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A description of self-similar Blow-up for dimensions $n \geqslant 3$

by

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ABSTRACT. — A precise description of the asymptotic behavior near the blowup singularity for solutions of $u_t - \Delta u = f(u)$ which blowups in finite time T is given.

Key words : Blowup, self similar, nonlinear parabolic equation, thermal runaway.

RÉSUMÉ. — On établie une description précise de la conduite asymptotique autour de la singularité de l'explosion totale pour la solution de l'équation $u_t - \Delta u = f(u)$.

Classification A.M.S. : 35 B 05, 35 K 55, 35 K 60, 34 C 15.

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0. INTRODUCTION

The purpose of this paper is to give a precise description of the asymptotic behavior for solutions $u(z, t)$ of

$$u_t = \Delta u + f(u) \tag{0.1}$$

which blow-up in finite positive time T. We assume $f(u)=u^p\,(p>1)$ or $f(u)=e^u$, and $z\in B_R=\{z\in\mathbb{R}^n : |z|<R\}$ where R is sufficiently large to guarantee blow-up.

Giga and Kohn ([8], [11]) recently characterized the asymptotic behavior of solutions $u(z, t)$ of (0.1) with $f(u)=u^p$ near a blow-up singularity assuming a suitable upper bound on the rate of blow-up and provided $n=1$, 2, or $n\geqq 3$ and $p\leqq \dfrac{n+2}{n-2}$. For $B_R\subseteqq\mathbb{R}^n$ using recent *a priori* bounds established by Friedman-McLeod [7], this implies that solutions $u(z, t)$ of (0.1) with suitable initial-boundary conditions satisfy

$$(T-t)^{\beta}u(z, t)\to\beta^{\beta} \quad \text{as } t\to T^- \tag{0.2}$$

provided $|z|\leqq C(T-t)^{1/2}$ for arbitrary $C\geqq 0$ and where $\beta=\dfrac{1}{p-1}$.

For $f(u)=e^u$ and $n=1$ or 2, Bebernes, Bressan, and Eberly [1] proved that solutions $u(z, t)$ of (0.1) satisfy

$$u(z, t)+\ln(T-t)\to 0 \quad \text{as } t\to T^- \tag{0.3}$$

provided $|z|\leqq C(T-t)^{1/2}$ for arbitrary $C\geqq 0$.

The real remaning difficulty in understanding how the single point blow-up occurs for (0.1) rests on determining the nonincreasing globally Lipschitz continuous solutions of an associated steady-state equation

$$y''+\left(\frac{n-1}{x}-\frac{x}{2}\right)y'+F(y)=0, \qquad 0<x<\infty \tag{0.4}$$

where $F(y)=y^p-\beta y$ or e^y-1 for $f(y)=y^p$ or e^y respectively and where $y(0)>0$ and $y'(0)=0$.

For $F(y)=y^p-\beta y$ in the cases $n=1$, 2, or $n\geqq 3$ and $p\leqq\dfrac{n}{n-2}$, we give a new proof of a special case of a known result ([8], Theorem 1) that the only such positive solution of (0.4) is $y(x)\equiv\beta^{\beta}$. For $F(y)=e^y-1$ and $n=1$, Bebernes and Troy [3] proved that the only such solution is $y(x)\equiv 0$.

Eberly [5] gave a much simpler proof showing $y(x) \equiv 0$ is the only solution for the same nonlinearity valid for $n=1$ and 2.

For $3 \leqq n \leqq 9$, Troy and Eberly [6] proved that (0.4) has infinitely many nonincreasing globally Lipschitz continuous solutions on $[0, \infty)$ for $F(y)=e^y-1$. Troy [10] proved a similar multiplicity result for (0.4) with $F(y)=y^p-\beta y$ for $3 \leqq n \leqq 9$ and $p>\frac{n+2}{n-2}$.

This multiple existence of solutions complicates the stability analysis required to precisely describe the evolution of the time-dependent solutions $u(z, t)$ of (0.1) near the blow-up singularity.

In this paper we extend the results of Giga-Kohn [8] and Bebernes-Bressan-Eberly [1] to the dimensions $n \geqq 3$ by proving that, in spite of the multiple existence of solutions of (0.4), the asymptotic formulas (0.2) and (0.3) remain the same as in dimensions 1 and 2. The key to unraveling these problems is a precise understanding of the behavior of the nonconstant solutions relative to a singular solution of (0.4) given by

$$S_e(x)=\ln \frac{2(n-2)}{x^2} \tag{0.5}$$

for $f(u)=e^u$ and $n \geqq 3$, and

$$S_p(x)=\left\{-4\beta\left[\beta+\frac{1}{2}(2-n)\right]/x^2\right\}^{\beta} \tag{0.6}$$

for $f(u)=u^p$ and $\beta+\frac{1}{2}(2-n)<0$, $n \geqq 3$. This will be accomplished by counting how many times the graphs of a nonconstant self-similar solution crosses that of the singular solution.

1. STATEMENT OF THE RESULTS

We consider the initial value problem

$$\left.\begin{aligned} u_t-\Delta u=f(u), \quad & (z, t) \in \Omega \times (0, T) \\ u(z, 0)=\varphi(z), \quad & z \in \Omega \\ u(z, t)=0, \quad & (z, t) \in \partial\Omega \times (0, T) \end{aligned}\right\} \tag{1.1}$$

where $\Omega = B_R = \{z \in \mathbb{R}^n : |z|z| < R\}$, φ is nonnegative, radially symmetric, nonincreasing ($\varphi(z) \geqq \varphi(x)$ for $|z| \leqq |x| \leqq R$), and $\Delta\varphi + f(\varphi) \geqq 0$ on Ω. The two nonlinearities considered are

$$f(u) = e^u \tag{1.2}$$

or

$$f(u) = u^p, \qquad u \geqq 0, \quad p > 1. \tag{1.3}$$

We assume $R > 0$ and $\varphi(z) \geqq 0$ are such that the radially symmetric solution $u(z, t)$ blows-up in finite positive time T. By the maximum principle, $u(., t)$ is radially decreasing for each $t \in [0, T)$ and $u_t(z, t) > 0$ for $(z, t) \in \Omega \times (0, T)$.

Friedman and McLeod [7] proved that blow-up occurs only at $z = 0$. The following arguments are essentially those used in [7] to obtain the needed *a priori* bounds.

Let $U(t) = u(0, t)$. Since $\Delta u(0, t) \leqq 0$ because u is radially symmetric and decreasing, from (1.1) it follows that $U'(t) \leqq f(U(t))$. Integrating, we have

$$-\ln(T - t) \leqq u(0, t), \qquad t \in [0, T) \tag{1.4}$$

for $f(u) = e^u$, and

$$\beta^\beta (T - t)^{-\beta} \leqq u(0, t), \qquad t \in [0, T) \tag{1.5}$$

for $F(u) = u^p$

Define the radially symmetric function $J(z, t) = u_t - \delta f(u)$ where $\delta > 0$ is to be determined. Then $J_t - \Delta J - f'(u) J \geqq 0$. For $0 < \eta < \min(R, T)$, let $\Omega_\eta = B_{R-\eta}$ be the ball of radius $R - \eta$ centered at $0 \in \mathbb{R}^n$. Let $\Pi_\eta = \Omega_\eta \times (\eta, T)$. Since blow-up occurs only at $z = 0$, $u(z, t)$ is bounded on the parabolic boundary of Π_η and $f(u) \leqq C_0 < \infty$ there. Since $u_t > 0$ on $\Omega \times (0, T)$, we have $u_t \geqq C > 0$ on the parabolic boundary of Π_η. Hence, for $\delta > 0$ sufficiently small, $J \geqq C - \delta C_0 > 0$ there. By the maximum principle, $J > 0$ on Π_η. An integration yields the following upper bound on $u(0, t)$:

$$u(0, t) \leqq -\ln[\delta(T - t)], \qquad t \in [\eta, T) \tag{1.6}$$

for $f(u) = e^u$, and

$$u(0, t) \leqq \left(\frac{\beta}{\delta}\right)^\beta (T - t)^{-\beta}, \qquad t \in [\eta, T) \tag{1.7}$$

for $f(u) = u^p$. In fact, since $u_t(., t) \geqq 0$ for $t \in [0, T)$, these bounds are true for all $t \in [0, T)$.

As in [7], we also have the existence of $\bar{t} < T$ such that

$$|\nabla u(z, t)| \leqq [2 e^{u(0, t)}]^{1/2}, \qquad (z, t) \in \bar{\Omega} \times [\bar{t}, T) \tag{1.8}$$

for $f(u)=e^u$, and

$$|\nabla u(z,t)| \leqq \left[\frac{2}{p+1}[u(0,t)]^{p+1}\right]^{1/2}, \qquad (z,t)\in\bar{\Omega}\times[\bar{t},\mathrm{T}) \tag{1.9}$$

for $f(u)=u^p$.

In this paper we prove the following two theorems which describe the asymptotic self-similar blow-up of $u(z,t)$.

THEOREM 1. — *(a) For $n\geqq 3$, the solution $u(z,t)$ of (1.1)-(1.2) satisfies $u(z,t)+\ln(\mathrm{T}-t)\to 0$ uniformly on $\{(z,t):|z|\leqq \mathrm{C}(\mathrm{T}-t)^{1/2}\}$ for arbitrary $\mathrm{C}\geqq 0$ as $t\to \mathrm{T}^-$.*

(b) For $n\geqq 3$ and $p>\dfrac{n}{n-2}$, the solution $u(z,t)$ of (1.1)-(1.3) satisfies $(\mathrm{T}-t)^\beta u(z,t)\to\beta^\beta$ uniformly on $\{(z,t):|z|\leqq \mathrm{C}(\mathrm{T}-t)^{1/2}\}$ for arbitrary $\mathrm{C}\geqq 0$ as $t\to\mathrm{T}^-$.

THEOREM 2. — *Let $r=|z|$ and $v(r,t)=u(z,t)$. There is a value $r_1\in(0,\mathrm{R})$ such that the following properties hold.*

(a) $v(r_1,0)=\mathrm{S}_*(r_1)$ where S_* is the singular solution given in (0.5) *or* (0.6).

(b) $v(r,0)<\mathrm{S}_*(r)$ for $0<r<r_1$.

(c) For each $r\in(0,r_1)$ there is a $\bar{t}=\bar{t}(r)\in(0,\mathrm{T})$ such that $v(r,t)>\mathrm{S}_*(r)$ for $t\in(\bar{t},\mathrm{T})$.

2. THE SELF-SIMILAR PROBLEM

Since the solution $u(z,t)$ of (1.1) is radially symmetric, the initial-boundary value problem can be reduced to a problem in one spatial dimension.

Let $\Pi'=\{(r,t):\ 0<r<\mathrm{R},\ 0<t<\mathrm{T}\}$. If $r=|z|$, then $v(r,t)=u(z,t)$ is well-defined on Π' and satisfies

$$v_t=v_{rr}+\frac{n-1}{r}v_r+f(v), \qquad (r,t)\in\Pi' \tag{2.1}$$

$$\begin{aligned} &v(r,0)=\varphi(r), \qquad r\in(0,\mathrm{R})\\ &v_r(0,t)=0, \qquad v(\mathrm{R},t)=0, \qquad t\in(0,\mathrm{T})\end{aligned} \tag{2.2}$$

To analyze the behavior of v as $t \to \mathrm{T}^-$, we make the following change of variables:

$$\sigma = \ln [\mathrm{T}/(\mathrm{T}-t)], \qquad x = r(\mathrm{T}-t)^{-1/2}$$

Then Π' transforms into Π where

$$\Pi = \{(x, \sigma) : \sigma > 0,\ 0 < x < \mathrm{R}\mathrm{T}^{-1/2} e^{1/2\,\sigma}\}.$$

If $f(u) = e^u$, set

$$w(x, \sigma) = v(r, t) + \ln(\mathrm{T}-t).$$

If $f(u) = u^p$, set

$$w(x, \sigma) = (\mathrm{T}-t)^\beta v(r, t).$$

Then $w(x, \sigma)$ solves

$$w_\sigma = w_{xx} + c(x) w_x + \mathrm{F}(w), \qquad (x, \sigma) \in \Pi \tag{2.3}$$

$$w_x(0, \sigma) = 0, \qquad \sigma \in (0, \infty) \tag{2.4}$$

where $c(x) = (n-1)/x - x/2$; if $f(u) = e^u$, then

$$\begin{gathered} \mathrm{F}(w) = e^w - 1 \\ w(\mathrm{R}\mathrm{T}^{-1/2} e^{1/2\,\sigma}, \sigma) = -\sigma + \ln \mathrm{T}, \qquad \sigma \in (0, \infty) \\ w(x, 0) = \varphi(x\mathrm{T}^{1/2}) + \ln \mathrm{T}, \qquad x \in (0, \mathrm{R}\mathrm{T}^{-1/2}) \end{gathered} \tag{2.5}$$

and if $f(u) = u^p$, then

$$\left.\begin{gathered} \mathrm{F}(w) = w^p - \beta w \\ w(\mathrm{R}\mathrm{T}^{-1/2} e^{1/2\,\sigma}, \sigma) = 0, \qquad \sigma \in (0, \infty) \\ w(x, 0) = \mathrm{T}^\beta \varphi(x\mathrm{T}^{1/2}), \qquad x \in (0, \mathrm{R}\mathrm{T}^{-1/2}) \end{gathered}\right\} \tag{2.6}$$

Using the *a priori* bounds established in section I for $u(z, t)$ using the ideas of [7], we have the following *a priori* estimates for $w(x, \sigma)$. For $\mathrm{F}(w) = e^w - 1$, from (1.4) and (1.6)

$$0 \leqq w(0, \sigma) \leqq -\ln \delta, \qquad \sigma \geqq 0. \tag{2.7}$$

For $\mathrm{F}(w) = w^p - \beta w$, from (1.5) and (1.7)

$$\beta^\beta \leqq w(0, \sigma) \leqq (\beta/\delta)^\beta, \qquad \sigma \geqq 0. \tag{2.8}$$

The estimates (1.8) and (1.9) imply that

$$-\gamma \leqq w_x(x, \sigma) \leqq 0 \quad \text{on } \bar{\Pi} \tag{2.9}$$

for some positive constant γ, and combining this with (2.7) and (2.8) yields

$$-\gamma x \leqq w(x, \sigma) \leqq \mu \quad \text{on } \bar{\Pi} \tag{2.10}$$

where γ and μ are positive constants depending on δ. In fact, for $F(w)=w^p-\beta w$, $w(x, \sigma)=(T-t)^\beta v(r, t)\geqq 0$ since $v(r, 0)\geqq 0$ and $v_t(r, t)\geqq 0$.

3. BEHAVIOR NEAR SINGULAR SOLUTIONS

The partial differential equation (2.3) has a time-independent solution fro certain choices of n and p. More precisely, if $n>2$ and $F(w)=e^w-1$, then

$$S_e(x)=\ln[2(n-2)/x^2] \tag{3.1}$$

is a singular solution of (2.3). If $F(w)=w^p-\beta w$, $n>2$ and $p>\dfrac{n}{n-2}$, then

$$S_p(x)=\left\{-4\beta\left[\beta+\frac{1}{2}(2-n)\right]/x^2\right\}^\beta \tag{3.2}$$

is a singular solution of (2.3). These solutions are in fact singular solutions of (2.1) because

$$1+\frac{1}{2}xS_e'=0, \qquad S_e''+\frac{n-1}{x}S_e'+\exp(S_e)=0 \tag{3.3}$$

and

$$\beta S_p+\frac{1}{2}xS_p'=0, \qquad S_p'=0, \qquad S_p''+\frac{n-1}{x}S_p'+(S_p)^p=0 \tag{3.4}$$

for $0<x<\infty$.

Consider first the singular solution $S_e(x)$ of (2.3) with $F(w)=e^w-1$. Then $S_e(0^+)=\infty>w(0, 0)$ and

$$S_e(RT^{-1/2})=\ln[2(n-2)TR^{-2}]<\ln T=w(RT^{-1/2}, 0)$$

since $2(n-2)<R^2$ for blow-up in finite time (Lacey [9], Bellout [4]). This proves that $w(x, 0)$ intersects $S_e(x)$ at least once for $0<x<RT^{-1/2}$.

Similarly for $F(w)=w^p-\beta w$ and $S_p(x)$, we can make the following observations: $S_p(0^+)=\infty>w(0, 0)$ and $S_p(RT^{-1/2})>0=w(RT^{-1/2}, 0)$. If $w(x, 0)\leqq S_p(x)$ on $[0, RT^{-1/2}]$, we conclude by the maximum principle that $w(x, \sigma)\leqq S_p(x)$ on $\bar{\Pi}$. By the result of Troy [10] (*see* part b of Lemma 4.4), any positive global nonincreasing time-independent solution $y(x)$ associated with (2.3) must interest $S_p(x)$ transversally at least once. By the argument given in Giga-Kohn [8] (or see our theorem 5.1),

$w(x, \sigma) \to 0$ as $\sigma \to \infty$ for each $x \geqq 0$. In particular, $w(0, \sigma) \to 0$, a contradiction to (2.8).

In either case, we can conclude that there exists a first $x_1 \in (0, \mathrm{RT}^{-1/2})$ such that $w(x_1, 0) = \mathrm{S}_*(x_1)$ and $w(x, 0) < \mathrm{S}_*(x)$ on $(0, x_1)$.

LEMMA 3.1. — *There is a continuously differentiable function $x_1(\sigma)$ with domain $[0, \infty)$ such that $x_1(0) = x_1$ and $w(x_1(\sigma), \sigma) = \mathrm{S}_*(x_1(\sigma))$ for all $\sigma \geqq 0$.*

Proof. — Define $\mathrm{D}(x, \sigma) = w(x, \sigma) - \mathrm{S}_*(x)$. We first claim that $\nabla \mathrm{D} \neq (0, 0)$ whenever $\mathrm{D} = 0$. We had $v_t(r, t) > 0$ on Π'. For $f(v) = e^v$,

$$v_t = (\mathrm{T} - t)^{-1}\left(w_\sigma + 1 + \frac{1}{2} x w_x\right),$$

and for $f(v) = v^p$,

$$v_t = (\mathrm{T} - t)^{-\beta - 1}\left(w_\sigma + \beta w + \frac{1}{2} x w_x\right).$$

If $\nabla \mathrm{D} = (0, 0)$ at a point in Π where $\mathrm{D} = 0$, then $\mathrm{D}_\sigma = 0$ implies that $w_\sigma = 0$. For $f(v) = e^v$, $\mathrm{D}_x = 0$ implies that $1 + \frac{1}{2} x w_x = 0$. For $f(v) = v^p$, $\mathrm{D}_x = 0$ implies that $\beta w + \frac{1}{2} x w_x = 0$. In either case, $v_t = 0$ is forced at some point in Π', a contradiction.

Secondly, we claim that $\mathrm{D}_x \neq 0$ at any value $(\bar{x}, \bar{\sigma}) \in \Pi$ where $\mathrm{D}(\bar{x}, \bar{\sigma}) = 0$ and $\mathrm{D}(x, \bar{\sigma}) < 0$ in a left neighborhood of $\bar{x}$.

If $\mathrm{D}(\bar{x}, \bar{\sigma}) = 0$ and $\mathrm{D}_x(\bar{x}, \bar{\sigma}) = 0$, then equations (2.3), (3.3), and (3.4) imply that $\mathrm{D}_{xx}(\bar{x}, \bar{\sigma}) = \mathrm{D}_\sigma(\bar{x}, \bar{\sigma})$. In addition, since $v_t > 0$ we have $\mathrm{D}_\sigma(\bar{x}, \bar{\sigma}) > 0$. Thus $\mathrm{D}_{xx}(\bar{x}, \bar{\sigma}) > 0$, which implies that $(\bar{x}, \bar{\sigma})$ is a local minimum point for D, a contradiction to $\mathrm{D} < 0$ on a left neighborhood of $\bar{x}$. Thus, $\mathrm{D}_x(\bar{x}, \bar{\sigma}) > 0$.

Recall that $v(r, 0) = \varphi(r)$ where $\Delta\varphi + f(\varphi) \geqq 0$. This implies

$$\mathrm{D}_{xx}(x, 0) + \frac{n-1}{x} \mathrm{D}_x(x, 0) + \mathrm{F}(w(x, 0)) - \mathrm{F}(\mathrm{S}_*(x)) \geqq 0$$

for x in a left neighborhood of x_i. On a left neighborhood of x_1, this in turn yields $(x^{n-1} \mathrm{D}_x(x, 0))_x \geqq 0$. An integration yields $\mathrm{D}_x(x_1, 0) > 0$. By the implicit function theorem, there is a continuously differentiable function $x_1(\sigma)$ such that $x_1(0) = x_1$ and $\mathrm{D}(x_1(\sigma), \sigma) = 0$ for some maximal interval $[0, \sigma_0)$. If $\sigma_0 < \infty$, then by continuity $\mathrm{D}(x_1(\sigma_0), \sigma_0) = 0$.

But $D_x(x_1(\sigma_0), \sigma_0)>0$, so the implicit function theorem allows an extension of the domain past σ_0, a contradiction to the maximality of $[0, \sigma_0)$. Thus, $\sigma_0=\infty$. □

For $f(u)=u^p$, since $w(0, 0)<S_p(0^+)$, $w(RT^{-1/2}, 0)<S_p(RT^{-1/2})$, and $w(x_1, 0)=S_p(x_1)$ transversally, there must be a last point of intersection between $w(x, 0)$ and $S_p(x)$, say $x_L\in(x_1, RT^{-1/2})$. A construction similar to Lemma 3.1 leads to the existence of a continuously differentiable function $x_L(\sigma)$ with domain $[0, \infty)$ such that $x_L(0)=x_L$ and $w(x_L(\sigma), \sigma)=S_p(x_L(\sigma))$ for $\sigma\geqq 0$.

Let $\Pi_1=\{(x, \sigma): \sigma>0, 0<x<x_1(\sigma)\}$. We can now prove the following comparison result on this set.

LEMMA 3.2. – *$D(x, \sigma)<0$ for $(x, \sigma)\in\Pi_1$.*

Proof. – By Lemma 3.1, we have shown that $D\leqq 0$ on the parabolic boundary of Π_1. Since $F(w)$ is a local one-sided Lipschitz continuous function, we can apply the Nagumo-Westphal comparison result to obtain $D\leqq 0$ on $\bar{\Pi}_1$.

If $D(x_0, \sigma_0)=0$ for some $(x_0, \sigma_0)\in\Pi_1$, then $D_x(x_0, \sigma_0)=0$, $D_{xx}(x_0, \sigma_0)\leqq 0$ and $D_\sigma(x_0, \sigma_0)\neq 0$ [since $\nabla D\neq(0, 0)$ when $D=0$]. But $D_\sigma(x_0, \sigma_0)\neq 0$ implies $D(x_0, \sigma)$ is positive for some σ near σ_0. This contradicts $D\leqq 0$ on $\bar{\Pi}_1$.

Let $x_2=\sup\{x\in(x_1, RT^{-1/2}]: D(s, 0)\geqq 0$ for $s\in[x_1, 0)=0$ and $D_x(x_1, 0)>0$, the supremum exists. For $f(u)=e^u$, $x_2\leqq RT^{-1/2}$, and for $f(u)=u^p$, $x_2\leqq x_L<RT^{-1/2}$. Define $x_2(\sigma)=x_2 e^{1/2\,\sigma}$ and $\Pi_2=\{(x, \sigma): \sigma>0, x_1(\sigma)<x<x_2(\sigma)\}$.

LEMMA 3.3. – *$D(x_2(\sigma), \sigma)\geqq 0$ for all $\sigma\geqq 0$. Moreover, $D(x, \sigma)>0$ for $(x, \sigma)\in\Pi_2$.*

Proof. – Let $E(\sigma)=D(x_2(\sigma), \sigma)$. By definition of x_2, $E(0)=D(x_2, 0)\geqq 0$. Also, $E'(\sigma)=D_\sigma(x_2(\sigma), \sigma)+\frac{1}{2}x_2(\sigma)D_x(x_2(\sigma), \sigma)$.

We had earlier that $v_t(r, t)\geqq 0$ on $\bar{\Pi}'$. Via the change of variables $(r, t)\to(x, \sigma)$, this implies $E'(\sigma)\geqq 0$ in the case $f(v)=e^v$ and $e^{-\beta\sigma}\frac{d}{d\sigma}[e^{\beta\sigma}E(\sigma)]=E'(\sigma)+\beta E(\sigma)\geqq 0$ in the case $f(v)=v^p$. An integration yields $E(\sigma)\geqq 0$ for $\sigma\geqq 0$.

On the parabolic boundary of Π_2, we now have that $D\geqq 0$. By the Nagumo-Westphal comparison theorem, $D\geqq 0$ on $\bar{\Pi}_2$. A similar argument as in Lemma 3.2 shows that $D>0$ on Π_2. □

COROLLARY 3.4. — *For each* $N>0$ *there is a* $\sigma_N>0$ *such that for each* $\sigma>\sigma_N$, $w(x, \sigma)$ *intersects* $S_*(x)$ *at most once for* $x\in[0, N]$.

Proof. — For each $N>0$ choose σ_N such that $N=x_2\exp\left(\frac{1}{2}\sigma_N\right)$.
Lemma 3.2 guarantees that $D(x, \sigma)<0$ for $x\in[0, x_1(\sigma))$ and Lemma 3.3 guarantees that $D(x, \sigma)>0$ for $\in(x_1(\sigma), x_2(\sigma)]$. For $\sigma>\sigma_N$, $[0, N]\subseteqq[0, x_2(\sigma)]$ by definition of σ_N, so $D=0$ at most once on this interval. □

In section 5 we will see that $x_1(\sigma)\to l$ as $\sigma\to\infty$ where $S_e(l)=0$ or $S_p(l)=\beta^\beta$.

4. ANALYSIS OF THE STEADY-STATE PROBLEM

The time-independent solutions of (2.3)-(2.4) satisfy

$$y''+c(x)y'+F(y)=0, \qquad 0<x<\infty \tag{4.1}$$

$$y(0)=\alpha, \qquad y'(0)=0 \tag{4.2}$$

In this section we will analyze the behavior of a particular class of solutions of (4.1) which are possible members of the ω-limit set for the initial-boundary value problems (2.3)-(2.4)-(2.5) or (2.3)-(2.4)-(2.6).

By the *a priori* bounds stated in section 2, we have that $w(0, \sigma)$ is bounded for $\sigma\geqq 0$. More precisely for $F(w)=e^w-1$, $w(0, \sigma)\in[0, -\ln\delta]$, and for $F(w)=w^p-\beta w$, $w(0, \sigma)\in[\beta^\beta, (\beta/\delta)^\beta]$, for $\sigma\geqq 0$. We also had $-\gamma\leqq w_x(x, \sigma)\leqq 0$ on $\bar{\Pi}$ and, for $F(w)=w^p-\beta w$, $w\geqq 0$ on $\bar{\Pi}$.

If $F(w)=e^w-1$, we need to consider those solutions $y(x)$ of (4.1)-(4.2) which satisfy

$$y(0)=\alpha\geqq 0, \qquad y'(x)\leqq 0 \quad \text{for } x\geqq 0, \quad y'(x) \text{ bounded below.} \tag{4.3}$$

For $n=1$ or 2, (4.1)-(4.2)-(4.3) has only the solution $y(x)\equiv 0$ ([3], [5]). For $3\leqq n\leqq 9$, (4.1)-(4.2)-(4.3) has infinitely many nonconstant solutions [6]. In this section we prove that all nonconstant solutions of (4.1)-(4.2)-(4.3) must intersect the singular solution $S_e(x)$ at least twice. Hence, the only solution intersecting $S_e(x)$ exactly once is $y(x)\equiv 0$.

For $F(w)=w^p-\beta w$, we consider those solutions $y(x)$ of (4.1)-(4.2) which satisfy

$$y(0)=\alpha\geqq\beta^\beta, \qquad y'(x)\leqq 0 \qquad \text{and} \qquad y(x)>0 \quad \text{for } x\geqq 0. \tag{4.4}$$

For $n=1$, 2, or $n \geqq 3$ with $p \leqq \dfrac{n}{n-2}$ we prove a special case of the known result [8] that the only solution to (4.1)-(4.2)-(4.4) is $y(x) \equiv \beta^\beta$. Troy [10] showed that, for $n \geqq 3$ and $p > \dfrac{n+2}{n-2}$, (4.1)-(4.2)-(4.4) has infinitely many nonconstant solutions. In this section we show that any nonconstant solution $y(x)$ of (4.1)-(4.2)-(4.4) must intersect $S_p(x)$ at least twice. Hence, the only solution intersecting $S_p(x)$ *exactly once is* $y(x) \equiv \beta^\beta$.

LEMMA 4.1. — *Consider initial value problem* (4.1)-(4.2).

(*a*) *Any solution to* (4.1)-(4.2)-(4.3) *must satisfy* $y(\sqrt{2n}) \leqq 0$.

(*b*) *Any solution to* (4.1)-(4.2)-(4.4) *must satisfy* $y(\sqrt{2n}) \leqq \beta^\beta$.

Proof. — (*a*) In this case, $F(y) = e^y - 1 \geqq y$, so equation (4.1) implies that $y'' + c(x) y' + y \leqq 0$. Let $u(x) = \alpha(1 - x^2/2n)$. Then $u'' + c(x) u' + u = 0$, $u(0) = y(0)$, and $u'(0) = y'(0)$. Define $W(x) = u(x) y'(x) - u'(x) y(x)$. While $u(x) > 0$, $W' + c(x) W \leqq 0$ and $W(0) = 0$, so an integration yields that $W(x) \leqq 0$. But $(y/u)'(x) = W(x)/[u(x)]^2 \leqq 0$, so integrating from 0 to $\sqrt{2n}$ yields $y(\sqrt{2n}) \leqq u(\sqrt{2n}) = 0$.

Note that for $\alpha > 0$, if $y(z) = 0$, then $y'(z) < 0$ by uniqueness to initial value problems, so $y(x) < 0$ for $x > z$.

(*b*) The function $F(y) = y^p - \beta y$ in convex, so $F(y) \geqq y - \beta^\beta$ and equation (4.1) implies that $v'' + c(x) v' + v \leqq 0$ where $v(x) = y(x) - \beta^\beta$. A similar argument as in part (*a*) shows that $v(\sqrt{2n}) \leqq 0$, thus, $y(\sqrt{2n}) \leqq \beta^\beta$.

Note that for $\alpha > \beta^\beta$, if $y(z) = \beta^\beta$, then $y'(z) < 0$ by uniqueness to initial value problems, so $y(x) < \beta^\beta$ for $x > z$. □

Define $h(x) = y'' + \dfrac{n-1}{x} y'$. For $F(y) = e^y - 1$, define $g(x) = 1 + \dfrac{1}{2} x y'$ and for $F(y) = y^p - \beta y$, define $g(x) = \beta y + \dfrac{1}{2} x y'$. It can be shown that h and g satisfy the following equations:

$$g'' + c(x) g' + [F'(y) - 1] g = 0, \qquad g(0) > 0, \quad g'(0) = 0. \tag{4.5}$$

$$h'' + c(x) h' + [F'(y) - 1] h = -F''(y)(y')^2, \qquad h(0) \leqq 0, \quad h'(0) = 0. \tag{4.6}$$

For $F(y) = e^y - 1$,

$$g' - \frac{1}{2} x g = -\frac{1}{2} x e^y + \frac{1}{2}(2-n) y'. \tag{4.7}$$

For $F(y) = y^p - \beta y$,

$$g' - \frac{1}{2} x g = -\frac{1}{2} x y^p + \left[\beta + \frac{1}{2}(2-n)\right] y'. \tag{4.8}$$

Also define $W(x)=g(x)h'(x)-g'(x)h(x)$. Then

$$W'+c(x)W=-F''(y)(y')^2 g, \quad W(0)=0,$$

and

$$W(x)=-x^{1-n}e^{(1/4)x^2}\int_0^x s^{n-1}e^{-(1/4)s^2}F''[y(s)][y'(s)]^2 g(s)\,ds \qquad (10)$$

$$=: -x^{1-n}e^{(1/4)x^2}I(x)$$

where $I(x)\geqq 0$, while $g>0$ on $(0, x)$. Note that $\left(\frac{h}{g}\right)'(x)=W(x)/[g(x)]^2$, so while $g>0$ on $(0, x)$, we have

$$h(x)=\frac{h(0)}{g(0)}g(x)-g(x)\int_0^x t^{1-n}e^{(1/4)t^2}I(t)[g(t)]^{-2}\,dt \qquad (4.9)$$

LEMMA 4.2. — *Consider initial value problem* (4.1)-(4.2).

(*a*) *If* $y(x)$ *is a solution to* (4.1)-(4.2)-(4.3) *with* $\alpha>0$, *then* $g(x)$ *must have a zero.*

(*b*) *If* $y(x)$ *is a solution to* (4.1)-(4.2)-(4.4) *with* $\alpha>\beta^\beta$, *then* $g(x)$ *must have a zero.*

Proof. — Suppose that $g(x)\geqq\varepsilon>0$ for all $x\geqq 0$. Note that $h(0)<0$ because $\alpha>0$ [part (*a*)] or $\alpha>\beta^\beta$ [part (*b*)]. Then (4.9) implies that $h(x)\leqq[h(0)/g(0)]g(x)\leqq-\delta<0$ since $h(0)/g(0)<0$ and since $I(x)\geqq 0$. Multiplying by x^{n-1} and integrating yields $y'(x)\leqq-\frac{\delta}{n}x$. This contradicts the boundedness of y' in equation (4.3) and forces y to be negative eventually, contradicting equation (4.4). Thus, $g(x)$ cannot be bounded away from zero.

Suppose that $g(x)>0$ for $x\geqq 0$ and that g is not bounded away from zero. Suppose there is an increasing unbounded sequence $\{x_k\}_1^\infty$ such that $g'(x_k)=0$. Equation (4.5) implies that $g''(x_k)=[1-F'(y(x_k))]g(x_k)$. However, Lemma 4.1 implies that $1-F'(y(x_k))>0$ for k sufficiently large. This forces $g''(x_k)>0$ for k sufficiently large, a contradiction, since this would imply that g has two local minimums without a local maximum between. It must be the case that $g'(x)<0$ for x sufficiently large and $g(x)\to 0$ as $x\to\infty$.

Suppose there is an increasing unbounded sequence $\{x_k\}_1^\infty$ such that $g''(x_k)=0$ and $g'(x_k)\leqq -L<0$. Then equation (4.5) implies that $0=c(x_k)g'(x_k)+[F'(y(x_k))-1]g(x_k)$ where $c(x_k)\to-\infty$, $g'(x_k)\leqq-L$, $F'(y(x_k))-1$ is bounded, and $g(x_k)\to 0$. But then the right-hand side of the last equality must become infinite, a contradiction. Thus, $g'(x)<0$ for x large and $g'(x) \to 0$.

In equation (4.9), take the limit as $x \to \infty$ to obtain

$$\lim_{x \to \infty} h(x) = - \lim_{x \to \infty} g(x) \int_0^x t^{1-n} e^{(1/4)t^2} \mathrm{I}(t) [g(t)]^{-2} dt$$

$$= \lim_{x \to \infty} x^{1-n} e^{(1/4)x^2} \mathrm{I}(x) [g'(x)]^{-1} = -\infty$$

where we have used L'Hôpital's rule. This implies that $h(x) \leqq -\delta < 0$ for x sufficiently large. Multiplying by x^{n-1} and integrating yields $y'(x) \leqq \mathrm{K} - \frac{\delta}{n} x$ for x sufficiently large. As before, this contradicts the boundedness of y' in equation (4.3) and forces y to be negative eventually, contradicting equation (4.4).

In all of the above cases, we arrived at contradictions, so there must be a value x_0 such that $g(x_0)=0$, $g'(x_0)<0$, and $g(x)>0$ on $[0, x_0)$. □

LEMMA 4.3. — *Consider problem* (4.1)-(4.2)-(4.3).

(*a*) *If* $1 \leqq n \leqq 2$, *then the only solution is* $y(x) \equiv 0$.

(*b*) *If* $n>2$, *then the only solution which intersects* $\mathrm{S}_e(x)$ *exactly once is* $y(x) \equiv 0$.

Proof. — (*a*) Let $1 \leqq n \leqq 2$, then $\frac{1}{2}(2-n) \geqq 0$. Let x_0 be the first zero for $g(x)$. Suppose there is an $x_1 > x_0$ such that $g'(x_1)=0$ and $g(x)<0$ on $(x_0, x_1]$. Equation (4.7) implies that

$$0 < -\frac{1}{2} x_1 g(x_1) = g'(x_1) - \frac{1}{2} x_1 g(x_1) = -\frac{1}{2} x_1 e^{y(x_1)} + \frac{1}{2}(2-n) y'(x_1) < 0$$

which is a contradiction. Thus, $g'(x)<0$ for $x \geqq x_0$ and so $g(x) \leqq -\varepsilon < 0$ for $x \geqq \bar{x} > x_0$. But $h(x) = g(x) - e^{y(x)} \leqq g(x) \leqq -\varepsilon$. Multiplying by x^{n-1} and integrating yields $y'(x) \leqq \mathrm{K} - \frac{\varepsilon}{n} x$, contradicting equation (4.3). As a result, the only solution of (4.1)-(4.2)-(4.3) for these values of n is $y(x) \equiv 0$.

(*b*) Let $n>2$. Define $\mathrm{D}(x) = y(x) - \mathrm{S}_e(x)$ where S_e is the singular solution discussed in section 3. Then

$$\left.\begin{aligned} \mathrm{D}'' + c(x)\mathrm{D}' + \frac{2(n-2)}{x^2}(e^{\mathrm{D}}-1) = 0, \qquad 0<x<\infty, \\ \mathrm{D}(0^+) = -\infty, \qquad \mathrm{D}'(0^+) = \infty. \end{aligned}\right\} \tag{4.10}$$

Note that $D'>0$ while $D<0$ on $(0, x]$. Suppose that $D(x)<0$ for all $x \geqq 0$. Then $e^D - 1 < 0$ and $D'' + c(x) D' \geqq 0$. Integrating this last equation yields

$$x^{n-1} e^{-(1/4)x^2} D'(x) \geqq \bar{x}^{n-1} e^{-(1/4)\bar{x}^2} D'(\bar{x}) =: p > 0.$$

Consequently,

$$D(x) \geqq D(\bar{x}) + \int_{\bar{x}}^{x} p t^{1-n} e^{(1/4)t^2} dt.$$

But the right-hand side of this inequality must be positive for x sufficiently large, contradicting our assumption. Thus, $D(x)$ must have a first zero x_1 and $D'(x)>0$ on $(0, x_1]$.

By Lemma 4.2, $g(x)$ must have a zero x_0. But then $D'(x_0) = \frac{2}{x_0} g(x_0) = 0$ and $x_0 > x_1$. If $D(x_0)<0$, then there must have been a second zero x_2 for D. Otherwise, $D(x)>0$ on $(x_1, x_0]$, Suppose that $D>0$ for all $x \geqq x_0$. Then there is an $\bar{x}$ sufficiently large such that $D(\bar{x})>0$, $D'(\bar{x})<0$, $D''(\bar{x})>0$, and $c(\bar{x})<0$. Evaluating equation (4.10) at $\bar{x}$ yields $0<(D''+cD'+e^D-1)(\bar{x})=0$, a contradiction. Thus, D must have a second zero x_2.

We have shown that there are at least two points of intersection between the graphs of $y(x)$ and $S_e(x)$ for $\alpha>0$. Thus, the only solution to (4.1)-(4.2)-(4.3) which intersects $S_e(x)$ exactly once is $y(x) \equiv 0$. □

LEMMA 4.4. — *Consider initial value problem* (4.1)-(4.2)-(4.4).

(a) *If* $1 \leqq n \leqq n \leqq 2$, *or if* $n>2$ *and* $\beta + \frac{1}{2}(2-n) \geqq 0$, *then the only solutions is* $y(x) \equiv \beta^\beta$.

(*b*) *If* $n>2$ *and* $\beta + \frac{1}{2}(2-n) < 0$, *then the only solution which intersects* $S_p(x)$ *exactly once is* $y(x) \equiv \beta^\beta$.

Proof. — (*a*) In this case, $\beta + \frac{1}{2}(2-n) \geqq 0$. Let x_0 be the first zero for $g(x)$. Suppose there is an $x_1 > x_0$ such that $g'(x_1)=0$ and $g(x)<0$ on $(x_0, x_1]$. Equation (4.8) implies that

$$0 < -\frac{1}{2} x_1 g(x_1) = g'(x_1) - \frac{1}{2} x_1 g(x_1)$$

$$= -\frac{1}{2} x_1 [y(x_1)]^p + \left[\beta + \frac{1}{2}(2-n)\right] y'(x_1) \leqq 0$$

which is a contradiction. Thus $g'(x_0)<0$ for $x \geqq x_0$ and so $g(x) \leqq -\varepsilon<0$ for $x \geqq \bar{x}>x_0$. But $h(x)=g(x)-[y(x)]^p \leqq g(x) \leqq -\varepsilon$. Multiplying by x^{n-1} and integrating yields $y'(x) \leqq \mathrm{K}-\dfrac{\varepsilon}{n} x$, which forces $y(x)$ to have a zero. This contradicts equation (4.4). As a result, the only solution for these cases is $y(x) \equiv \beta^\beta$.

(*b*) Let $n>2$ and $\mathrm{f}+\dfrac{1}{2}(2-n)<0\left(p>\dfrac{n}{n-2}\right)$. The result for the cases $p>\dfrac{n+2}{n-2}$ is proved by Troy [10]. For the larger range $p>\dfrac{n}{n-2}$ we have the following proof. Define $\mathrm{W}(x)=y(x)\mathrm{S}_p'(x)-y'(x)\mathrm{S}_p(x)$ and $\mathrm{Q}(u)=\mathrm{F}(u)/u$. Then $\mathrm{W}'+c(x)\mathrm{W}=y\mathrm{S}_p[\mathrm{Q}(y)-\mathrm{Q}(\mathrm{S}_p)]$. Note that $\mathrm{Q}(u)$ is an increasing function. Also note that $\mathrm{W}(x)=-2\mathrm{K}x^{-2\beta-1}g(x)$ where $\mathrm{S}_p(x)=\mathrm{K}x^{-2\beta}$. Thus, $x^{n-1}\mathrm{W}(x)=-2\mathrm{K}x^{n-2-2\beta}g(x)$ where $n-2-2\beta>0$. As a result, $x^{n-1}\mathrm{W}(x) \to 0$ as $x \to 0^+$. Integrating the equation for $\mathrm{W}(x)$, we obtain

$$x^{n-1}e^{-(1/4)x^2}\mathrm{W}(x)=\int_0^x t^{n-1}e^{-(1/4)t^2}y(t)\mathrm{S}_p(t)[\mathrm{Q}(y(t))-\mathrm{Q}(\mathrm{S}_p(t))]\,dt.$$

If $0<y<\mathrm{S}_p$ for all $x \geqq 0$, then since $\mathrm{Q}(u)$ is increasing, $\mathrm{W}(x)<0$ for all x. But then $g(x)>0$ for all x is forced, a contradiction to Lemma 4.2. Consequently, there must be a value z such that $y(z)=\mathrm{S}_p(z)$.

Also, $\mathrm{W}(x)<0$ for $x \in [0, x_0)$. At x_0, $0<\mathrm{W}'(x_0)$ which implies that $y(x_0)>\mathrm{S}_p(x_0)$. [Note that $\mathrm{W}'(x_0)=0$ and $y(x_0)=\mathrm{S}_p(x_0)$ imply that $y'(x_0)=\mathrm{S}'(x_0)$ which in turn would imply, by uniqueness to initial value problems, that $y(x) \equiv \mathrm{S}_p(x)$, a contradiction.] So $z<x_0$ is necessary.

Let $x_1>x_0$ be small enough so that $\mathrm{W}(x_1)>0$. Suppose that $y>\mathrm{S}_p$ for all $x>z$. Then integrating the equation for $\mathrm{W}(x)$, we have $\mathrm{W}'+c(x)\mathrm{W} \geqq 0$ and

$$x^{n-1}e^{-(1/4)x^2}\mathrm{W}(x) \geqq x_1^{n-1}e^{-(1/4)x_1^2}\mathrm{W}(x_1)=:p>0.$$

But $(\mathrm{S}_p/y)'(x)=\mathrm{W}(x)/[y(x)]^2$, so

$$(\mathrm{S}_p/y)(x) \geqq (\mathrm{S}_p/y)(x_1)+p\int_{x_1}^x t^{1-n}e^{(1/4)t^2}[y(t)]^{-2}\,dt.$$

For x sufficiently large, the right-hand side must become larger than 1, in which case $(\mathrm{S}/y)(x) \geqq 1$. That is, there is another value q where $y(q)=\mathrm{S}_p(q)$.

We have shown that there are at least two points of intersection between the graphs of $y(x)$ and $S_p(x)$ for $\alpha > \beta^\beta$. Thus, the only solution to (4.1)-(4.2)-(4.4) which intersects $S_p(x)$ exactly once is $y(x) \equiv \beta^\beta$. □

5. THE CONVERGENCE RESULTS

We are now able to precisely describe how the blowup asymptotically evolves in dimensions $n \geqq 3$. Let $w(x, \sigma)$ be the solution of (2.3)-(2.4)-(2.5) or (2.3)-(2.4)-(2.6) depending on the nonlinearity being considered. By Corollary 3.4 we know that for each $N > 0$ there is a $\sigma_N > 0$ such that $w(x, \sigma)$ intersects $S_*(x)$ at most once on $[0, N]$ for each $\sigma > \sigma_N$. By Lemmas 4.3 and 4.4, the only possible steady-state solution of (2.3) with $F(w) = e^w - 1$ which intersects $S_e(x)$ at most once is $y(x) \equiv 0$, and for $F(w) = w^p - \beta w$, the only possible steady-state solution of (2.3) intersecting $S_p(x)$ at most once is $y(x) \equiv \beta^\beta$.

Because of these observations we are now able to prove a convergence or stability result similar to those given in [8] and [1] which prove that the ω-limit set for (2.3)-(2.4)-(2.5) consists of the singleton critical point $y(x) \equiv 0$, and for (2.3)-(2.4)-(2.6), $y(x) \equiv \beta^\beta$.

For the sake of completeness, we include the proof of the following theorem which is influenced by the ones given in [1] and [8].

THEOREM 5.1. – *Let* $n \geqq 3$.

(*a*) *As* $\sigma \to \infty$, *the solution* $w(x, \sigma)$ *of* (2.3)-(2.4)-(2.5) *converges to* $y(x) \equiv 0$ *uniformly in x on compact subsets of* $[0, \infty)$.

(*b*) *As* $\sigma \to \infty$, *the solution* $w(x, \sigma)$ *of* (2.3)-(2.4)-(2.6) *converges to* $y(x) \equiv \beta^\beta$ *uniformly in x on compact subsets of* $[0, \infty)$.

Proof. – Define $w^\tau(x, \sigma) := w(x, \sigma + \tau)$ as the function obtained by shifting w in time by the amount τ. We will show that as $\tau \to \infty$, $w^\tau(x, \sigma)$ converges to the solution $y(x)$ uniformly on compact subsets of $\mathbb{R}^+ \times \mathbb{R}$. Provided that the limiting function is unique, it is equivalent to prove that given any unbounded increasing sequence $\{n_j\}$, there exists a subsequence $\{n_j\}$ such that w^{n_j} converges to $y(x)$ uniformly on compact subsets of $\mathbb{R}^+ \times \mathbb{R}$.

Let $N \in \mathbb{Z}^+$. For i sufficiently large, the rectangle given by $Q_{2N} = \{(x, \sigma): 0 \leqq x \leqq 2N, |\sigma| \leqq 2N\}$ lies in the domain of w^{n_i}. The radially symmetric function $\tilde{w}(\zeta, \sigma) = w^{n_i}(|\zeta|, \sigma)$ solves the parabolic equation

$$\tilde{w}_\sigma = \Delta\tilde{w} - \frac{1}{2}\langle \zeta, \nabla\tilde{w} \rangle + F(\tilde{w})$$

on the cylinder given by $\Gamma_{2N} = \{(\zeta, \sigma) \in \mathbb{R}^n \times \mathbb{R}: |\zeta| \leqq 2N, |\sigma| \leqq 2N\}$ with $-2N\gamma \leqq \tilde{w}(\zeta, \sigma) \leqq \mu$ using (2.10).

By Schauder's interior estimates, all partial derivatives of $\tilde{w}$ can be uniformly bounded on the subcylinder $\Gamma_N \subseteqq \Gamma_{2N}$. Consequently, w^{n_i}, $w^{n_i}_\sigma$, and $w^{n_i}_{xx}$ are uniformly Lipschitz continuous on $Q_N \subseteqq Q_{2N}$. Their Lipschitz constants depend on N but not on i. By the Arzela-Ascoli theorem, there is a subsequence $\{n_j\}_1^\infty$ and a function $\bar{w}$ such that w^{n_j}, $w^{n_j}_\sigma$, $w^{n_j}_{xx}$ converge to $\bar{w}$, $\bar{w}_\sigma$, and $\bar{w}_{xx}$, respectively, uniformly on Q_N.

Repeating the construction for all N and taking a diagonal subsequence, we can conclude that $w^{n_j} \to \bar{w}$, $w^{n_j}_\sigma \to \bar{w}$, and $w^{n_j}_{xx} \to \bar{w}_{xx}$ uniformly on every compact subset in $\mathbb{R}^+ \times \mathbb{R}$. Clearly $\bar{w}$ satisfies (2.3)-(2.4) with $-\gamma \leqq \bar{w}_x \leqq 0$. For $n \geqq 3$ and $F(w) = e^w - 1$, the limiting function $\bar{w}$ intersects $S_e(x)$ at most once since, by Corollary 3.4, $w^{n_j}(x, \sigma)$ intersects $S_e(x)$ at most once on [0, N] for each $\sigma > \sigma_N$, and $0 \leqq \bar{w}(0, \sigma) \leqq -\ln\delta$ for $\sigma \geqq 0$. For $n \geqq 3$, $\beta + \frac{1}{2}(2-n) < 0$, and

$$F(w) = w^p - \beta w,$$

Corollary 3.4 guarantees that $\bar{w}$ intersects $S_p(x)$ at most once. By (2.8) we have $\beta^\beta \leqq w(0, \sigma) \leqq (\beta/\delta)^\beta$ for $\sigma \geqq 0$.

We now prove that $\bar{w}$ is independent of σ. For the solution $w(x, \sigma)$ of (2.3)-(2.4)-(2.5) or (2.6), define the energy functional

$$\left.\begin{aligned} E(\sigma) &= \int_0^v \rho(x)\left[\frac{1}{2}w_x^2 - G(w)\right]dx, \\ v = RT^{-1/2}e^{1/2\,\sigma}, &\qquad \rho(x) = x^{n-1}e^{-(1/4)x^2} \end{aligned}\right\} \tag{5.1}$$

where $G(w) = e^w - w$ if $F(w) = e^w - 1$, and $G(w) = w^{p+1}/(p+1) - \frac{1}{2}\beta w^2$ if $F(w) = w^p - \beta w$.

Multiplying equation (2.3) by ρw_σ and integrating from 0 to v yields the equation

$$\int_0^v \rho w_\sigma^2 \, dx = \int_0^v w_\sigma (\rho w_x)_x \, dx + \int_0^v \frac{\partial}{\partial \sigma} [\rho G(w)] \, dx$$

$$= \int_0^v \frac{\partial}{\partial \sigma} \left[\rho G(w) - \frac{1}{2} \rho w_x^2 \right] dx + \rho w_\sigma w_x \Big|_{x=0}^{x=v} \qquad (5.2)$$

Moreover,

$$E'(\sigma) = \int_0^v \frac{\partial}{\partial \sigma} \left[\frac{1}{2} \rho w_x^2 - \rho G(w) \right] dx$$

$$+ \frac{1}{2} v \left\{ \rho(v) \left[\frac{1}{2} w_x^2(v, \sigma) - G(w(v, \sigma)) \right] \right\} \qquad (5.3)$$

Therefore, for all a, b with $0 \leqq a < b$, integrating (5.2) with respect to σ from a to b, and using (5.3), we have

$$\int_a^b \int_0^v \rho w_x \, dx \, d\sigma = -\int_a^b E'(\sigma) \, d\sigma + \int_a^b \rho(v) w_\sigma(v, \sigma) w_x(v, \sigma) \, d\sigma$$

$$+ \frac{1}{2} \int_a^b \rho(v) \left[\frac{1}{2} w_x^2(v, \sigma) - G(w(v, \sigma)) \right] d\sigma$$

$$=: E(a) - E(b) + \psi(a, b) \qquad (5.4)$$

Recalling that $|w_x| \leqq \gamma$ and observing that

$$w_\sigma(v, \sigma) = -1 - R u_r(R, T(1 - e^{-\sigma}))$$

for $f(u) = e^u$, or $w_\sigma(v, \sigma) = -R u_r(R, T(1 - e^{-\sigma}))$ for $f(u) = u^p$, we see that in either case the quantity is uniformly bounded as $\sigma \to \infty$. We conclude that

$$\lim_{a \to \infty} \{ \sup_{b > a} \psi(a, b) \} = 0 \qquad (5.5)$$

For any fixed N, we shall prove that

$$\int_{Q_N} \int \rho \bar{w}_\sigma^2 \, dx \, d\sigma = \lim_{n_j \to \infty} \int_{Q_N} \int \rho (w_\sigma^{n_j})^2 \, dx \, d\sigma = 0.$$

Note that it is not a restriction to assume that $\lim_{j \to \infty} (n_{j+1} - n_j) = \infty$. For all j large enough, $N \leqq RT^{-1/2} \exp\left[\frac{1}{2}(n_j - N)\right]$ and $n_{j+1} - n_j \geqq 2N$. Hence,

$$\int_{-N}^{N} \int_{0}^{N} \rho (w_\sigma^{n_j})^2 \, dx \, d\sigma \leqq \int_{-N}^{-N+n_{j+1}-n_j} \int_{0}^{RT^{-1/2} \exp(1/2\, n_j)} \rho (w_\sigma^{n_j})^2 \, dx \, d\sigma$$

$$= E(n_j - N) - E(n_{j+1} - N) + \psi(n_j - N, n_{j+1} - N)$$

by (5.4). As a consequence of (5.5), we have

$$\int_{Q_N}\int \rho \, \bar{w}_\sigma^2 \, dx \, d\sigma \leqq \limsup_{j \to \infty} [E(n_j - N) - E(n_{j+1} - N)]. \tag{5.6}$$

Fix any K arbitrarily large. For j sufficiently large, we have

$$E(n_j - N) - E(n_{j+1} - N)$$

$$= \int_0^K \frac{1}{2} \rho \{ [w_x^{n_j}(x, -N)]^2 - [w_x^{n_{j+1}}(x, -N)]^2 \} \, dx$$

$$- \int_0^K \rho [G(w^{n_j}(x, -N) - G(w^{n_{j+1}}(x, -N)] \, dx$$

$$+ \int_K^{RT^{-1/2} \exp[1/2 (n_j - N)]} \rho \left\{ \frac{1}{2} [w_x^{n_j}(x, -N)]^2 - G(w^{n_j}(x, -N)) \right\} dx$$

$$\int_K^{RT^{-1/2} \exp[1/2 (n_j - N)]} \rho \left\{ \frac{1}{2} [w_x^{n_{j+1}}(x, -N)]^2 - G(w^{n_{j+1}}(x, -N)) \right\} dx \tag{5.7}$$

In (5.7), the first two integrals on the right-hand side converge to zero as $j \to \infty$. Recalling that $|w_x^{n_j}(x, -N)| \leqq \gamma$ and $-\gamma x \leqq w^{n_j}(x, -N) \leqq \mu$, we see that the sum of the absolute values of the last two integrals is bounded by $M \int_K^{\infty} x^{n-1} e^{-(1/4) x^2} \, dx$ where M is a positive constant. This integral can be made arbitrarily small by choosing K large enough.

This proves that $\int_{-N}^{N} \rho \, \bar{w}_\sigma^2 \, dx \, d\sigma = 0$ and hence $\bar{w}_\sigma = 0$. Thus, $\bar{w}(x, \sigma) = \bar{w}(x, 0) = y(x)$ where $y(x)$ is a nonincreasing globally Lipschitz continuous solution of (4.1)-(4.2) which intersects $S_*(x)$ at most once. If $f(u) = e^u$, then $y(0) \in [0, -\ln \delta]$ and so $y(x) \equiv 0$ is the only solution which intersects $S_e(x)$ exactly (and thus at most) once on $[0, \infty)$. Similarly for $f(u) = u^p$, $y(0) \in [\beta^\beta, (\beta/\delta)^\beta]$ and the only possible solution is $y(x) \equiv \beta^\beta$.

Since the limiting solution $y(x)$ is unique in either case, $\omega^\tau(x, \sigma) \to y(x)$ as $\tau \to \infty$ and we have the result asserted. □

Proof of Theorem 1. – The last theorem shows that $w(x, \sigma) \to y(x)$ uniformly in x on compact subsets of $[0, \infty)$ as $\sigma \to \infty$.

(*a*) In the case $f(u) = e^u$, changing back to the variables (r, t), we have that $v(r, t) + \ln(T-t) \to 0$ as $t \to T^-$ provided $r \leqq C(T-t)^{1/2}$ for arbitrary $C \geqq 0$.

In particular, $v(0, t) + \ln(T-t) \to 0$ as $t \to T^-$.

(*b*) In the case $f(u) = u^p$ we obtain $(T-t)^\beta v(r, t) \to \beta^\beta$ as $t \to T^-$ provided $r \leqq C(T-t)^{1/2}$ for arbitrary $C \geqq 0$. In particular, $(T-t)^\beta v(0, t) \to \beta^\beta$ as $t \to T^-$.

Proof of Theorem 2. – Theorem 5.1 guarantees that the first branch of zeros $x_1(\sigma)$ of $D(x, \sigma) = w(x, \sigma) - S_*^*(x)$ is bounded and converges to l where $S_e(l) = 0$ or $S_p(l) = \beta^\beta$.

Define $r_1 = x_1 T^{1/2}$. Then $D(x_1, 0) = 0$ implies that $v(r_1, 0) = S_*(r_1)$. In addition, $v(r, 0) < S_*(r)$ for $r \in (0, r_1)$.

Since $x_1(\sigma)$ is bounded and since $\dfrac{d}{d\sigma} D(r T^{-1/2} e^{1/2\sigma}, \sigma) \geqq 0$ for each $r \in (0, r_1)$, there is a value $\bar{\sigma} > 0$ such that

$$r T^{-1/2} e^{1/2\bar{\sigma}} = x_1(\bar{\sigma}) D(x_1(\bar{\sigma}), \bar{\sigma}) = 0,$$

and $D(r T^{-1/2} e^{1/2\sigma}, \sigma) > 0$ for $\sigma > \bar{\sigma}$. Changing back to the variables (r, t) with $\bar{\sigma} = \ln[T//(T-\bar{t})]$, we obtain $v(r, t) > S_*(r)$ for $t \in (\bar{t}, T)$.

Remark. – After this paper was completed we received the preprint [11] of Giga and Kohn. In the introduction there is a detailed discussion of self-similar solutions and their importance in describing the behavior of solutions near a blow up point. The referee pointed out a number of papers ([12] to [18]) which are related to the ideas used in this paper. Their relevance is discussed in [11]. The referee also pointed out a briefer proof of Lemma 4.1 which we have used.

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