

## Ljusternik-Schnirelmann theory on $C^1$ -manifolds

by

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**ABSTRACT.** — Let  $M$  be a complete Finsler manifold of class  $C^1$ . It is shown that if  $M$  contains a compact subset of category  $k$  (in  $M$ ), then each function  $f \in C^1(M, \mathbb{R})$  which is bounded below and satisfies the Palais-Smale condition must necessarily have  $k$  critical points. This should be compared with the known result that  $f$  has at least  $\text{cat}(M)$  critical points provided  $M$  is of class  $C^2$ . An application is given to an eigenvalue problem for a quasilinear differential equation involving the  $p$ -Laplacian  $-\text{div}(|\nabla u|^{p-2} \nabla u)$ ,  $1 < p < \infty$ .

*Key words:* Finsler manifold, critical point, Ljusternik-Schnirelmann theory, Ekeland's variational principle, category, genus, eigenvalue problem,  $p$ -Laplacian.

**RÉSUMÉ.** — Soit  $M$  une variété de Finsler complète de classe  $C^1$ . On démontre que si  $M$  contient un sous-ensemble compacte de catégorie  $k$  (dans  $M$ ), alors toute fonction  $f \in C^1(M, \mathbb{R})$  qui est bornée inférieurement et satisfait à la condition de Palais-Smale doit nécessairement avoir  $k$  point critique. Ce résultat est à rapprocher du théorème connu selon lequel  $f$  a au moins  $\text{cat}(M)$  point critique lorsque  $M$  est de classe  $C^2$ . On donne une application à un problème de valeur propre pour une équation

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différentielle quasi-linéaire faisant intervenir le  $p$ -Laplacien  
 $-\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ ,  $1 < p < \infty$ .

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## 1. INTRODUCTION

Let  $M$  be a compact  $C^2$ -manifold without boundary and  $f: M \rightarrow \mathbb{R}$  a continuously differentiable function. A classical result by Ljusternik and Schnirelmann [14], *cf.* also ([8], [21]), asserts that if  $M$  is of category  $k$  [denoted  $\operatorname{cat}(M) = k$ ], then  $f$  has at least  $k$  distinct critical points (all definitions will be given in the next section). This result has been generalized by Palais ([16], [17]) who proved the following

1.1. THEOREM. — *Let  $M$  be a  $C^2$  Finsler manifold (without boundary) and  $f \in C^1(M, \mathbb{R})$  a function which is bounded below and such that for each  $c \in \mathbb{R}$  the set  $f_c = \{x \in M : f(x) \leq c\}$  is complete in the Finsler metric for  $M$ . If  $f$  satisfies the Palais-Smale condition and if  $\operatorname{cat}(M) = k$ , then  $f$  has at least  $k$  distinct critical points.*

The key ingredient in the proof of Theorem 1.1 is a deformation lemma which in its simplest form says that if  $c$  is not a critical value of  $f$  and if  $\varepsilon > 0$  is small enough, then there exists a mapping  $\eta: [0, 1] \times M \rightarrow M$  satisfying  $\eta(0, x) = x$ ,  $f(\eta(t, x)) \leq f(x)$  for all  $t$  and  $x$ , and  $\eta(1, f_{c+\varepsilon}) \subset f_{c-\varepsilon}$  (i. e.,  $\eta$  deforms  $f_{c+\varepsilon}$  to  $f_{c-\varepsilon}$ ). The deformation is constructed by letting  $\eta(t, x)$  move along the integral lines of a pseudogradient vector field for  $f$  as  $t$  varies from 0 to 1. As is well known from the theory of ordinary differential equations, integral lines may not exist unless the vector field is locally Lipschitz continuous. To carry out the above construction it seems therefore necessary to assume that  $M$  is at least of class  $C^{2-}$  (a mapping is of class  $C^{2-}$  if it is differentiable and the derivative is locally Lipschitz continuous).

In this paper we will be concerned with a generalization of Theorem 1.1 to  $C^1$ -manifolds. Ideally, one would like to show that the conclusion remains valid if  $M$  is a  $C^1$  Finsler manifold. Our result is slightly weaker, yet it seems to be sufficient for most of practical purposes. It asserts that

if  $M$  is  $C^1$  and contains a compact set of category  $k$  (in  $M$ ), other assumptions being as in Theorem 1.1, then the conclusion still holds. Note in particular that if  $M$  contains compact sets of arbitrarily large category, then  $f$  has infinitely many critical points.

The proof of Theorem 1.1 is carried out as follows. Set

$$c_j = \inf_{\text{cat}_M(A) \geq j} \sup_{x \in A} f(x),$$

where  $1 \leq j \leq k$  and  $\text{cat}_M(A)$  denotes the category of  $A$  in  $M$ . Assume for simplicity that all  $c_j$  are finite and distinct. Then, if  $c_j$  is not a critical value, one finds an  $A$  with  $\text{cat}_M(A) \geq j$  and  $f(x) \leq c + \varepsilon \forall x \in A$ . By the deformation lemma, if  $B = \eta(1, A)$ , then  $\text{cat}_M(B) \geq j$  and  $f(x) \leq c - \varepsilon \forall x \in B$ , a contradiction. So all  $c_j$  are critical and  $f$  has at least  $k$  critical points. As we pointed out earlier, this argument is not readily applicable if  $M$  is only of class  $C^1$ . Our proof is therefore quite different. We define

$$c_j = \inf_{A \in \Lambda_j} \sup_{x \in A} f(x),$$

where  $\Lambda_j = \{A \subset M : \text{cat}_M(A) \geq j \text{ and } A \text{ is compact}\}$ . On  $\Lambda_j$  we introduce the Hausdorff metric  $dist$  and set  $\Pi(A) = \sup_{x \in A} f(x)$ . Again, assume for simplicity that all  $c_j$  are distinct. By Ekeland's variational principle (see the next section), there exists an  $A \in \Lambda_j$  such that

$$\Pi(B) - \Pi(A) \geq -\varepsilon dist(A, B), \quad \forall B \in \Lambda_j,$$

If  $c_j$  is not a critical value, then, by slightly deforming  $A$ , we find  $B \in \Lambda_j$  with  $dist(A, B) \leq s$  and  $\Pi(B) - \Pi(A) < -\varepsilon s$  for all small  $s > 0$ . So  $-\varepsilon s \leq \Pi(B) - \Pi(A) < -\varepsilon s$ , a contradiction. The idea of using Ekeland's principle to show the existence of critical points other than local minima may be found in [2] (Section 5.5), and an argument similar to the above one (using Ekeland's principle on the space of subsets) — in [20].

Suppose now that  $X$  is a Banach space and  $f, g \in C^1(X, \mathbb{R})$  are two even functions. Consider the eigenvalue problem

$$Find (x, \lambda) \in X \times \mathbb{R} \text{ such that } f'(x) = \lambda g'(x) \text{ and } g(x) = b. \quad (1)$$

Problems of this type have been studied by several authors. See e. g. [1], [3], [5], [10], [11], [19], [23]. If  $b \neq g(0)$  is a regular value of  $g$ , then  $M = g^{-1}(b)$  is a  $C^1$ -manifold,  $0 \notin M$  and there is a one-to-one correspondence between solutions of (1) and critical points of  $f|_M$ . Assuming in addition that  $f|_M$  is bounded below and  $g \in C^2(X, \mathbb{R})$ , we may pass to the quotient space  $\tilde{M} = M/\sim$ , where  $\sim$  is the equivalence relation identifying

$x$  with  $-x$ , and use Theorem 1.1 in order to obtain a lower bound for the number of solutions of (1). The assumption that  $g \in C^2(X, \mathbb{R})$  [or even  $g \in C^{2-}(X, \mathbb{R})$ ] turns out to be too restrictive for some applications, *cf.* Browder [5]. One possible approach to (1) when  $g \in C^1(X, \mathbb{R})$  is by using the Galerkin approximations (*see e. g.* [5]). However, in order to carry out the limiting procedure it seems necessary to put some restrictions on  $f$  and  $g$  which are not needed in the case of  $g \in C^2(X, \mathbb{R})$ . A different approach has been taken by Amann [1] who has shown that the deformation lemma remains valid whenever  $g \in C^1(X, \mathbb{R})$  and  $M$  is bounded and homeomorphic to the unit sphere by the radial projection mapping. From this he has derived results on (1) which generalize those in [5]. As a corollary to our generalization of Theorem 1.1 we shall show that it is neither necessary to assume that  $g \in C^{2-}(X, \mathbb{R})$  nor that  $M$  is bounded and homeomorphic to the unit sphere.

The paper is organized as follows. In Section 2 we collect some definitions and facts which will be useful later. In Section 3 we state and prove the main theorem. Some of its consequences and extensions are given in Section 4. In Section 5 we present an application to the boundary value problem

$$A u + f(u) = \lambda g(u) \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0,$$

where  $\Omega \subset \mathbb{R}^N$  is bounded and  $A$  is the  $p$ -Laplacian,  $1 < p < \infty$  (in particular,  $A = -\Delta$  if  $p = 2$ ).

I would like to thank Ivar Ekeland for bringing to my attention the problem of generalizing the Ljusternik-Schnirelmann theory to  $C^1$ -manifolds.

## 2. PRELIMINARIES

Let  $M$  be a  $C^1$  Banach manifold (without boundary). Denote the tangent bundle of  $M$  by  $T(M)$  and the tangent space of  $M$  at  $x$  by  $T_x(M)$ . Let  $\|\cdot\|: T(M) \rightarrow [0, +\infty)$  be a continuous function such that

(i) For each  $x \in M$ , the restriction of  $\|\cdot\|$  to  $T_x(M)$ , denoted by  $\|\cdot\|_x$  (or sometimes simply by  $\|\cdot\|$ ), is an admissible norm on  $T_x(M)$ ;

(ii) For each  $x_0 \in M$  and  $k > 1$  there is a trivializing neighbourhood  $U$  of  $x_0$  such that

$$\frac{1}{k} \| \cdot \|_x \leq \| \cdot \|_{x_0} \leq k \| \cdot \|_x, \quad \forall x \in U.$$

The function  $\| \cdot \|$  is called a *Finsler structure* for  $T(M)$ . A regular manifold together with a fixed Finsler structure for  $T(M)$  is called a *Finsler manifold*. Every paracompact  $C^1$  Banach manifold admits a Finsler structure [16] (Theorem 2.11). For a  $C^1$ -path  $\sigma: [a, b] \rightarrow M$  define the *length* of  $\sigma$  by

$$l(\sigma) = \int_b^a \| \sigma'(t) \| dt.$$

If  $x, y$  are two points in the same connected component of  $M$ , let the *distance*  $\rho(x, y)$  be defined as the infimum of  $l(\sigma)$  over all  $\sigma$  joining  $x$  and  $y$ . Then  $\rho$  is a metric for each component of  $M$  (called the *Finsler metric*), and it is consistent with the topology of  $M$  [17] (Section 2).

Let  $M$  be a Finsler manifold and  $f \in C^1(M, \mathbb{R})$ . Denote the differential of  $f$  at  $x$  by  $df(x)$ . Then  $df(x)$  is an element of the cotangent space of  $M$  at  $x$ ,  $T_x(M)^*$ . A point  $x \in M$  is said to be a *critical point* of  $f$  if  $df(x) = 0$ . The corresponding value  $c = f(x)$  will be called a *critical value*. Values other than critical are *regular*. We shall repeatedly use the following notation:

$$K = \{ x \in M : df(x) = 0 \}, \quad K_c = K \cap f^{-1}(c),$$

$$f_c = \{ x \in M : f(x) \leq c \}.$$

If  $M$  is a Finsler manifold, then the cotangent bundle  $T(M)^*$  has a dual Finsler structure given by

$$\| w \| = \sup \{ \langle w, v \rangle : v \in T_x(M), \| v \|_x = 1 \},$$

where  $w \in T_x(M)^*$  and  $\langle , \rangle$  is the duality pairing between  $T_x(M)^*$  and  $T_x(M)$ . It follows that the mapping  $x \mapsto \| df(x) \|$  is well defined and continuous for  $f \in C^1(M, \mathbb{R})$ . A function  $f \in C^1(M, \mathbb{R})$  is said to satisfy the *Palais-Smale condition at the level c*,  $c \in \mathbb{R}$ , [(PS) $_c$  in short] if each sequence  $(x_n) \subset M$  such that  $f(x_n) \rightarrow c$  and  $\| df(x_n) \| \rightarrow 0$  has a convergent subsequence. This is a local version of the following compactness condition due to Palais and Smale: If  $f(x_n)$  is bounded and  $\| df(x_n) \| \rightarrow 0$ , then a subsequence of  $(x_n)$  converges.

Let  $x_0 \in M - K$ . There exists a vector  $V_0 \in T_{x_0}(M)$  such that  $\|V_0\| = 1$  and  $\langle df(x_0), V_0 \rangle > \frac{2}{3} \|df(x_0)\|$ . Set  $v_0 = \frac{3}{2} \|df(x_0)\| V_0$ . Then

$$\|v_0\| < 2 \|df(x_0)\| \quad \text{and} \quad \langle df(x_0), v_0 \rangle > \|df(x_0)\|^2.$$

Such  $v_0$  is called *pseudogradient vector* ([16], [17]). It is easily seen that  $\|v_0\| > \|df(x_0)\|$  and  $\langle df(x_0), v_0 \rangle > \frac{1}{4} \|v_0\|^2$ . Let  $\varphi: U \rightarrow T_{x_0}(M)$  be a chart at  $x_0$ . Denote

$$g = f \circ \varphi^{-1}: \varphi(U) \subset T_{x_0}(M) \rightarrow \mathbb{R}.$$

Then  $df(x_0)$  is locally represented by  $g'(\varphi(x_0))$ , where  $g'$  is the Fréchet derivative of  $g$ . Therefore  $\langle g'(\varphi(x_0)), v_0 \rangle > \frac{1}{4} \|v_0\|^2$ . Since  $g'$  is continuous,

$$\|v_0\| > \|df(x_0)\| \quad \text{and} \quad \langle g'(y), v_0 \rangle > \frac{1}{4} \|v_0\|^2, \quad \forall y \in \varphi(U) \quad (2)$$

provided  $U$  is small enough. We have proved

**2.1. PROPOSITION.** — *For each  $x_0 \in M - K$  there exist a chart  $\varphi: U \rightarrow T_{x_0}(M)$  at  $x_0$  and a vector  $v_0 \in T_{x_0}(M)$  such that (2) is satisfied (with  $g = f \circ \varphi^{-1}$ ).*

In what follows we shall need the notions of Ljusternik-Schnirelmann category and genus. Let  $M$  be a topological space. A set  $A \subset M$  is said to be of *category  $k$  in  $M$*  [denoted  $\text{cat}_M(A) = k$ ] if it can be covered by  $k$  but not  $k - 1$  closed sets which are contractible to a point in  $M$ . If such  $k$  does not exist,  $\text{cat}_M(A) = +\infty$ . Let  $X$  be a real Banach space and  $\Sigma$  the collection of all symmetric subsets of  $X - \{0\}$  which are closed in  $X$  ( $A$  is symmetric if  $A = -A$ ). A nonempty set  $A \in \Sigma$  is said to be of *genus  $k$*  [denoted  $\gamma(A) = k$ ] if  $k$  is the smallest integer with the property that there exists an odd continuous mapping from  $A$  to  $\mathbb{R}^k - \{0\}$ . If there is no such  $k$ ,  $\gamma(A) = +\infty$ , and if  $A = \emptyset$ ,  $\gamma(A) = 0$ . Below we summarize pertinent properties of category and genus.

- 2.2. PROPOSITION.** — *Let  $M$  be a topological space and  $A, B \subset M$ . Then*
- (a)  $\text{cat}_M(A) = 0$  if and only if  $A = \emptyset$ .
  - (b)  $\text{cat}_M(A) = 1$  if and only if  $\bar{A}$  is contractible to a point in  $M$ .
  - (c) If  $A \subset B$ , then  $\text{cat}_M(A) \leq \text{cat}_M(B)$ .
  - (d)  $\text{cat}_M(A \cup B) \leq \text{cat}_M(A) + \text{cat}_M(B)$ .

- (e) If  $\text{cat}_M(B) < \infty$ , then  $\text{cat}_M(A - B) \geq \text{cat}_M(A) - \text{cat}_M(B)$ .
- (f) If  $A$  is closed in  $M$  and  $\alpha: [0, t_0] \times A \rightarrow M$  is a deformation of  $A$  (i. e.,  $\alpha(0, x) = x \forall x \in A$ ), then  $\text{cat}_M(A) \leq \text{cat}_M(\alpha(t_0, A))$ .
- (g) If  $M$  is a Finsler manifold and  $A \subset M$ , then there is a neighbourhood  $U$  of  $A$  such that  $\text{cat}_M(\bar{U}) = \text{cat}_M(A)$ .
- (h) If  $M$  is a connected Finsler manifold and  $A$  is a closed subset of  $M$ , then  $\text{cat}_M(A) \leq \dim(A) + 1$ , where  $\dim$  denotes the covering dimension.

Properties (a)–(d) follow directly from the definition, (e) follows from (c) and (d) because  $A \subset (A - B) \cup B$ , (f) is Theorem 6.2(3) in [16] and (vii) on p. 191 in [17], and (g), (h) follow from Theorems 6.3, 6.4 in [16] upon observing that each Finsler manifold is necessarily an absolute neighbourhood retract (ANR) [15] (Theorem 5).

2.3. PROPOSITION. — Let  $A, B \in \Sigma$ . Then

- (a) If there exists an odd continuous mapping  $f: A \rightarrow B$ , then  $\gamma(A) \leq \gamma(B)$ .
- (b) If  $A \subset B$ , then  $\gamma(A) \leq \gamma(B)$ .
- (c)  $\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$ .
- (d) If  $\gamma(B) < \infty$ ,  $\gamma(\overline{A - B}) \geq \gamma(A) - \gamma(B)$ .
- (e) If  $A$  is compact, then  $\gamma(A) < \infty$  and there exists a neighbourhood  $N$  of  $A$ ,  $N \in \Sigma$ , such that  $\gamma(N) = \gamma(A)$ .
- (f) If  $N$  is a symmetric and bounded neighbourhood of the origin in  $\mathbb{R}^k$  and if  $A$  is homeomorphic to the boundary of  $N$  by an odd homeomorphism, then  $\gamma(A) = k$ .

(g) If  $X_0$  is a subspace of  $X$  of codimension  $k$  and if  $\gamma(A) > k$ , then  $A \cap X_0 \neq \emptyset$ .

Properties (a)–(f) may be found e. g. in [8], [19], [21] and (g) in [19].

In the proof of the main theorem we shall employ the following variational principle due to Ekeland [2] (Corollary 5.3.2), [9].

2.4. PROPOSITION. — Let  $(Z, d)$  be a complete metric space and  $\Pi: Z \rightarrow (-\infty, +\infty]$  a proper (i. e.,  $\Pi \neq +\infty$ ) lower semicontinuous function which is bounded below. If  $\varepsilon > 0$  and  $x \in Z$  satisfy

$$\Pi(x) \leq \inf_{z \in Z} \Pi(z) + \varepsilon^2,$$

then there exists a  $y \in Z$  such that

$$\Pi(y) \leq \Pi(x), \quad d(x, y) \leq \varepsilon$$

and

$$\Pi(z) - \Pi(y) \geq -\varepsilon d(y, z), \quad \forall z \in Z.$$

### 3. THE MAIN THEOREM

3.1. THEOREM. — Suppose that  $M$  is a  $C^1$  Finsler manifold and  $f \in C^1(M, \mathbb{R})$  is bounded below and such that  $f_c$  is complete in the metric  $\rho$  for each  $c \in \mathbb{R}$ . Define

$$c_j = \inf_{A \in \Lambda_j} \sup_{x \in A} f(x),$$

where  $\Lambda_j = \{A \subset M : \text{cat}_M(A) \geq j \text{ and } A \text{ is compact}\}$ . If  $\Lambda_k \neq \emptyset$  for some  $k \geq 1$  and if  $f$  satisfies  $(PS)_c$  for all  $c = c_j$ ,  $j = 1, \dots, k$ , then  $f$  has at least  $k$  distinct critical points.

*Proof.* — Assume that  $M$  is connected. This causes no loss of generality because if  $M = \bigcup_i M_i$ , where  $M_i$  are the connected components of  $M$ , then

it follows from the definition of category that  $\text{cat}_M(A) = \sum_i \text{cat}_{M_i}(A \cap M_i)$ .

Since  $\Lambda_{j+1} \subset \Lambda_j$  for  $j = 1, \dots, k-1$  and the sets  $A \in \Lambda_j$  are compact,

$$-\infty < c_1 \leq c_2 \leq \dots \leq c_k < +\infty.$$

Given  $j$ , suppose  $c_j = \dots = c_{j+p} \equiv c$  for some  $p \geq 0$ . It suffices to show that

$$\text{cat}_M(K_c) \geq p + 1. \quad (3)$$

Indeed, it follows from (3) that  $\text{cat}_M(K_{c_j}) \geq 1$ , so  $K_{c_j} \neq \emptyset$ . This gives the correct number of critical points if all  $c_j$  are distinct. If they are not,  $p > 0$  for some  $j$ . Therefore  $\dim(K_{c_j}) \geq \text{cat}_M(K_{c_j}) - 1 \geq 1$  according to (h) of Proposition 2.2, so  $K_{c_j}$  is an infinite set.

Let  $b > c_k$  be a real number. Define

$$\Gamma_j = \{A \subset f_b : \text{cat}_M(A) \geq j \text{ and } A \text{ is compact}\}.$$

It is easy to see that

$$c_j = \inf_{A \in \Gamma_j} \sup_{x \in A} f(x).$$

Let  $\mathcal{S}$  be the collection of all nonempty closed and bounded subsets of  $f_b$ . In  $\mathcal{S}$  we introduce the Hausdorff metric  $dist$  [12] (§15. VII) given by

$$dist(A, B) = \max \left\{ \sup_{a \in A} \rho(a, B), \sup_{b \in B} \rho(b, A) \right\},$$

where  $\rho(a, B) = \inf_{b \in B} \rho(a, b)$ . Since  $f_b$  is complete, so is the space  $(\mathcal{S}, dist)$  [12] (§29. IV).

In order to continue the proof we shall need two lemmas.

3. 2. LEMMA. —  $(\Gamma_j, dist)$  is a complete metric space.

*Proof.* — It suffices to show that  $\Gamma_j$  is closed in  $\mathcal{S}$ . Let  $(A_n)$  be a sequence in  $\Gamma_j$  and let  $A_n \rightarrow A$ . It is easy to see that  $A$  is compact. Let  $U$  be a neighbourhood of  $A$  in  $M$  such that  $cat_M(U) = cat_M(A)$  [cf. Proposition 2. 2 (g)]. Since  $A_n \rightarrow A$ ,  $A_n \subset U$  for almost all  $n$ . Hence  $cat_M(A) = cat_M(U) \geq cat_M(A_n) \geq j$ , so  $A \in \Gamma_j$ .  $\square$

3. 3. LEMMA. — The function  $\Pi: \Gamma_j \rightarrow \mathbb{R}$  defined by

$$\Pi(A) = \sup_{x \in A} f(x)$$

is lower semicontinuous.

*Proof.* — Let  $A_n \rightarrow A$ . For each  $x \in A$  there is a sequence  $(x_n)$  such that  $x_n \rightarrow x$  and  $x_n \in A_n$ . Therefore

$$f(x) = \lim f(x_n) \leq \liminf \Pi(A_n).$$

Since  $x$  was chosen arbitrarily,  $\Pi(A) \leq \liminf \Pi(A_n)$ .  $\square$

*Proof of Theorem 3. 1 continued.* — Recall that we want to show that  $cat_M(K_c) \geq p + 1$  [cf. (3)]. Suppose  $cat_M(K_c) \leq p$ . Denote

$$N_\delta(K_c) = \{x \in M: \rho(x, K_c) \leq \delta\}.$$

Since  $f$  satisfies (PS)<sub>c</sub>,  $K_c$  is compact. It is therefore possible [via Proposition 2. 2 (g)] to choose  $\delta > 0$  so that  $cat_M(N_{2\delta}(K_c)) = cat_M(K_c) \leq p$ .

Let  $k \in \left(1, \frac{3}{2}\right)$  be a fixed number. Using (PS)<sub>c</sub>, we may find an arbitrarily small  $\varepsilon > 0$  with the property that

$$\|df(x)\| \geq 6\varepsilon, \quad \forall x \in f^{-1}([c - \varepsilon, c + \varepsilon]) - N_\delta(K_c). \tag{4}$$

Suppose  $\varepsilon < \delta < 1$ . Choose an  $A_1 \in \Gamma_{j+p}$  such that  $\Pi(A_1) \leq c + \varepsilon^2$ . Let  $A_2 = A_1 - N_{2\delta}(K_c)$ . Then  $\Pi(A_2) \leq c + \varepsilon^2$  and, by Proposition 2. 2,  $cat_M(A_2) \geq cat_M(A_1) - cat_M(N_\delta(K_c)) \geq j + p - p = j$ . So  $A_2 \in \Gamma_j$ . By Propo-

sition 2.4 and Lemmas 3.2 and 3.3, there is an  $A \in \Gamma_j$  such that

$$\Pi(A) \leq c + \varepsilon^2, \quad \text{dist}(A, A_2) \leq \varepsilon$$

and

$$\Pi(B) - \Pi(A) \geq -\varepsilon \text{dist}(A, B), \quad \forall B \in \Gamma_j. \quad (5)$$

Since  $\varepsilon < \delta$  and  $\text{dist}(A, A_2) \leq \varepsilon$ ,

$$A \cap N_\delta(K_c) = \emptyset. \quad (6)$$

Our goal now is to obtain a contradiction by constructing a  $B \in \Gamma_j$  which will fail to satisfy (5). Denote

$$S = A \cap \left\{ x \in M : f(x) \geq c - \frac{1}{2}\varepsilon \right\}.$$

Then  $S \neq \emptyset$  because  $A \in \Gamma_j$ . Given  $x_i \in S$ , choose a chart  $\varphi_i : U_i \rightarrow T_{x_i}(M)$  at  $x_i$  such that

$$\frac{1}{k} \| \|_x \leq \| \|_{x_i} \leq k \| \|_x, \quad \forall x \in U_i. \quad (7)$$

It follows from (6) and Proposition 2.1 that if  $U_i$  is sufficiently small, then

$$U_i \subset f^{-1}([c - \varepsilon, c + \varepsilon]) - N_\delta(K_c) \quad (8)$$

and there exists a vector  $v_i$  satisfying (2)  $\forall y \in \varphi(U_i)$ . Let  $V_i \subset U_i$  be an open neighbourhood of  $x_i$  such that

$$\rho(x, M - U_i) \geq \delta_i \quad \text{and} \quad \rho_i(\varphi_i(x), T_{x_i}(M) - \varphi_i(U_i)) \geq \delta_i, \quad \forall x \in V_i,$$

where  $\delta_i > 0$  and

$$\rho_i(\varphi_i(x), T_{x_i}(M) - \varphi_i(U_i)) = \inf \{ \| \varphi_i(x) - z \|_{x_i} : z \in T_{x_i}(M) - \varphi_i(U_i) \}.$$

Proceeding in this way for each  $x_i \in S$  we obtain an open covering  $(V_i)$  of  $S$ . Since  $S$  is compact, there exists a finite subcovering  $V_1, \dots, V_m$ , to which we may subordinate a continuous partition of unity  $\xi_1, \dots, \xi_m$ . Let  $\chi : M \rightarrow [0, 1]$  be a continuous function such that  $\chi \equiv 1$  on  $S$  and  $\chi \equiv 0$  on  $M - \bigcup_{i=1}^m V_i$ , and let  $\psi_i = \chi \xi_i$ . The sets  $U_1, \dots, U_m$  cover  $S$ , and by construction,

$$\sum_{i=1}^m \psi_i(x) \equiv 1 \quad \text{on } S, \quad \psi_1 = \dots = \psi_m \equiv 0 \quad \text{on } M - \bigcup_{i=1}^m V_i$$

and

$$\begin{aligned} \rho(x, M - U_i) \geq \delta_0, \quad \rho_i(\varphi_i(x), T_{x_i}(M) - \varphi_i(U_i)) \geq \delta_0 \\ \forall x \in \text{supp } \psi_i, \end{aligned} \tag{9}$$

where  $\delta_0 = \min \{ \delta_1, \dots, \delta_m \}$ .

Fix a number  $t \in (0, \delta_0 / (1 + k^2))$  and let

$$\alpha_1(t, x) = \begin{cases} \varphi_1^{-1} \left( \varphi_1(x) - t \psi_1(x) \frac{v_1}{\|v_1\|} \right) & \text{if } x \in U_1 \\ x & \text{otherwise.} \end{cases}$$

According to (9),  $\alpha_1$  is well defined and continuous. For an arbitrary point  $x \in U_1$ , let  $\sigma_1(s) = \alpha_1(s, x)$ ,  $0 \leq s \leq t$ . Since  $\sigma_1$  is a path joining  $x$  to  $\alpha_1(t, x)$ ,

$$\rho(x, \alpha_1(t, x)) \leq \int_0^t \|\sigma_1'(s)\| ds \leq k \int_0^t \left\| \frac{d}{ds} \varphi_1(\sigma_1(s)) \right\|_{x_1} ds = k \psi_1(x) t \tag{10}$$

according to (7) and the definition of  $\alpha_1$ . Denote  $g = f \circ \varphi_1^{-1}$  and let  $x \in U_1$ . Then it follows from the mean value theorem and Proposition 2.1 that for some  $\theta \in (0, 1)$ ,

$$\begin{aligned} f \circ \alpha_1(t, x) - f(x) &= g \left( \varphi_1(x) - t \psi_1(x) \frac{v_1}{\|v_1\|} \right) - g(\varphi_1(x)) \\ &= -t \psi_1(x) \left\langle g' \left( \varphi_1(x) - \theta t \psi_1(x) \frac{v_1}{\|v_1\|} \right), \frac{v_1}{\|v_1\|} \right\rangle \leq -\frac{1}{4} t \psi_1(x) \|v_1\| \\ &\leq -\frac{1}{4} t \psi_1(x) \|df(x_1)\|. \end{aligned} \tag{11}$$

Therefore, employing (8) and (4),

$$f \circ \alpha_1(t, x) - f(x) \leq -\frac{3}{2} \varepsilon t \psi_1(x). \tag{12}$$

Note that (10) and (12) are satisfied for all  $x \in M$  because  $\psi_1(x) = 0$  whenever  $x \notin U_1$ . Note also that  $\alpha_1(t, A) \in \Gamma_j$  according to (f) of Proposition 2.2.

Let

$$\alpha_2(t, x) = \begin{cases} \varphi_2^{-1} \left( \varphi_2(\alpha_1(t, x)) - t \psi_2(x) \frac{v_2}{\|v_2\|} \right) & \text{if } \alpha_1(t, x) \in U_2 \\ \alpha_1(t, x) & \text{otherwise.} \end{cases}$$

We need to show that  $\alpha_2$  is well defined and continuous. Let  $x \in \text{supp } \psi_2$  and let  $\sigma$  be a path joining  $x$  to  $\alpha_1(t, x)$ . Since  $\rho(x, M - U_2) \geq \delta_0$  and

$$\rho(x, \alpha_1(t, x)) \leq kt < \frac{\delta_0 k}{1+k^2} \leq \frac{1}{2} \delta_0 \quad (13)$$

according to (9) and (10),  $\alpha_1(t, x) \in U_2$  and  $l(\sigma) \geq \delta_0$  if  $\sigma$  leaves  $U_2$ . Therefore

$$\rho(x, \alpha_1(t, x)) = \inf \{ l(\sigma) : \sigma \text{ joins } x \text{ to } \alpha_1(t, x) \text{ and } \sigma \subset U_2 \}.$$

Furthermore, if  $\sigma \subset U_2$ ,

$$\begin{aligned} l(\sigma) &= \int_a^b \|\sigma'(s)\| ds \geq \frac{1}{k} \int_a^b \left\| \frac{d}{ds} \varphi_2(\sigma(s)) \right\|_{x_2} ds \\ &\geq \frac{1}{k} \left\| \int_a^b \frac{d}{ds} \varphi_2(\sigma(s)) ds \right\|_{x_2} = \frac{1}{k} \|\varphi_2(\alpha_1(t, x)) - \varphi_2(x)\|_{x_2}. \end{aligned}$$

Combining these two facts,

$$\|\varphi_2(\alpha_1(t, x)) - \varphi_2(x)\|_{x_2} \leq k \rho(x, \alpha_1(t, x)) \leq k^2 t.$$

Hence by the triangle inequality,

$$\begin{aligned} \left\| \varphi_2(\alpha_1(t, x)) - t \psi_2(x) \frac{v_2}{\|v_2\|} - \varphi_2(x) \right\|_{x_2} &\leq \|\varphi_2(\alpha_1(t, x)) - \varphi_2(x)\|_{x_2} \\ &\quad + \left\| t \psi_2(x) \frac{v_2}{\|v_2\|} \right\|_{x_2} \leq k^2 t + t < \delta_0. \end{aligned}$$

So it follows from (9) that  $\varphi_2(\alpha_1(t, x)) - t \psi_2(x) \frac{v_2}{\|v_2\|} \in \varphi(U_2)$  and  $\alpha_2$  is well defined. If  $\alpha_1(t, x)$  is sufficiently close to the boundary of  $U_2$ , then  $x \notin \text{supp } \psi_2$  according to (9) and (13). Hence for such  $x$ ,  $\alpha_2(t, x) = \alpha_1(t, x)$ . Therefore  $\alpha_2$  is continuous. Set

$$\sigma_2(s) = \varphi_2^{-1} \left( \varphi_2(\alpha_1(t, x)) - s \psi_2(x) \frac{v_2}{\|v_2\|} \right), \quad 0 \leq s \leq t.$$

Then  $\rho(\alpha_1(t, x), \alpha_2(t, x)) \leq l(\sigma_2) \leq k \psi_2(x) t$  [cf. the argument of (10)]. This and (10) yield

$$\rho(x, \alpha_2(t, x)) \leq k(\psi_1(x) + \psi_2(x)) t.$$

The same argument as in (11) and (12) implies that

$$f \circ \alpha_2(t, x) - f \circ \alpha_1(t, x) \leq -\frac{3}{2} \varepsilon t \psi_2(x).$$

So by (12),

$$f \circ \alpha_2(t, x) - f(x) \leq -\frac{3}{2} \varepsilon t (\psi_1(x) + \psi_2(x)).$$

Since  $\alpha_2(t, A)$  was obtained from  $\alpha_1(t, A)$  by continuous deformation,  $\alpha_2(t, A) \in \Gamma_j$ .

Proceeding as above, we eventually define

$$\alpha_m(t, x) = \begin{cases} \Phi_m^{-1} \left( \Phi_m(\alpha_{m-1}(t, x)) - t \Psi_m(x) \frac{v_m}{\|v_m\|} \right) & \text{if } \alpha_{m-1}(t, x) \in U_m \\ \alpha_{m-1}(t, x) & \text{otherwise,} \end{cases}$$

and show that

$$\rho(x, \alpha_m(t, x)) \leq k(\psi_1(x) + \dots + \psi_m(x)) t \leq kt, \tag{14}$$

$$f \circ \alpha_m(t, x) - f(x) \leq -\frac{3}{2} \varepsilon t (\psi_1(x) + \dots + \psi_m(x)) \tag{15}$$

and  $\alpha_m(t, A) \in \Gamma_j$ . Let  $B = \alpha_m(t, A)$ . By (14),  $dist(A, B) \leq kt$ . Since  $\Pi(B) \geq c$  and  $f \circ \alpha_m(t, x) \leq f(x)$ ,

$$\sup_{x \in A} f \circ \alpha_m(t, x) = \sup_{x \in S} f \circ \alpha_m(t, x). \tag{16}$$

Recall that  $k < \frac{3}{2}$  and  $\psi_1(x) + \dots + \psi_m(x) = 1$  on  $S$ . Using this, (5), (15)

and (16), we obtain

$$-\frac{3}{2} \varepsilon t < -\varepsilon kt \leq -\varepsilon dist(A, B) \leq \Pi(B) - \Pi(A)$$

$$= \sup_{x \in S} f \circ \alpha_m(t, x) - \sup_{x \in S} f(x) \leq \sup_{x \in S} (f \circ \alpha_m(t, x) - f(x)) \leq -\frac{3}{2} \varepsilon t,$$

a contradiction.  $\square$

3.4. REMARK. — Condition  $(PS)_c$  in Theorem 3.1 may be replaced with the following weaker one: If there is a sequence  $(x_n)$  such that  $f(x_n) \rightarrow c$  and  $\|df(x_n)\| \rightarrow 0$ , then  $c$  is a critical value and  $\inf_{n \in \mathbb{N}} \rho(x_n, K_c) = 0$ . This is

seen by verifying that the previous argument applies if (3) is modified to

$$(3') \quad \text{either } K_c \text{ is not compact or } cat_{\mathbb{M}}(K_c) \geq p + 1.$$

A still weaker (but insufficient for our purposes) version of  $(PS)_c$  has been introduced in [4]. It says that  $c$  is a critical value whenever there exists a sequence  $(x_n)$  such that  $f(x_n) \rightarrow c$  and  $\|df(x_n)\| \rightarrow 0$ .

#### 4. RELATED RESULTS

4.1. COROLLARY. — Suppose that  $M$  is a closed symmetric  $C^1$ -submanifold of a real Banach space  $X$  and  $0 \notin M$ . Suppose also that  $f \in C^1(M, \mathbb{R})$  is even and bounded below. Define

$$c_j = \inf_{A \in \Gamma_j} \sup_{x \in A} f(x),$$

where  $\Gamma_j = \{A \subset M : A \in \Sigma, \gamma(A) \geq j \text{ and } A \text{ is compact}\}$ . If  $\Gamma_k \neq \emptyset$  for some  $k \geq 1$  and if  $f$  satisfies  $(PS)_c$  for all  $c = c_j, j = 1, \dots, k$ , then  $f$  has at least  $k$  distinct pairs of critical points.

*Proof.* — Let  $\tilde{M} = M/\sim$ , where  $\sim$  is the equivalence relation identifying  $x$  with  $-x$ , and let  $\tilde{f}: \tilde{M} \rightarrow \mathbb{R}$  be the function induced by  $f$ . It is clear that  $\tilde{M}$  and  $\tilde{f}$  satisfy the hypotheses of Theorem 3.1. Furthermore, if  $A \in \Gamma_k$ , then, setting  $\tilde{X} = (X - \{0\})/\sim$  and  $\tilde{A} = A/\sim$ ,  $\text{cat}_{\tilde{X}}(\tilde{A}) = \gamma(A) \geq k$  [18] (Theorem 3.7). Since  $\tilde{M} \subset \tilde{X}$ , it follows from the definition of category that  $\text{cat}_{\tilde{M}}(\tilde{A}) \geq \text{cat}_{\tilde{X}}(\tilde{A}) \geq k$ . So  $\tilde{A} \in \Lambda_k$ . Hence  $\tilde{f}$  possesses at least  $k$ , and  $f$  at least  $k$  pairs of critical points.  $\square$

A different (and perhaps more natural) proof of the corollary may be obtained by modifying the argument of Theorem 3.1 (category should be replaced with genus and the mappings  $\alpha_i$  should be odd in  $u$ ).

The assumption that  $f$  is bounded below in Theorem 3.1 was used only in order to assure that  $c_1 > -\infty$ . It is therefore easy to see that the following stronger results are valid.

4.2. COROLLARY. — Suppose that  $M$  is a  $C^1$  Finsler manifold and  $f \in C^1(M, \mathbb{R})$  is such that  $f_c$  is complete in the metric  $\rho$  for each  $c \in \mathbb{R}$ . Let  $c_j$  and  $\Lambda_j$  be defined as in Theorem 3.1. If  $\Lambda_k \neq \emptyset$  for some  $k \geq 1$ ,  $c_m > -\infty$  for some  $m$ ,  $1 \leq m \leq k$ , and  $(PS)_c$  is satisfied for all  $c = c_j, m \leq j \leq k$ , then  $f$  has at least  $k - m + 1$  distinct critical points.

4.3. COROLLARY. — Suppose that  $M$  is a closed symmetric  $C^1$ -submanifold of a real Banach space  $X$  and  $0 \notin M$ . Suppose also that  $f \in C^1(M, \mathbb{R})$  is an even function. Let  $c_j$  and  $\Gamma_j$  be defined as in Corollary 4.1. If  $\Gamma_k \neq \emptyset$  for some  $k \geq 1$ ,  $c_m > -\infty$  for some  $m$ ,  $1 \leq m \leq k$ , and  $(PS)_c$  is satisfied for all  $c = c_j, m \leq j \leq k$ , then  $f$  has at least  $k - m + 1$  distinct pairs of critical points.

A function  $f: X \rightarrow \mathbb{R}$ , where  $X$  is a Banach space, is said to be Gâteaux differentiable if for each  $x \in X$  there exists a linear mapping  $f'(x) \in X^*$  such

that

$$\frac{d}{dt}f(x+ty)|_{t=0} = \langle f'(x), y \rangle, \quad \forall y \in X. \quad (17)$$

Let us remark that there is a different—weaker—definition of Gâteaux differentiability in which it is not required that the left-hand side of (17) be linear in  $y$  (see e. g. [3], [8]). A function  $f: M \rightarrow \mathbb{R}$ , where  $M$  is a Finsler manifold, will be called *Gâteaux differentiable* if for each  $x \in M$  and each chart  $\varphi: U \rightarrow T_x(M)$  at  $x$ ,  $f \circ \varphi^{-1}$  is Gâteaux differentiable. The Gâteaux derivative  $df$  is *strong-to-weak\** continuous if for each  $x \in M$ , each sequence  $x_n \rightarrow x$  and each chart  $\varphi: U \rightarrow T_x(M)$  at  $x$ ,

$$(f \circ \varphi^{-1})'(\varphi(x_n)) \rightarrow (f \circ \varphi^{-1})'(\varphi(x))$$

in the weak\* topology of  $T_x(M)^*$ .

4. 4. REMARK. — Theorem 3. 1 and Corollaries 4. 1-4. 3 remain valid if  $f$ , instead of being  $C^1$ , is continuous and Gâteaux differentiable with the derivative strong-to-weak\* continuous. This follows by observing that in the proofs of Proposition 2. 1 and Theorem 3. 1 only the above weaker smoothness assumption has been used.

## 5. AN APPLICATION

Let  $\Omega \subset \mathbb{R}^N$  be a bounded domain with smooth boundary  $\partial\Omega$  and let  $f, g$  be two continuous real-valued functions. Fix a number  $p \in (1, \infty)$  and denote

$$F(t) = \int_0^t f(s) ds \quad \text{and} \quad G(t) = \int_0^t g(s) ds.$$

We will be concerned with the following eigenvalue problem: *Given  $b \in \mathbb{R}$ , find a function  $u$  and a real number  $\lambda$  such that*

$$-\operatorname{div}(|\nabla u|^{p-2} \nabla u) + f(u) = \lambda g(u) \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0$$

and

$$\frac{1}{p} \int_{\Omega} |\nabla u|^p dx + \int_{\Omega} F(u) dx = b. \quad (18)$$

Suppose that  $f$  and  $g$  satisfy the growth restriction

$$|f(t)|, |g(t)| \leq a_1 + a_2 |t|^r, \quad (19)$$

where  $1 \leq r < \frac{Np}{N-p} - 1$  if  $N > p$ ,  $1 \leq r < \infty$  otherwise. Let  $H \equiv H_0^{1,p}(\Omega)$  be the usual Sobolev space (of real-valued functions) with the norm  $\|u\| = \left( \int_{\Omega} |\nabla u|^p \right)^{1/p}$ . Define

$$\Phi(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p dx + \int_{\Omega} F(u) dx \equiv \frac{1}{p} \|u\|^p + \Phi_1(u),$$

$$\Psi(u) = - \int_{\Omega} G(u) dx.$$

The following result follows from standard arguments in Sobolev spaces.

5.1. PROPOSITION. — Suppose that  $f, g$  satisfy (19). Then

(i)  $\Phi, \Psi \in C^1(H, \mathbb{R})$  and

$$\langle \Phi'(u), v \rangle = \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v dx + \int_{\Omega} f(u) v dx,$$

$$\langle \Psi'(u), v \rangle = - \int_{\Omega} g(u) v dx, \quad \forall v \in H.$$

(ii)  $\Phi'_1$  and  $\Psi'$  are completely continuous (i. e., they map weakly convergent sequences to strongly convergent ones).

(iii)  $\Phi_1$  and  $\Psi$  are continuous with respect to weak convergence in  $H$ .

The proof for  $p=2$  may be found e. g. in [19] (Appendix B). If  $p \neq 2$ , the argument of [19] applies upon observing that the Sobolev embedding  $H \hookrightarrow L^{p+1}(\Omega)$  is compact.

5.2. LEMMA. — (i)  $\Phi'$  maps bounded sets to bounded sets.

(ii) If  $u_n \rightarrow \bar{u}$  weakly in  $H$  and  $\Phi'(u_n)$  converges strongly, then  $u_n \rightarrow \bar{u}$  strongly.

*Proof.* — Let  $A : H \rightarrow H^*$  be the mapping given by

$$\langle A u, v \rangle = \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v dx.$$

Then  $\Phi' = A + \Phi'_1$ . Since  $|\langle A u, v \rangle| \leq \|u\|^{p-1} \|v\|$  and  $\Phi'_1$  is completely continuous, the conclusion follows.

(ii) It is easy to verify that there is a constant  $\alpha > 0$  such that

$$\langle Au - Av, u - v \rangle \geq \alpha \|u - v\|^p. \tag{20}$$

Suppose that  $u_n \rightarrow \bar{u}$  weakly and  $\Phi'(u_n)$  is strongly convergent. Then also  $Au_n$  is strongly convergent. It follows therefore from (20) with  $u = u_n$  and  $v = \bar{u}$  that  $u_n \rightarrow \bar{u}$  strongly.  $\square$

A much more general version of Proposition 5.1 and Lemma 5.2 may be found in [5].

Suppose that  $b$  is a regular value of  $\Phi$ . Then  $M = \Phi^{-1}(b)$  is a  $C^1$ -manifold and  $u \in M$  is a critical point of  $\tilde{\Psi} = \psi|_M$  if and only if  $\psi'(u) = \mu \Phi'(u)$  for some  $\mu \in \mathbb{R}$ . Assume also that  $\psi'(u) \neq 0$  on  $M$ . It follows that  $\mu \neq 0$  for such  $\psi$  and, according to Proposition 5.1, there is a one-to-one correspondence between critical points of  $\tilde{\Psi}$  and weak solutions of (18) [ $\lambda = \mu^{-1}$  in (18)].

5.3. THEOREM. — *Suppose that  $f, g \in C(\mathbb{R}, \mathbb{R})$  are two odd functions satisfying (19). Suppose also that  $G(t) > 0$  for almost all  $t$  and there exist positive constants  $d_1, d_2, d_3$  such that  $F(t) \geq -d_1|t|^p - d_2$  and  $G(t) \geq -F(t) - d_3$ . If  $b > 0$  is a regular value of  $\Phi$ , then (18) has infinitely many weak solutions.*

Note that under the hypotheses of Theorem 5.3  $\Phi$  need not be in  $C^{2-}(\mathbb{H}, \mathbb{R})$ . In particular, if  $1 < p < 2$ , then  $\Phi'$  cannot be locally Lipschitz continuous [cf. (20)]. If  $p \geq 2$  and  $f \in C^1(\mathbb{R}, \mathbb{R})$ , then  $\Phi \in C^2(\mathbb{H}, \mathbb{R})$  provided  $f'$  satisfies the growth restriction

$$|f'(t)| \leq a_4 + a_5|t|^{r-1},$$

where  $r$  is as in (19). Note also that under the present hypotheses  $M = \Phi^{-1}(b)$  need neither be bounded nor radially homeomorphic to the unit sphere in  $\mathbb{H}$ .

*Proof of Theorem 5.3.* — We shall keep the notation introduced earlier in this section. First we show that  $\psi'(u) \neq 0$  if  $u \neq 0$ . Assume without loss of generality that  $\text{ess sup } u > 0$ . Since  $G(t) > 0$  for almost all  $t$ , there exist  $\beta > \alpha > 0$  such that  $\text{meas } \{x \in \Omega : u(x) \geq \beta\} > 0$ ,  $\text{meas } \{x \in \Omega : u(x) \leq \alpha\} > 0$  and  $g(t) > 0$  on  $[\alpha, \beta]$ . By [7] (Theorem 1),  $\text{meas } \{x \in \Omega : \alpha \leq u(x) \leq \beta\} > 0$ . Hence  $g \circ u > 0$  on a set of positive measure. So  $\psi'(u) \neq 0$  and there is a one-to-one correspondence between critical points of  $\tilde{\Psi}$  and weak solutions of (18).

It is clear that  $M$  is symmetric. We claim that it contains compact subsets of arbitrarily large genus, i. e.,  $\Gamma_k \neq \emptyset$  for any  $k \geq 1$  ( $\Gamma_k$  was defined in Corollary 4.1). Since  $\mathbb{H}$  is separable, there exists a biorthogonal system

$(e_m, e_n^*)_{m, n \in \mathbb{N}}$  such that  $e_m \in H$ ,  $e_n^* \in H^*$ , the  $e_m$ 's are linearly dense in  $H$  and the  $e_n^*$ 's are total for  $H$  [13] (Proposition 1. f. 3). Let us remark that we could in particular choose  $(e_m)_{m \in \mathbb{N}}$  to be a Schauder basis for  $H$  [which exists according to [22] (Section 4. 9. 4)], and then find biorthogonal functionals  $e_n^*$ . Denote

$$H_m = \text{span} \{ e_1, \dots, e_m \}, \quad H_m^\perp = \text{cl } \text{span} \{ e_{m+1}, e_{m+2}, \dots \}$$

(cl is the closure). If  $m$  is large enough,

$$\inf \left\{ \frac{1}{p} \|u\|^p - d_1 \int_\Omega |u|^p dx : u \in H_m^\perp, \|u\| = 1 \right\} > 0. \tag{21}$$

Indeed, otherwise there would exist a sequence  $(u_m)$  such that  $u_m \in H_m^\perp$ ,  $\|u_m\| = 1$  and

$$\frac{1}{p} \|u_m\|^p - d_1 \int_\Omega |u_m|^p dx \leq \frac{1}{m}. \tag{22}$$

Since  $\langle e_n^*, u_m \rangle = 0$  for all  $m \geq n$  and the  $e_n^*$ 's are total,  $u_m \rightarrow 0$  weakly in  $H$ . Therefore  $u_m \rightarrow 0$  strongly in  $L^p(\Omega)$ , a contradiction to (22). It follows from (21) and the assumption on  $F$  that for some  $\alpha > 0$ ,

$$\Phi(u) \geq \frac{1}{p} \|u\|^p - \int_\Omega (d_1 |u|^p + d_2) dx \geq \alpha \|u\|^p - \int_\Omega d_2 dx, \quad \forall u \in H_m^\perp.$$

In particular, the sets  $H_m^\perp \cap M$  and  $H_m^\perp \cap \Phi_b$  are bounded. Let  $E_k = \text{span} \{ e_{m+1}, \dots, e_{m+k} \}$ . Then  $\dim E_k = k$  and  $E_k \cap \Phi_b$  is a bounded and symmetric neighbourhood of  $0 \in E_k$ . Therefore  $\gamma(E_k \cap M) = k$  according to (f) of Proposition 2. 3. Since  $E_k \cap M$  is compact,  $\Gamma_k \neq \emptyset$ .

Let  $j > m$ . Since  $G(t) > 0$  for almost all  $t$ ,  $G \circ u > 0$  on a set of positive measure for any  $u \in M$  (cf. the argument at the beginning of the proof). Therefore  $\tilde{\Psi} < 0$  on  $M$  and

$$c_j = \inf_{A \in \Gamma_j} \sup_{u \in A} \tilde{\Psi}(u) < 0. \tag{23}$$

Let  $A \in \Gamma_j$ . Then  $A \cap H_m^\perp \neq \emptyset$  by (g) of Proposition 2. 3. Since  $H_m^\perp \cap M$  is a bounded set,  $\tilde{\Psi}|_{H_m^\perp \cap M}$  is bounded below. Hence  $c_j > -\infty$  whenever  $j > m$ .

We shall show that  $\tilde{\Psi}$  satisfies  $(PS)_c$  for any  $c < 0$ . The conclusion of the theorem will then follow from Corollary 4. 3. Suppose  $\tilde{\Psi}(u_n) \rightarrow c < 0$  and  $\|d\tilde{\Psi}(u_n)\| \rightarrow 0$ . Since

$$\tilde{\Psi}(u) = - \int_\Omega G(u) dx \leq \int_\Omega (F(u) + d_3) dx = b + \int_\Omega d_3 dx - \frac{1}{p} \|u\|^p, \tag{24}$$

the sequence  $(u_n)$  is bounded. Assume after passing to a subsequence that  $u_n \rightarrow \bar{u}$  weakly in  $H$ . Since  $\tilde{\Psi}(\bar{u}) = c < 0$  [by (iii) of Proposition 5.1],  $\bar{u} \neq 0$ . Let  $J : H \rightarrow H^*$  be the duality mapping [see e. g. [6] (Section 4)]. Recall that  $J$  and  $J^{-1}$  are continuous,  $\|Ju\| = \|u\|$  and  $\langle Ju, u \rangle = \|u\|^2, \forall u \in H$ . Define the projection mapping  $P_u : H \rightarrow T_u(M)$  by

$$P_u v = v - \frac{\langle \Phi'(u), v \rangle}{\|\Phi'(u)\|^2} J^{-1} \Phi'(u).$$

Note that  $\langle \psi'(u), v \rangle = \langle d\tilde{\Psi}(u), v \rangle$  whenever  $v \in T_u(M)$ . Since

$$|\langle \psi'(u), P_u v \rangle| = |\langle d\tilde{\Psi}(u), P_u v \rangle| \leq \|d\tilde{\Psi}(u)\| \|P_u v\| \leq 2 \|d\tilde{\Psi}(u)\| \|v\|$$

and  $d\tilde{\Psi}(u_n) \rightarrow 0$ ,

$$\sup_{\|v\| \leq 1} \left\{ \langle \psi'(u_n), v \rangle - \frac{\langle \Phi'(u_n), v \rangle}{\|\Phi'(u_n)\|^2} \langle \psi'(u_n), J^{-1} \Phi'(u_n) \rangle \right\} \rightarrow 0.$$

Therefore

$$\psi'(u_n) - \frac{\langle \psi'(u_n), J^{-1} \Phi'(u_n) \rangle}{\|\Phi'(u_n)\|^2} \Phi'(u_n) \rightarrow 0. \tag{25}$$

Since  $\psi'(u_n) \rightarrow \psi'(\bar{u}) \neq 0$  and  $\Phi'(u_n)$  is bounded [by (i) of Lemma 5.2],  $\Phi'(u_n)$  (or a subsequence of it) is strongly convergent. By (ii) of Lemma 5.2,  $u_n \rightarrow \bar{u}$  strongly.  $\square$

5.4. REMARKS. — (i) It is well known that functionals of the type considered here do not satisfy  $(PS)_0$ . To see this for  $\tilde{\Psi}$ , let  $(u_n) \subset M$  be a sequence converging weakly to 0. Then  $\tilde{\Psi}(u_n) \rightarrow 0$  and  $\psi'(u_n) \rightarrow 0$ . Since  $0 \notin M$ , no subsequence of  $(u_n)$  converges strongly, and it follows from (ii) of Lemma 5.2 that  $\Phi'(u_n) \neq 0$  for almost all  $n$ . Therefore  $d\tilde{\Psi}(u_n) \rightarrow 0$  (recall that  $d\tilde{\Psi}(u_n) \rightarrow 0$  if and only if (25) is satisfied).

(ii) The numbers  $c_j$  in (23) tend to zero as  $j \rightarrow \infty$ . Indeed, observe that  $A \cap H_{j-1}^+ \neq \emptyset$  for any  $A \in \Gamma_j$  [by (g) of Proposition 2.3]. Furthermore,

$$\epsilon_j = \inf \{ \tilde{\Psi}(u) : u \in H_j^+ \cap M \} \rightarrow 0 \quad \text{as } j \rightarrow \infty$$

[because if  $(u_j)$  is a sequence such that  $u_j \in H_j^+ \cap M$ , then  $u_j \rightarrow 0$  weakly in  $H$ ; therefore  $\tilde{\Psi}(u_j) \rightarrow 0$ ]. Since  $\epsilon_j \leq c_j < 0, c_j \rightarrow 0$ .

(iii) Suppose that  $f, g \in C(\mathbb{R}, \mathbb{R})$  are odd, satisfy (19),  $G(t) > 0$  for almost all  $t$  and  $G(t) \geq -F(t) - d_3$ . If  $b < 0$  is a regular value of  $\Phi$ , then (18) has at least  $k$  pairs of weak solutions provided  $M = \Phi^{-1}(b)$  contains a compact subset of genus  $k$ . The proof is obtained by applying Corollary 4.1 to the functional  $-\tilde{\Psi}$ . Since  $-\tilde{\Psi}$  may not satisfy  $(PS)_0$ , we must show that  $c_j > 0$ . Suppose  $\tilde{\Psi}(u_n) \rightarrow 0$ . By (24), the sequence  $(u_n)$  is bounded, so we may

assume that  $u_n \rightarrow \bar{u}$  weakly in  $H$ . Since

$$\Phi(u_n) = \frac{1}{p} \|u_n\|^p + \Phi_1(u_n) = b < 0$$

and  $\Phi$  is weakly lower semicontinuous,  $\Phi(\bar{u}) \leq b < 0$ . It follows that  $\bar{u} \neq 0$ . On the other hand,  $\tilde{\Psi}(u_n) \rightarrow \tilde{\Psi}(\bar{u}) = 0$ . So  $\bar{u} = 0$ . This contradiction shows that  $-\tilde{\Psi}$  is bounded away from 0. Therefore  $c_j > 0$ . A related result in a geometrically simpler situation (in which it is easy to compute  $k$ ) may be found in Zeidler [23] (Proposition 11).

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