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A continuous version of Liapunov's convexity theorem

by

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ABSTRACT. — Given a continuous map $s \mapsto \mu_s$, from a compact metric space into the space of nonatomic measures on T, we show the existence of a family $(A_\alpha^s)_{\alpha \in [0, 1]}$, increasing in α and continuous in s, such that

$$\mu_s(A_\alpha^s) = \alpha \mu_s(T) \qquad (\alpha \in [0, 1]).$$

Key words : Liapunov's convexity theorem - Measure theory - Selections.

RÉSUMÉ. — Étant donnée une application continue $s \mapsto \mu_s$, d'un espace métrique compact dans l'espace des mesures nonatomiques sur T, nous montrons l'existence d'une famille $(A_\alpha^s)_{\alpha \in [0, 1]}$, croissante avec α et continue en s, telle que

$$\mu_s(A_\alpha^s) = \alpha \mu_s(T) \qquad (\alpha \in [0, 1]).$$

1. INTRODUCTION

Let μ be a non-atomic finite measure on a measurable space T. A result of measure theory states the existence of a family $(A_\alpha)_\alpha$ of subsets of T, increasing with α in [0, 1] and such that

$$\mu(A_\alpha) = \alpha \mu(T).$$

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According to Liapunov's Convexity Theorem on the range of vector measures (*see* Halmos [2], [3] and Liapunov [4]) the above result holds for a finite family of nonatomic measures μ_i, $i=1, \ldots, n$: there exists an increasing family $(A_\alpha)_\alpha$ such that

$$\mu_i(A_\alpha)=\alpha\mu_i(T), \qquad i=1, \ldots, n.$$

In general, the above is not true for an infinite family $(\mu_s)_s$ of measures (*see* Liapunov [5]). In this paper we consider a map $s \to \mu_s$, continuous for s in a compact metric space S. Denoting by $\mathscr{A}(\mu_s)$ the set of increasing families $(A_\alpha^s)_\alpha$ satisfying

$$\mu_s(A_\alpha^s)=\alpha\mu_s(T),$$

we show the existence of a selection $(\tilde{A}_\alpha^s)_\alpha$ of the multivalued map $\mathscr{A}(\mu_s)$ continuously depending on s in the sense of Definition 2 of the following section.

2. NOTATIONS AND PRELIMINARY RESULTS

We consider a measure space $(T, \mathscr{F}, \mu_0)$ where μ_0 is a non-atomic positive measure on a σ-algebra $\mathscr{F}$ and $\mu_0(T)=1$. Denote by $\mathscr{M}$ the set of positive finite measures μ on T which are absolutely continuous with respect to μ_0, hence non-atomic. The metric in $\mathscr{M}$ is induced by the norm $\|\mu\|$ given by the variation of μ.

DEFINITION 1. — A family $(A_\alpha)_{\alpha\in[0, 1]}$, $A_\alpha \in \mathscr{F}$, is called *increasing* if

$$A_\alpha \subseteq A_\beta \quad \text{when}\, \alpha \leqq \beta.$$

An increasing family is called *refining* $A \in \mathscr{F}$ with respect to the measure $\mu=(\mu_1, \ldots, \mu_n) \in \mathscr{M}^n$ if $A_0=\varnothing$, $A_1=A$ and

$$\mu(A_\alpha)=\alpha\mu(A) \qquad (\alpha \in [0, 1]).$$

The set of the families refining T with respect to μ is denoted by $\mathscr{A}(\mu)$.

The proofs of Lemmas 1 and 2 are based on Liapunov's theorem (*see* Fryszkowski [1]).

LEMMA 1. — *Consider a vector measure* $\mu \in \mathscr{M}^n$. *For each* $A \in \mathscr{F}$ *there exists a family* $(A_\alpha)_{\alpha\in[0, 1]}$ *refining* A *with respect to* μ. *In particular, the set* $\mathscr{A}(\mu)$ *is nonempty.*

In what follows, S is a compact metric space with distance d.

LEMMA 2. — *Let* $s \to \mu_s$ *be a continuous map from* S *into* $\mathcal{M}^n$. *Then for every* $\varepsilon > 0$ *there exists an increasing family* $(A_\alpha)_\alpha$ *satisfying*

(i) $\mu_0(A_\alpha) = \alpha$ $(\alpha \in [0, 1])$;

(ii) $|\mu_s(A_\alpha) - \alpha\mu_s(T)| < \varepsilon$ $(\alpha \in [0, 1], s \in S)$.

DEFINITION 2. — A map $s \to (A_\alpha^s)_\alpha$ is called *continuous* on S if for every $s^0 \in S$ and $\varepsilon > 0$ there exists a $\delta > 0$ such that: s, s' and s'' in $B(s'^0, \delta)$ implies

$$\sup_{\alpha \in [0, 1]} \mu_s(A_\alpha^{s'} \triangle A_\alpha^{s''}) < \varepsilon.$$

Analogously we set

DEFINITION 3. — The set valued map $s \to \mathcal{A}(\mu_s)$ is called *continuous* if for every $s^0 \in S$ and $\varepsilon > 0$ there exists a $\delta > 0$ such that: s, s' and s'' in $B(s^0, \delta)$ implies $\forall (A'_\alpha) \in \mathcal{A}(\mu_{s'}), \exists (A''_\alpha) \in \mathcal{A}(\mu_{s''})$ such that

$$\sup_{\alpha \in [0, 1]} \mu_s(A'_\alpha \triangle A''_\alpha) < \varepsilon.$$

We will use the symbol $\dot{\bigcup}$ to denote the union of disjoint sets. Finally, we recall that $\rho(.,.)$ defined as $\rho(A, B) = \mu(A \Delta B)$ $(\mu \in \mathcal{M})$ is a pseudometric on $\mathcal{F}$.

Remarks. — (*a*) In [5], Liapunov considers a sequence μ_n of measures on $[0, 2\pi]$ defined by a family of densities f_n converging strongly in L^1 to zero. He shows that there cannot exist any Borel subset A of $[0, 2\pi]$ such that for every n, $\mu_n(A) = \frac{1}{2}\mu_n([0, 2\pi])$. By associating μ_n to the point $1/n$ and $\mu_\infty = 0$ to the point 0, we have a map $s \mapsto \mu_s$ from the compact metric space $S = \{1/n : n \in N\} \cup \{0\}$ into the space of nonatomic measures. The continuity at 0 follows from the strong convergence of (f_n). This example shows that the assumptions of Theorem 1 below do not guarantee the existence of a constant selection.

(*b*) A further example is taken from Valadier [7]. Let S and T be the real interval [0, 1], and set $\mu_s(A) = \int_A e^{-st}\,dt$. Assume there exists a set $\bar{A} \subseteq T$ such that

$$\forall s, \quad \mu_s(A) = \frac{1}{2}\mu_s(T).$$

Then

$$\int_{-\infty}^{+\infty} \chi_A(t)\, e^{-st}\, dt = \int_{-\infty}^{+\infty} \frac{1}{2}\chi_T(t)\, e^{-st}\, dt.$$

Since the Laplace transformations of χ_A and $\frac{1}{2}\chi_T$, both of compact support, are analytic and coincide on [0,1], they are identical. By the injectivity of the Lapalce Transformation, we have

$$\chi_A = \frac{1}{2}\chi_T,$$

a contradiction. Hence again we have an example where there exist no constant selections.

(*c*) It seems more natural to express the continuity in terms of the pseudometric $\rho(A, B) = \mu_0(A, B)$. However, Definition 2 is not necessarily equivalent to the continuity with respect to this pseudometric when μ_0 is not absolutely continuous with respect to μ_{s^0}.

3. MAIN RESULTS

In order to prove our main theorem we need three additional Lemmas.

LEMMA 3. — *Consider a* 1-*dimensional measure* $\mu \in \mathcal{M}$ *and an increasing family* $(A_\alpha^1)_\alpha$ *such that for some* $\varepsilon > 0$,

$$|\mu(A_\alpha^1) - \alpha\mu(T)| < \varepsilon \qquad (\alpha \in [0, 1]).$$

There exists an increasing family $(A_\alpha^2)_\alpha$ *such that*

(i) $\mu(A_\alpha^2) = \alpha\mu(T) \quad (\alpha \in [0, 1])$

(ii) $\mu(A_\alpha^1 \triangle A_\alpha^2) < 6\varepsilon \quad (\alpha \in [0,1])$.

Proof. — Fix M so that $\frac{1}{M} \geqq \frac{\varepsilon}{\mu(T)} \geqq \frac{1}{M+1}$. We begin by defining recursively an increasing family $(A_\alpha^2)_\alpha$ for $\alpha = i/M$, $i = 0, \ldots, M$, such that (i) holds and $A_{i/M}^2 \subseteq A_{(i+1)/M}^1$. Set $A_0^2 = \emptyset$ and assume $A_{i/M}^2$ has been defined for $i = 0, \ldots, n < M$.

Case 1. — When $\mu(A_{(n+1)/M}^1) \geqq \frac{n+1}{M}\mu(T)$, define $A_{(n+1)/M}^2$, by Lemma 1, as a set such that $A_{n/M}^2 \subseteq A_{(n+1)/M}^2 \subseteq A_{(n+1)/M}^1$ and

$$\mu(A_{(n+1)/M}^2) = \frac{n+1}{M}\mu(T).$$

Case 2. — When $\mu(A_{(n+1)/M}^1) < \frac{n+1}{M}\mu(T)$, we first notice that by the choice of M we have that $\mu(A^1_{(n+2)/M}) \geqq \frac{n+1}{M}\mu(T)$; hence we can define $A^2_{(n+1)/M}$ as a set such that $A^1_{(n+1)/M} \subseteq A^2_{(n+1)/M} \subseteq A^1_{(n+2)/M}$ and

$$\mu(A^2_{(n+1)/M}) = \frac{n+1}{M}\mu(T).$$

Notice that $A^2_{(n+1)/M} \supseteq A^2_{n/M}$, since $A^1_{(n+1)/M} \supseteq A^2_{n/M}$ by the inductive hypothesis.

In either case, we have

$$\begin{aligned}\mu(A^2_{(n+1)/M} \triangle A^1_{(n+1)/M}) &= \left|\mu(A^2_{(n+1)/M}) - \mu(A^1_{(n+1)/M})\right| \\ &\leqq \left|\mu(A^2_{(n+1)/M}) - \frac{n+1}{M}\mu(T)\right| \\ &\quad + \left|\mu(A^1_{(n+1)/M}) - \frac{n+1}{M}\mu(T)\right| \\ &< \varepsilon.\end{aligned}$$

By Lemma 1 it is now easy to define a family $(A^2_\alpha)_{\alpha \in [0,1]}$ such that

(a) $A^2_{i/M} \subseteq A^2_\alpha \subseteq A^2_\beta \subseteq A^2_{(i+1)/M}$ for $\frac{i}{M} \leqq \alpha \leqq \beta \leqq \frac{i+1}{M}$;

(b) $\mu(A^2_\alpha) = \alpha\mu(T)$.

Now we check that (ii) holds for $\frac{i}{M} \leqq \alpha \leqq \frac{i+1}{M}$. We can as well assume that $\mu(T) \geqq 6\varepsilon$ otherwise (ii) trivially holds.

$$\begin{aligned}\mu(A^1_\alpha \triangle A^2_\alpha) &= \mu(A^1_\alpha \setminus A^2_\alpha) + \mu(A^2_\alpha \setminus A^1_\alpha) \\ &\leqq \mu(A^1_{(i+1)/M} \setminus A^1_{i/M}) + \mu(A^1_{i/M} \setminus A^2_{i/M}) \\ &\quad + \mu(A^2_{(i+1)/M} \setminus A^2_{i/M}) + \mu(A^2_{i/M} \setminus A^1_{i/M}) \\ &\leqq \frac{1}{M}\mu(T) + 2\varepsilon + \frac{1}{M}\mu(T) + \varepsilon \\ &\leqq 2\frac{\varepsilon\mu(T)}{\mu(T) - \varepsilon} + 3\varepsilon = \frac{2\varepsilon}{1 - (\varepsilon/\mu(T))} + 3\varepsilon \\ &\leqq \left(\frac{12}{5} + 3\right)\varepsilon < 6\varepsilon. \quad \blacksquare\end{aligned}$$

COROLLARY. — *The set-valued map* $s \to \mathscr{A}(\mu_s)$ *is continuous.*

Proof. — Choose s^0 and $\varepsilon>0$. Let $\delta>0$ be such that $d(s, s^0)<\delta$ implies $\|\mu_s-\mu_{s^0}\|<\varepsilon/26$. Fix s, s' and s'' in $B(s^0, \delta)$ and $A'_\alpha \in \mathscr{A}(\mu_{s'})$. Since

$$\begin{aligned}|\mu_{s^0}(A'_\alpha)-\alpha\mu_{s^0}(T)| &= |\mu_{s^0}(A'_\alpha)-\mu_{s'}(A'_\alpha)+\mu_{s'}(A'_\alpha)\\ &\quad -\alpha\mu_{s'}(T)+\alpha\mu_{s'}(T)-\alpha\mu_{s^0}(T)|\\ &\leqq 2\|\mu_{s'}-\mu_{s^0}\|<\varepsilon/13,\end{aligned}$$

by Lemma 3 there exists $A^0_\alpha \in \mathscr{A}(\mu_{s^0})$ such that $\mu_{s^0}(A'_\alpha \triangle A^0_\alpha) \leqq 6\varepsilon/13$. Analogously, given A^0_α, there exists $A''_\alpha \in \mathscr{A}(\mu_{s''})$ such that $\mu_{s''}(A^0_\alpha \triangle A''_\alpha) \leqq 6\varepsilon/13$.

Hence

$$\begin{aligned}\mu_s(A'_\alpha \triangle A''_\alpha) &\leqq |\mu_s(A'_\alpha \triangle A''_\alpha)-\mu_{s^0}(A'_\alpha \triangle A''_\alpha)|+\mu_{s^0}(A'_\alpha \triangle A''_\alpha)\\ &\leqq \|\mu_s-\mu_{s^0}\|+\mu_{s^0}(A'_\alpha \triangle A^0_\alpha)+\mu_{s^0}(A^0_\alpha \triangle A''_\alpha)\\ &\leqq \varepsilon/26+6\varepsilon/13+\|\mu_{s^0}-\mu_{s''}\|+\mu_{s''}(A^0_\alpha \triangle A''_\alpha)\\ &\leqq \varepsilon. \quad \blacksquare\end{aligned}$$

In the following Lemmas, the symbol $\sup\limits_{\lambda_j(s)>0}$ is a shorthand notation for $\sup\limits_{\{j \in \mathbb{N} : \lambda_j(s)>0\}}$.

LEMMA 4. — *Let $s \to \mu_s$ be a continuous map from a metric space* S *into the space* $\mathscr{M}$ *and let $(B(s_j, \eta_j))_{j=1,\ldots,N}$ be a finite open covering of* S. *Let $(\lambda_j(\,.\,))_{j=1,\ldots,N}$ be a continuous partition of unity subordinate to it such that $\lambda_j(s_j)=1$.*

For any center s_j, $j=1, \ldots, N$, let be defined a finite increasing family $(\bar{A}^{s_j}_{i/M})_{i=0,\ldots,M}$ such that

$$\mu_{s_j}(\bar{A}^{s_j}_{i/M})=\frac{i}{M}\mu_{s_j}(T) \qquad (i \in \{0, \ldots, M\}).$$

Then for each $s \in S$ there exists an increasing family $(A^s_\alpha)_\alpha$ that extends the family $(\bar{A}^{s_j}_{i/M})_i$ in the sense that $A^{s_j}_{i/M}=\bar{A}^{s_j}_{i/M}$ for every i and j, and such that the following properties hold:

(i) $|\mu_s(A^s_\alpha)-\alpha\mu_s(T)| \leqq 6 \sup\limits_{\lambda_j(s)>0} \|\mu_s-\mu_{s_j}\| \ (\alpha \in [0, 1])$;

(ii) *for* $\alpha \in \left[\dfrac{i}{M}, \dfrac{i+1}{M}\right]$ *and any center* s_j,

$$\mu_{s_j}(A^s_\alpha \triangle A^{s_j}_\alpha) \leqq \sup_{\lambda_k(s)>0} \mu_{s_j}(\bar{A}^{s_k}_{(i+1)/M} \triangle \bar{A}^{s_j}_{(i+1)/M})$$

$$+\sup_{\lambda_k(s)>0}\|\mu_{s_j}-\mu_{s_k}\|+\frac{1}{M}(\sup_{\lambda_k(s)>0}\mu_{s_k}(T)+\mu_{s_j}(T));$$

(iii) $\lim_{s\to s^*}\sup_{\alpha\in[0,1]}\mu_0(A^s_\alpha\triangle A^{s^*}_\alpha)=0.$

Proof. — For each $s\in S$, first we will define the sets $(A^i_{i/M})_i$ by interpolating among the given families $(\bar{A}^{s_j}_{i/M})_i$, taking from each set a subset having measure proportional to the corresponding $\lambda_i(s)$. Then we extend the construction for $\alpha\in]i/M,(i+1)/M[$. Finally we check that (i)-(iii) hold.

I. For any set $A\subseteq T$, we define $A^1=A$ and $A^0=T\setminus A$. We denote by $\mathscr{K}$ the set of all $N\times(M-1)$ matrices $\Gamma=(\gamma_{ij})$ whose elements are in $\{0,1\}$.

Now we define

$$\begin{aligned}A(\Gamma)=&(\bar{A}^{s_1}_{1/M})^{\gamma_{11}}\cap\ldots\cap(\bar{A}^{s_N}_{1/M})^{\gamma_{1N}}\\&\cap(\bar{A}^{s_1}_{2/M})^{\gamma_{21}}\cap\ldots\cap(\bar{A}^{s_N}_{2/M})^{\gamma_{2N}}\\&\ldots\ldots\ldots\ldots\ldots\ldots\\&\cap(\bar{A}^{s_1}_{(M-1)/M})^{\gamma_{M-1,1}}\cap\ldots\cap(\bar{A}^{s_N}_{(M-1)/M})^{\gamma_{M-1,N}}.\end{aligned}$$

Note that:

(*a*) since the family $(\bar{A}^{s_j}_{i/M})_i$ is increasing in i, $A(\Gamma)=\varnothing$ if $\exists\, i,j:\gamma_{ij}=1$, $\gamma_{i+1,j}=0$; moreover, if $\Gamma_1\neq\Gamma_2$, then $A(\Gamma_1)\cap A(\Gamma_2)=\varnothing$;

(*b*) for any i, j

$$\bar{A}^{s_j}_{i/M}=\dot{\bigcup_{\substack{\Gamma\in\mathscr{K}\\ \gamma_{ij}=1}}}A(\Gamma),$$

i. e. the family at the r. h. s. is a partition of $\bar{A}^{s_j}_{i/M}$;

(*c*) $$\bigcup_{\substack{\Gamma\in\mathscr{K}\\ \gamma_{ij}=0,\,\gamma_{ik}=1}}A(\Gamma)=A^{s_k}_{i/M}\setminus A^{s_j}_{i/M},\qquad\bigcup_{\substack{\Gamma\in\mathscr{K}\\ \gamma_{ij}=1,\,\gamma_{ik}=1}}A(\Gamma)=A^{s_j}_{i/M}\cap A^{s_k}_{i/M}.$$

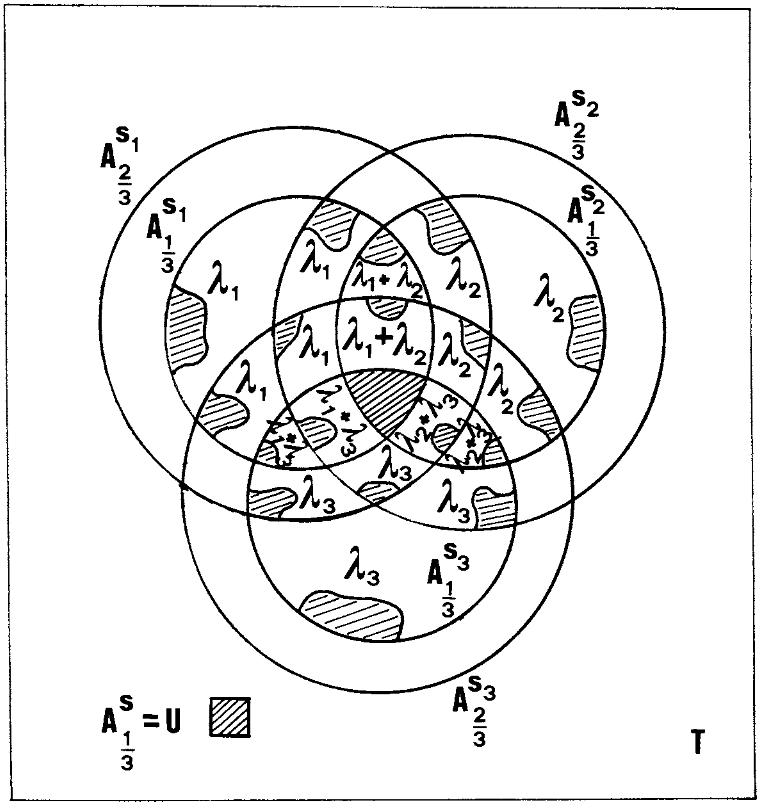
By lemma 1, for each $\Gamma\in\mathscr{K}$ there exists a family $(A(\Gamma)_\alpha)_{\alpha\in[0,1]}$ refining $A(\Gamma)$ with respect to the measure $(\mu_0,\mu_{s_1},\ldots,\mu_{s_N})$. Define

$$\beta^i_\Gamma(s)=\sum_{k=1}^{N}\gamma_{ik}\lambda_k(s)$$

and

$$A^s_{i/M}=\dot{\bigcup_{\Gamma\in\mathscr{K}}}A(\Gamma)_{\beta^i_\Gamma(s)}\qquad(1)$$

(see *Fig.*, where the case $N=M=3$ is described).



The family $(A^{s}_{i/M})_i$ coincides with $(\bar{A}^{s_j}_{i/M})_i$ for $s=s_j$; in fact we have $\beta^i_\Gamma(s_j)=\gamma_{ij}$ so that, by (*b*),

$$A^{s_j}_{i/M} = \bigcup_{\Gamma\in\mathcal{K}} A(\Gamma)_{\gamma_{ij}} = \bigcup_{\substack{\Gamma\in\mathcal{K}\\ \gamma_{ij}=1}} A(\Gamma) = \bar{A}^{s_j}_{i/M}.$$

Next we have:

$$\begin{aligned}\mu_{s_j}(A^{s}_{i/M}) &= \sum_{\Gamma\in\mathcal{K}} \mu_{s_j}(A(\Gamma)_{\beta^i_\Gamma(s)}) = \sum_{\Gamma\in\mathcal{K}} \beta^i_\Gamma(s)\,\mu_{s_j}(A(\Gamma))\\ &= \sum_{\Gamma\in\mathcal{K}} \left(\sum_{k=1}^{N} \gamma_{ik}\lambda_k(s)\right)\mu_{s_j}(A(\Gamma))\\ &= \sum_{k=1}^{N} \lambda_k(s) \sum_{\Gamma\in\mathcal{K}} \gamma_{ik}\,\mu_{s_j}(A(\Gamma)) \qquad (2)\end{aligned}$$

$$= \sum_{k=1}^{N} \lambda_k(s)\, \mu_{s_j}\Big(\bigcup_{\substack{\Gamma \in \mathscr{K} \\ \gamma_{ik}=1}} A(\Gamma)\Big)$$

$$= \sum_{k=1}^{N} \lambda_k(s)\, \mu_{s_j}(A^{s_k}_{i/M}).$$

II. Set, for $\alpha = (1-t)\, i/M + t\,(i+1)/M$ ($t \in [0, 1]$) and $s \in S$,

$$A^s_\alpha = \bigcup_{\Gamma \in \mathscr{K}} A(\Gamma)_{(1-t)\, \beta^i_\Gamma(s) + t\, \beta^{i+1}_\Gamma(s)}.$$

Remark that by the above definition and (1), it follows that

$$\mu_{s_j}(A^s_\alpha) = (1-t)\, \mu_{s_j}(A^s_{i/M}) + t\, \mu_{s_j}(A^s_{(i+1)/M}).$$

We claim that

$$\mu_{s_j}(A^s_\alpha) = \sum_{k=1}^{N} \lambda_k(s)\, \mu_{s_j}(A^{s_k}_\alpha) \qquad (j = 1, \ldots, N;\ \alpha \in [0, 1];\ s \in S).$$

In fact, for α as above, we have:

$$\mu_{s_j}(A^s_\alpha) = \sum_{\Gamma \in \mathscr{K}} [(1-t)\, \beta^i_\Gamma(s) + t\, \beta^{i+1}_\Gamma(s)]\, \mu_{s_j}(A(\Gamma))$$

$$= (1-t) \sum_{k=1}^{N} \lambda_k(s) \sum_{\Gamma \in \mathscr{K}} \gamma_{ik}\, \mu_{s_j}(A(\Gamma))$$

$$+ t \sum_{k=1}^{N} \lambda_k(s) \sum_{\Gamma \in \mathscr{K}} \gamma_{i+1,\, k}\, \mu_{s_j}(A(\Gamma))$$

$$= (1-t) \sum_{k=1}^{N} \lambda_k(s)\, \mu_{s_j}(A^{s_k}_{i/M}) + t \sum_{k=1}^{N} \lambda_k(s)\, \mu_{s_j}(A^{s_k}_{(i+1)/M})$$

$$= \sum_{k=1}^{N} \lambda_k(s)\, [(1-t)\, \mu_{s_j}(A^{s_k}_{i/M}) + t\, \mu_{s_j}(A^{s_k}_{(i+1)/M})]$$

$$= \sum_{k=1}^{N} \lambda_k(s)\, \mu_{s_j}(A^{s_k}_\alpha).$$

III. We are now in the position of proving (i). Fix $s \in S$ and $\alpha \in [0, 1]$ and set $\omega_s = \sup \{ \| \mu_s - \mu_{s_j} \| : \lambda_j(s) > 0 \}$. We have:

$$| \mu_s(A^s_\alpha) - \alpha \mu_s(T) | \leqq | \mu_s(A^s_\alpha) - \mu_{s_j}(A^s_\alpha) |$$

$$+ | \mu_{s_j}(A^s_\alpha) - \alpha \mu_{s_j}(T) | + \alpha | \mu_{s_j}(T) + \mu_s(T) |$$

$$\leqq 2\,\omega_s + \left| \sum_{k=1}^{N} \lambda_k(s)\,\mu_{s_j}(A_\alpha^{s_k}) - \alpha\mu_{s_j}(T) \right|$$

$$\leqq 2\,\omega_s + \sum_{k=1}^{N} \lambda_k(s)\,[\,|\,\mu_{s_j}(A_\alpha^{s_k}) - \mu_{s_k}(A_\alpha^{s_k})\,|$$

$$+ \alpha\,|\,\mu_{s_k}(T) - \mu_{s_j}(T)\,|\,]$$

$$\leqq 6\,\omega_s.$$

In order to prove (ii), note first that

$$\begin{aligned} A_{i/M}^{s} \triangle A_{i/M}^{s_j} &= \Big(\dot{\bigcup_{\Gamma \in \mathscr{K}}} A(\Gamma)_{\beta_\Gamma^i(s)} \Big) \triangle \Big(\dot{\bigcup_{\Gamma \in \mathscr{K}}} A(\Gamma)_{\beta_\Gamma^i(s_j)} \Big) \\ &= \bigcup_{\Gamma \in \mathscr{K}} \big(A(\Gamma)_{\beta_\Gamma^i(s)} \triangle A(\Gamma)_{\beta_\Gamma^i(s_j)} \big) \end{aligned} \tag{3}$$

and that, by a calculation similar to (2) and by (c),

$$\mu_{s_j}\Big(\bigcup_{\substack{\Gamma \in \mathscr{K} \\ \gamma_{ij}=0}} A(\Gamma)_{\beta_\Gamma^i(s)} \Big) = \sum_{k=1}^{N} \lambda_k(s)\,\mu_{s_j}(A_{i/M}^{s_k} \setminus A_{i/M}^{s_j}), \tag{4}$$

$$\mu_{s_j}\Big(\bigcup_{\substack{\Gamma \in \mathscr{K} \\ \gamma_{ij}=1}} (A(\Gamma) \setminus A(\Gamma)_{\beta_\Gamma^i(s)}) \Big) = \sum_{k=1}^{N} \lambda_k(s)\,\mu_{s_j}(A_{i/M}^{s_j} \setminus A_{i/M}^{s_k}). \tag{5}$$

Therefore, for any i, j, from (3) and recalling that $\beta_\Gamma^i(s_j) = \gamma_{ij}$, we have

$$\mu_{s_j}(A_{i/M}^{s} \triangle A_{i/M}^{s_j}) = \mu_{s_j}\Big(\bigcup_{\substack{\Gamma \in \mathscr{K} \\ \gamma_{ij}=0}} A(\Gamma)_{\beta_\Gamma^i(s)} \Big) + \mu_{s_j}\Big(\bigcup_{\substack{\Gamma \in \mathscr{K} \\ \gamma_{ij}=1}} (A(\Gamma) \setminus A(\Gamma)_{\beta_\Gamma^i(s)} \Big)$$

and from (4), (5) this last expression is

$$\begin{aligned} \sum_{k=1}^{N} \lambda_k(s)\,\mu_{s_j}(A_{i/M}^{s_k} \setminus A_{i/M}^{s_j}) &+ \sum_{k=1}^{N} \lambda_k(s)\,\mu_{s_j}(A_{i/M}^{s_j} \setminus A_{i/M}^{s_k}) \\ &= \sum_{k=1}^{N} \lambda_k(s)\,\mu_{s_j}(A_{i/M}^{s_k} \triangle A_{i/M}^{s_j}) \\ &\leqq \sup\{\, \mu_{s_j}(A_{i/M}^{s_k} \triangle A_{i/M}^{s_j}) : \lambda_k(s) > 0 \,\}. \end{aligned}$$

Hence (ii) holds for $\alpha = i/M$.

In order to prove (ii) for α in $]i/M, (i+1)/M[$, let us note that

$$\begin{aligned} A_\alpha^{s} \setminus A_\alpha^{s'} &\subseteq [(A_{(i+1)/M}^{s} \setminus A_{(i+1)/M}^{s_j}) \setminus A_\alpha^{s'}] \cup [A_{(i+1)/M}^{s_j} \setminus A_\alpha^{s'}] \\ &\subseteq (A_{(i+1)/M}^{s} \setminus A_{(i+1)/M}^{s_j}) \cup (A_{(i+1)/M}^{s_j} \setminus A_{i/M}^{s'}), \end{aligned}$$

so that

$$\mu_{s_j}(A_\alpha^s \setminus A_\alpha^{s_j}) \leqq \mu_{s_j}(A_{(i+1)/M}^s \setminus A_{(i+1)/M}^{s_j}) + \mu_{s_j}(A_{(i+1)/M}^{s_j} \setminus A_{i/M}^{s_j})$$

and

$$\mu_{s_j}(A_\alpha^{s_j} \setminus A_\alpha^s) \leqq \mu_{s_j}(A_{(i+1)/M}^{s_j} \setminus A_{(i+1)/M}^s) + \mu_{s_j}(A_{(i+1)/M}^s \setminus A_{i/M}^s).$$

Hence

$$\begin{aligned}
\mu_{s_j}(A_\alpha^s \triangle A_\alpha^{s_j}) &\leqq \mu_{s_j}(A_{(i+1)/M}^s \triangle A_{(i+1)/M}^{s_j}) \\
&\quad + \mu_{s_j}(A_{(i+1)/M}^{s_j} \setminus A_{i/M}^{s_j}) + \mu_{s_j}(A_{(i+1)/M}^s \setminus A_{i/M}^s) \\
&\leqq \sup \{ \mu_{s_j}(A_{(i+1)/M}^{s_k} \triangle A_{(i+1)/M}^{s_j}) : \lambda_k(s) > 0 \} \\
&\quad + (1/M)\, \mu_{s_j}(T) + \sum_{k=1}^{N} \lambda_k(s)\, \mu_{s_j}(A_{(i+1)/M}^{s_k} \setminus A_{i/M}^{s_k}) \\
&\leqq \sup_{\lambda_k(s)>0} \mu_{s_j}(A_{(i+1)/M}^{s_k} \triangle A_{(i+1)/M}^{s_j}) + (1/M)\, \mu_{s_j}(T) \\
&\quad + \sum_{k=1}^{N} \lambda_k(s) \left| \mu_{s_j}(A_{(i+1)/M}^{s_k} \setminus A_{i/M}^{s_k}) - \mu_{s_k}(A_{(i+1)/M}^{s_k} \setminus A_{i/M}^{s_k}) \right| \\
&\quad + \sum_{k=1}^{N} \lambda_k(s)\, \mu_{s_k}(A_{(i+1)/M}^{s_k} \setminus A_{i/M}^{s_k}) \\
&\leqq \sup_{\lambda_k(s)>0} \mu_{s_j}(\bar{A}_{(i+1)/M}^{s_k} \triangle A_{(i+1)/M}^{s_j}) \\
&\quad + \sup_{\lambda_k(s)>0} \| \mu_{s_j} - \mu_{s_k} \| \\
&\quad + (1/M) \sup_{k=1,\ldots,N} \mu_{s_k}(T).
\end{aligned}$$

This proves (ii).

Finally we prove (iii); for $\alpha = (1-t)\, i/M + t\,(i+1)/M$ we have

$$\begin{aligned}
\mu_0(A_\alpha^s \triangle A_\alpha^{s^*}) &= \sum_{\Gamma \in \mathscr{K}} \mu_0(A(\Gamma)_{(1-t)\beta_\Gamma^i(s) + t\beta_\Gamma^{i+1}(s)} \\
&\qquad \triangle A(\Gamma)_{(1-t)\beta_\Gamma^i(s^*) + t\beta_\Gamma^{i+1}(s^*)}) \\
&= \sum_{\Gamma \in \mathscr{K}} \{ | [(1-t)\beta_\Gamma^i(s) + t\beta_\Gamma^{i+1}(s)] \\
&\qquad - [(1-t)\beta_\Gamma^i(s^*) + t\beta_\Gamma^{i+1}(s^*)] |\, \mu_0(A(\Gamma)) \} \\
&\leqq (1-t) \sum_{\Gamma \in \mathscr{K}} | \beta_\Gamma^i(s) - \beta_\Gamma^i(s^*) |\, \mu_0(A(\Gamma)) \\
&\quad + t \sum_{\Gamma \in \mathscr{K}} | \beta_\Gamma^{i+1}(s) - \beta_\Gamma^{i+1}(s^*) |\, \mu_0(A(\Gamma)).
\end{aligned}$$

By taking the limit as s tends to s^* we conclude the proof. ■

LEMMA 5. — *Let $s \to \mu_s$ be a continuous map from a compact metric space* S *into the space* $\mathcal{M}$ *and, for each* $s \in S$, *let* $(\bar{A}^s_\alpha)_\alpha$ *be an increasing family, continuous with respect to s and such that, for some* $\varepsilon > 0$,

$$|\mu_s(\bar{A}^s_\alpha) - \alpha\,\mu_s(T)| < \varepsilon \qquad (\alpha \in [0, 1], s \in S).$$

For every $s \in S$ *there exists an increasing family* $(A^s_\alpha)_\alpha$ *continuous with respect to s and such that*

(i) $|\mu_s(A^s_\alpha) - \alpha\mu_s(T)| < \varepsilon/10 \qquad (\alpha \in [0, 1])$;

(ii) $\sup_{\alpha \in [0, 1]} \mu_s(\bar{A}^s_\alpha \triangle A^s_\alpha) < 10\,\varepsilon$.

Proof. — By continuity, for each $s \in S$ there is a $\eta_s > 0$ such that $d(s, s') < 2\eta_s$ implies $\|\mu_s - \mu_{s'}\| < \varepsilon/60$ and $\mu_{s'}(\bar{A}^s_\alpha \triangle \bar{A}^{s'}_\alpha) < \varepsilon$. The open balls $B(s, \eta_s)$ cover S. Let $\{B(s_j, \eta_j) : j = 1, \ldots, N\}$ be a finite sub-covering and $\{\lambda_j : j = 1, \ldots, N\}$ be a continuous partition of unity subordinate to it and such that $\lambda_j(s_j) = 1$, $j = 1, \ldots, N$.

Let $(A^{s_j}_\alpha)_\alpha$ be the families defined by Lemma 3 by taking $\mu = \mu_{s_j}$.

Fix j such that $\mu_{s_j}(T) = \max\{\mu_{s_k}(T) : k = 1, \ldots, N\}$ and choose $M \geqq 2\mu_{s_j}(T)/\varepsilon$. By Lemma 4, extend the collection $(A^{s_k}_{i/M})_{i=0, \ldots, M}$ $(k = 1, \ldots, N)$ to the family $(A^s_\alpha)_{\alpha \in [0, 1]} (s \in S)$.

The continuity of $s \to (A^s_\alpha)_{\alpha \in [0, 1]}$ follows from (iii) of Lemma 4, recalling that $\mu_s \ll \mu_0$ for each $s \in S$.

The choice of η_s and (i) of Lemma 4 imply that (i) holds. Moreover

$$\mu_{s_j}(\bar{A}^s_\alpha \triangle A^s_\alpha) \leqq \mu_{s_j}(\bar{A}^s_\alpha \triangle \bar{A}^{s_j}_\alpha) + \mu_{s_j}(\bar{A}^{s_j}_\alpha \triangle A^{s_j}_\alpha) + \mu_{s_j}(A^{s_j}_\alpha \triangle A^s_\alpha).$$

By the choice of η_s and (ii) of Lemma 3, the r. h. s. is bounded by

$$\varepsilon + 6\,\varepsilon + \mu_{s_j}(A^{s_j}_\alpha \triangle A^s_\alpha),$$

which, by (ii) of Lemma 4 and the choice of M, yields

$$\mu_{s_j}(\bar{A}^s_\alpha \triangle A^s_\alpha) \leqq \left(9 + \frac{1}{60}\right)\varepsilon.$$

Since $\|\mu_{s_j} - \mu_s\| < \varepsilon/60$, (ii) follows. ■

The following theorem shows the existence of a selection $(\tilde{A}^s_\alpha)$ from $\mathcal{A}(\mu_s)$, continuously depending on s.

THEOREM 1. — *Let $s \to \mu_s$ be a continuous map from a compact metric space* S *into the space* $\mathcal{M}$. *For every* $s \in S$ *there an increasing family* $(\tilde{A}^s_\alpha)_\alpha$ *of measurable subsets of* T *satisfying*

$$\mu_s(\tilde{A}^s_\alpha) = \alpha\mu_s(T) \qquad (\alpha \in [0, 1]) \tag{6}$$

and such that the map $s \to (\tilde{A}_\alpha^s)_\alpha$ *is continuous.*

Proof. — We assume that we have defined for s in S an increasing family $(A_\alpha^{s,n})_\alpha$ which is continuous with respect to s and satisfies

$$|\mu_s(A_\alpha^{s,n}) - \alpha\mu_s(T)| < 10^{-n}.$$

By Lemma 2, the above is true for $n=1$ taking a family $(A_\alpha^{s,1})_\alpha$ constant with respect to s.

We obtain the existence of an increasing family $(A_\alpha^{s,n+1})_\alpha$ continuous with respect to s and such that

$$|\mu_s(A_\alpha^{s,n+1}) - \alpha\mu_s(T)| < 10^{-(n+1)} \tag{7}$$

and

$$\mu_s(A_\alpha^{s,n+1} \triangle A_\alpha^{s,n}) < 10^{-(n-1)}. \tag{8}$$

In fact, set in Lemma 5 $\bar{A}_\alpha^s$ to be $A_\alpha^{s,n}$ and ε to be 10^{-n} to infer the existance of a family, denoted by $(A_\alpha^{s,n+1})_\alpha$, satisfying (7) and (8).

Consider now the sequence $((A_\alpha^{s,n})_\alpha)_{n\in\mathbb{N}}$ defined by the above recursive procedure: we wish to show that it converges to a family $(\tilde{A}_\alpha^s)_\alpha$ which is continuous with respect to s and satisfies (6).

Property (8) implies that the sequence $(A_\alpha^{s,n})_n$ (s and α fixed) is a Cauchy sequence in $\mathscr{F}$ supplied with the pseudometric $\rho_s(A, B) = \mu_s(A \triangle B)$. The procedure in Oxtoby [6], Chap. 10, defines a limit family $(\tilde{A}_\alpha^s)_\alpha$, which is increasing: $\tilde{A}_\alpha^s = \bigcup_{n\in\mathbb{N}} \bigcap_{m \geqq n} A_\alpha^{s,m}$.

By the inequality

$$|\mu_s(A) - \mu_s(B)| \leqq \mu_s(A \triangle B)$$

and (7) we have

$$\mu_s(\tilde{A}_\alpha^s) = \lim_{n\to\infty} \mu_s(A_\alpha^{s,n}) = \alpha\mu_s(T).$$

In order to check the continuity of the map $s \to (\tilde{A}_\alpha^s)_\alpha$, fix $\varepsilon > 0$ and $s^0 \in S$. Since the inequality (8) is uniform with respect to s and α, there exists an $\bar{n}$ such that $\mu_s(A_\alpha^{s,\bar{n}} \triangle \tilde{A}_\alpha^s) < \varepsilon/5$ for every s in S and α in [0, 1]. Let $\delta > 0$ be such that

$$\|\mu_s - \mu_{s^0}\| < \varepsilon/10 \qquad [s \text{ in } B(s^0, \delta)]$$

and

$$\sup_{\alpha\in[0,1]} \mu_s(A_\alpha^{s',\bar{n}} \triangle A_\alpha^{s'',\bar{n}}) < \varepsilon/5 \qquad [s, s' \text{ and } s'' \text{ in } B(s^0, \delta)].$$

Then for every $\alpha \in [0, 1]$, s, s' and s'' in $B(s^0, \delta)$, we have:

$$\begin{aligned}\mu_s(\tilde{A}_\alpha^{s'} \triangle \tilde{A}_\alpha^{s''}) &\leqq \mu_s(\tilde{A}_\alpha^{s'} \triangle A_\alpha^{s',\bar{n}}) + \mu_s(A_\alpha^{s',\bar{n}} \triangle \tilde{A}_\alpha^{s''}) \\ &\leqq \mu_s(\tilde{A}_\alpha^{s'} \triangle A_\alpha^{s',\bar{n}}) + \mu_s(A_\alpha^{s',\bar{n}} \triangle A_\alpha^{s'',\bar{n}}) + \mu_s(A_\alpha^{s'',\bar{n}} \triangle \tilde{A}_\alpha^{s''})\end{aligned}$$

$$\leqq \| \mu_s - \mu_{s'} \| + \mu_{s'}(\tilde{A}_\alpha^{s'} \triangle A_\alpha^{s', \bar{n}}) + \mu_s(A_\alpha^{s', \bar{n}} \triangle A_\alpha^{s'', \bar{n}})$$
$$+ \| \mu_s - \mu_{s''} \| + \mu_{s''}(A_\alpha^{s'', \bar{n}} \triangle \tilde{A}_\alpha^{s''})$$
$$< \varepsilon. \quad \blacksquare$$

COROLLARY. — *Under the same assumptions, for every* $\eta > 0$ *and for every increasing family* $(A_\alpha)_\alpha$ *satisfying*

$$| \mu_s(A) - \alpha . \mu_s(T) | < \eta \qquad (\alpha \in [0, 1], s \in S),$$

the family $(\tilde{A}_\alpha^s)_\alpha$ *of Theorem* 1 *can be chosen as to satisfy, in addition,*

$$\mu_s(\tilde{A}_\alpha^s \triangle A_\alpha) < \eta \qquad (\alpha \in [0, 1], s \in S).$$

Proof. — Set $A_\alpha^{s, 1}$ to be A_α in the proof of Theorem 1. ■

REFERENCES

[1] A. FRYSZKOWSKI, Continuous Selections for a Class of Non-Convex Multivalued Maps, *Studia Mathematica*, T. **LXXVI**, 1983, pp. 163-174.

[2] P. HALMOS, The Range of a Vector Measure, *Bull. Am. Math. Soc.*, Vol. **54**, 1948, pp. 416-421.

[3] P. HALMOS, *Measure Theory*, Van Nostrand, Princeton, 1950.

[4] A. LIAPUNOV, Sur les fonctions-vecteurs complètement additives, *Bull. Acad. Sci. U.R.S.S., Ser. Math.*, Vol. 4, 1940, pp. 465-478 (Russian).

[5] A. LIAPUNOV, same as above, Vol. **10**, 1946, pp. 277-279 (Russian); for an English version, *see* J. DIESTEL and J. J. UHL, *Vector Measures*, 1977, A.M.S., Providence, Rhode Island, p. 262.

[6] J. OXTOBY, *Measure and Category*, Second Edition, Springer Verlag, New York, 1980.

[7] M. VALADIER, Une mesure vectorielle sans atoms dont l'ensemble des valeurs est non convexe, *Boll. U.M.I.*, Serie VI, Vol. **II-C**, No. 1, 1983, pp. 293-296.

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