal properties; that is, under appropriate conditions on $g(u)$, for each nonnegative integer K there exists a radial solution of (1A.1) with precisely K zeroes.

Recently, Brezis and Lieb [4] have shown how the variational approach can be generalized to systems of elliptic equations. They consider systems of the form

$$-\Delta u_i(x) = f^i(u(x)), \qquad i = 1, \ldots, m, \tag{1A.2}$$

where the m functions $f^i : \mathbb{R}^m \to \mathrm{R}$ are gradients of some function $\mathrm{F} \in \mathrm{C}^1(\mathbb{R}^m)$; that is, for each i,

$$f^i(u) = \partial \mathrm{F}(u) / \partial u_i, \qquad f^i(0) = 0. \tag{1A.3}$$

They prove that there is a function $u(x)$ which minimizes the action associated with (1A.2).

In this paper we show how topological methods can be used to prove the existence of infinitely many radial solutions for a system of equations. We consider a system of the form (1A.2), (1A.3) and find conditions of F (U) which guarantee the existence of infinitely many radial solutions. We are also able to define a notion of winding number for each radial solution, and prove that for each positive integer K there exists a radial solution with winding number K.

B. Precise statement of the problem and the first main result

The system we consider is

$$\left.\begin{aligned} \Delta u_1 + f_1(u_1, u_2) = 0 \\ \Delta u_2 + f_2(u_1, u_2) = 0 \end{aligned}\right\} \tag{1B.1}$$

where u_1 and u_2 are functions of $x \in \mathbb{R}^n$, $n > 1$. We assume that there exists a function $F \in C^2(\mathbb{R}^2)$ such that for $i = 1, 2$,

$$f_i(u_1, u_2) = \frac{\partial F}{\partial u_i}(u_1, u_2) \tag{1B.2}$$

for each $(u_1, u_2) \in \mathbb{R}^2$. By a radial solution of (1B.1) we mean a nonconstant bounded solution of the form

$$(u_1(x), u_2(x)) = (U_1(r), U_2(r)), \qquad r = \|x\|.$$

We always assume that a radial solution satisfies, for $i = 1, 2$,

(a) $U_i(r) \in C^2(\mathbb{R})$;

(b) $\lim_{r \to \infty} U_i(r) = 0$.

If $(U_1(r), U_2(r))$ is a radial solution of (1B.1), and

$$V_i(r) = U_i'(r), \qquad i = 1, 2,$$

then (U_1, U_2, V_1, V_2) satisfies the first order systems of ordinary differential equations, for $r>0$,

$$\left.\begin{aligned} U_1' &= V_1 \\ V_1' &= -\frac{n-1}{r} V_1 - F_{U_1}(U_1, U_2) \\ U_2' &= V_2 \\ V_2' &= -\frac{n-1}{r} V_2 - F_{U_2}(U_1, U_2) \end{aligned}\right\} \tag{1B.3}$$

together with the boundary conditions

$$\left.\begin{aligned} &(a)\ (U_1(0), U_2(0), V_1(0), V_2(0)) = (U^1, U^2, 0, 0) \\ &(d)\ \lim_{r\to\infty} (U_1(r), U_2(r), V_1(r), V_2(r)) = (0, 0, 0, 0) \end{aligned}\right\} \tag{1B.4}$$

for some real numbers U^1 and U^2. The equalities in (1B.4) are meant to be taken componentwise. We prove the existence of radial solutions of (1B.1) by proving the existence of solutions of (1B.3) and (1B.4).

NOTATION. – For convenience we set

$$U(r) = (U_1(r), U_2(r)),$$
$$V(r) = (V_1(r), V_2(r)),$$

and

$$\mathcal{O} = (0,0).$$

ASSUMPTIONS ON F. – We wish to assume that F looks someting like what is shown in Figure 1. The precise assumptions on F are:

(F1) $F \in C^2(\mathbb{R}^2)$.

(F2) F has at least three nondegenerate local maxima. These are at $A=(A_1, A_2)=(0, 0)$, $B=(B_1, B_2)$, and $C=(C_1, C_2)$. F also has two saddles. These are at $D=(D_1, D_2)$ and $E=(E_1, E_2)$.

(F3) $F(A)<F(B)<F(C)$ and $B_1<D_1<A_1<E_1<C_1$. Moreover, there exists α_0 such that if α is any critical point of F with $\alpha \notin \{A, B, C\}$, then $F(\alpha) < F(A) - \alpha_0$. For convenience, we assume that $F(A)=0$.

(F4) There exists W such that if $K<W$, then the level set $\{U : F(U) \geqq K\}$ is convex.

(F5) If $U_1 = D_1$ or E_1, then $\partial F / \partial U_1 (U_1, U_2) = 0$ for all $U_2 \varepsilon \mathbb{R}$.

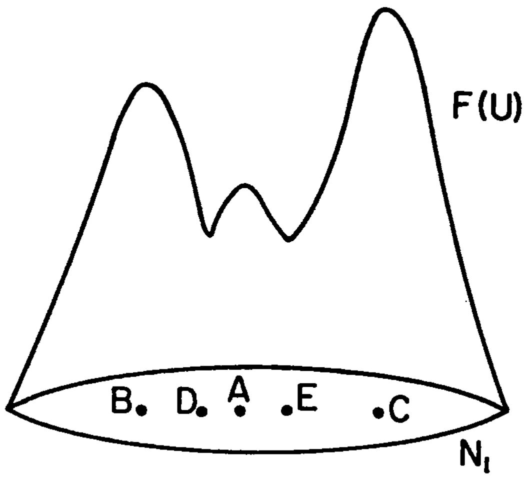

FIG. 1

(F6) Let

$$N_1 = \{U \varepsilon \mathbb{R}^2 : F(U) \geqq W\},$$
$$X_1 = \{(U_1, U_2) \in N_1 : U_1 < D_1\}, \qquad (1B.5)$$
$$X_2 = \{(U_1, U_2) \in N_1 : D_1 < U_1 < E_1\}$$

and

$$X_3 = \{(U_1, U_2) \in N_1 : E_1 < U_1\}.$$

Suppose that (U (r), V (r)) is a bounded solution of (1B.3) with $n=1$ which satisfies, for $i=1$, 2, or 3,

(*a*) $$U(r) \in X_i \quad \text{for all } r > 0$$

and

(*b*) $$F(U(r)) > F(A) - \alpha_0 \quad \text{for some } r > 0.$$

Then U (r) is identically equal to one of the critical points A, B, or C, and V $(r) = (0,0)$ for all $r > 0$.

Remarks concerning these assumptions are certainly in order. These remarks will be given shortly. We first state our first main result.

THEOREM 1. — *Assume that* f_1 *and* f_2 *satisfy* (1B.2) *where* F (U_2, U_2) *satisfies* (F1)-(F6). *Then there exists infinitely many radial solutions of* (1B.1).

We also prove another theorem which is stated explicitly in Section 1E. For that Theorem we define a notion of winding number for each radial solution. Our second Theorem states that for each positive integer K there exists a radial solution with winding number K.

The proof of these two results is split into three papers. In this paper we introduce a family of equations which depend on a small parameter ε. As $\varepsilon \to 0$, the family of equations approach (1B.1). We prove, in this paper, that for each ε there exists infinitely many solutions of the new equations. In [10] we reduce the problem of proving that for each $\varepsilon > 0$ and positive integer K there exists a solution of the new equation with winding number K to an algebraic problem. In [9] we solve the algebraic problem. Hence, [9] together with [10] prove that for each $\varepsilon > 0$ and positive integer K there does indeed exist a solution of the new equations with winding number K. In Section 6 of this paper we prove that as $\varepsilon \to 0$, some sequence of the set of solutions with winding number K for the new equations converge to a solution of (1B.1) with winding number K.

C. Remarks on the assumptions on F

Remark 1. — (F4) will be used to prove that the set of bounded solutions of (1B.3) is compact.

Remark 2. — (F6) guarantees that the set of bounded solutions of (1B.3) is not too bizarre. One may think of (1B.3) with $n=1$ as describing the motion of a ball rolling along the landscape defined by the graph of F without friction. There may exist bounded solutions of (1B.3) because the ball may roll back and forth between the mountain peaks given by F (A), F (B), and F (C). Assumption (F6) implies that these are the only bounded solutions of (1B.3) with $n=1$, besides the critical points, which lie above F (A) $-\alpha_0$ for some r. We are not interested in bounded solutions which lie below F (A) $-\alpha_0$ for all r, because if $U(r) = (U_1(r), U_2(r))$ is a radial solution, then $\lim_{r \to \infty} U(r) = A$ implies that $F(U(r)) > A - \alpha_0$ for r sufficiently large.

Remark 3. – (F5) is the most unreasonable assumption. It can be weakened slightly as follows. Let

$$\left.\begin{aligned} l_D &= \{U_1, U_2) \in N_1 : U_1 = D_1\} \\ &\text{and} \\ l_E &= \{(U_1, U_2) \in N_1 : U_1 = E_1\}. \end{aligned}\right\} \tag{1C.1}$$

Then (F5) implies that if $U \in l_D \cap N_1$ or $U \in l_E \cap N_1$, then $\nabla F(U)$ is tangent to l_D or l_E, respectively. Our result remains valid if this property holds for some line l_D and l_E through D and E, not necessarily the ones given by (1C.1). We choose l_D and l_E as in (1C.1) only for convenience. We do feel that our method of proof should carry over to more general assumptions than (F5). The main place where (F5) is used is to define the notion of winding number. In Appendix B we comment how one should be able to weaken this assumption.

D. The winding number

Recall that for the scalar equation (1A.1) it has been proven that under appropriate conditions on $g(u)$, for each nonnegative integer K there exists a radial solution with K zeroes. We wish to prove an analogous result for the system (1B.1). However, instead of counting the number of zeroes we introduce a notion of winding number which measures how many times a trajectory $(U_1(r), U_2(r), V_1(r), V_2(r))$ winds around in phase space.

For the scalar equation, the notion of winding number is simple because it makes sense to count the number of times a trajectory winds around a point (the origin, for example). For (1B.3) the phase space is four dimensional (or five dimensional if one includes r as a dependent variable as we shall do shortly), and it does not make sense to count how many times a trajectory winds around a point. Instead, we define two, two dimensional planes, P_D and P_E, and count how many times the solutions wind around these objects. P_D and P_E are defined as follows:

$$P_D = \{(U, V) : U_1 = D_1 \text{ and } V_1 = 0\}$$

and

$$P_E = \{(U, V) : U_1 = E_1 \text{ and } V_1 = 0\}.$$

Clearly, P_D and P_E are two dimensional. Perhaps the most important property of P_D and P_E is

PROPOSITION 1.D.1. — *P_D and P_E are invariant with respect to the flow defined by* (1B.3). *That is, if*

$$(U(r_0), V(r_0)) \in P_D \ (P_E)$$

for some r_0, then

$$(U(r), V(r)) \in P_D \ (P_E)$$

for all r.

Proof. — From (1B.3) and (F5) we conclude that on P_D or P_E,

$$U_1' = V_1 = 0$$

and

$$V_1' = -\frac{n-1}{r} V_1 - F_{U_1}(U_1, U_2) = 0.$$

These two equalities prove the proposition.

COROLLARY 1D.2. — *If* $(U(r), V(r))$ *is a radial solution of* (1B.1), *then* $(U(r), V(r)) \notin P_D \cup P_E$ *for all r.*

Proof. — If $(U(r_0), V(r_0)) \in P_D \cup P_E$ for some r_0, then, by Proposition 1D.1, $(U(r), V(r)) \in P_D \cup P_E$ for all r. This is impossible because $\lim_{r \to \infty} (U(r), V(r)) = (A, \mathcal{O}) \notin P_D \cup P_E$.

It now makes sense to count the number of times a radial solution winds around P_D and P_E. This is done as follows. Let

$$Q_D = \{(U, V) : U \in N_1,\ U_1 = D_1, \text{and } V_1 < 0\},$$

and

$$Q_E = \{(U, V) : U \in N_1,\ U_1 = E_1, \text{and } V_1 > 0\}.$$

DEFINITION. — Suppose that $U(r)$ is a radial solution of (1B.1). Then the winding number of U is defined by

$$h(U) = \text{card}\{r : (U(r), V(r)) \in Q_D \cup Q_E\}. \qquad (1D.1)$$

By card X we mean the cardinality of the set X.

Remark. — The notion of winding number may seem complicated because it involves trajectories in four dimensional phase space which is difficult to picture. However, one can easily compute the winding number by just considering $U(r)=(U_1(r), U_2(r))$ in the two dimensional state space. Recall that $h(U)$ equals to the number of times $U(z)$ intersects Q_D or Q_E. Now $U(z_0)\in Q_D$ if and only if $U(z_0)\in l_D$ and $V_1(z_0)<0$. Since $V_1=U'_1$ this implies that $U(z_0)\in Q_D$ if and only if $U(z_0)\in l_D$, and at $z=z_0$, $U(z)$ crosses l_D from right to left. Similarily, $U(z_0)\in Q_E$ if and only if $U(z_0)\in l_E$ and at $z=z_0$, $U(z)$ crosses l_E from left to right. Of course, this remark depends on our assumption (F5). In Appendix B, we will discuss how one should be able to weaken (F5). Under the weaker assumptions we will still be able to define two surfaces, P_D and P_E, which do not intersect the radial solutions. Hence, we will still be able to define the notion of winding number. However, we will not be able to compute the winding number by just considering the trajectories in the state space.

E. The second main result

THEOREM 2. — *Let* K *be any positive integer. Then there exists a radial solution,* $U(r)$, *of* (1B.1) *such that either* $h(U)=K$ *or* $h(U)=K+1$.

Remark 1. — The fact that we have either $h(U)=K$ or $h(U)=K+1$ may be disturbing because one would expect there to exist a radial solution with $h(U)=K$. The reason that we obtain the weaker result is that we are counting the number of times solutions wind around two objects, namely P_D and P_E.

Remark 2. — We actually prove that for each integer K there exists at least two radial solutions of (1B.1), each with winding number K or $K+1$. We explain why this is true in Appendix C.

F. Reduction to a connection problem

The boundary conditions (1B.4) state that we must find a trajectory in phase space which begins (at $r=0$) on the U-plane and ends (at $r=\infty$) at the origin. These boundary conditions are awkward to work with because we don't really know where to begin on the U-plane. It will be more convenient to transform (1B.3), (1B.4) to a problem where we look for a trajectory which connects two critical points. This is done as follows.

We first consider r as a dependent variable by introducing $z=r$ as the new independent variable. We compactify by letting

$$r=1/2\ln\frac{1+\rho}{1-\rho}.$$

Then (1B.3) becomes

$$\left.\begin{aligned} U_1' &= V_1 \\ V_1' &= -\varphi(\rho)V_1 - F_{U_1}(U) \\ U_2' &= V_2 \\ V_2' &= -\varphi(\rho)V_2 - F_{U_2}(U) \\ \rho' &= 1-\rho^2 \end{aligned}\right\} \quad (1F.1a)$$

where

$$\varphi(\rho)=(n-1)\left[\frac{1}{2}\ln\frac{1+\rho}{1-\rho}\right]^{-1} \quad \text{for} \quad 0<\rho\leqq 1.$$

A radial solution must satisfy

$$\left.\begin{aligned} &(U, V, \rho)(0)=(U_0, \mathcal{O}, 0) \\ &\text{and} \\ &(U, V, \rho)(+\infty)=(A, \mathcal{O}, 1) \end{aligned}\right\} \quad (1F.1b)$$

for some U_0. Now for $\varepsilon>0$ let

$$\varphi_\varepsilon(\rho)=\begin{cases} 1/\varepsilon & \text{for} \quad -1\leqq\rho\leqq\rho_\varepsilon \\ \varphi(\rho) & \text{for} \quad \rho_\varepsilon\leqq\rho\leqq 1 \end{cases}$$

where ρ_ε satisfies

$$1/\varepsilon=\varphi(\rho_\varepsilon).$$

Consider the system

$$\left.\begin{aligned} U_1' &= V_1 \\ V_1' &= -\varphi_\varepsilon(\rho) V_1 - F_{U_1}(U) \\ U_2' &= V_2 \\ V_2' &= -\varphi_\varepsilon(\rho) V_2 - F_{U_2}(U) \\ \rho' &= 1-\rho^2 \end{aligned}\right\} \qquad (1F.2)$$

together with the boundary conditions

$$(U, V, \rho)(-\infty) = (B, \mathcal{O}, -1)$$

and

$$(U, V, \rho)(+\infty) = (A, \mathcal{O}, +1). \qquad (1F.3)$$

It will be convenient to make the change in variables

$$\left.\begin{aligned} \tilde{U}_i(z) &= U_i(-z), \qquad i=1,2, \\ \tilde{V}_i(z) &= -V_i(-z), \qquad i=1,2, \\ \tilde{\rho}(z) &= -\rho(-z) \\ \tilde{\varphi}_\varepsilon(\rho) &= \varphi_\varepsilon(-\rho). \end{aligned}\right\} \qquad (1F.3)$$

Then (1F.2), (1F.3) become, after dropping the hats,

$$\left.\begin{aligned} U_1' &= V_1 \\ V_1' &= +\varphi_\varepsilon(\rho) V_1 - F_{U_1}(U) \\ U_2' &= V_2 \\ V_2' &= +\varphi_\varepsilon(\rho) V_2 - F_{U_2}(U) \\ \rho' &= 1-p^2 \end{aligned}\right\} \qquad (1F.5)$$

and

$$\left.\begin{aligned} (U, V, \rho)(-\infty) &= (A, \mathcal{O}, -1) \\ (U, V, \rho)(+\infty) &= (B, \mathcal{O}, +1). \end{aligned}\right\} \qquad (1F.6)$$

In this paper we prove

THEOREM 3. — *There exists* ε_0 *such that if* $0<\varepsilon<\varepsilon_0$, *then there exists infinitely many solutions of* (1F.5), (1F.6).

Of course, if we make the change in variables (1F.4), then this implies that there exists infinitely many solutions of (1F.2), (1F.3).

In two subsequent papers, [9] and [10], we prove

THEOREM 4. — *There exists* ε_0 *such that if* $0<\varepsilon<\varepsilon_0$ *and* K *is any positive integer, then there exists a solution*

$$W_\varepsilon(z)=(U_\varepsilon(z), V_\varepsilon(z), \rho_\varepsilon(z))$$

of (1F.2), (1F.3) *such that either* $h(U_\varepsilon)=K$ *or* $h(U_\varepsilon)=K+1$. In Section 6 of this paper we prove, assuming Theorem 4,

THEOREM 5. — *Let* K *be a positive integer. Then there exists a sequence* $\{\varepsilon_k\}$ *such that as* $k\to\infty$, $\{U_{\varepsilon_k}(z)\}$ *converges to a radial solution,* U (z), *of* (1B.1). *Moreover, either* $h(U)=K$ *or* $h(U)=K+1$.

G. Description of the proof

The proof of Theorem 3 is quite geometrical. The purpose of this subsection is to introduce the basic geometrical features of the proof. Each solution of (1F.5) corresponds to a trajectory in five dimensional phase space. The boundary conditions (1F.6) imply that we are looking for a trajectory which approaches $(A, \mathcal{O}, -1)$ as $z\to-\infty$ and $(B, \mathcal{O}, +1)$ as $z\to+\infty$. Hence, we are looking for a trajectory which lies in both W_A, the unstable manifold at $(A, \mathcal{O}, -1)$, and W_B^s, the stable manifold at $(B, \mathcal{O}, +1)$.

The first step in the proof of Theorem 3 is to obtain *a priori* bounds on the bounded solutions of (1F.5). This is done in Section 2. We construct a five dimensional box, N, which contains all of the bounded solutions of (1F.5). We then construct, in Section 4B, a subset $\mathscr{E}$ of the boundary of N with the property that each nontrivial trajectory in W_A can only leave N through $\mathscr{E}$. The most interesting feature of $\mathscr{E}$ is that it has four topological holes.

We then analyze the unstable manifold at $(A, \mathcal{O}, -1)$. Because the dimension of W_A is three, we show that it is possible to parametrize the nontrivial trajectories in W_A be the points in the disc

$$\mathscr{D}=\{d\,\varepsilon\,\mathbb{R}^2:\|d\|<1\}.$$

That is, to each $d\,\varepsilon\,\mathscr{D}$ there corresponds a unique trajectory, $\gamma(d, z)$, which lies in W_A. This parametrizations is defined in Section 3A. To each trajectory in W_A there is a winding number, as described in Section 1D. Hence, to each $d \varepsilon \mathcal{D}$ we can assign the integer $h(d)$ which is equal to the winding number of $\gamma(d, z)$.

Now the solutions of (1F.5), (1F.6) correspond to a certain subset of $\mathcal{D}$, which we denote by X. That is,

$$X = \{d \in \mathcal{D} : \gamma(d, z) \to (B, \mathcal{O}, +1) \text{ as } z \to \infty\}.$$

Let $Y = D \setminus X$. We shall prove that for each $y \in Y$, $\gamma(y, z)$ must leave N (*see* Proposition 2C.1). Because $\gamma(y, z)$ can only leave N through $\mathcal{E}$, we have a (continuous) mapping $\Lambda : Y \to \mathcal{E}$ defined by, $\Lambda(y)$ is equal to the place where $\gamma(y, z)$ leaves N.

Suppose that $g(s)$ is a continuous function from $I = [0,1]$ to Y such that $g(0) = g(1)$. The $(\Lambda . g)(I)$ defines a continuous, closed curve in $\mathcal{E}$. In Section 4B we define an algebraic object, $\Gamma(g)$, which describes how $(\Lambda . g)(I)$ winds around the four holes in $\mathcal{E}$. $\Gamma(g)$ will be an element of F_4, the free group on four elements. It will have the following important property:

Proposition A (*see* Proposition 4 B). — *If g_1 is homotopic to g_2 relative to* Y, *then* $\Gamma(g_1) = \Gamma(g_2)$.

The next step in the proof of Theorem 3 is to assign to each element $\Gamma \in F_4$ a positive integer $|\Gamma|$. We prove

Proposition B (*see* Proposition 5 A.1). — *Let* M *be a positive integer. There exists a continuous function $g : I \to Y$ such that $g(0) = g(1)$ and* $|\Gamma(g)| > M$.

Theorem 3 will then follow from Propositions A and B.

The key steps in the proof of Proposition B are Propositions 3B.1 and 4C.1. In Proposition 4C.1 we derive a relationship between $|\Gamma(g)|$ and the winding number of the various trajectories $\gamma(g(s), .)$, $0 \leqq s \leqq 1$. In Proposition 3B.1 we show that there must exist trajectories in W_A with arbitrarily high winding number.

2. THE ISOLATING NEIGHBORHOOD

A. Basic definitions

Until stated otherwise we fix $\varepsilon>0$. Our immediate goal is to define a set N in phase space which contains all of the bounded solutions of (1F.5). Recall the set N_1 defined in (1B.5). Let

$$\left.\begin{aligned}\bar{P}_D&=\{(U, V, \rho):(U, V)\in P_D, |\rho|\leqq 1\},\\ \bar{P}_E&=\{(U, V, \rho):(U, V)\in P_E, |\rho|\leqq 1\},\\ N_2&=\{(U, V, \rho):U\in N_1, \|V\|\leqq \bar{V}, |\rho|\leqq 1\}\end{aligned}\right\}\qquad (2A.1)$$

where $\bar{V}$ is a large number to be determined, and

$$N=N_2\setminus(\bar{P}_D\cup\bar{P}_E).$$

Remark. — N is topologically a five dimensional box with two "tubes", $\bar{P}_D$ and $\bar{P}_E$, removed. This is topologically equivalent to a two dimensional disc with two points removed.

We wish to prove

PROPOSITION 2A.1. — *If* $\bar{V}$, *appearing in* (2A.1), *is sufficiently large, then all solutions of* (1F.5), (1F.6) *lie in* N.

This result is proved in the next section. The proof is broken up into a number of lemmas.

B. Proof of Proposition 2A.1

LEMMA 2B.1. — *The projection onto* U-*space of every bounded solution of* (1F.5) *lies in* N_1. *Moreover, there cannot exist a solution of* (1F.5) *whose projection onto* U-*space is internally tangent to* ∂N_1, *the boundary of* N_1.

Proof. — The proof follows Conley [5]. Choose $K\leqq W$ and suppose that $(U(z), V(z), \rho(z))$ is a solution of (1F.5) which satisfies for some z_0,

$$F(U(z_0))=K$$

and

$$\left.\frac{\partial F}{\partial z}\right|_{z_0} = \langle \nabla F(U(z_0)), V(z_0) \rangle = 0.$$

Then

$$\left.\frac{d^2 F}{d z^2}\right|_{z_0} = d^2 F(U) + \varphi_\varepsilon(z_0) \langle \nabla F(U(z_0)), V(z_0) \rangle - \langle \nabla F(U(z_0)), \nabla F(U(z_0)) \rangle < 0 \quad (2B.1)$$

since the assumption that the level set $\{F(U)=K\}$ is convex implies that

$$d^2 F(\xi) < 0 \quad \text{if } \xi \neq 0 \text{ and } \langle \nabla F, \xi \rangle = 0.$$

This implies that there cannot be any internal tangencies on the level set $\{F(U)=K\}$ for all $K \leqq W$.

On any solution which leaves the set where $F > W$ there is a point where $F < W$ and either $dF/dz < 0$ or $dF/dz > 0$. Suppose that

$$F(U(z_0)) < W \quad \text{and} \quad \left.\frac{dF}{dz}\right|_{z_0} < 0.$$

Then (2B.1) implies that $F(U(z))$ is strictly decreasing for $z \geqq z_0$. Therefore, if the solution were bounded, it would have to go to a rest point where $F < W$. Since there aren't any such rest points, the solution must be unbounded in forward time. A similar argument shows that if

$$F(U(z_0)) < W \quad \text{and} \quad \left.\frac{dF}{dz}\right|_{z_0} > 0,$$

then the solution is unbounded in backward time.

Remark. — The proof of this last result shows that if $U(z)$ leaves N_1 in forwards or backwards time, it can never return to N_1.

LEMMA 2B.2. — *$\bar{V}$, as in* (2A.1), *can be chosen so that if*

$$U(z_0) \in N_1 \quad \textit{and} \quad \|V(z_0)\| > \bar{V},$$

then $U(z)$ *leaves* N_1 *in backwards time.*

Proof. — Suppose that $\| V(z_0) \| > \bar{V}$, where $\bar{V}$ is to be determined. Then either

$$V_1(z_0) > \frac{1}{2}\bar{V}, \qquad V_1(z_0) < -\frac{1}{2}\bar{V},$$

$$V_2(z_0) > \frac{1}{2}\bar{V}, \qquad \text{or} \qquad V_2(z_0) < -\frac{1}{2}\bar{V}. \tag{2B.2}$$

We assume that $V_1(z_0) > 1/2\,\bar{V}$, and, for convenience, $z_0 = 0$. Choose M_1 so that $\| \nabla F(U) \| < M_1$ in N_1. Then, from (1F.5),

$$V_1' = \varphi_\varepsilon(\rho) V_1 - F_{U_1}(U) \leqq \frac{1}{\varepsilon} V_1 + M_1$$

as long as $V_1 \geqq 0$. Therefore, if $V_1 \geqq 0$, then

$$[e^{-1/\varepsilon z} V_1]' \leqq e^{-1/\varepsilon z} M_1.$$

Integrate, $-z_1 \leqq z \leqq 0$, to obtain

$$V_1(-z_1) \geqq e^{-1/\varepsilon z_1} V_1(0) - \varepsilon M_1 (1 - e^{-1/\varepsilon z_1}) \geqq 1/2\, e^{-1/\varepsilon} \bar{V} - \varepsilon_1 M$$

as long as $0 \leqq z_1 \leqq 1$ and $V_1(z) \geqq 0$ for $-1 \leqq z \leqq 0$. This last statement is true if

$$\frac{1}{2} e^{-1/\varepsilon} \bar{V} - \varepsilon M_1 \geqq 0$$

or

$$\bar{V} \geqq 2\varepsilon M_1 e^{1/\varepsilon}$$

which we assume to be true. Therefore,

$$U_1'(z) = V_1(z) \geqq \frac{1}{2} e^{-1/\varepsilon} \bar{V} - \varepsilon M_1$$

for $-1 \leqq z \leqq 0$, which implies, upon integration, that

$$U_1(-1) \leqq U_1(0) - \left(\frac{1}{2} e^{-1/\varepsilon} \bar{V} - \varepsilon M_1\right). \tag{2B.3}$$

Let M_2 = diameter of N_1 and choose $\bar{V}$ so that

$$\frac{1}{2} e^{-1/\varepsilon} \bar{V} - \varepsilon M_1 > M_2. \tag{2B.4}$$

Then (2B.3) and (2B.4) imply that $U(-1)\notin N_1$. Similar arguments hold for the other cases in (2B.2).

We assume throughout the rest of the paper that $\bar{V}$ is chosen so that Lemma 2B.2 is valid. We can now present the

Completion of the proof of Proposition 2A.1. — Suppose that $(U(z), V(z), \rho(z))$ is a solution of (1F.5), (1F.6). From Lemma 2B.1 and 2B.2 we conclude that

$$(U(z),\ V(z),\ \rho(z))\in N_2 \tag{2B.5}$$

for all z. From Corollary 1D.2 it follows that

$$(U(z),\ V(z))\notin P_D\cup P_E$$

for all z. From the definitions this implies that

$$(U(z),\ V(z),\ \rho(z))\notin \bar{P}_D\cup \bar{P}_E$$

for all z. Together with (2B.5) and the definition of N, this implies the desired result.

C. The energy H and the critical point C

Consider the function

$$H(U,\ V)=\frac{1}{2}\langle V,\ V\rangle^2+F(U) \tag{2C.1}$$

where $\langle V,\ V\rangle$ is the usual inner product in $\mathbb{R}^2$. If $(U(z), V(z), \rho(z))$ is a solution of (1F.5) we write $H(z)=H(U(z), V(z))$. Note that on a solution of (1F.5),

$$H'(z)=\varphi_\varepsilon(\rho(z))\langle V(z),\ V(z)\rangle^2. \tag{2C.2}$$

Therefore,

$$H'(z)\geqq 0, \tag{2C.3}$$

and $H(z)$ is increasing on solutions. An immediate consequence of this is

PROPOSITION 2C.1. — *The only bounded solutions of* (1F.5) *which lies in the set* $\{\rho\neq -1\}$ *are the critical points and trajectories which connect the critical points.*

If $(U(z), V(z), \rho(z))$ is a solution of (1F.5), (1F.6), then

$$\lim_{z \to -\infty} H(z) = F(A) \quad \text{and} \quad \lim_{z \to +\infty} H(z) = F(B). \tag{2C.4}$$

We conclude from (2C.3) and (2C.4) that on a solution of (1F.5), (1F.6),

$$F(U(z)) \leqq H(z) < F(B) < F(C).$$

This immediately implies

LEMMA 2C.2. — *There exists* δ *such that if* $(U(z), V(z), \rho(z))$ *is a solution of* (1F.5), (1F.6), *then* $\|U(z) - C\| > \delta$ *for all* z.

This lemma demonstrates that, since we are only interested in solutions of (1F.5), (1F.6), the values of F in $C_\delta \equiv \{U : \|U - C\| < \delta\}$ do not matter. In particular, F(U) may be chosen to be arbitrarily large in C_δ. We change F(U) in C_δ so that if $(U(z), V(z), \rho(z)) \in W_A$, the unstable manifold at $(A, \mathcal{O}, -1)$, then $U(z) \neq C$ for all z. This is possible for the following reason. Suppose that $(U(z), V(z), \rho(z)) \in W_A$ and $U(z_0) = C$ for some z_0. If F(C) is very large, then we must have that $\|U'(z_1)\| = \|V(z_1)\|$ is very large for some $z_1 < z_0$. However, as Lemma 2B.2 shows, if $\|V(z_1)\|$ is too large, then $U(z)$ will leave N_1 in backwards time. The remark following Lemma 2B.1 implies that $U(z)$ can then never return to N_1, in backwards time, after leaving N_1. This contradicts the assumption that $(U(z), V(z), \rho(z)) \in W_A$. To make this all precise would be very tedious so we do not give the details.

3. THE LOCAL UNSTABLE MANIFOLD AT $\bar{A}$

A. A parametrization of the trajectories in the unstable manifold at $(A, \mathcal{O}, -1)$

Let W_A be the unstable manifold at $\bar{A} = (A, \mathcal{O}, -1)$. We assume throughout that if $(U, V, \rho) \in W_A$, then $\rho > -1$. The behavior of W_A near $\bar{A}$ is determined by linearizing (1F.5) at $\bar{A}$. If we set

$$\left.\frac{\partial^2 F}{\partial U_1^2}\right|_A = a, \qquad \left.\frac{\partial^2 F}{\partial U_1 \partial U_2}\right|_A = b, \qquad \left.\frac{\partial^2 F}{\partial U_2^2}\right|_A = c,$$

then the linear system at $\bar{A}$ is

$$\begin{bmatrix} U_1 \\ U_2 \\ V_1 \\ V_2 \\ \rho \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -a & -b & 0 & 0 & 0 \\ -c & -d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ V_1 \\ V_2 \\ \rho \end{bmatrix} \qquad (3A.1)$$

To compute the eigenvalues and eigenvectors of this system let

$$M = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

be the Hessian matrix of F at A. Since F has a local maximum at A, it follows that M has negative eigenvalues, which we denote by $-\lambda_1$ and $-\lambda_2$, with corresponding eigenvectors $\omega^1 \in \mathbb{R}^2$ and $\omega^2 \in \mathbb{R}^2$. The eigenvalues of (3A.1) are then

$$\sigma_1, \quad -\sigma_1, \quad \sigma_2, \quad -\sigma_2, \quad \sigma_3 \qquad (3A.2)$$

where, for $i=1, 2$,

$$\sigma_i = \lambda_i^{1/2} \quad \text{and} \quad \sigma_3 = 2. \qquad (3A.3)$$

Eigenvectors corresponding to $\pm\sigma_i$, $i=1, 2$, are

$$p_i^{\pm} = (\omega^i, \ \pm\sigma_i \omega^i). \qquad (3A.4)$$

An eigenvector corresponding to σ_3 is

$$p_3 = (0, 0, 0, 0, 1). \qquad (3A.5)$$

We conclude from the Stable Manifold Theorem (*see* [6]),

THEOREM 3A.1. — *Near $\bar{A}$, W_A is a C^2 injectively immersed, three dimensional manifold. Moreover, the tangent space to W_A at $\bar{A}$ is the linear subspace spanned by p_1^+, p_2^+, and p_3.*

Let

$$A_\delta = \{ (U, \rho) : \| U - A \|^2 + (\rho+1)^2 = \delta^2, \ \rho > -1 \},$$

and let $\mathrm{loc}\, W_A$ equal to the local unstable manifold at $\bar{A}$. An important consequence of Theorem 3A.1 is

PROPOSITION 3A.2. — *δ can be chosen so that* (*a*) *for each* $(U_0, \rho_0) \in A_\delta$ *there exists a unique* $V_0 \in \mathbb{R}^2$ *such that* $(U_0, V_0, \rho_0) \in \operatorname{loc} W_A$, *and*

(*b*) *if* $(U(z), V(z), \rho(z))$ *is any nontrivial trajectory in* W_A *which satisfies* $\rho(z) \geqq -1$ *for some* z, *then there exists a unique* z_0 *such that* $(U(z_0), V(z_0), \rho(z_0)) \in \operatorname{loc} W_A$ *and* $(U(z_0), \rho(z_0)) \in A_\delta$.

Proof. — This result follows immediately from Theorem 3A.1 and the fact that not each of the U or ρ components of p_1^+, p_2^+, and p_3^+ are zero.

We assume throughout that δ is chosen so that the proposition is true. The result implies that we may parametrize the nontrivial trajectories in W_A by the points on the hemisphere A_δ. Let

$$G = \{(U, V, \rho) : (U, \rho) \in A_\delta, (U, V, \rho) \in \operatorname{loc} W_A\}, \qquad (3A.6)$$

and

$$L_1 : \quad A_\delta \to G \qquad (3A.7)$$

the bijection given by the Proposition. Let

$$\mathscr{D}_1 = \{U : \|U - A\| \leqq \delta\}$$

and let

$$L_2 : \quad A_\delta \to \mathscr{D}_1$$

be the projection map. That is

$$L_2(U, \rho) = U.$$

Let $L_3 : \mathscr{D}_1 \to G$ be defined by

$$L_3 = L_1 . L_2^{-1}.$$

We have now parametrized the nontrivial trajectories in W_A by the disc $\mathscr{D}_1$. Instead of $\mathscr{D}_1$, it will be more convenient to work with the unit disc $\mathscr{D}$ defined by

$$\mathscr{D} = \{(x, y) \in \mathbb{R}^2 : \|(x, y)\| \leqq 1\}.$$

Let $L_4 : \mathscr{D} \to \mathscr{D}_1$ be the map

$$L_4(x, y) = (A_1 + \delta x, A_2 + \delta y),$$

and $L_0 : \mathscr{D} \to G$ the map

$$L_0 = L_3 . L_4. \qquad (3A.8)$$

Then to each point $d \in \mathscr{D}$ there corresponds a point $L_0(d) \in \text{loc}\, W_A$. Let

$$\gamma(d, z) = (U(d, z), V(d, z), \rho(d, z))$$

be the trajectory in W_A which passes through the point $L_0(d)$. For $(x, y) \in \mathscr{D}$, define the polar coordinates

$$x = \tau \cos\theta \quad \text{and} \quad y = \tau \sin\theta. \tag{3A.9}$$

Suppose that $d \in \mathscr{D}$. From Proposition 2G.1, either

(*a*) $\gamma(d, z)$ is a solution of (1F.6)

or

(*b*) $\gamma(d, z)$ eventually leaves N.

We write

$$\mathscr{D} = X \cup Y \tag{3A.10}$$

where

$$X = \{ d \in \mathscr{D} : \gamma(d, z) \text{ is a solution of } (1F.6) \} \tag{3A.11}$$

and

$$Y = \{ d \in \mathscr{D} : \gamma(d, z) \text{ leaves } N \}.$$

For $d \in \mathscr{D}$ we define

$$h(d) = h(U(d, z)) \tag{3A.12}$$

where $h(U(d, z))$ is defined as in (1D.1).

Note that Theorem 3 is equivalent to proving that X is an infinite set.

B. τ close to 1

The following result is crucial to the proof of the theorems.

PROPOSITION 3B.1. — *Given* $M > 0$ *there exists* $\tau_M \in (0, 1)$ *such that if*

$$\tau_M < \tau < 1, \qquad 0 \leqq \theta \leqq 2\pi, \qquad d = (\tau, \theta), \qquad \text{and} \qquad U(d, z) = B$$

for some z, then $h(d) \geqq M$. *Moreover,* τ_M *does not depend on* ε.

Because the proof of this result is extremely long and technical we save the proof for Appendix A.

4. AN ALGEBRAIC INVARIANT

A. A preliminary result

PROPOSITION 4A.1. — *There exists $\varepsilon_0>0$ such that if $0<\varepsilon<\varepsilon_0$, then the following is true. Fix $\tau\in(0, 1)$ and $q\in\partial X_2$. Then there exists $\theta=\theta(q, \tau)$ such that $U((\tau, \theta), z_0)=q$ for some z_0 and $U((\tau, \theta), z)\in X_2$ for $z<z_0$. Moreover, $\theta(q, \tau)$ can be chosen to depend continuously on q and τ.*

Remark. — X_2 was defined in (1B.5).

Proof. — Let I equal to the set of $\tau\in(0, 1)$ for which the first part of the lemma is true. That is, for each $\tau\in I$ and $q\in\partial X_2$ there exists $\theta(q, \tau)$ such that if $d=(\tau, \theta(q, \tau))$, then $U(d, z_0)=q$ for some z_0 and $U(d, z)\in X_2$ for $z<z_0$. We prove that I is open, closed, and nonempty.

We first demonstrate why I is open. Note that if $\tau\in(0, 1)$, then $U((\tau, \theta), z)$ must leave X_2. This is because of (2C.2). Moreover, if we set $Q(\tau, \theta)$ equal to the place where $U((\tau, \theta), z)$ leaves X_2, then $Q(\tau, \theta)$ is continuous. This is because assumption (F5) and Lemma 2B.1 imply that $U((\tau, \theta), z)$ cannot be tangent to ∂X_2. It now easily follows that I is open. This also shows that $\theta(q, \tau)$ can be chosen to be continuous.

It is trivial to show that I is closed, so it remains to prove that I is nonempty. We prove that $\tau\in I$ if τ and ε are sufficiently small. We will prove that there exists $\varepsilon_0>0$ and $\tau_0\in(0,1)$ such that if $0<\varepsilon<\varepsilon_0$, $0<\tau<\tau_0$, and $0\leqq\theta\leqq 2\pi$, then $\|U((\tau, \theta), z)-A\|$ is increasing as long as $U((\tau, \theta), z)\in X_2$. Because $U((\tau, \theta), z)$ must eventually leave X_2 this implies the desired result.

Recall that $A=(0,0)$. Because A is a nondegenerate local maximum of $F(U)$, there exists $\delta_0>0$ such that

$$\langle U, \nabla F(U)\rangle<0 \qquad \text{if} \quad \|U\|<\delta_0.$$

We shall now show that if $\tau\in(0, 1)$ and $\|U((\tau, \theta), z)\|<\delta_0$ for $z<z_0$, then $\|U(\tau, \theta), z)\|$ is increasing for $z<z_0$. Fix $(\tau, \theta)\in\mathscr{D}$, set $U(z)=U((\tau, \theta), z)$, and assume that $\|U(z)\|<\delta_0$ for $z<z_0$. For $z<z_0$,

$$\langle U, V\rangle'=\langle V, V\rangle+\varphi_\varepsilon(\rho)\langle U, V\rangle-\langle U, \nabla F(U)\rangle>\varphi_\varepsilon(\rho)\langle U, V\rangle.$$

Integrating this equation from $-\infty$ to z we find that

$$\langle U(z), V(z)\rangle>0 \qquad \text{for} \quad z<z_0.$$

It then follows that

$$\langle U, U \rangle' = 2 \langle U, V \rangle > 0$$

for $z < z_0$. Hence, $\| U(z) \|$ is increasing as long as $\| U(z) \| < \delta_0$.

Since we are trying to prove something about τ very small, it is natural to consider what happens when $\tau = 0$. If $\tau = 0, 0 \leqq \theta \leqq 2\pi$, and $d = (\tau, \theta)$, then, from (1F.5),

$$(U(d, z), V(d, z)) = A \quad \text{for all } z$$

and $\rho(d, z)$ is governed by the equation

$$\rho' = 1 - \rho^2.$$

In particular,

$$\lim_{z \to \infty} (U(d, z), V(d, z), \rho(d, z)) = (A, \mathcal{O}, +1) \equiv A'.$$

Note that if $\tau = 0$ then all values of θ determine the same trajectory $\gamma((0, \theta), z)$. From continuous dependence of solutions of ordinary differential equations on initial data we conclude

LEMMA 1. — *Let $\mathcal{N}$ be any neighborhood of* A′. *Then there exists τ_0 such that if $0 < \tau < \tau_0$, and $0 \leqq \theta \leqq 2\pi$, then $\gamma((\tau, \theta), z_0) \in \mathcal{N}$ for some z_0.*

This lemma demonstrates that if τ is small, then $\gamma((\tau, \theta), z)$ will pass close to the critical point A′. After passing close to A′, $\gamma((\tau, \theta), z)$ will leave a given neighborhood of A′ close to the unstable manifold of A′. This will be made precise shortly. We first discuss the behavior of $W_{A'}$, the unstable manifold at A′. We first prove

LEMMA 2. — *Given λ_0, there exists ε_0 such that if $0 < \varepsilon < \varepsilon_0$,*

$$\gamma(z) = (U(z), V(z), \rho(z)) \in W_{A'},$$

$\| U(z_0) \| = \delta_0$, *and* $\| U(z) \| < \delta_0$ *for* $z < z_0$, *then* $\| V(z_0) \| > \lambda_0$.

Remark. — Note that if $(U(z), V(z), \rho(z)) \in W_{A'}$, then $\rho(z) = 1$ for all z.

Proof. — Let

$$\lambda = \lambda_0 / \delta_0$$

and

$$S = \{ (U, V, \rho) : | V_1 | \geqq \lambda | U_1 |, | V_2 | \geqq \lambda | U_2 |, \rho = 1 \}.$$

We prove that there exists ε_0 such that if $0<\varepsilon<\varepsilon_0$, then $\gamma(z)\in S'$ for each z. This certainly implies the desired result.

We first prove that there exists ε_0 such that if $0<\varepsilon<\varepsilon_0$, then S is positively invariant; that is, if $\gamma(z_1)\in S$ for some z_1, then $\gamma(z)\in S$ for all $z>z_1$. This is proven by showing that on the boundary of S, the vector field given by the right side of (1F.5) points into S.

There are many cases to consider. Suppose, for example, that $V_1=\lambda U_1$, $U_1>0$, and $V_2\geqq\lambda U_2$. Let $n=(\lambda, -1)$ be a vector outwardly normal to $\{(U_1, U_1): V_1=\lambda U_1\}$. Then

$$n.(U_1', V_1')=V_1-\varphi_\varepsilon(\rho)V_1+F_{U_1}(U)$$
$$=(1-1/\varepsilon)V_1+F_{U_1}(U)\leqq(1-1/\varepsilon_0)\lambda U_1+F_{U_1}(U)<0$$

if ε_0 is sufficiently small. A similar proof works in the other cases.

To complete the proof of Lemma 2, we show that ε_0 can be chosen so that if $0<\varepsilon<\varepsilon_0$ and $\gamma(z)\in W_{A'}$ then there exists z_1 such that $\gamma(z)\in W_{A'}$ for $z<z_1$. This is proved by linearizing (1F.5) at A′ and showing that the eigenvectors corresponding to the positive eigenvalues point into S. These eigenvectors are given as follows. Let

$$\left.\frac{\partial^2 F}{\partial U_1^2}\right|_A=a,\qquad \left.\frac{\partial^2 F}{\partial U_1\,\partial U_2}\right|_A=b,\qquad \frac{\partial^2 F}{\partial U_2^2}=c$$

and

$$M=\begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

be the Hessian matrix of F at A. Because F(U) has a nondegenerate local maximum at A it follows that M has negative eigenvalues, which we denote by $-\lambda_1$ and $-\lambda_2$. The positive eigenvalues of (1F.5) linearized at A′ are σ_1 and σ_2 where σ_i, $i=1, 2$, is the positive root of the polynomial

$$\sigma^2-1/\varepsilon\sigma-\lambda_i=0.$$

Let w_i, $i=1, 2$, be the eigenvector of M corresponding to $-\lambda_i$. Then the eigenvector of (1F.5) corresponding to σ_i is

$$p_i=(w_i, \sigma_i w_i),\qquad i=1, 2.$$

Because each $\sigma_i\to\infty$ as $\varepsilon\to 0$, it follows that each p_i points into S if ε is sufficiently small. Because $W_{A'}$ is tangent to the linear space spanned by p_1 and p_2 this completes the proof of Lemma 2.

LEMMA 3. — *There exists* ε_0 *such that if* $0<\varepsilon<\varepsilon_0$, *and* $\gamma(z)=(U(z), V(z), \rho(z))\in W_{A'}$, *then for some* z_1, $U(z)\in X_2$ *for* $z<z_1$, $U(z_1)\in\partial X_2$, *and* $\|U(z)\|$ *is increasing for* $z<z_1$.

Proof. — Let

$$M_1=\sup\{\|\nabla F(U)\|: U\in X_2\},$$
$$M_2=\sup\{|\langle U, \nabla F(U)\rangle|: U\in X_2\},$$
$$\lambda_0=\max\{2M_1, 2M_2^{1/2}\}.$$

By Lemma 2, there exists ε_0 such that if $0<\varepsilon<\varepsilon_0$,

$$\gamma(z)=(U(z), V(z), \rho(z))\in W_{A'},$$

$\|U(z_0)\|=\delta_0$, and $\|U(z)\|<\delta_0$ for $z<z_0$, then $\|V(z_0)\|>\lambda_0$. As before, because $\langle U, \nabla F(U)\rangle<0$ for $\|U\|<\delta_0$, it follows that $\|U(z)\|$ is increasing for $z<z_0$. We assume that $\varepsilon_0<1$. Then for $z>z_0$, $\varepsilon<\varepsilon_0$,

$$\frac{d\|V\|^2}{dz}=2\varphi_\varepsilon(\rho)\|V\|^2-2\langle V, \nabla F(U)\rangle\geqq 2/\varepsilon\|V\|^2-2\|V\|M_1$$
$$\geqq 2\|V\|(1/\varepsilon_0\|V\|-M_1)\geqq 2\|V\|(1/\varepsilon_0\lambda_0-M_1)>0 \quad (4A.1)$$

as long as $\|V\|>\lambda_0$ and $U(z)\in X_2$. However, at $z=z_0$, $\|V\|>\lambda_0$ and (4A.1) implies that $\|V\|$ is then increasing. Hence, $\|V(z)\|>\lambda_0$ for $z>z_0$, $U(z)\in X_2$. Another computation shows that if $z\geqq z_0$, $\varepsilon<\varepsilon_0$, and $U(z)\in X_2$, then

$$\langle U, V\rangle'=\langle V, V\rangle+1/\varepsilon\langle U, V\rangle-\langle U, \nabla F(U)\rangle$$
$$>\lambda_0^2+1/\varepsilon_0\langle U, V\rangle-M_2>3M_2+1/\varepsilon_0\langle U, V\rangle. \quad (4A.2)$$

Since at z_0,

$$\langle U, V\rangle=\frac{1}{2}\frac{d}{dz}\|U\|^2>0,$$

we conclude, by integrating (4A.2), that $\langle U, V\rangle$ is increasing for $z>z_0$, $U(z)\in X_2$. Therefore, $\langle U, V\rangle>0$ for $z>z_0$, $U(z)\in X_2$. Finally, if $z>z_0$, $U(z)\in X_2$, then

$$\frac{d}{dz}\|U\|^2=2\langle U, V\rangle>0,$$

which completes the proof of the lemma.

We now return to the proof of Proposition 4A.1. Introduce new variables $(x_1, x_2, y_1, y_2, \rho)$ so that near A′,

$$W_{A'}=\{x_1=x_2=0,\ \rho=1\},$$
$$W_S=\{y_1=y_2=0,\ \rho<1\},$$

where W_S is the stable manifold at A′. In these new coordinates, the equations become near A′,

$$\left.\begin{aligned} x_1'&=-\gamma_1 x_1+g_1(X, Y, \rho)\\ y_1'&=\gamma_2 y_1+g_2(X, Y, \rho)\\ x_2'&=-\gamma_3 x_2+g_3(X, Y, \rho)\\ y_2'&=\gamma_4 y_2+g_4(X, Y, \rho)\\ \rho'&=1-\rho^2 \end{aligned}\right\} \tag{4A.3}$$

where $X=(x_1, x_2)$, $Y=(y_1, y_2)$,

$$g_i=O(\|(X, Y)\|^2+|1-\rho|^2), \qquad i=1, 2, 3, 4,$$

and $-\gamma_1$, γ_2, $-\gamma_3$, and γ_4 are the eigenvalues of (1F.5) at A′.

Choose δ_1 so that if $\|(X, Y)\|<\delta_1$, then in the old coordinates, $\|(U, V)\|<\delta_0$.

For $\lambda\in(0, \delta_1)$, let

$$\mathcal{N}_\lambda=\{(X, Y, \rho):\|Y\|\leqq\delta_1,\ \|X\|\leqq\lambda,\ 1-\lambda\leqq\rho\leqq 1\},$$
$$\mathcal{N}^+=\{(X, Y, \rho)\in\mathcal{N}_\lambda:\|Y\|=\delta_1\}.$$

By choosing δ_1 smaller, if necessary, and choosing λ small, we may assume, from (4A.3), that trajectories can only leave $\mathcal{N}_\lambda$ through $\mathbb{N}^+$.

Now points in $\mathcal{N}^+$ lie close to points in $W_{A'}$. It then follows from Lemma 3 and the continuous dependence of solutions on initial data.

LEMMA 4. — *Assume that* $0<\varepsilon<\varepsilon_0$ *where* ε_0 *is given in Lemma* 3. *Then* λ *can be chosen so that if* $\gamma(z)=(U(z), V(z), \rho(z))$ *is a solution of* (1F.5) *and* $\gamma(z_1)\in\mathcal{N}^+$, *then there exists* $z_2>z_1$ *such that* $U(z)\in X_2$ *for* $z\in(z_1, z_2)$, $U(z_2)\in\partial X_2$, *and* $\|U(z)\|$ *is increasing for* $z\in(z_1, z_2)$.

We are now ready to complete the proof of Proposition 4A.1. Choose λ so small that trajectories can only leave $\mathcal{N}_\lambda$ through $\mathcal{N}^+$, and that Lemma 4 holds. From Lemma 1, there exists τ_0 such that if $0<\tau<\tau_0$, $0\leqq\theta\leqq 2\pi$, and $d=(\tau, \theta)$, then $\gamma(d, z_1)\in\mathcal{N}_\lambda$ for some z_1.

Now fix $\tau \in (0, \tau_0)$, $\theta \in [0, 2\pi]$, and let $d = (\tau, \theta)$. We prove that there exists z_0 such that $U(z_0) \in \partial X_2$, $U(z) \in X_2$ for $z < z_0$, and $\| U(z) \|$ is increasing for $z < z_0$. Choose z_1 so that $\gamma(d, z_1) \in \mathcal{N}_\lambda$. We have already proven that $\| U(z) \|$ is increasing as long as $\| U(z) \| < \delta$. By assumption,

$$\mathcal{N}_\lambda \subset \{ (U, V, \rho) : \| U \| \leqq \delta_0 \}.$$

Hence, $\gamma(d, z)$ must leave $\mathcal{N}_\lambda$. Because $\gamma(d, z)$ can only leave $\mathcal{N}_\lambda$ through $\mathcal{N}^+$, there exists $z_2 > z_1$ such that $\gamma(d, z_2) \in \mathcal{N}^+$, and $\| U(z) \|$ is increasing for $z < z_2$. The result now follows from Lemma 4.

Let q_1 be a point on the top side of X_2, and q_2 a point on the bottom side of X_2. Let $\theta_1(\tau)$ and $\theta_2(\tau)$ be continuous functions such that for each $\tau \in (0, 1)$, there exists z_i, $i = 1, 2$, such that

$$U((\tau, \theta_i(\tau)), z_i) = q_i$$

and

$$U((\tau, \theta_i(\tau)), z) \in X_2 \quad \text{for} \quad z < z_i.$$

We assume, without loss of generality, that for each τ, $\theta_1(\tau) < \theta_2(\tau)$.

Let

$$\mathcal{D}_0 = \{ (\tau, \theta) : \theta_1(\tau) \leqq \theta \leqq \theta_2(\tau) \},$$
$$X_0 = X \cap \mathcal{D}_0,$$
$$Y_0 = Y \cap \mathcal{D}_0.$$

B. Γ and Γ^*

Recall that if $y \in Y$, then $\gamma(y, z)$ leaves N. Let

$$\mathscr{E} = \{ (U, V, \rho) \in \partial N : \| V \| < \bar{V} \}.$$

It follows from Lemma 2B.2 that if $y \in Y$, then $\gamma(y, z)$ must leave N through $\mathscr{E}$. Hence, we have a mapping

$$\Lambda : \quad Y \to \mathscr{E}$$

defined by $\Lambda(y)$ = place where $\gamma(y, z)$ leaves N. From Lemma 2B.1 it follows that Λ is continuous.

Let I be the unit interval and $\mathscr{G}$ the set of functions $g : I \to Y_0$ such that

(a) g is continuous;

(*b*) $g(0)\in\{(\tau, \theta):\theta=\theta_1(\tau)\}$;

(*c*) $g(1)\in\{(\tau, \theta):\theta=\theta_2(\tau)\}$.

Note that $\mathscr{E}$ is topologically equivalent to an annulus with four holes removed. For $g\in\mathscr{G}$, we shall define two algebraic objects, $\Gamma^*(g)$ and $\Gamma(g)$, which indicate how the curve $(\Lambda.g)(\mathrm{I})$ winds around the four holes. They will be elements of F_4, the set of words on the four letters α, β, γ, and δ.

We begin with the same notation. For convenience we assume that N_1 is the square:

$$\mathrm{N}_1=\{(\mathrm{U}_1, \mathrm{U}_2):|\mathrm{U}_1|\leqq \mathrm{W}, |\mathrm{U}_2|\leqq \mathrm{W}\}.$$

Let

$$\begin{gathered}\mathscr{E}_1=\{(\mathrm{U}, \mathrm{V}, \rho)\in\mathscr{E}:\mathrm{U}_1>\mathrm{E}_1\},\\ \mathscr{E}_2=\{(\mathrm{U}, \mathrm{V}, \rho)\in\mathscr{E}:\mathrm{D}_1<\mathrm{U}_1<\mathrm{E}_1, \mathrm{U}_2=\mathrm{W}\},\\ \mathscr{E}_3=\{(\mathrm{U}, \mathrm{V}, \rho)\in\mathscr{E}:\mathrm{U}_1<\mathrm{D}_1\},\\ \mathscr{E}_4=\{(\mathrm{U}, \mathrm{V}, \rho)\in\mathscr{E}:\mathrm{D}_1<\mathrm{U}_1<\mathrm{E}_1, \mathrm{U}_2=-\mathrm{W}\},\end{gathered}$$

$$\begin{gathered}l_1=l_\alpha^+=\{(\mathrm{U}, \mathrm{V}, \rho):\mathrm{U}_1=\mathrm{E}_1, \mathrm{U}_2=\mathrm{W}, 0<\mathrm{V}_1\leqq\bar{\mathrm{V}}, \mathrm{V}_2=0, |\rho|\leqq 1\},\\ l_2=l_\alpha^-=\{(\mathrm{U}, \mathrm{V}, \rho):\mathrm{U}_1=\mathrm{E}_1, \mathrm{U}_2=\mathrm{W}, -\bar{\mathrm{V}}\leqq\mathrm{V}_1<0, \mathrm{V}_2=0, |\rho|\leqq 1\},\\ l_3=l_\beta^+=\{(\mathrm{U}, \mathrm{V}, \rho):\mathrm{U}_1=\mathrm{D}_1, \mathrm{U}_2=\mathrm{W}, 0<\mathrm{V}_1\leqq\bar{\mathrm{V}}, \mathrm{V}_2=0, |\rho|\leqq 1\},\\ l_4=l_\beta^-=\{(\mathrm{U}, \mathrm{V}, \rho):\mathrm{U}_1=\mathrm{D}_1, \mathrm{U}_2=\mathrm{W}, -\bar{\mathrm{V}}\leqq\mathrm{V}_1<0, \mathrm{V}_2=0, |\rho|\leqq 1\},\\ l_5=l_\gamma^+=\{(\mathrm{U}, \mathrm{V}, \rho):\mathrm{U}_1=\mathrm{D}_1, \mathrm{U}_2=-\mathrm{W}, 0<\mathrm{V}_1\leqq\bar{\mathrm{V}}, \mathrm{V}_2=0, |\rho|\leqq 1\},\\ l_6=l_\gamma^-=\{(\mathrm{U}, \mathrm{V}, \rho):\mathrm{U}_1=\mathrm{D}_1, \mathrm{U}_2=-\mathrm{W}, -\bar{\mathrm{V}}\leqq\mathrm{V}_1<0, \mathrm{V}_2=0, |\rho|\leqq 1\},\\ l_7=l_\delta^+=\{(\mathrm{U}, \mathrm{V}, \rho):\mathrm{U}_1=\mathrm{E}_1, \mathrm{U}_2=-\mathrm{W}, 0<\mathrm{V}_1\leqq\bar{\mathrm{V}}, \mathrm{V}_2=0, |\rho|\leqq 1\},\\ l_8=l_\delta^-=\{(\mathrm{U}, \mathrm{V}, \rho):\mathrm{U}_1=\mathrm{E}_1, \mathrm{U}_2=-\mathrm{W}, -\bar{\mathrm{V}}\leqq\mathrm{V}_1<0, \mathrm{V}_2=0, |\rho|\leqq 1\}.\end{gathered}$$

Assume that $g\in\mathscr{G}$. Choose $\eta_k\in[0, 1]$, $k=1, 2, \ldots, \mathrm{K}$, such that

$$\left.\begin{array}{l}(a)\ \eta_1=0,\ \eta_\mathrm{K}=1;\\ (b)\ \eta_k<\eta_{k+1}\text{ for all }k;\\ (c)\ (\Lambda.g)(\eta_k)\in\mathrm{U}_{l=1}^4\,\mathrm{E}_i\text{ for all }k;\\ (d)\ (\Lambda.g)(\eta_k, \eta_{k+1})\text{ intersects at most one of the line segments}\\ l_i,\ i=1-8,\text{ for all }k.\end{array}\right\}\quad(4\mathrm{B}.1)$$

We refer to $\eta^*=\{\eta_1, \ldots, \eta_k\}$ as a g-partition. It is not hard to prove that a g-partition does exist.

We now define $\Gamma^*(g, \eta^*)$. First we define

$$\lambda_k = \lambda(\eta_k) \in \{\alpha, \beta, \gamma, \delta\}$$

and

$$e_k = e(\eta_k) \in \{-1, 0, 1\}.$$

These quantities are determined by the following Table. In the Table, we let, for $s \in [0, 1]$,

$$\Phi(s) \equiv (\Lambda . g)(s).$$

$\Phi(\eta_k) \in$	$\Phi(\eta_{k+1}) \in$	$\Phi(\eta)$ crosses for $\eta \in (\eta_k, \eta_{k+1})$	$\lambda(\eta_{k+1})$	$e(\eta_{k+1})$
$\mathscr{E}_2$	$\mathscr{E}_1$	l_α^+	α	1
$\mathscr{E}_1$	$\mathscr{E}_2$	l_α^+	α	-1
$\mathscr{E}_2$	$\mathscr{E}_3$	l_β^-	β	1
$\mathscr{E}_3$	$\mathscr{E}_2$	l_β^-	β	-1
$\mathscr{E}_3$	$\mathscr{E}_4$	l_γ^-	γ	-1
$\mathscr{E}_4$	$\mathscr{E}_3$	l_γ^-	γ	1
$\mathscr{E}_4$	$\mathscr{E}_1$	l_δ^+	δ	1
$\mathscr{E}_1$	$\mathscr{E}_4$	l_δ^+	δ	-1

For each case not shown in the chart we let $e_k = e(\eta_k) = 0$. For this case we do not define $\lambda_k = \lambda(\eta_k)$ because, as we shall see, since $e_k = 0$ the choice of λ_k doesn't matter. Then define

$$\Gamma^*(g, \eta^*) = \prod_{i=1}^{K} \lambda_i^{e_i} = \lambda_1^{e_1} \lambda_2^{e_2} \ldots \lambda_K^{e_K}.$$

Here is an example.

In Figure 2, $\xi_k = \Phi(\eta_k)$. For this example

$$\Gamma^*(g, \eta^*) = \alpha\alpha^{-1}\alpha\beta\gamma^{-1}\delta^{-1}.$$

Note that there may be cancellations in $\Gamma^*(g, \eta^*)$. By $\Gamma(g)$ we mean the element of F_4 obtained by making all cancellations in $\Gamma^*(g, \eta^*)$. We shall see later that $\Gamma(g)$ does not depend on η^*. In the above example

$$\Gamma(g) = \alpha\beta\gamma^{-1}\delta^{-1}.$$

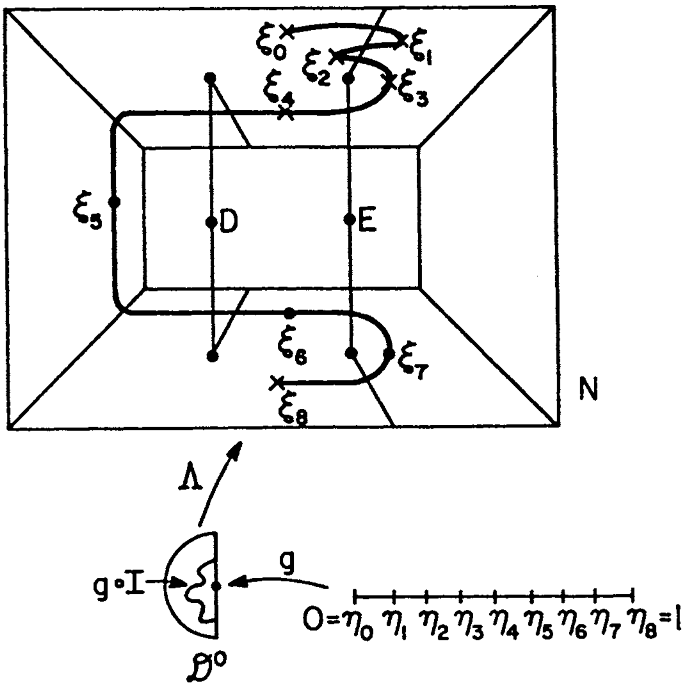

FIG. 2

By $\Gamma^*(g)$ we mean the subset of F_4 consisting of all elements which yield, after all cancellations, $\Gamma(g)$. Note that, for each η^*,

$$\Gamma^*(g, \eta^*) \in \Gamma^*(g) \qquad \text{and} \qquad \Gamma(g) \in \Gamma^*(g).$$

For the above example, $\Gamma^*(g)$ includes the elements

$$\alpha\beta\delta\delta^{-1}\gamma^{-1}\delta^{-1} \qquad \text{and} \qquad \alpha\beta\beta^{-1}\gamma\delta\delta^{-1}\gamma^{-1}\beta\gamma^{-1}\delta^{-1}.$$

The following two Propositions will be important for the rest of the paper. Their proofs are tedious but straightforward. We do not give the details.

PROPOSITION 4B.1. — *$\Gamma(g)$ does not depend on the choice of* η^*.

Before stating the next proposition we need the following definition.

DEFINITION. — Suppose that $g_1, g_2 \in \mathcal{G}$. We say that g_1 *is homotopic to* g_2 *relative to* Y_0, and write $g_1 \sim g_2$, if there exists a continuous map

$$\Phi: \quad I \times I \to Y_0$$

such that

(a) $\Phi(s, 0)=g_1(s)$ for $s\in \mathrm{I}$

(b) $\Phi(s, 1)=g_2(s)$ for $s\in I$

(c) $\Phi(., t)\in \mathscr{D}$ for each $t\in \mathrm{I}$.

PROPOSITION 4B.2. — *If $g_1 \sim g_2$, then $\Gamma(g_1)=\Gamma(g_2)$.*

C. A formula for the Winding number

Recall that for each $d=(\tau, \theta)\in\mathscr{D}_0$ there corresponds a trajectory $\gamma(d, z)\in \mathrm{W_A}$. Moreover, there is a winding number, $h(d)$, which was defined in (3A.12). Suppose that $g(s)\in\mathscr{D}$ and $s_0\in \mathrm{I}$. In this section we derive a formula for

$$h_1(s_0)=h(g(s_0))$$

in terms of $\Gamma^*(g, \eta^*)$ for some g-partition η^*. First we need some notation.

Suppose that $\Gamma\in \mathrm{F}_4$ is given by

$$\Gamma=\lambda_1^{e_1}\dots\lambda_J^{e_J}$$

where each $\lambda_i\in\{\alpha, \beta, \gamma, \delta\}$ and $e_i\in\{-1,1\}$. Let

$$\omega(\Gamma)=\sum_{i=1}^{J} e_i.$$

Let $g\in\mathscr{G}$ and η^* be a g-partition. Define the map

$$\Lambda_0(g, \eta^*): \mathrm{I}\to \mathrm{F}_4$$

as follows. Suppose that $\eta^*=\{\eta_1, \dots, \eta_k\}$ and $\eta^k\leqq s<\eta_{k+1}$. Then define

$$\Lambda_0(g, \eta^*)(s)=\lambda_1^{e_1}\dots\lambda_k^{e_k},$$

where the λ_i and e_i were defined in Table. Finally, define

$$\Lambda_1(g, \eta^*): \mathrm{I}\to \mathrm{Z}^+,$$

where Z^+ is the set of nonnegative integers, by

$$\Lambda_1(g, \eta^*)(s)=[\omega . \Lambda_0(g, \eta^*)](s).$$

We can now state the main result of this section.

PROPOSITION 4C.1. — *Assume that* $g \in \mathcal{G}$ *and* $\eta^* = \{\eta_1, \ldots, \eta_k\}$ *is a* g*-partition. If* $\eta_k \leqq s_0 < \eta_{k+1}$, *then either*

$$h_1(s_0) = \Lambda_1(g, \eta^*)(\eta_k) \qquad \text{or} \qquad h_1(s_0) = \Lambda_1(g, \eta^*)(\eta_{k+1}).$$

Proof. — The proof is by induction on k. First assume that $k=1$. That is,

$$0 = \eta_1 \leqq s_0 < \eta_2. \tag{4C.1}$$

By assumption, $\eta_1 = 0$ and $(\Lambda . g)(\eta_1) \in \mathcal{E}_2$ where $\mathcal{E}_2$ was defined in the previous section. There are a number of cases to consider. Suppose, for example, that

$$(\Lambda . g)(s_0) \in \mathcal{E}_2 \qquad \text{and} \qquad (\Lambda . g)(\eta_2) \in \mathcal{E}_2. \tag{4C.2}$$

Because $(\Lambda . g)(\eta_2) \in \mathcal{E}_2$, it follows from Table, that $e_1 = 0$ and, therefore,

$$\Lambda_1(g, \eta^*)(\eta_1) = \Lambda_1(g, \eta^*)(\eta_2) = 0.$$

Hence, we need to prove that $h_1(s_0) = 0$. Suppose that $h_1(s_0) > 0$.

Let Q_D and Q_E be as in Section 1, and

$$\hat{Q}_D = \{(U, V, \rho) : (U, V) \in Q_D, \ |\rho| \leqq 1\},$$
$$\hat{Q}_E = \{(U, V, \rho) : (U, V) \in Q_E, \ |\rho| \leqq 1\}.$$

If $h_1(s_0) > 0$, then $\gamma(g(s_0), z)$ must intersect $\hat{Q}_D \cup \hat{Q}_E$ at least once, suppose, for example, that $\gamma(g(s_0), z)$ intersects $\hat{Q}_D$. Because $(\Lambda . g)(s_0) \in \mathcal{E}_2$, it follows that $\gamma(g(s_0), z)$ must also intersect

$$Q_D^- = \{(U, V, \rho) : U_1 = D_1, \ V_1 < 0, \ U \in N_1, \ |\rho| \leqq 1\}.$$

Let

$$s_1 = \inf\{s : \gamma(g(s), z) \text{ intersects } \hat{Q}_D\},$$
$$s_2 = \inf\{s : \gamma(g(s), z) \text{ intersects } Q_D^-\}.$$

Clearly, $0 \leqq s_1 < s_2 < \eta_2$. Moreover,

$$(\Lambda . g)(s_1) \in l_\beta^+ \qquad \text{and} \qquad (\Lambda . g)(s_2) \in l_\beta^-.$$

This, however, contradicts (4B.1*d*) in the definition of a g-partition.

There are other cases to consider besides (4C.2). We only consider one more. The rest are similar. Suppose that

$$(\Lambda . g)(s_0) \in \mathscr{E}_1 \quad \text{and} \quad (\Lambda . g)(\eta_2) \in \mathscr{E}_1.$$

Then, using Table, we find that $\Lambda_1(g, \eta^*)(\eta_1)=0$ and $\Lambda_1(g, \eta^*)(\eta_2)=1$. We claim that $h_1(s_0)=1$.

Because $(\Lambda . g)(s_0) \in \mathscr{E}_1$, it is clear that $\gamma(g(s_0), z)$ must intersect $\hat{Q}_D$ at least once. Hence, $h_1(s_0) \geqq 1$. Suppose that $h_1(s_0) > 1$. Then $\gamma(g(s_0), z)$ must intersect $\hat{Q}_D \cup \hat{Q}_E$ at least twice. Because $\gamma(g(0), z)$ does not intersect $\hat{Q}_D \cup \hat{Q}_E$ at all, this implies that there exists s_1, s_2 with

$$\eta_1 < s_1 < s_2 < s_0 < \eta_2$$

such that $(\Lambda . g)(s_1) \in l_i$ and $(\Lambda . g)(s_2) \in l_j$ with $i \neq j$. This, again, contradicts (4B.1*d*) in the definition of a g-partition.

To complete the proof we must prove the induction step. That is, we assume that the proposition is true if $\eta_j \leqq s_0 \leqq \eta_{j+1}$ for $j<k$, and then prove the result for $j=k$. The proof of this is very similar to the proof just given so we do not include the details.

5. COMPLETION OF THE PROOF OF THEOREM 3

A. Preliminaries

Suppose that $g \in \mathscr{D}_1$ and

$$\Gamma(g) = \lambda_1^{e_1} \lambda_2^{e_2} \dots \lambda_K^{e_K}.$$

Let

$$|\Gamma(g)| = \sup_{1 \leqq J \leqq K} \sum_{i=1}^{J} e_i.$$

In the next subsection we prove

PROPOSITION 5A.1. — *Let* M *be a positive integer and* τ_M *be as in Proposition* 3B.1. *Suppose that* $g \in \mathscr{G}$ *is given by* $g(s) = (\tau(s), \theta(s))$ *and* $\tau(s) > \tau_M$ *for each s. Then* $|\Gamma(g)| > M$.

We now show how this proposition is used to prove Theorem 3.

Suppose that Theorem 3 is not true; that is X_0 is a finite set. Suppose that $X_0=\{d_1, \ldots, d_L\}$ where, for each j, $d_j=(\tau_j, \theta_j)$. Let

$$\tau_0=\frac{1}{2}(1+\sup_{1\leq j\leq L}\tau_j).$$

Then $\tau_0<1$. Choose $g_0(s)=(\tau^0(s), \theta^0(s))\in\mathscr{D}$ so that $\tau^0=\tau_0$ for each s. From Proposition 5A.1 there exists $\tau^*>\tau_0$ such that if $g_1(s)=(\tau^1(s), \theta^1(s))\in\mathscr{G}$ satisfies $g_1(s)>\tau^*$ for each s, then $|\Gamma(g_1)|>|\Gamma(g_0)|+1$. But $g_0(s)$ and $g_1(s)$ are clearly homotopic relative to Y_0. From Proposition 4B.2 it follows that $\Gamma(g_0)=\Gamma(g_1)$. Hence

$$|\Gamma(g_0)|=|\Gamma(g_1)|>|\Gamma(g_0)|+1.$$

This is clearly impossible, thus proving the Theorem.

B. Proof of Proposition 5A.1

Let $\gamma(g(s), z)=(U(g(s), z), V(g(s), z), \rho(g(s), z))$. Now $U(g(0), z)$ leaves X_2 through its top side and $U(g(1), z)$ leaves X_2 through its bottom side. Moreover, by the remarks in Section 2C, $U(g(s), z)\neq C$ for all s and z. Since the curves $U(g(s), .)$ vary continuously with s, this implies that there exists s_0 and z_0 such that $U(g(s_0), z_0)=B$. Because $\tau(s_0)>\tau_M$, Proposition 3B.1 implies that $h_1(s_0)=h(g(s_0))>M$. By Proposition 4C.1, if η^* is any g-partition, and $\eta_j\leqq s_0<\eta_{j+1}$, then either

$$h_1(s_0)=\Lambda_1(g, \eta^*(\eta_j) \qquad \text{or} \qquad h_1(s_0)=\Lambda_1(g, \eta^*)(\eta_{j+1}).$$

From the definitions, this implies that

$$|\Gamma^*(g, \eta^*)|>M.$$

We must show that this implies that $|\Gamma(g)|>M$.

Note that $\Gamma(g)$ is obtained from $\Gamma^*(g, \eta^*)$ by a finite number of cancellations. We show that after each cancellation the index, $|.|$, is still greater than M. More precisely, suppose that $\Gamma^*(g, \eta^*)$ is of the form

$$\Gamma^*(g, \eta^*)=\Gamma_A\lambda_k^{e_k}\lambda_{k+1}^{e_{k+1}}\Gamma_B \tag{5B.1}$$

where $\{\Gamma_A, \Gamma_B\} \subset F_4$, $\lambda_k = \lambda_{k+1} \in \{\alpha, \beta, \gamma, \delta\}$, and $e_k = -e_{k+1} \in \{-1, 1\}$. Let

$$\Gamma' = \Gamma_A \Gamma_B.$$

We prove that $|\Gamma'| > M$. Since $\Gamma(g)$ is obtained from $\Gamma^*(g, \eta^*)$ by a finite number of such cancellations, this will prove the desired result.

If $e_k = -1$, then $|\Gamma'| = |\Gamma^*(g, \eta^*)| > M$. Hence, we assume that $e_k = +1$.

There are four cases to consider. These are, either $\lambda_k = \alpha$, β, γ, or δ. First assume that $\lambda_k = \alpha$. We then consider two subcases. These are

$$\left.\begin{array}{lll} (a) & h(g(s_0)) > M & \text{for some } s_0 < \eta_{k-1}, \\ (b) & h(g(s)) \leqq M & \text{for all } s < \eta_{k-1}. \end{array}\right\} \qquad (5B.2)$$

Suppose (5B.2*a*). Choose $j < k-1$ so that $\eta_j \leqq s_0 < \eta_{j+1}$. Then, from Proposition 4C.1, either

$$h(g(\eta_0)) = \sum_{i=1}^{j} e_i > M, \qquad \text{or} \qquad h(g(\eta_0)) = \sum_{i=1}^{j+1} e_i > M.$$

In either case, it follows that $|\Gamma'| > M$.

Now suppose that (5B.2*b*) holds. We first show that (5B.2*b*) implies that $U(g(s), z) \neq B$ for all z and $s < \eta_{k+1}$. Suppose, for the sake of a contradiction, that $U(g(s_0), z_0) = B$ for some z_0 and $s_0 < \eta_{k+1}$. Let

$$\psi(s_0) = \sup\{z < z_0 : U(g(s_0), z) \in l_D\}.$$

Because $\lim_{z \to -\infty} U(g(s_0), z) = A$ and $U(g(s_0), z_0) = B$, it is clear that $\psi(s_0$ is well defined. For s close to s_0 there exists a continuous function $\psi(s)$ such that $U(g(s), \psi(s)) \in l_D$. Let J be the maximal subset of [0,1] such that $\psi(s)$ is a well defined, continuous function. Let

$$\zeta = \inf\{s : s \in J\}.$$

Because $U(g(s_0), z_0) = B$ and $\tau(s_0) > \tau_M$ we conclude that $h(g(s_0)) > M$. It follows that $h(g(s)) > M$ for all $s \in J$. In particular, $h(g(\zeta)) > M$. If $\zeta < \eta_{k-1}$, then (5B.2*b*) is contradicted. So assume that $\eta_{k-1} \leqq \zeta < \eta_{k+1}$.

Clearly $U(g(\zeta), z)$ leaves N_1 through $l_D \cap \partial N_1$. Hence, $(\Lambda.g)(\zeta) \in l_4 \cup l_6$ where l_4 and l_6 were defined in Section 4B. However, from (4B.1*d*), $(\Lambda.g)(s)$ can cross at most one of the lines $l_1 - l_8$ *for* $s \in (\eta_k, \eta_{k+1})$. Because $\lambda_k = \lambda_{k+1} = \alpha$ we find, from Table, that $(\Lambda.g)(s)$ crosses l_1 for some $s \in [\eta_{k-1}, \eta_k)$ and another $s \in [\eta_k, \eta_{k+1})$. This gives the desired contradiction. We have now shown that if $\lambda_k = \alpha$ and (5B.2b) holds, then $U(g(s), z) \neq B$ for all z and $s < \eta_{k+1}$.

However, $U(g(s_0), z_0) = B$ for some s_0 and some z_0. If (5B.2b) holds, then $\eta_0 \geqq \eta_{k+1}$. Suppose that $\eta_j \leqq \eta_0 < \eta_{j+1}$ for some $j \geqq k+1$. Because $U(g(s_0), z_0) = B$, it follows that $h(g(s_0)) > M$. From Proposition 4C.1 we conclude that either $\Lambda_1(g, \eta^*)(\eta_j) > M$ or $\Lambda_1(g, \eta^*)(\eta_{j+1}) > M$. Because $j \geqq k+1$, this implies that $|\Gamma'| > M$.

It remains to consider the cases $\lambda_k = \beta$, γ, and δ. The proofs of each of these cases is similar to the one just given so we do not give the details.

6. PROOF OF THEOREM 5

We now prove Theorem 5. We assume that there exists $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$ and K is any positive integer, then there exists a solution

$$W_\varepsilon(z) = (U_\varepsilon(z), V_\varepsilon(z), \rho_\varepsilon(z))$$

of (1F.5), (1F.6) such that either $h(U_\varepsilon) = K$ or $h(U_\varepsilon) = K+1$. Fixing K, we prove that there exists a sequence $\{\varepsilon_k\}$ such that, as $k \to \infty$, $\{U_{\varepsilon_k}(-z)\}$ converges to a solution, $U(z)$, of (1B.1). Moreover, either $h(U) = K$ or $h(U) = K+1$.

The first step is to obtain an *a priori* estimate on $W_\varepsilon(z)$, independent of ε. We do know that for each ε, $W_\varepsilon(z) \in N$, however $\bar{V}$ appearing in the definition of N, depends on ε. Note that for each ε, $U_\varepsilon(z) \in N_1$, which does not depend on ε, and $|\rho_\varepsilon(z)| \leqq 1$. Hence, we need to obtain an *a priori* bound on $V_\varepsilon(Z)$.

LEMMA 6.1. – *There exists a constant* V^* *such that* $\| V_\varepsilon(z) \| < V^*$ *for each* ε *and* z.

Proof: Recall that $U_\varepsilon(z) \in N_1$ for each ε and z. We prove that there exists a constant V^* such that if $\gamma(z) = (U(z), V(z), \rho(z))$ is a solution of (1F.5) for any $\varepsilon > 0$, $U(z_0) \in N_1$, and $\| V(z_0) \| > V^*$, then $U(z) \notin N_1$ for some $z > z_0$.

Suppose that $\| V(z_0) \| > V^*$ where V^* is to be determined. Then either

$$\left.\begin{array}{lll} V_1(z_0) > 1/2\, V^*, & & V_1(z_0) < -1/2\, V^* \\ V_2(z_0) > 1/2\, V^*, & \text{or} & V_2(z_0) < -1/2\, V^*. \end{array}\right\} \tag{6.1}$$

We assume that $V_1(z_0) > 1/2\, V^*$, and, for convenience, $z_0 = 0$. Choose M_1 so that $\|\nabla F(U)\| < M_1$ in N_1. Then, from (1F.5),

$$V_1' = \varphi_\varepsilon(\rho)\, V_1 - F_{U_1}(U) \geqq -M_1$$

as long as $V_1(z) > 0$. Therefore,

$$V_1(z) \geqq V^* - M_1 z \geqq 1/2\, V^*$$

as long as $0 \leqq z \leqq 1$, and $V^* > 2 M_1$, which we assume to be true. For $0 \leqq z \leqq 1$

$$U_1'(z) = V_1(z) \geqq 1/2\, V^*.$$

Hence,

$$U_1(1) \geqq U_1(0) + 1/2\, V^*.$$

If we let

$$V^* > 2\, \text{diameter}(N_1),$$

the result follows. A similar argument holds for the other cases in (6.1).

It is now clear that some subsequence $\{W^k(z)\}$, of $\{W_\varepsilon(z)\}$ converges to a function $W(z) = (U(z), V(z), \rho(z))$. Recalling (1F.4), we let $(\hat{U}(z), \hat{V}(z)) = (U(-z), -V(-z))$. We shall then prove

$$\left.\begin{array}{ll} (a) & \hat{V}(z) = 0 \quad \text{for} \quad z < 0, \\ (b) & (\hat{U}(z), \hat{V}'(z)) \ \text{ is a solution of (1B.3) for } z > 0, \\ (c) & (\hat{U}(z), \hat{V}(z)) \ \text{ satisfies (1B.4)}, \\ (d) & h(\hat{U}) = K \quad \text{or} \quad K+1. \end{array}\right\} \quad (6.2)$$

To prove (6.2*a*) we show

LEMMA 6.2. — *Given* δ, *there exists* $\varepsilon_\delta > 0$, $\rho_\delta < 0$, *such that if* $0 < \varepsilon < \varepsilon_\delta$, *then*

$$\|V_\varepsilon(z)\| < \delta \qquad \textit{if} \quad \rho_\delta < \rho_\varepsilon(z) < 1.$$

Proof. — Note that $\varphi_\varepsilon(\rho) \to \infty$ as $\rho \to 0$ and $\varepsilon \to 0$. Assume that $\|\nabla F(U)\| < M_1$ for $U \in X_2$, and choose $\varepsilon_\delta > 0$, $\rho_\delta < 0$ so that

$$\varphi_\varepsilon(\rho) > 2 M_1/\delta \qquad \text{for} \quad 0 < \varepsilon < \varepsilon_\delta, \quad \rho_\delta < \rho \leqq 1.$$

Assume that $\|V_\varepsilon(z_0)\| > \delta$ for some $0<\varepsilon<\varepsilon_\delta$, $\rho(z_0)>\rho_\delta$. We assume that $V_{1\varepsilon}(z)>\delta/2$. The other cases are similar. Then, for $z>z_0$,

$$V'_{1\varepsilon} = \varphi_\varepsilon(\rho)\, V_{1\varepsilon} - F_{U_1}(U) > \frac{2\,M_1}{\delta} V_{1\varepsilon} - M_1 > 0 \tag{6.3}$$

as long as $V_{1\varepsilon}(z)>\delta/2$. However, we are assuming that $V_{1\varepsilon}(z_0)>\delta/2$. Hence, (6.3) implies that $V'_{1\varepsilon}(z)>0$ for all $z>z_0$. This, in turn, implies that $V_{1\varepsilon}(z)>\delta/2$ for all $z>z_0$. This is impossible because $\lim_{z\to\infty} V_{1\varepsilon}(z)=0$.

An immediate consequence of this result is

COROLLARY 6.3: $\lim_{z\to 0^-} V(z)=(0, 0)$.

Because $\hat{V}(z) = -V(-z)$, we have

COROLLARY 6.4 : $\lim_{z\to 0^+} \hat{V}(z)=(0, 0)$.

We now prove (6.2*b*). That is,

LEMMA 6.5. – *($\hat{U}(z)$, $\hat{V}(z)$) is a solution* (1B.3) *for* $z>0$.

Proof. – Fix $z_0>0$. Then $\varphi_\varepsilon(\hat{\rho}(z)) = -(n-1)/z$ is uniformly bounded for $z>z_0$. Hence,

$$V'_\varepsilon = -(n-1)/z\, V_\varepsilon - \nabla F(U_\varepsilon)$$

is uniformly bounded in ε for $z>z_0$. Passing to the limit, $\varepsilon\to 0$, in (1F.2) for $z>z_0$ gives the desired result.

LEMMA 6.6. – *Either* $h(\hat{U})=K$ *or* $K+1$.

Proof. – There are two things that can go wrong. These are

(*a*) Some oscillations in $U_\varepsilon(z)$ run off to $z=\infty$ as $\varepsilon\to 0$,

(*b*) Some oscillations become smaller and smaller and then disappear at some finite value of z as $\varepsilon\to 0$. (6.4)

Now, (6.4*b*) is impossible because if it did happen, then there must exist a z_0 such that

$$\hat{U}_1(z_0)=D_1 \quad \text{or} \quad E_1 \quad \text{and} \quad \hat{V}_1(z_0)=0.$$

However, because $(\hat{U}, \hat{V})$ is a solution of (1B.3), this together with assumption (F5) implies that $U_1(z)=D_1$ or E_1 for all z. This is impossible, because we will prove in the next lemma that $\lim_{z\to\infty} \hat{U}(z)=A$.

Now suppose (6.4*a*) were true. This implies that some oscillations in $(U_\varepsilon(z), V_\varepsilon(z), \rho_\varepsilon(z))$ approach the surface $\rho = -1$ as $\varepsilon \to 0$. Recall that each trajectory $W_\varepsilon(z)$ is parametrized by a point $(\tau_\varepsilon, \theta_\varepsilon)$ in $\mathscr{D}$. Because $h(U_\varepsilon) = K$ or $K+1$, Proposition 3B.1 implies that there exists $\tau_K \in (0, 1)$, which does not depend on ε, such that $\tau_\varepsilon < \tau_K$. This certainly implies the desired result, because if some of the oscillations did accumulate on the surface $\rho = -1$ as $\varepsilon \to 0$, then we must have that $\tau_\varepsilon \to 1$ as $\varepsilon \to 0$.

We now verify (6.2C), thus completing the proof of Theorem 5.

LEMMA 6.7. — $(\hat{U}(z), \hat{V}(z))$ *satisfies* (1B.4).

Proof. — (1B.4*a*) follows from Corollary 6.4. In order to prove (1B.4*b*), we show that $\lim_{z \to -\infty} (U(z), V(z)) = (\mathcal{O}, \mathcal{O})$. This however follows from the fact, which was proved in the preceeding Lemma, that there exists a constant $\tau_K \in (0, 1)$. which does nor depend on ε, such that $\tau_\varepsilon < \tau_K$.

APPENDIX A.
PROOF OF PROPOSITION 3B.1

Idea of the Proof. — Suppose that

$$\gamma(z) = (U(z), V(z), \rho(z)) \in W_A$$

satisfies $U(z_0) = B$ for some z_0. Then

$$\lim_{z \to -\infty} H(z) = F(A) < F(B) \leqq H(z_0).$$

Therefore, while $\gamma(z) \in N$, $H(z)$ must increase from $F(A)$ to $F(B)$. Recall from (2C.2) that

$$H'(z) = \varphi_\varepsilon(\rho(z)) \| V(z) \|^2.$$

Since $\| V(z) \| \leqq \bar{V}$ as long as $U(z) \in N$ we have that

$$H'(z) \leqq \varphi_\varepsilon(\rho(z)) \bar{V}^2. \tag{A.1}$$

If $\tau = 1$, $0 \leqq \theta \leqq 2\pi$, and $d = (\tau, \theta)$, then $\rho(d, z) = -1$ for all z. Hence, $\varphi_\varepsilon(\rho(z)) = 0$ for all z. If τ is close to one, then, in some sense, $\gamma(d, z)$ will remain close to the hyperplane $\{\rho = -1\}$ for a long time, and $\varphi_\varepsilon(\rho(z))$ will remain small. From (A.1) this implies that $H(z)$ will increase very slowly. Since $H(z)$ must increase from $F(A)$ to $F(B)$ this implies that $\gamma(z)$ must spend a long time in N. We then use hypothesis (F6) to conclude that $U(d, z)$ must move back and forth between the mountain peaks defined by $F(A)$, $F(B)$ and $F(C)$ a large number of times. Together with the remark in Section 1.D this implies the desired result.

Remark. — Many of the ideas in the proof of this result may be found in the paper of Jones and Küpper [7].

Proof of Proposition 3B.1. — Introduce new variables $(x_1, x_2, y_1, y_2, \rho)$ so that near $\bar{A}$,

$$W_A = \{ x_1 = x_2 = 0, \ \rho > -1 \}$$

and

$$W^s = \{ y_1 = y_2 = 0, \ \rho = -1 \},$$

where W^s is the stable manifold at $\bar{A}$. In these new coordinates, the equations become, near $\bar{A}$,

$$\begin{aligned} x_1' &= -\sigma_1 x_1 + g_1(X, Y, \rho) \\ y_1' &= +\sigma_1 y_1 + g_2(X, Y, \rho) \\ x_2' &= -\sigma_2 x_2 + g_3(X, Y, \rho) \\ y_2' &= +\sigma_2 y_2 + g_4(X, Y, \rho) \\ \rho' &= 1 - \rho^2 \end{aligned}$$

where $X = (x_1, x_2)$, $Y = (y_1, y_2)$, σ_i is defined in (3A.3), and

$$g_i = O(\|(X, Y)\|^2 + |\rho + 1|^2), \qquad i = 1, 2, 3, 4.$$

In what follows we assume that $\gamma(z) = (U(z), V(z), \rho(z))$ is a solution of (1F.5). In the new variables we use the same notation; that is,

$$\gamma(z) = (X(z), Y(z), \rho(z))$$

and

$$H(z) = H(X, Y, \rho) = H(U(z), V(z)).$$

Let

$$\sigma_0 = \min\{ \sigma_1, \sigma_2 \} \qquad \text{and} \qquad h(\rho) = (1 + \rho)^{\sigma_0/8}. \qquad \text{(A.2)}$$

For each $\delta>0$, $0<r<\delta$, $a>0$, let

$$M_\delta=\{(X, Y, \rho) : \|X\|<\delta, \|Y\|<\delta, -1<\rho<-1+\delta\},$$
$$D_r^-=\{(X, Y, \rho) : \|X\|=\delta, -1<\rho<-1+r, 0<H(X, Y, \rho)<r\},$$
$$D_r^+=\{(X, Y, \rho) : \|Y\|=\delta, -1<\rho<-1+r, 0<H(X, Y, \rho)<r\},$$
$$D_{a,r}^-=D_r^-\setminus\{(X, Y, \rho) : H(X, Y, \rho)<ah(\rho)\}.$$

In the definitions of D_r^- and D_r^+, recall that $F(A)=0$. From (2C.2) we conclude that on the sets

$$\{(X, Y, \rho) : \|Y\|=0, \rho=-1\} \quad \text{and} \quad \{(X, Y, \rho) : \|X\|=0, \rho=-1\}$$

we have that $H(X, Y, \rho)=0$.

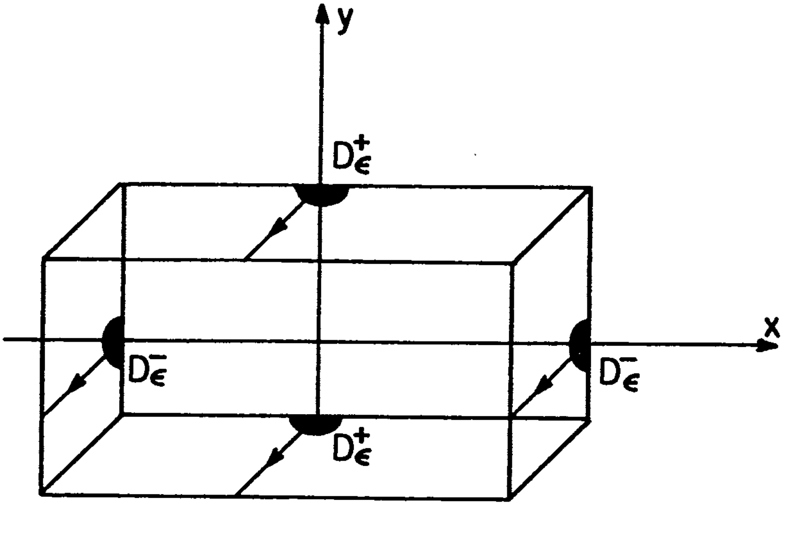

FIG. 3

LEMMA 1. — *There exists $k>0$ with the following property. Given $a>0$, there exists $\delta(a)$ such that if*

$$(X, Y, \rho)\in M_{\delta(a)} \quad \textit{and} \quad H(X, Y, \rho)>a,$$

then

$$\|X\|\geqq ak \quad \textit{and} \quad \|Y\|\geqq ak.$$

Proof. — First consider the old (U, V, ρ) coordinates. There exist smooth functions

$$G_i : \mathbb{R}^3 \to R, \qquad i=1, 2,$$

such that if $(U, V, \rho) \in W_A$ and $\|(U, V, \rho) - \bar{A}\|$ is sufficiently small, then

$$V_1 = G_1(U_1, U_2, \rho) \qquad \text{and} \qquad V_2 = G_2(U_1, U_2, \rho).$$

Note that $G_i(A, \rho) = 0$ for each ρ, $i=1, 2$. Hence, at $(U, \rho) = (A, -1)$,

$$\frac{\partial G_i}{\partial \rho} = 0, \qquad i=1, 2.$$

Moreover, at (A, −1),

$$\left|\frac{\partial G_i}{\partial U_j}\right| \leqq \sigma \equiv \max(\sigma_1, \sigma_2)$$

for $i=1, 2$ and $j=1, 2$. Hence, there exists a constant δ_1 such that if

$$\|(U, \rho) - (A, -1)\| \leqq \delta_1, \tag{A.3}$$

then for $i=1, 2$ and $j=1, 2$,

$$\left|\frac{\partial G_i}{\partial \rho}\right| < 1 \qquad \text{and} \qquad \left|\frac{\partial G_i}{\partial U_j}\right| < 2\sigma.$$

It follows that if (A.3) holds, and

$$G(U, \rho) = (G_1(U, \rho), G_2(U, \rho)),$$

then

$$\|(U, G(U, \rho), \rho) - (U, G(U, -1), -1)\| \leqq \rho + 1. \tag{A.4}$$

Now suppose (A.3) holds and

$$H(U, V) \geqq a. \tag{A.5}$$

That is,

$$\frac{\|V\|^2}{2} + F(U) \geqq a. \tag{A.6}$$

From (2C.2) and the assumption that $F(A)=0$, it follows that

$$H(U, G(U, -1)) = \frac{\|G(U, -1)\|^2}{2} + F(U) = 0. \qquad (A.7)$$

Subtracting (A.7) from (A.6) we obtain

$$\|V\|^2 - \|G(U, -1)\|^2 \geqq a,$$

or

$$\|V\| - \|G(U, -1)\| \geqq \frac{a}{\|V\| + \|G(U, -1)\|} \geqq \frac{a}{K} \qquad (A.8)$$

where K is chosen so that

$$\|V\| + \|G(U, -1)\| < K$$

in M_{δ_1}. Together with (A.4), (A.8) implies that if (A.3), (A.5) hold, then

$$\|(U, V, \rho) - (U, G(U, \rho), \rho)\| \geqq \frac{a}{k} - \delta_1.$$

Assume that $\delta_1 < a/2K$. Therefore, if (A.4), (A.5) hold, then

$$\|(U, V, \rho) - (U, G(U, \rho), \rho)\| \geqq \frac{a}{2K}.$$

Because $|\partial G_i/\partial U_j| < 2\sigma$ for each i, j in M_δ, this implies that there exists k_1, which depends on K and σ, such that if (A.4), (A.5) hold, then

$$\|(U, V, \rho) - W_A\| \geqq a k_1.$$

Since W_A is given in the new coordinates by $\{\|X\| = 0\}$, this implies that for some k which depends on k_1,

$$\|Y\| \geqq ak.$$

A similar argument shows that k can be chosen so that

$$\|X\| \geqq ak,$$

which completes the proof.

LEMMA 2. — *Given* a, r_1 *there exists* r_2, δ_1 *such that if* $\gamma(0)\in D^-_{a,\,r_2}\cap M_{\delta_1}$, *then for some* $z\geqq 0$, $\gamma(z_0)\in D^+_{r_1}$ *and* $\gamma(z)\in M_{\delta_1}$ *for* $z\in[0, z_0]$. *We write* $r_2=E_1(a, r_1)$.

Proof. — Let $\delta(a)$ be as in Lemma 1. We assume that

$$\gamma(0)=(X(0), Y(0), \rho(0))\in M_{\delta(a)},$$

and $H(\gamma(0))>ah(\rho(0))$. From Lemma 1 it follows that

$$\|Y(0)\|>akh(\rho(0)).$$

Therefore, either

$$|y_1(0)|>\frac{ak}{2}h(\rho(0)) \qquad \text{or} \qquad |y_2(0)|>\frac{ak}{2}h(\rho(0)).$$

We assume, without loss of generality, that

$$y_1(0)>\frac{ak}{2}h(\rho(0)).$$

At $y_1=0$, $g_2(X, Y, \rho)=\partial g_2/\partial y_2(X, Y, \rho)=0$. Therefore, we may assume that δ is so small that in M_δ,

$$|g_2(X, Y, \rho)|<\frac{\sigma_1}{2}y_1.$$

Therefore, in M_δ, if $y_1>0$,

$$y_1'=\sigma_1 y_1+g_2(X, Y_1, \rho)>\sigma_1/2\,y_1,$$

or integrating,

$$y_1(z)\geqq y_1(0)\exp\left(\frac{1}{2}\sigma_1 z\right)\geqq\frac{akh(\rho(0))}{2}\exp\left(\frac{1}{2}\sigma_1 z\right).$$

If we let

$$T=2/\sigma_1\,\text{lin}\frac{2\delta}{akh(\rho(0))}, \tag{A.9}$$

then there exists a $z_0<T$ such that $\|Y(z_0)\|=\delta$ and $\|Y(z)\|<\delta$ for $0<z<z_0$.

On the other hand, in M_δ, if $y_1 > 0$,

$$y_1' < 2\sigma_1 y_1 \qquad \text{or} \qquad y_1(z) \leqq y_1(0) \exp(2\sigma_1 z).$$

If $\gamma(0) \in D^-_{a, r_2}$, then $y_1(0) < r_2$ so

$$y_1(z) \leqq r_2 \exp(2\sigma_1 z).$$

A similar argument shows that if $\gamma(0) \in D^-_{a, r_2}$, then in M_δ, for δ small,

$$y_2(z) \leqq r_2 \exp(2\sigma_2 z).$$

This demonstrates that $z_0 \to \infty$ as $r_2 \to 0$. Choose r_2 so small that

$$\delta e^{-\sigma/2 z_0} < 1/2 k r_1. \qquad \text{(A.10)}$$

We assume that δ is chosen so that in M_δ,

$$|g_1(X, Y, \rho)| < \sigma/2 x_1 \qquad \text{and} \qquad |g_3(X, Y, \rho)| < \sigma/2 x_2.$$

Hence, if $x_i > 0$, $i = 1, 2$, then

$$x_i' < -\sigma/2 x_i,$$

or

$$x_i(z) < x_i(0) e^{-\sigma/2 z}.$$

If $0 < x_i(0) < \delta$, then from (A.10),

$$x_i(z_0) < \delta e^{-\sigma/2 z_0} < 1/2 k r_1.$$

A similar argument holds if $x_i < 0$ for $i = 1$ or 2. Therefore if $\|\gamma(0)\| < \delta$, then $\|X(z_0)\| < k r_1$, which implies, by Lemma 1, that $H(z_0) < r_1$.

It remains to prove $0 < 1 + \rho(z_0) < r_1$ if r_2 is sufficiently small. However,

$$\rho' = 1 - \rho^2 < 2(1 + \rho),$$

which implies that

$$1 + \rho(z_0) \leqq (1 + \rho(0)) e^{2 z_0}.$$

Since $z_0 < T$, it follows from (A.9) that

$$1 + \rho(z_0) \leqq (1 + \rho(0)) \left[\frac{2\delta}{akh(\rho(0))}\right]^{4/\sigma_0} \to 0 \qquad \text{as} \quad \rho(0) \to -1$$

because of our choice of $h(\rho)$ given in (A.2).

LEMMA 3. — *Given* δ, *there exists* ρ_δ, T_δ *such that if*

$$\left.\begin{array}{ll}(a) & 0<H(0)<1/2\,F(B);\\ (b) & |\rho(0)+1|<\rho_\delta;\\ (c) & \gamma(z)\notin M_\delta \quad \textit{for} \quad 0\leqq z\leqq T_\delta;\\ (d) & U(0)\in X_i \quad \textit{for} \quad i=1,\ 2,\ \textit{or}\ 3;\end{array}\right\}\quad (A.11)$$

then $U(z)\notin X_i$ *for some* $z\in(0,\ T_\delta)$.

Proof. — First suppose that $\rho(0)=-1$, $0<H(0)<1/2\,F(B)$, and $U(0)\in X_1$. From assumption F(6), there exists T_1 such that $\gamma(z)\notin X_1$ for some $z\in(0,\ T_1)$. From continuous dependence of solutions on initial data we can choose ρ_0 such that if $-1<\rho(0)<-1+\rho_0$, $0<H(0)<1/2\,F(B)$, and $U(0)\in X_1$, then $U(z)\notin X_1$ for some $z\in(0,\ 2\,T_1)$. By a similar argument, ρ_0 and T_1 can be chosen so that if $-1<\rho(0)<-1+\rho_0$, $0<H(0)<1/2\,F(B)$, and $U(0)\in X_3$, then $U(z)\notin X_3$ for some $z\in(0,2\,T_1)$.

Finally, if $\gamma(0)\in X_2$, then the only problem is if $\gamma(z)$ gets close to $\bar{A}$. We rule this out, however, by (A.11*c*). Hence, an argument similar to the one just given shows that the lemma is true if $U(0)\in X_2$.

COROLLARY 4. — *Given* δ, M, *there exists* ρ_0, T_M *such that if*

$$\begin{array}{ll}(a) & 0<H(0)<1/2\,F(B);\\ (b) & |\rho(0)+1|<\rho_0;\\ (c) & \gamma(z)\notin M_\delta \quad \textit{for} \quad 0\leqq z\leqq T_M,\\ (d) & \gamma(z)\in N \quad \textit{for} \quad 0\leqq z\leqq T_M,\end{array}$$

then $U(z)$ *crosses* l_D *and* l_E *as least* M times for $0\leqq z\leqq T_M$.

Proof. — Let T_δ and ρ_δ be as in Lemma 3, and

$$T_M=MT_\delta \qquad \text{and} \qquad \rho_0=\rho_\delta\exp(-2\,T_M).$$

If $|\rho(0)+1|<\rho_0$, then the equation $\rho'=1-\rho^2$ implies that

$$1+\rho(z)<\rho_\sigma \qquad \text{for} \quad 0\leqq z\leqq T_M.$$

The result now follows from Lemma 3.

Recall that we are assuming that $A=(0,\ 0)$. For $\lambda>0$, let

$$A_\lambda=\{U : \|U\|=\lambda|\} \qquad \text{and} \qquad \bar{A}_\lambda=\{U : \|U\|<\lambda\}.$$

LEMMA 5. — *There exists* $\lambda_0 > 0$ *such that if* $\lambda < \lambda_0$ *and*

(*a*) $$U(0) \in A_\lambda,$$

(*b*) $$\langle U(0), V(0) \rangle > 0,$$

(*c*) $$H(0) = 0,$$

(*d*) $$\rho(0) = -1,$$

then there exists z_0 *such that* $U(z_0) \notin X_2$ *and* $U(z) \notin \bar{A}_\lambda$ *for* $0 < z < z_0$.

Proof. — Because A is assumed to be nondegenerate local maximum of F, there exists λ_1 such that if $U \in \bar{A}_{\lambda_1}$, then $\langle U, \nabla F(U) \rangle < 0$. It then follows that if $\lambda > \lambda_1$, $U(0) \in A_\lambda$, and $\langle U(0), V(0) \rangle > 0$, then $\| U(z) \|$ is increasing for $z > 0$, at least until $\| U(z) \| = \lambda_1$.

If the lemma is not true, then there exist sequences $\{ q_n \}$, $\{ p_n \}$, and $\{ z_n \}$, $z_n > 0$ for each n, such that

$$\| q_n \| < \lambda_1 / n, \qquad H(q_n, p_n) = 0, \qquad \langle q_n, p_n \rangle > 0,$$

and if $\gamma^n(z)$ is the solution of (1F.5) satisfying $\gamma^n(0) = (q_n, p_n, -1)$, then $\| U(z_n) \| = \lambda_1 / n$, and $U(z) \in X_2$ for $0 \leqq z \leqq z_n$. Since $H(q_n, p_n) = 0$ and $\| q_n \| < \lambda_1 / n$ it follows that $p_n \to 0$ as $n \to \infty$. Let

$$\gamma_n = \inf \{ F(U) : 0 \leqq z \leqq z_n \},$$

$$\gamma = \sup \{ \gamma_n \}.$$

From the remarks at the beginning of the proof we know that

$$\gamma < \inf \{ F(U) : U \in A_{\lambda_1} \} < 0.$$

Choose λ_2 so that $F(U) > 1/2 \gamma$ if $U \in \bar{A}_{\lambda_2}$.

Suppose that $\gamma^n(z) = (U^n(z), V_n(z), -1)$. We reparametrize $\gamma^n(z)$ so that $\| U^n(0) \| = \lambda_2$, and choose $\{ \zeta_n \}$ and $\{ \eta_n \}$ so that $\zeta_n < 0 < \eta_n$,

$$\gamma^n(\zeta_n) = (q_n, p_n, -1),$$

$$\| U^n(\eta_n) \| = \lambda_1 / n,$$

and

$$\| U_n(z) \| > \| q_n \| \qquad \text{for} \quad z \in (\zeta_n, \eta_n).$$

Let

$$\bar{\gamma}^n(z)\begin{cases}(q_n, p_n, -1) & \text{if} \quad z<\zeta_n\\ \gamma^n(z) & \text{if} \quad \zeta_n \leqq z \leqq \eta_n\\ \gamma^n(\eta_n) & \text{if} \quad z>\eta_n.\end{cases}$$

Then at least some subsequence of $\{\bar{\gamma}(z)\}$ converges. We suppose, for convenience, that it is the entire sequence. Let

$$\gamma(z)=\lim_{n\to\infty} \bar{\gamma}^n(z).$$

One easily checks that

(*a*) $$\lim_{z\to\pm\infty} \gamma(z)=\bar{A};$$

(*b*) $\gamma(z)$ lies entirely in X_2;

(*c*) $\gamma(z)$ is a solution of (1F.5),

(*d*) $$\|\gamma(0)\|=\lambda_2.$$

This all contradicts assumption (F6), thus proving the Lemma.

By continuous dependence of solutions on a parameter we conclude

COROLLARY 6. — *There exists* λ_0, ρ_0 *such that if* $\lambda<\lambda_0$, $\rho_1<\rho_0$ *and*

(*a*) $$U(0)\in A_\lambda;$$

(*b*) $$\langle U(0), V(0)\rangle>0;$$

(*c*) $$H(0)<\lambda;$$

(*d*) $$-1\leqq\rho(0)\leqq-1+\rho_1,$$

then there exists z_0 *such that* $U(z_0)\notin X_2$ *and* $U(z)\notin \bar{A}_\lambda$ *for* $0\leqq z\leqq z_0$.

LEMMA 7. — *Let* ρ_0 *and* λ_0 *be as in Corollary* 6. *There exists* $\tau_1\in(0, 1)$ *and* $\lambda_1\in(0, \lambda_0)$ *such that if* $\tau_1<\tau\leqq 1$, $0\leqq\theta\leqq 2\pi$, *and* $d=(\tau, \theta)$, *then*

$\gamma(d, z)=(U(z), V(z), \rho(z))$ *satisfies, for some* η_0,

(*a*) $$\|U(\eta_0)\|=\lambda_1;$$

(*b*) $$\|U(z)\|<\lambda_1 \quad \textit{for} \quad z<\eta_0;$$

(*c*) $$-1<\rho(0)\leqq -1+\rho_0;$$

(*d*) $$\langle U(\eta_0), V(\eta_0)\rangle>0.$$

Proof. — Near $\bar{A}$, the behavior of W_A is determined by the linearized system (3A). One can solve the linearized system explicitly in terms of the eigenvalues and eigenvectors given in (3A.3), (3A.4). The lemma follows because the U components of the eigenvalues p_i^*, $i=1, 2$, are nonzero.

Combining Corollary 6 and Lemma 7 we conclude

COROLLARY 8. — *There exists* $\tau_1\in(0, 1)$, $\lambda_1>0$ *such that if* $\tau_1<\tau\leqq 1$, $0\leqq\theta\leqq 2\pi$, *and* $d=(\tau, \theta)$, *then* $\gamma(d, z)=(U(z), V(z), \rho(z))$ *satisfies, for some* $\eta_0<\eta_1$,

(*a*) $$\|U(\eta_0)\|=\lambda_1;$$

(*b*) $$\|U(z)\|<\lambda_1 \quad \text{for} \quad z<\eta_0;$$

(*c*) $$\|U(z)\|>\lambda_1 \quad \text{for} \quad \eta_0<z<\eta_1;$$

(*d*) $$U(\eta_1)\in\partial X_2.$$

Remark 1. — To each $\gamma(z)\in W_A$ there corresponds $d=(\tau, \theta)\in\mathscr{D}$ such that $\gamma(z)=\gamma(d, z)$. Surely, λ can be chosen so that if $\gamma(z)\in W_A\cap D_\lambda^-$, then $\tau_1<\tau<1$. This is because λ small and τ close to one both describe trajectories close to the surface $\{\rho=-1\}$.

Let λ_1 be as in Corollary 8,

$$l_1=\min\{\|U^1-U^2\|: U^1\in\partial X_2, \|U^2\|=\lambda_1\},$$
$$\hat{X}_2=\{U\in X_2: \|U\|>\lambda_1\},$$
$$h_1=-\sup\{F(U): U\in\hat{X}_2\}.$$

Note that $h_1>0$. Let

$$V^1=h_1^{1/2} \qquad \text{and} \qquad T_1=l_1/\bar{V}.$$

LEMMA 9. — *Let* M *be a positive integer. Then a and* δ_3 *can be chosen so that given* r_1, *there exists* r_2 *such that if* $\gamma(0)\in D_{r_2}^+$, $\gamma(z)\notin M_{\delta_2}$ *for* $z\in(0, z_0)$, $\gamma(z_0)\in M_{\delta_2}$, *and* $\gamma(z)$ *crosses* l_D *and* l_E, *fewer than* M *times for* $z\in(0, z_0)$, *then* $\gamma(z_0)\in D_{a, r_1}^-$. *We write* $r_2=E_2(r_1)$.

Proof. — Choose δ_2 so that if $\gamma(z) \in M_\delta$, then $H(z) < r_1$. By Corollary 4, there exists T_M, which depends on M and δ_2, such that $z_0 < T_M$. From Proposition 2A.1 we conclude that

$$\|V(z)\| < \bar{V} \quad \text{for} \quad 0 \leqq z \leqq z_0. \tag{A.12}$$

Now

$$\rho' = 1 - \rho^2 \leqq 2(1+\rho).$$

Therefore,

$$\rho(z_0) \leqq (1+\rho(0))\, e^{2 z_0} - 1 \leqq (1+\rho(0))\, e^{2 T_M} - 1. \tag{A.13}$$

We now wish to obtain a lower bound on $H(z_0)$. Now $\gamma(z)$ corresponds to a trajectory $\gamma(d, z)$ for some $d \in \mathscr{D}$. Suppose that $d = (\tau, \theta)$. By choosing r_2 sufficiently small we may assume that $\tau_1 < \tau < 1$ where τ_1 was defined in Corollary 8. Then Corollary 8 implies that there exists $z_1 < z_2$ such that

(*a*) $\|U(z_1)\| = \lambda_1$;

(*b*) $U(z_2) \in \partial X_2$;

(*c*) $U(z) \in X_2$ for $z \in (z_1, z_2)$.

It is clear that $z_1 < z_2 < z_0$. For $z \in (z_1, z_2)$,

$$F(U(z)) \leqq h_1.$$

Moreover, because

$$\lim_{z \to -\infty} H(z) = F(A) = 0 \quad \text{and} \quad H'(z) > 0,$$

it follows that, for all z,

$$H(z) = 1/2 \|V(z)\|^2 + F(U(z)) > 0.$$

Therefore, if $z \in (z_1, z_2)$, then

$$\|V(z)\| \geqq 2 h_1^{1/2}. \tag{A.14}$$

Because $\|U(z_2) - U(z_1)\| > l_1$, (A.12) implies that

$$z_2 - z_1 > l_1/\bar{V}. \tag{A.15}$$

Recall that

$$\varphi_\varepsilon(\rho)=2(n-1)\left[\frac{1}{2}\ln\frac{1+\rho}{1-\rho}\right]^{-1}$$

for $\rho<\rho_\varepsilon$ which we assume to be true. Therefore,

$$\varphi_\varepsilon(\rho)>2(n-1)\left[\ln\frac{2}{1+\rho}\right]^{-1}>2(n-1)\left[\ln\frac{2}{1+\rho(0)}\right]^{-1}. \quad (A.16)$$

Recall, from (2C.2), that

$$H'(z)=\varphi_\varepsilon(\rho(z))\langle V(z), V(z)\rangle^2. \quad (A.17)$$

Then (A.14), (A.15), (A.16) and (A.17) imply that

$$H(z_0)>H(z_2)-H(z_1)\geqq\frac{8h_1 l_1(n-1)}{\bar{V}}\left[\ln\frac{2}{1+\rho(0)}\right]^{-1}. \quad (A.18)$$

Together with (A.2) it follows that

$$\frac{h(\rho(z_0))}{H(z_0)}<\frac{(1+\rho(0))^{\sigma_0/8}e^{z_0/4}\bar{V}}{8h_1 l_1(n-1)}\ln\left[\frac{2}{1+\rho(0)}\right]\to 0 \qquad \text{as} \quad \rho(0)\to -1.$$

Choose $r_3<r_1$ such that if

$$-1<\rho(z_0)<-1+r_3, \qquad \text{then} \quad H(z_0)>ah(\rho(z_0)). \quad (A.19)$$

From (A.13) it follows that there exists r_2 such that

$$-1<\rho(0)<-1+r_2, \qquad \text{then} \quad -1<\rho(z_0)<-1+r_3<-1+r_1.$$

Together with (A.19) and the definition of D^-_{a,r_l} this implies the desired result.

LEMMA 10. — *Fix* $\delta>0$ *and let* T_M *be as in Corollary* 4. *Then there exists* r_0 *such that if* $\gamma(0)\in D_\varepsilon$, *then* $H(z)<1/2\,F(B)$ *for* $0\leqq z\leqq T_M$.

Proof. — Recall that

$$H'(z)=\varphi_\varepsilon(\rho)\|V\|^2\leqq\varphi_\varepsilon(\rho)(\bar{V})^2. \quad (A.20)$$

From (A.13) we have that if $0\leqq z\leqq T_M$, then

$$\rho(z)\to -1 \qquad \text{as} \quad \rho(0)\to -1.$$

Since $\varphi_\varepsilon(\rho) \to -1$ as $\rho \to -1$ this implies that if r is sufficiently small and $-1 < \rho(0) < -1 + r$, then

$$\varphi_\varepsilon(\rho) < \mathrm{F}(\mathrm{B})/4\,\bar{\mathrm{V}}^2\,\mathrm{T}_\mathrm{M}.$$

Together with (A.20) this implies that if $0 \leqq z \leqq \mathrm{T}_\mathrm{M}$, then

$$\mathrm{H}'(z) < \mathrm{F}(\mathrm{B})/4\,\mathrm{T}_\mathrm{M}.$$

Therefore, it suffices to prove that if r is sufficiently small and $\gamma(0) \in \mathrm{D}_\varepsilon$, then $\mathrm{H}(0) < 1/4\,\mathrm{F}(\mathrm{B})$.

However, $\mathrm{H}(0)$ depends continuously on $\gamma(0)$. As $r \to 0$, $\gamma(z)$ approaches a trajectory in the unstable manifold of $\bar{\mathrm{A}}$ lying entirely in the surface $\{\rho = -1\}$. On such a trajectory, $\mathrm{H}(z) = 0$ for all z. This is because when $\rho = -1$, $\varphi(\rho) = 0$, and therefore $\mathrm{H}'(z) = 0$. By continuity we conclude that $\mathrm{H}(0)$ is as small as we please by choosing r close to 0.

Completion of the proof of Proposition 3*B*.1. — Let a be as in Lemma 9, and $\delta = \inf\{\delta_1, \delta_2, \lambda\}$ where δ_1 appeared in Lemma 2, δ_2 appeared in Lemma 9, and λ appeared in Remark 1. Let ρ_0 be as in Corollary 4, and r_0 as in Lemma 10. Define r_k, $k = 1, \ldots, 2\mathrm{M}$ by

$$r_1 = \min\{r_0, \rho_0, \lambda\}, \qquad r_2 = \mathrm{E}_2(r_1),$$
$$r_3 = \mathrm{E}_1(r_2), \ldots, r_{2\mathrm{M}} = \mathrm{E}_2(r_{2\mathrm{M}-1}).$$

That is, if r_k has been defined and k is odd, then $r_{k+1} = \mathrm{E}_2(r_k)$. If k is even, then $r_{k+1} = \mathrm{E}_1(r_k)$. Let $r = r_{2\mathrm{M}}$ and assume that $\gamma(0) \in \mathrm{D}_\varepsilon^+$. From Corollary 4, Lemma 10, and the assumption that $\mathrm{U}(z_0) = \mathrm{B}$ for some z_1, it follows that either

$$\begin{aligned} &(a) \qquad \mathrm{U}(z) \text{ crosses } l_\mathrm{D} \text{ and } l_\mathrm{E} \text{ at least M times} \\ &\text{or} \\ &(b) \qquad \gamma(z_1) \in \mathrm{M}_\delta \quad \text{for some } z_1 < \mathrm{T}_\mathrm{M}. \end{aligned} \tag{A.21}$$

If (A.21 a) holds, then we are done. So suppose that (A.21 b) holds.

Corollary 8 implies that $\mathrm{U}(z)$ must cross either l_D or l_E at least once for some $z \in (0, \mathrm{T}_\mathrm{M})$.

From Lemma 9, $\gamma(z_1) \in \mathrm{D}^-_{a, r_{2\mathrm{M}-1}}$. From Lemma 2, there exists $z_2 > z_1$ such that $\gamma(z) \in \mathrm{M}_\delta$ for $z_1 < z < z_2$ and $\gamma(z_2) \in \mathrm{D}^+_{r_{2\mathrm{M}-2}}$.

Continuing in this way we conclude that for each k, $1 \leqq k \leqq \mathrm{M}$, there exists z_k such that either $\mathrm{U}(z)$ crosses l_D and l_E at least M times for $z < z_k$, or $\gamma(z_k) \in \mathrm{D}_{r_{2\mathrm{M}-2k}}$ and $\mathrm{U}(z)$ crosses l_D and l_E at least k times for $z \in (0, z_k)$.

Setting $k=\mathrm{M}$ we conclude that $\mathrm{U}(z)$ must cross l_{D} and l_{E} at least M times. From the Remark in Section 1 D we conclude that $h(\mathrm{U})\geqq\mathrm{M}$.

To complete the proof we note that τ_{M} can be chosen so that if $\tau_{\mathrm{M}}<\tau<1$, $0\leqq\theta\leqq 2\pi$, and $d=(\tau, \theta)$, then $\gamma(d, z)\in \mathrm{D}_r^+$ for some z.

It is clear that τ_{M} does not depend on ε, because all of our analysis was near $\rho=-1$ where the equations do not depend on ε.

APPENDIX B.
WEAKENING ASSUMPTION (F5)

The main place where assumption (F5) was used was to define the notion of winding number. That is, if we let $\mathrm{P_D}$ and $\mathrm{P_E}$ be as in Section 1D and

$$\hat{\mathrm{P}}_{\mathrm{D}}=\{(\mathrm{U}, \mathrm{V}, \rho):(\mathrm{U}, \mathrm{V})\in \mathrm{P}_{\mathrm{D}}, |\rho|\leqq 1\},$$
$$\hat{\mathrm{P}}_{\mathrm{E}}=\{(\mathrm{U}, \mathrm{V}, \rho):(\mathrm{U}, \mathrm{V})\in \mathrm{P}_{\mathrm{E}}, |\rho|\leqq 1\},$$

then our theorem states that for each positive integer K there exists a solution of (1F. 1) which winds around $\hat{\mathrm{P}}_{\mathrm{D}}$ and $\hat{\mathrm{P}}_{\mathrm{E}}$, K or K + 1 times. We now demonstrate how one can significantly weaken (F5) and still define a notion of winding number. We then comment how one must alter the proof of the theorems if one uses the weaker assumptions.

Note that $\hat{\mathrm{P}}_{\mathrm{D}}$ and $\hat{\mathrm{P}}_{\mathrm{E}}$ are three dimensional manifolds which do not intersect $\mathrm{W_A}$, the unstable manifold at $\bar{\mathrm{A}}$. Because the phase space is five dimensional it then makes sense to count the number of times trajectories wind around $\hat{\mathrm{P}}_{\mathrm{D}}$ and $\hat{\mathrm{P}}_{\mathrm{E}}$. We shall now show how to weaken (F5) so that it is still possible to define three dimensional manifolds which do not intersect $\mathrm{W_A}$. We first motivate the construction as follows.

Let us think of the graph of F(U) as a landscape with three mountain peaks given by F(A), F(B), and F(C). Then (1F. 1) describes the motion of a ball rolling along the landscape with a certain friction $\varphi(\rho)$. A consequence of (F5) is the following. Suppose we place, at $z=0$, the ball on the mountainside with

$$\mathrm{U}(0)\in l_{\mathrm{D}}\cup l_{\mathrm{E}} \qquad \text{and} \qquad \mathrm{V}(0)=(0,0).$$

Recall that l_{D} and l_{E} were defined in (1C. 1). If $\mathrm{U}(0)\neq \mathrm{D}\cup\mathrm{E}$, then the ball will fall off the mountainside in forward and backwards time. Moreover,

$U(z) \in l_D \cup l_E$ for all z. Our new assumption will state that there exist two curves l'_D and l'_E, which lie in N_1, pass through D and E, and each divides N_1 into two regions, such that if

$$U(0) \in l'_D \cup l'_E, \qquad V(0) = (0, 0), \qquad \text{and} \qquad U(0) \neq D \cup E,$$

then the ball will roll off the mountainside in a reasonable fashion. By a "reasonable fashion" we mean that there exists $z_0 > 0$ such that $U(z_0) \notin N_1$, $U(-z_0) \notin N_1$, and $U(z) \notin \{A, B, C\}$ for $|z| < z_0$. We now proceed to make this all precise. We also show how this condition gives rise to two, three dimensional manifolds with which we can define the notion of winding number.

For convenience we assume that

$$N_1 = \{U : |U_1| \leqq W, |U_2| \leqq W\}.$$

Instead of (F5), we assume that there exists smooth curves $U_1 = \psi_D(U_2)$ and $U_1 = \psi_E(U_2)$ which, to begin with, satisfy

(*a*) $\psi_D(D_2) = D_1$ and $\psi_E(E_2) = E_1$;

(*b*) $-W < \psi_D(U_2) < \psi_E(U_2) < W$ for $|U_2| < W$;

(*c*) $B_1 < \psi_D(B_2)$, $\psi_D(A_2) < A_1 < \psi_E(A_2)$, $\psi_E(C_2) < C_1$.

Let l'_D be the curve given by $\{(\psi_D(U_2), U_2) : |U_2| \leqq W\}$ and l'_E the curve given by $\{(\psi_E(U_2), U_2) : |U_2| \leqq W\}$. Let

$$\mathcal{S}_D = \{(U, V, \rho) : U \in l'_D, U \neq D, \|V\| = 0, |\rho| \leqq 1\}$$
$$\cup \{(U, V, \rho) : U = D, V \text{ is tangent to } l'_D \text{ at } D, |\rho| \leqq 1\},$$
$$\mathcal{S}_E = \{(U, V, \rho) : U \in l'_E, U \neq E, \|V\| = 0, |\rho| \leqq 1\}$$
$$\cup \{(U, V, \rho) : U = E, V \text{ is tangent to } l'_E \text{ at } E, |\rho| \leqq 1\}.$$

We assume that if $\gamma(z) = (U(z), V(z), \rho(z))$ is a solution of (1F.1*a*), and $\gamma(0) \in \mathcal{S}_D \cup \mathcal{S}_E$, then there exists $z_0 > 0$ such that $U(z_0) \notin N_1$, $U(-z_0) \notin N_1$, and $U(z) \neq A$, B, or C for $|z| < z_0$.

We now show how this assumption is used to define two, three dimensional manifolds do not intersect W_A. In what follows, we assume that $\gamma(z) = (U(z), V(z), \rho(z))$ is a solution of (1F.1*a*). Let N_2 be as in (2A.1), and

$$P'_D = \{ \gamma(z) : \gamma(0) \in \mathscr{S}_D,\ z \in \mathbb{R} \} \cap N_2,$$

$$P'_E = \{ \gamma(z) : \gamma(0) \in \mathscr{S}_E,\ z \in \mathbb{R} \} \cap N_2.$$

It is not hard to see that P'_D and P'_E are smooth, three dimensional manifolds. Moreover, our new assumption implies that

$$(P'_D \cup P'_E) \cap W_A = \varnothing.$$

We feel that our theorems remain valid if we replace (F5) by this weaker assumption, if in the definition of winding number we replace $\hat{P}_D$ and $\hat{P}_E$ by P'_D and P'_E.

APPENDIX C.
EXPLANATION OF REMARK 2 IN SECTION 1E

We can actually prove that for each integer K there exists at least two radial solutions of (1B.1), each with winding number K or K+1. This is because the radial solutions we considered in this paper each corresponded to parameters $(\tau, \theta) \in \mathscr{D}_0$. Recall that in the definition of $\mathscr{D}_0$, given in Section 4A, we assumed that

$$\theta_1(\tau) \leqq \theta \leqq \theta_2(\tau). \qquad (C.1)$$

If we let $\mathscr{D}_1 = \{ (\tau, \theta) : \theta \notin [\theta_1(\tau), \theta_2(\tau)] \}$, then the same proof shows that for each integer K there exists a radial solution with winding number K or K+1 which corresponds to parameters $(\tau, \theta) \in \mathscr{D}_1$. Hence, we obtain another solution.

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